4

An Asymptotically Optimal Substructuring Method for the Stokes Equation

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1 Introduction

In this paper, we propose and analyze an asymptotically optimal Schur complement interface reduction for the Stokes equation on plane polygonal domains. It is based on using special Poincaré-Steklov (PS) operators, see also [QV91]. We refer to [KW96] for the related results based on a coupling of the stream function-vorticity formulation and the decomposition approach from [GP79]. The multigrid methods of finite elements (FE) for the Stokes and Navier-Stokes equations have been considered, e.g. in [Wit89].

The main ingredient of our method is an appropriate factorization of the matrix-valued traction operator $\mathbf{S}_T^{-1} : \mathbf{u} \to (\sigma_{nn}, \sigma_{n\tau})^T$ which maps the trace of the velocity vector into the normal and shear stress components σ_{nn} and $\sigma_{n\tau}$. We introduce a symmetric and positive definite ($s.p.d.$) Poincaré-Steklov operator S_{st} for the Stokes equation, see (10), which maps the trace of the pressure into the normal velocity component under the constraints $u_{\tau|\Gamma} = div\mathbf{u}_{|\Gamma} = 0$. This interface operator admits a stable FE approximation providing an asymptotically optimal stiffness matrix compression. We study the mapping properties of the continuous PS operator and briefly discuss the corresponding discrete FE approximations. In the case of a rectangular domain, we apply the algorithm of the complexity $O(N \log^2 N)$ for the fast Schur complement matrix-vector multiplication, where N is the number of degrees of freedom on the (subdomain) boundary, see [KW97]. For domains composed of $M \geq 1$ rectangular substructures, our interface reduction is shown to have a complexity $O(MN \log^{q_r} N)$, where $q_r = 2$ for the multilevel BPX interface preconditioner [JHBX90] and $q_r = 3$ in the case of a BPS type [BPS86] preconditioner. Using an interface reduction by the refined skeleton in the case of polygonal boundaries, see [Kho96, KP95, KS96, KW96], yields an algorithm of the same complexity as above, where $q_r + 1$ must be substituted for q_r. The approach proposed may be extended to the $3D$ case.

Let $\Omega \in R^2$ be a bounded domain with either a smooth or convex polygonal

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boundary $\Gamma = \cup_{j=1}^{J} \Gamma_j$ composed of linear pieces Γ_j. For given $\alpha, \nu > 0$, $\mathbf{f} \in L^2(\Omega)^2$ and $\mathbf{g} \in \{\mathbf{u} \in H^{1/2}(\Gamma)^2 : (u_n, 1)_{L^2(\Gamma)} = 0\}$, consider the Stokes equation:
Find $(\mathbf{u}, p) \in X \times M$ *such that*

$$\begin{cases} \alpha\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} & in\, \Omega \in R^2 \\ div\mathbf{u} = 0 & in\ \ \Omega \\ \mathbf{u} = \mathbf{g} & on\ \Gamma\ , \end{cases} \tag{1}$$

where $M = L_0^2(\Omega) = \{p \in L^2(\Omega);\ \int\limits_\Gamma p dx = 0\}$, $X = H^1(\Omega)^2$.

For ease of presentation, consider the case $\alpha = 0$. Denote by $\mathbf{n} = (n_1, n_2)^T$ and $\tau = (-n_2, n_1)^T$ the unit outward normal and tangential vectors, respectively. We use the standard notations $X_0 = H_0^1(\Omega)^2$, $V = \{\mathbf{v} \in X : div\mathbf{v} = 0\}$ and $V_0 = V \cap X_0$ and define the continuous bilinear form $a : X \times X \to R$ by

$$a(\mathbf{u}, \mathbf{v}) := 2 \sum_{i,j=1}^{2} \int\limits_\Omega \varepsilon_{ij}(\mathbf{u}) : \varepsilon_{ij}(\mathbf{v}) dx, \qquad \varepsilon_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}). \tag{2}$$

The V_0-ellipticity of $a(\cdot,\cdot)$, the trace theorem and validity of the LBB *inf-sup* condition

$$\exists \beta > 0 : \qquad \sup_{\mathbf{v} \in X_0} \frac{(q, div\mathbf{v})_{L^2(\Omega)}}{|\mathbf{v}|_{1,\Omega}} \geq \beta \|q\|_{0,\Omega}, \quad \forall q \in M\ , \tag{3}$$

imply *a priori* estimate $\|\mathbf{u}\|_{1,\Omega} + \|p\|_{0,\Omega} \leq c\Big(\|\mathbf{f}\|_{-1,\Omega} + \|\mathbf{g}\|_{1/2,\Gamma}\Big)$, see [GR86, Lad69].

