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Defect correction for Boussinesq Flow

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1 Introduction

A defect correction method for the convection-diffusion equation is presented. In the domain decomposition context the presented technique can be used as a preconditioner on each subdomain. The discretization is performed by second-order finite difference schemes (β-schemes) where the second-order upstream scheme is combined with the standard central scheme. Higher order discretizations with spectral methods are also considered. For preconditioning the usual first-order upstream scheme is employed. The defect correction iteration is used for relaxation inside a multigrid procedure. It is shown that the smoothing analysis yields rather pessimistic results. In the practical computation the discretization error is reached in two V-cycles. Numerical results are presented which demonstrate the high efficiency of our treatment. The convection-diffusion equation yields a good model for the numerical solution of the Navier-Stokes equations with high Reynolds numbers. In many applications the second-order accuracy is necessary in order to get a realistic impression of the flow.

The β-schemes were previously analyzed by Desideri & Hemker [DH92] and Luh [Luh92]. For $\beta = 1$ we obtain the standard second-order upstream scheme and for $\beta = 0$ the central scheme. $\beta = \frac{1}{2}$ results in *Fromm's* scheme. For $\beta = \frac{1}{3}$ we obtain the *upwind biased* scheme which is of third-order accuracy. Since an iterative solver for these higher order schemes yields bad convergence factors we propose a defect correction procedure. We have had good experience with this method for spectral discretizations. Here the higher order scheme is also preconditioned by the standard first-order upwind scheme. We investigate the smoothing properties of this procedure for the convection equation. This method is used for relaxation in a multigrid cycle. In the spectral scheme the solution is approximated by Chebyshev polynomials. By a Fourier analysis it can be shown that the eigenvalues of the preconditioned operator are bounded but complex. Hence one has to employ a nonsymmetric matrix iteration for the solution. We recommend the GMRES iteration which belongs to the residual minimization methods. Clearly, for the general convection-diffusion problem the first derivatives have to be approximated according to the sign of the coefficients. Therefore

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for the iterative solution we recommend flow-directed schemes. Since the Chebyshev nodes are dense near the boundary it is necessary to use line Gauss-Seidel relaxation (in an alternating manner). Finally this iterative solver is applied to the Boussinesq flow problem in vorticity-streamfunction formulation with high Rayleigh numbers.

2 Convection-diffusion Problem

Here we consider convection-diffusion problems which can in scalar, constant-coefficient form be written as

$$\begin{aligned} -\epsilon\Delta u + au_x + bu_y &= f \text{ in } \Omega = (-1,1)^2, && (2.1)\\ u &= g \text{ on } \partial\Omega, && (2.2) \end{aligned}$$

where $\epsilon = \frac{1}{Re}$ and Re denotes the Reynolds number. f is defined in Ω, and g is defined on $\partial\Omega$. a and b denote given constants. Such problems arise after a linearization of the Navier-Stokes equations or Boussinesq flow problems (see Section 3). The part $-\epsilon\Delta u$ denotes the diffusive part and $au_x + bu_y$ denotes the convective part of the above equation. Here we are mainly interested in convection-dominated flows where $\epsilon \ll h$ or $\epsilon \ll N^{-2}$. Here h denotes the step size of the finite difference (FD) scheme and N the maximal degree of the polynomials in a spectral scheme. It is well known that discretizations for this type of problem are in general unstable. One possibility to avoid the phenomenon of instability is to use upstream discretization for u'. An obvious disadvantage of this scheme lies in the fact that the method now becomes only first-order accurate. Hence it makes sense to use the first-order upstream scheme only as a preconditioner for a higher order scheme. We analyze the preconditioning properties of this method for the following higher order schemes:

- β-schemes, $\beta \in [0,1]$,
- spectral methods.

The first-order upstream scheme is explicitly given by the upstream operator L_h^1 (here in 1D):

$$L_h^1 \;\cong\; \frac{1}{h}\,[-1\ \underline{1}\ 0]\,.$$

The second-order hybrid scheme is the β-scheme L_h^β, which is a combination of the standard second-order upstream scheme $L_h^{su,2}$ and the second-order central scheme $L_h^{ce,2}$:

$$L_h^\beta \;=\; \beta L_h^{su,2} \;+\; (1-\beta)L_h^{ce,2}, \quad \beta \in [0,1]. \tag{2.3}$$

For all β we obtain at least second-order discretizations. For $\beta = 0$ the method becomes unstable. Especially, for $\beta = \frac{1}{3}$ we obtain a third-order scheme. Here we study the preconditioning properties of L_h^1 for the 2D convection operator

$$au_x \;+\; bu_y.$$

The defect correction iteration is defined by the operator

$$M_h \;=\; I_h \;-\; \omega\left(L_h^1\right)^{-1} L_h^\beta, \quad \beta \in [0,1].$$

Here ω denotes a relaxation parameter which should accelerate the convergence speed. By a Fourier analysis for the Fourier components $\underline{\theta}$ ($\underline{\theta} = \theta_1$ in $1D$, $\underline{\theta} = (\theta_1, \theta_2)$ in $2D$) the above operator M_h leads to the *amplification factor*

$$\mu(\underline{\theta}) \;=\; 1 - \omega \frac{\lambda^\beta(\underline{\theta})}{\lambda^1(\underline{\theta})},$$

where $\lambda^\beta(\underline{\theta})$ and $\lambda^1(\underline{\theta})$ denote the factors of the Fourier analysis for the operators L_h^β and L_h^1. Now the convergence factor ρ of the defect correction procedure is defined as the supremum of $\mu(\underline{\theta})$ taken over all frequencies $\underline{\theta} \in (-\pi, \pi]$ in 1D and $\underline{\theta} \in (-\pi, \pi]^2$ in 2D:

$$\rho(M_h) \;:=\; \sup_{\underline{\theta} \in (-\pi,\pi]^2} |\mu(\underline{\theta})|.$$