2 Poincaré-Steklov Operators for the Stokes equation

Introduce the Poincaré-Steklov (traction) operator

$$\mathbf{S}_T^{-1}\mathbf{u} \equiv \begin{pmatrix} S_{nn} & S_{\tau n}^T \\ S_{\tau n} & S_{\tau\tau} \end{pmatrix} \begin{pmatrix} u_n \\ u_\tau \end{pmatrix} := \begin{pmatrix} \sigma_{nn} \\ \sigma_{n\tau} \end{pmatrix} : X_{n\tau} \to X'_{n\tau},$$

by the identity

$$(\mathbf{S}_T^{-1}\mathbf{u}, \mathbf{v})_\Gamma := \langle \sigma_{nn}(\mathbf{u}), v_n \rangle_{L^2(\Gamma)} + \langle \sigma_{n\tau}(\mathbf{u}), v_\tau \rangle_{L^2(\Gamma)} = \nu a(\Upsilon\mathbf{u}, \Upsilon\mathbf{v}), \tag{4}$$

$\forall\ \mathbf{u}, \mathbf{v} \in X_{n\tau}(\Gamma)$, where $\Upsilon : X_{n\tau}(\Gamma) \to V$ is the Stokes solution (extension) operator defined by (1) with $\mathbf{f} = 0$. Here, $X_{n\tau}(\Gamma)$ is a trace space of the normal and tangential velocity components

$$X_{n\tau}(\Gamma) := \{\mathbf{v}_{n\tau} = \begin{pmatrix} v_n \\ v_\tau \end{pmatrix} : \mathbf{v} \in H^{1/2}(\Gamma)^2, (v_n, 1)_{L^2(\Gamma)} = 0\}, \quad \|\mathbf{v}_{n\tau}\|_{X_{n\tau}} = \|\mathbf{v}\|_{1/2,\Gamma}.$$

Our purpose is the construction of an efficient FE approximation to the PS operator $\mathbf{S}_T^{-1}$. To that end, we construct such approximations for the inverse to the block-diagonal components S_{nn}^{-1} and $S_{\tau\tau}^{-1}$ defined as the PS operators on the subspaces

$V_n = \{\mathbf{v} \in X_{n\tau} : v_\tau = 0\}$ and $V_\tau := \{v \in X_{n\tau} : v_n = 0\}$ respectively, each of which may be identified with a certain subspace of Y', where

$$Y := \begin{cases} H^{-1/2}(\Gamma) = \left(H^{1/2}(\Gamma)\right)', & \text{if } \Gamma \in C^{1,1} \\ \prod_{j=1}^{J} H^{-1/2}(\Gamma_j) = \left(\prod_{j=1}^{J} \widetilde{H}^{1/2}(\Gamma_j)\right)' & \text{if } \Gamma \text{ is a polygon.} \end{cases}$$

Denote $Y_1' = \{u \in Y' : (u,1)_{L^2(\Gamma)} = 0\}$.

Consider more precisely the block structure of the 2×2 matrix valued-operator $\mathbf{S}_T^{-1}$. First introduce the basic PS operators associated with the Laplace and biharmonic equations (see [KW96, KS96] for the corresponding variational formulations)

$$S_\Delta^{-1} : \mu \to \frac{\partial}{\partial n} u_{|\Gamma} \in H^{-1/2}(\Gamma); \qquad \begin{cases} \Delta u = 0, \;\; u \in H^1(\Omega) \\ u_{|\Gamma} = \mu \in H^{1/2}(\Gamma), \end{cases} \tag{5}$$

$$S_{\Delta^2}^{-1} : \gamma \to -\Delta\psi_{|\Gamma} \in Y; \qquad \begin{cases} \Delta^2\psi = 0, \;\; \psi \in H^2(\Omega) \cap H_0^1(\Omega) \\ \frac{\partial\psi}{\partial n}_{|\Gamma} = \gamma \in Y'. \end{cases} \tag{6}$$

Introduce the operator $\widetilde{S}_\Delta : g \to \psi_{|_\Gamma}$, where $\psi \in H^1(\Omega)$ solves the equation

$$\begin{cases} \Delta\psi = \frac{1}{mes\,\Omega} \int\limits_\Gamma g ds \;\; \text{in } \Omega \\ \frac{\partial\psi}{\partial n} = g \;\; \text{on } \Gamma. \end{cases} \tag{7}$$

This operator coincides with S_Δ for $g \in H_1^{-1/2}(\Gamma) = \{u \in H^{-1/2}(\Gamma) : (u,1)_{L^2(\Gamma)} = 0\}$. Let $D = \frac{d}{d\tau}$ and $D^{-1}u = \int\limits_{\tau_0}^{\tau} u(s)ds$, $\forall u \in H_1^{-1/2}(\Gamma)$. Note that the operators S_Δ^{-1} and D provide isomorphisms from $H_1^{1/2}(\Gamma) = \{u \in H^{1/2}(\Gamma) : (u,1)_{L^2(\Gamma)} = 0\}$ onto $H_1^{-1/2}(\Gamma)$ and $Ker\, S_\Delta^{-1} = Ker\, D = span\{1\}$. The operator $S_\Delta^{-1} : H_1^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is *s.p.d.*, while $D = -D'$ is a skew-symmetric one. Due to [KS96], we know that the mapping $S_{\Delta^2}^{-1} : Y' \to Y$ is continuous and *s.p.d.*

Lemma 2.1 *The operator $\mathbf{S}_T^{-1} : X_{n\tau} \to X_{n\tau}'$ is continuous, symmetric and positive semidefinite. The representation*

$$\mathbf{S}_T^{-1}\mathbf{u} = \begin{pmatrix} -D^{-1}S_\Delta^{-1}S_{\Delta^2}^{-1}\widetilde{S}_\Delta^{-1}D^{-1} & -D^{-1}S_\Delta^{-1}S_{\Delta^2}^{-1} - 2D \\ S_{\Delta^2}^{-1}\widetilde{S}_\Delta^{-1}D^{-1} + 2D & S_{\Delta^2}^{-1} \end{pmatrix} \begin{pmatrix} u_n \\ u_\tau \end{pmatrix} \tag{8}$$

holds for $\forall \mathbf{u} \in X_{n\tau}(\Gamma)$.

Proof. The first assertion follows from definition (4) along the line of the proof of Theorem 3.1. To prove (8), we pass to the stream function-vorticity formulation $\mathbf{u} = \mathbf{curl}\psi$, $\psi \in H^2(\Omega)$, $\psi(x_0) = 0$, $x_0 \in \Gamma$

$$\begin{cases} \nu(\Delta\psi, \Delta\varphi)_{L^2(\Omega)} = \langle \mathbf{f}, \mathbf{curl}\varphi \rangle & \forall\varphi \in H_0^2(\Omega) \\ \psi = \int\limits_{x_0}^{x} g_n ds; \qquad \frac{\partial\psi}{\partial n} = -g_\tau & \text{on } \Gamma \end{cases} \tag{9}$$

using the properties of the biharmonic PS operator $S^{-1}_{\Delta^2}$ studied in [KS96]. More detailed analysis of the representation (8) may be found in [KW96]. □

Since FE discretization to the operators D, D^{-1} and S^{-1}_{Δ} is a rather standard topic, a crucial point in the implementation of the matrix-valued operator (8) is an efficient approximation to the operator $S^{-1}_{\Delta^2}$ associated with the bi-Laplacian. A mixed FE approximation $S_{\Delta^2_h}$ to S_{Δ^2} by $P_1 - P_1$ elements has been developed in [KS96]. It was shown to have the complexity $\mathcal{C}(S_{\Delta^2_h}) = O(N \log^q N)$, where $q = 2$ for a rectangular domain and $q = 3$ in the case of convex polygons. However, the corresponding mixed formulation turns out not to satisfy a uniform LBB condition with respect to the mesh parameter $h > 0$. Thus, an optimal error estimate was not achieved in [KS96].

3 A New Interface Reduction by the Trace of the Pressure

To overcome the above drawback and to develop an approach which may be potentially extended to the 3D problems, we introduce the new Poincaré-Steklov operator associated with the Stokes equation, which admits a stable FE approximation and provides a stiffness matrix compression scheme of the same complexity as for the biharmonic operator $S_{\Delta^2_h}$. Let Ω be either a convex polygon or a domain with a smooth boundary. Introduce the operator $S_{st} : Y \to Y'$ by

$$S_{st} : \lambda \to -(\mathbf{u}_\lambda)_{n_{|\Gamma}}, \quad \text{where} \quad \begin{cases} \Delta p_\lambda = 0, \;\; p_{\lambda|\Gamma} = \lambda \in Y \\ \nu \Delta \mathbf{u}_\lambda - \nabla p_\lambda = 0, \\ div \mathbf{u}_{\lambda|\Gamma} = 0; \quad (\mathbf{u}_\lambda)_{\tau|\Gamma} = 0, \end{cases} \tag{10}$$

which maps the trace of the pressure λ into the normal velocity component $(\mathbf{u}_\lambda)_n$ of the solution to (10) (cf. the decomposition approach developed in [GP79]).

Theorem 3.1 *The operator $S_{st} : Y \to Y'$ is continuous and s.p.d. on Y/R, such that $Ker S_{st} = span\{1\}$, implying*

$$S_{st} = -D\widetilde{S}_\Delta S_{\Delta^2} S_\Delta D \;\; \text{and} \;\; S_{\Delta^2} = -\widetilde{S}^{-1}_\Delta D^{-1} S_{st} D^{-1} S^{-1}_\Delta \quad \text{on } Y. \tag{11}$$

There exists continuous and s.p.d. pseudoinverse $S^{-1}_{st} : Y'_1 \to Y/R$. There holds

$$\mathbf{S}^{-1}_T \mathbf{u} = \begin{pmatrix} S^{-1}_{st} & S^{-1}_{st} D \widetilde{S}_\Delta - 2D \\ -\widetilde{S}_\Delta D S^{-1}_{st} + 2D & -S_\Delta D S^{-1}_{st} D \widetilde{S}_\Delta \end{pmatrix} \begin{pmatrix} u_n \\ u_\tau \end{pmatrix}. \tag{12}$$

Sketch of the proof. To prove the mapping properties of S_{st}, we first note that the constraint $div \mathbf{u}_{|\Gamma} = 0$ implies $div \mathbf{u} = 0$ in Ω for any $\mathbf{u} \in H^2(\Omega)$ satisfying (10). We then apply the basic variational formulation of the second equation in (10) (due to the corresponding Green's formula): $\mathbf{u}_\lambda \in X_\tau$,

$$\nu a(\mathbf{u}_\lambda, \mathbf{v}) - (p, div \mathbf{v}) = -\int_\Gamma \lambda v_n ds \qquad \forall\, \mathbf{v} \in X_\tau = \{\mathbf{z} \in X : z_{\tau|\Gamma} = 0\}, \tag{13}$$

which is valid since the conditions $div \mathbf{u}_{\lambda|\Gamma} = 0$ and $(\mathbf{u}_\lambda)_{\tau|\Gamma} = 0$ yield the representation

$$\sigma_{nn}(\mathbf{u}_\lambda) = -p_\lambda + 2\nu div \mathbf{u}_{\lambda|\Gamma} = -p_\lambda \qquad \text{on } \Gamma.$$

The symmetry and continuity of S_{st} is derived by the variational equation

$$\langle S_{st}\lambda, \mu\rangle_{L^2(\Gamma)} = \nu a(\mathbf{u}_\lambda, \mathbf{u}_\mu), \quad \forall\, \lambda,\, \mu \in Y. \tag{14}$$

Indeed, due to the trace theorem and Korn's inequality, it follows for $\mathbf{u}_\lambda \in X_\tau$

$$\|S_{st}\lambda\|^2_{Y'} = \|(\mathbf{u}_\lambda)_n\|^2_{Y'} \le c\|\mathbf{u}_{\lambda|\Gamma}\|^2_{H^{1/2}(\Gamma)} \le ca(\mathbf{u}_\lambda, \mathbf{u}_\lambda) = \tag{15}$$

$$= \frac{c}{\nu}\langle S_{st}\lambda, \lambda\rangle_{L^2(\Gamma)} \le \frac{c}{\nu}\|S_{st}\lambda\|_{Y'} \cdot \|\lambda\|_Y.$$

The positive definiteness of S_{st} follows from:
a) the norm equivalence (see [KS96])

$$\|\Lambda\mu\|_{L^2(\Omega)} \cong \|\mu\|_Y \qquad \forall \mu \in Y, \tag{16}$$

where the continuous mapping $\Lambda : Y \to L^2(\Omega)$, such that $\Lambda\mu = \varphi$ denotes a solution operator of the Dirichlet problem for the Laplace equation in a *very weak* form

$$\int\limits_\Omega \varphi \Delta z dx = \langle \mu, \frac{\partial z}{\partial n}\rangle_{L^2(\Gamma)} \qquad \forall z \in H^2(\Omega) \cap H^1_0(\Omega), \quad \mu \in Y;$$

b) *inf-sup* condition (3) for the subspace X_0.
In fact, we use (16), (3), the continuity of $a(\cdot,\cdot)$ and obtain

$$\|\lambda\|_Y \le c\|p_\lambda\|_{0,\Omega} \le \sup_{\mathbf{v}\in X_0} \frac{(p_\lambda, div\mathbf{v})}{|\mathbf{v}|_{1,\Omega}} = \tag{17}$$

$$= \nu \sup_{\mathbf{v}\in X_0} \frac{a(\mathbf{u}_\lambda, \mathbf{v})}{|\mathbf{v}|_{1,\Omega}} \le c\nu a(\mathbf{u}_\lambda, \mathbf{u}_\lambda)^{1/2} \le c\nu^{1/2}\langle S_{st}\lambda, \lambda\rangle^{1/2}_{L^2(\Gamma)}.$$

The representations (11) and (12) follow from (8) and from the equivalence between (19) and (7), see also the proof of Theorem 3.2. □

The operator S_{st} provides an alternative representation (12) to the matrix-valued PS operator $\mathbf{S}_T^{-1}$. In this case, we may avoid the stream function-vorticity formulation and construct a stable FE approximation to S_{st}. Moreover, the representation (12) involves only the operators in the normal-tangential (i.e., dimensionally invariant) form and provides a natural base for an extension of the underlying techniques to the $3D$ case. The operator S_{st} also provides an efficient boundary reduction to the Stokes equation (if $\mathbf{f} = 0$) with respect to the trace of the pressure.

Theorem 3.2 *The trace $\lambda = p_{|\Gamma}$ of the solution $(\mathbf{u}, p)$ to the Dirichlet problem (1) (with $\mathbf{f} = 0$) satisfies*

$$\lambda \in Y/R: \quad \langle S_{st}^{-1}\lambda, \mu\rangle_{L^2(\Gamma)} = \langle g_n - (\mathbf{u}_0)_n, \mu\rangle_{L^2(\Gamma)} \qquad \forall \mu \in Y/R, \tag{18}$$

where $\mathbf{u}_0$ solves the following mixed problem for the vector Laplace equation

$$-\nu\Delta\mathbf{u}_0 = 0 \qquad in\ \Omega; \qquad (\mathbf{u}_0)_\tau = g_\tau, \quad div\mathbf{u}_0 = 0 \ \ on\ \Gamma. \tag{19}$$

Proof. The unique solvability of (19) is checked by using the substitution $\mathbf{u}_0 = \mathbf{curl}\,\psi, \psi \in H^2(\Omega)$, $\psi(x_0) = 0$, where ψ satisfies (7) such that $\frac{\partial\psi}{\partial n} = g_\tau$ and $\frac{\partial\psi}{\partial\tau} = -(\mathbf{u}_0)_n$. Then the assertion follows from Theorem 3.1. □

Remark 3.1 *Equation (13) has an equivalent form*

$$\nu d(\mathbf{u}_\lambda, \mathbf{v}) + (\nabla p, \mathbf{v})_{L^2(\Omega)} = 0 \qquad \forall \mathbf{v} \in X_\tau, \tag{20}$$

where the bilinear form $d : X \times X \to R$ is defined by

$$d(\mathbf{u}, \mathbf{v}) := (curl\mathbf{u},\, curl\mathbf{v})_{L^2(\Omega)} + (div\mathbf{u},\, div\mathbf{v})_{L^2(\Omega)}. \tag{21}$$

Here, the operator curl : $X \to R$ is given by $curl\mathbf{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}$. For technical reasons, we further construct a discrete scheme on the base of above defined form $d(\cdot,\cdot)$.

4 A Stable FE Approximation to the Interface Operator

Let Ω be a rectangular domain. Assume $M_h \in M$ and $X_{\tau h} \in X_\tau$ to be the spaces of $P_1\, iso\, P_2/P_1$ FEs, see [Pir89], defined on the regular hierarchical triangulations $\mathcal{T}_h$ and $\mathcal{T}_{h/2}$ of $\overline{\Omega}$. Let $\mathbf{u}_{0h}$ be the discrete solution of (19) based on the FE approximation of the Poisson equation (7) with respect to M_h. Introduce the equations
Given $\lambda_h \in Y_h := M_{h|\Gamma}$, find $p_{\lambda h} \in M_h$, such that $p_{\lambda h} = \lambda_h$ on Γ and

$$(\nabla p_{\lambda h}, \nabla q_h)_{L^2(\Omega)} = 0 \qquad \forall q_h \in M_h \cap H_0^1(\Omega)\,; \tag{22}$$

Find $\mathbf{u}_{\lambda h} \in X_{\tau h}$, such that:

$$\nu d(\mathbf{u}_{\lambda h}, \mathbf{v}_h) - (p_{\lambda h}, div\mathbf{v}_h) = -(\lambda_h, (\mathbf{v}_h)_n)_{L^2(\Gamma)} \qquad \forall \mathbf{v}_h \in X_{\tau h}\,. \tag{23}$$

For any $\lambda_h \in Y_h$, define FE approximation P_h to S_{st} by

$$\langle P_h\lambda_h, \mu_h\rangle = \nu\, d(\mathbf{u}_{\lambda h}, \mathbf{u}_{\mu h}), \qquad \forall \mu_h \in Y_h\,. \tag{24}$$

The *s.p.d.* operator P_h admits a fast matrix-vector multiplication. The discrete system related to (18) can now be written as a boundary equation with respect to the trace of the pressure

$$\langle P_h\lambda_h, \mu_h\rangle = -((\mathbf{g} - \mathbf{u}_{0h}) \cdot \mathbf{n}\ , \mu_h)_{L^2(\Gamma)} \qquad \forall \mu_h \in Y_h\,. \tag{25}$$

With λ_h satisfying (25), the approximate velocity $\mathbf{u}_h$ and the pressure p_h are given by $\mathbf{u}_h = \mathbf{u}_{0h} + \mathbf{u}_{\lambda h}$, $\quad p_h = p_{\lambda h}$. Assuming $\mathcal{T}_h$ to be the uniform triangulation, we may prove the main result.

Theorem 4.1 *The operator $P_h : Y_h \to Y_h'$ is s.p.d. on Y_h/R providing the norm equivalence*

$$\nu\,\langle P_h\mu_h, \mu_h\rangle \simeq \|\mu_h\|^2_{Y/R} \qquad \forall \mu_h \in Y_h \tag{26}$$

with constants of equivalence not depending on h. There holds

$$\|\lambda - \lambda_h\|_Y \le c\, h\, (|\mathbf{u}|_{2,\Omega} + |p|_{1,\Omega})\,. \tag{27}$$

Sketch of the proof. Applying the trace theorem and Korn's inequality, we obtain

$$\langle P_h\lambda_h, \lambda_h\rangle = \nu\, d(\mathbf{u}_{\lambda h}, \mathbf{u}_{\lambda h}) \le c\,|\lambda_h|_{Y/R}\,\|(\mathbf{u}_{\lambda h})_n\|_{H^{1/2}(\Gamma)} \le |\lambda_h|_{Y/R}\, d(\mathbf{u}_{\lambda h}, \mathbf{u}_{\lambda h})^{1/2}.$$

The other direction follows from the norm equivalence $\|p_{\mu h}\|_{0,\Omega} \simeq \|\mu_h\|_Y$, $\forall \mu_h \in Y_h$, see [KS96], the discrete inf-sup condition and continuity of $d(\cdot,\cdot)$. Indeed,

$$p_{\lambda h} \in M_h: \qquad \|\lambda_h\|_{Y/R} \le c\,\|p_{\lambda h}\|_{0,\Omega} \le c \sup_{\mathbf{v}_h \in X_{0h}} \frac{(p_{\lambda h}, div\mathbf{v}_h)}{|\mathbf{v}_h|_{1,\Omega}} =$$

$$= \nu\, c \sup_{\mathbf{v}_h \in X_{0h}} \frac{d(\mathbf{u}_{\lambda h}, \mathbf{v}_h)}{|\mathbf{v}_h|_{1,\Omega}} \le \nu\, c\, d(\mathbf{u}_{\lambda h}, \mathbf{u}_{\lambda h})^{1/2} = \nu^{1/2}\, c\,\langle P_h\lambda_h, \lambda_h\rangle^{1/2}\,.$$

Now (27) follows from (26) and standard error estimates for (22) and (23), see [KW97] for more details. □

Finally, the symmetric and positive definite FE approximation to $\mathbf{S}_T^{-1}$ from (12) is obtained by a substitution of D_h, $\widetilde{S}_{\Delta_h}$ and P_h into (12) instead of the corresponding continuous operators.

Remark 4.1 *Using the discrete operator P_h, we immediately obtain an s.p.d. FE approximation to the biharmonic Poincaré-Steklov operator S_{Δ^2} by $S_{\Delta_h^2} = -S_{\Delta_h}^{-1} D_h^{-1} P_h D_h^{-1} \widetilde{S}_{\Delta_h}^{-1}$ yielding an optimal approximation error and efficient matrix compression. This means that our interface reduction for the Stokes equation provides an efficient solver for the stream function-vorticity formulation as well.*

5 An Interface Reduction by the Domain Decomposition

We consider the *s.p.d.* approximation of A_{Γ_0} by using the operator P_h. To fix the idea, we assume $\overline{\Omega} = \cup_i \overline{\Omega}_i$ to be composed of rectangular subdomains Ω_i. First derive an interface reduction to the equation (1) with the given right-hand side $\mathbf{f} \ne 0$ and $\mathbf{g} = 0$. For any subdomain Ω_i, assume the traction vector $\psi_{0i} = \binom{\sigma_{nn}(\mathbf{u}_{0i})}{\sigma_{n\tau}(\mathbf{u}_{0i})}_{|\Gamma_i}$ of the corresponding particular solution $\mathbf{u}_{0i} \in H_0^1(\Omega_i)^2$ to be given. Define the related trace space on the skeleton $\Gamma_0 = \cup_i \Gamma_i$ by

$$Y_{\Gamma_0} := \{\mathbf{u} = \mathbf{v}_{|\Gamma_0} : \mathbf{v} \in H_0^1(\Omega)^2,\, ((\mathbf{v}_i)_n, 1)_{\Gamma_i} = 0,\, i = 1, ..., M\} \tag{28}$$

and equip it with the norm $\|\mathbf{u}\|_{Y_{\Gamma_0}} = \inf_{\mathbf{z} \in V_0;\mathbf{z}_{|\Gamma_0} = \mathbf{u}} \|\mathbf{z}\|_{H^1(\Omega)}$. The interface reduction to (1) takes the form:

Find $\mathbf{u} \in Y_{\Gamma_0}$, such that $\mathbf{u} = \overline{\mathbf{u}}_{|\Gamma_0}$ ($\overline{\mathbf{u}}$ solves (1)) and satisfies

$$\langle A_{\Gamma_0}\mathbf{u}, \mathbf{v}\rangle_{\Gamma_0} := \sum_{i=1}^{M} (S_{iT}^{-1}\mathbf{u}_i, \mathbf{v}_i)_{\Gamma_i} = \sum_{i=1}^{M} (\psi_{0i}, \mathbf{v}_i)_{\Gamma_i} \quad \forall \mathbf{v} \in Y_{\Gamma_0}. \tag{29}$$

Due to V_0-ellipticity of $a(\cdot,\cdot)$, the continuous and symmetric operator $A_{\Gamma_0} : Y_{\Gamma_0} \to Y'_{\Gamma_0}$ is also positive definite. We approximate S_{iT}^{-1} given by (12) using the *s.p.d.* operator P_h. To avoid the divergence-free constraints $\big((\mathbf{u}_i)_n, 1\big)_{\Gamma_i} = 0$, $i = 1, \ldots M$ and then to apply the standard preconditioning techniques, we first extend the interface operator A_{Γ_0} to the constraints-free trace space $\widetilde{Y}_{\Gamma_0} := \{\mathbf{u} = \mathbf{v}_{|\Gamma_0} : \mathbf{v} \in H_0^1(\Omega)^2\}$ preserving the symmetry and the norm equivalence on Y_{Γ_0}. This extension is based on a scaling of the trace of the pressure on any subdomain boundary Γ_i (by an appropriate choice of the constants $p_i = (p, 1)_{L^2(\Gamma_i)}$) and on using of a special coarse mesh space Y_1 responsible for the divergence-free constraints on Γ_i.

Let $Y_1 = span\{\mathbf{g}^i\}_{i=1}^M \subset \widetilde{Y}_{\Gamma_0}$ be the coarse mesh space of the dimension $dim Y_1 = M$ (in general $\Gamma_i \subset supp\,\mathbf{g}^i$), where the normalized basis functions $\mathbf{g}^i$ and the corresponding Gram matrix $\mathcal{G}$ satisfy

$$det\,\mathcal{G} \neq 0, \;\; \mathcal{G} = \{g_{ij}\}_{i,j=1}^M, \quad g_{ij} = (\mathbf{g}^i, \mathbf{1}^j)_{\Gamma_0}, \quad \mathbf{1}^i = (1,0)^T \text{ on } \Gamma_i. \tag{30}$$

Then the following splitting into the direct sum $\widetilde{Y}_{\Gamma_0} = Y_{\Gamma_0} \oplus Y_1$ holds, such that $Y_1' = span\{\mathbf{1}^i\}_{i=1}^M = Y_{\Gamma_0}^{\perp}$. Let $\mathbf{S}_\Delta^{-1} : \widetilde{Y}_{\Gamma_0} \to \widetilde{Y}_{\Gamma_0}'$ be the Poincaré-Steklov operator corresponding to the weighted vector Laplacian. Define the operator $A_1 : Y_1 \to Y_1'$ on Y_1 (by an inexact h- harmonic extension of $\mathbf{g}^i$) providing the norm equivalence

$$\langle A_1 \mathbf{g}, \mathbf{g}\rangle_{\Gamma_0} \cong \langle \mathbf{S}_\Delta^{-1}\mathbf{g}, \mathbf{g}\rangle_{\Gamma_0} \qquad \forall\, \mathbf{g} \in Y_1. \tag{31}$$

We then obtain $\langle A_{\Gamma_0}\mathbf{u}, \mathbf{g}\rangle_{\Gamma_0} = 0$ and $\langle A_1\mathbf{g}, \mathbf{u}\rangle_{\Gamma_0} = 0$ $\forall \mathbf{u} \in Y_{\Gamma_0}$, $\mathbf{g} \in Y_1$ by an appropriate scaling of $p_{|\Gamma_i}$ and by the definition, respectively. The desired extension $\widetilde{A}_{\Gamma_0}$ is now defined for any $\mathbf{u}, \mathbf{v} \in Y_{\Gamma_0}$ and $\mathbf{g}_u, \mathbf{g}_v \in Y_1$ by

$$\langle \widetilde{A}_{\Gamma_0}(\mathbf{u} + \mathbf{g}_u), \mathbf{v} + \mathbf{g}_v\rangle_{\Gamma_0} = \langle A_{\Gamma_0}\mathbf{u}, \mathbf{v}\rangle_{\Gamma_0} + \langle A_1 \mathbf{g}_u, \mathbf{g}_v\rangle_{\Gamma_0}. \tag{32}$$

If we assume the right-hand sides ψ_{0i} to satisfy the compatibility conditions $(\psi_{0i}, \mathbf{g}^i)_{\Gamma_i} = 0$ (by an appropriate scaling of $\sigma_{nn}(\mathbf{u}_{0i})$), then (29) becomes equivalent to the equation

$$\mathbf{u} \in Y_{\Gamma_0} : \quad \langle \widetilde{A}_{\Gamma_0}(\mathbf{u} + \mathbf{g}_u), \mathbf{v}\rangle_{\Gamma_0} = \sum_{i=1}^M (\psi_{0i}, \mathbf{v}_i)_{\Gamma_i} \qquad \forall \mathbf{v} \in \widetilde{Y}_{\Gamma_0} \tag{33}$$

posed on the constraints-free trace space $\widetilde{Y}_{\Gamma_0}$ and providing $\mathbf{g}_u = 0$. Clearly, the operator $\widetilde{A}_{\Gamma_0}$ is symmetric and positive definite. It may be shown to be spectrally equivalent to $\mathbf{S}_\Delta^{-1}$. Thus, one may apply any standard preconditioners (which remain verbatim for the piecewise Laplacian) to solve the equation (33). In particular, the BPS, balancing type and multilevel BPX preconditioners may be constructed for the iterative solving of the interface equation (33). More detailed analysis of the abovementioned preconditioning techniques (also in the presence of right triangular subsructures) may be found in [KW97]. An efficient computation of the residual for the equations (29) and (33) is based on a fast matrix-vector multiplication for the local Schur complement matrices $\mathcal{S}_{iT}^{-1}$ associated with S_{iT}^{-1}. In the case of rectangular domains, the corresponding matrix compression scheme of the complexity $O(N \log^2 N)$ was presented in [KW97]. Here N denotes the number of degrees of freedom on the subdomain boundary. With such compression algorithm, we arrive at the estimate $O(MN \log^{q_r} N)$ for the overall computational complexity of the PCG methods applied to the system (33). Here, $q_r = 2$ for the multilevel BPX preconditioner on the interface and $q_r = 3$ in the case of the BPS preconditioner.

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REFERENCES

[GP79] Glowinski R. and Pironneau O. (1979) On mixed finite element approximation of the stokes problem i. convergence of the approximate solution. *Numer. Math.* 33: 397–424.

[GR86] Girault V. and Raviart P.-A. (1986) *Finite element methods for Navier-Stokes equations.* Springer-Verlag, Berlin.

[JHBS86] J. H. Bramble J. E. P. and Schatz A. H. (1986) The construction of preconditioners for elliptic problems by substructuring. *Math. Comp.* 47: 103–134.

[JHBX90] J. H. Bramble J. E. P. and Xu J. (1990) Parallel multilevel preconditioners. *Math. Comp.* 55: 1–22.

[Kho96] Khoromskij B. N. (1996) On fast computations with the inverse to harmonic potential operators via domain decomposition. *Numer. Linear Algebra with Applications* 3: 91–111.

[KP95] Khoromskij B. N. and Prössdorf S. (1995) Multilevel preconditioning on the refined interface and optomal boundary solvers for the laplace equation. *Advances in Comp. Math.* 4: 331–355.

[KS96] Khoromskij B. N. and Schmidt G. (1996) Asymptotically optimal interface solver for the biharmonic dirichlet problem on convex polygonal domains. *ZAMM* 76, Suppl. 1: 231–234.

[KW96] Khoromskij B. N. and Wittum G. (1996) An asymptotically optimal schur complement reduction for the stokes equation. Technical report, ICA III, Stuttgart University.

[KW97] Khoromskij B. N. and Wittum G. (1997) Sparse interface reduction for two-dimensional stokes equation. Technical report, ICA III, Stuttgart University.

[Lad69] Ladyzhenskaya O. A. (1969) *The mathematical theory of viscous incompressible flow.* Gordon and Breach, New York.

[Pir89] Pironneau O. (1989) *Finite Element Methods for Fluids.* J. Wiley & Sons Ltd., Masson, Paris.

[QV91] Quarteroni A. and Valli A. (1991) Theory and application of steklov-poincaré operators for boundary-value problems. In *Applied and Industrial Mathematics*, pages 179–203. Kluwer AP.

[Wit89] Wittum G. (1989) Multigrid methods for stokes and navier-stokes equation. *Numer. Math.* 54: 543–563.