Furthermore, we are interested in defect correction as a smoother in a multigrid procedure. The efficiency of a smoother can be measured by means of the *smoothing rate* μ_β. This rate can be obtained by taking the above supremum only for the high frequencies $|\underline{\theta}| := \max(|\theta_1|, |\theta_2|) \in (\frac{\pi}{2}, \pi]$:

$$\mu_\beta \;:=\; \sup_{\underline{\theta} \in (\frac{\pi}{2},\pi]} |\mu(\underline{\theta})|.$$

The Fourier analysis of Luh [Luh92] makes it clear that for all $\beta \in [0,1]$, ω and independent of the *alignment* $\frac{b}{a}$ the prediction

$$\rho(M_h) \;=\; 1$$

holds.

However, in the multigrid procedure we are more interested in the damping only of the high frequencies. Here we consider the special cases $\beta = 1$ and $\beta = \frac{1}{2}$. From the analysis in [Luh92] we observe:

- $\beta = 1$, $\omega_{opt} = 0.68$: $\mu_1 \leq 0.77$,
- $\beta = \frac{1}{2}$, $\omega_{opt} = 1.00$: $\mu_{\frac{1}{2}} \leq 0.72$.

For $\beta = 1$ the optimal parameter ω is about 0.68 for all alignments $\frac{b}{a}$. The corresponding smoothing rate μ_1 is decreasing for increasing alignment. The maximum is attained at $\frac{b}{a} = 0.1$ where $\mu_1 = 0.77$. For $\beta = \frac{1}{2}$ the value $\omega_{opt} = 1$ yields a quite good choice for all alignments $\frac{b}{a}$. For $\beta = 0$ the smoothing rate μ_0 is always equal to 1 (independent of the parameter choice of ω). From these considerations it becomes clear that the smoothing rates are rather bad compared to the usual rates for symmetric problems (e.g., the Poisson problem).

Here we apply a standard multigrid method. The transfer operators are given by *full weighting* restriction and *bilinear* interpolation. For relaxation we choose the already mentioned *Richardson* iteration with *defect correction*. It is explicitly defined as follows:

$$u_h^{j+1} \;=\; u_h^j \;-\; \omega \left(L_h^1\right)^{-1} \left(L_h^\beta u_h^j - f_h\right)$$

for $j = 0, 1, 2, \ldots$. u_h^0 denotes an initial approximation, which is chosen to be identically zero. ω denotes the relaxation parameter. Optimal choices are given by the smoothing analysis. Instead of solving the first-order problem relative to L_h^1 exactly we employ a *lexicographic* Gauß-Seidel step, which nearly yields an exact solver if $a,\ b > 0$. If a and b have different sign then after a renumbering of the grid points the same effect can be achieved. In case of variable coefficients a and b which change sign one has to use the *flow directed point relaxation.* Here we consider a multigrid method with 7 grids and corresponding step sizes $h_\nu = \frac{1}{2^\nu},\ \nu = 1, \ldots, 7$. In general, we employ a *V–cycle.* Other cycle structures as *W–cycle, F–cycle* or the *full multigrid* technique could not improve the convergence speed significantly. We employ two relaxations before and one after the coarse grid correction. The results are compared with the pure Richardson iteration without using multigrid. The absolute errors between the exact solution and the iterates are measured in the discrete L^1 and L^2 norms: $L1,\ L2$. By $Q(L1),\ Q(L2)$ we denote the quotient of the errors for two successive iterates. Numerical results are provided for the example where the exact solution is given by

$$u(x,y) = \sin\left(8\pi(y - \frac{b}{a}x)\right),$$

where $a = 4,\ b = 1$ and $\epsilon = 10^{-6}$ (see Y. Luh [Luh92]). The right hand side f and the Dirichlet boundary conditions are determined by u. We present results for $\beta = 1$. The corresponding numerical results are given in Table 1.

Table 1. $\beta = 1$, V–cycle

IT	$L1$	$L2$	$Q(L1)$	$Q(L2)$
1	$4.28 \cdot 10^{-2}$	$5.42 \cdot 10^{-2}$	0.067	0.076
2	$2.44 \cdot 10^{-2}$	$3.04 \cdot 10^{-2}$	0.569	0.560
3	$2.42 \cdot 10^{-2}$	$3.01 \cdot 10^{-2}$	0.994	0.989

The numerical results show the highly improved efficiency of the V-cycle. There is nearly no improvement by using other cycle structures. In particular, the first rate of the multigrid scheme is very small and yields already an error which is nearly equal to the discretization error. Hence in general it can be seen that 2 or 3 V-cycles are enough to reach the truncation error. So the smoothing analysis gives rather pessimistic results. In practice, the discretization error is reached very fast. By comparing the results for different step sizes we also confirm at least second-order accuracy. For $\beta = \frac{1}{3}$ we obtain a third-order method. Here we obtain the most precise results. For increasing β the results become less accurate.

Let us now consider the case of variable coefficients $a,\ b$, i.e., $a = a(x,y),\ b = b(x,y)$. For the iterative solution we recommend *flow directed schemes.* For smoothing it is recommended to use alternate iterations of FDHI (Flow Directed Horizontal Iterations) and FDVI (Flow Directed Vertical Iterations). In the literature this combination is called FDHVI (see [DM92], [HIK88]). The iterative scheme FDHI is a variant of line Gauss-Seidel relaxation. Han et al. [HIK88] describe a procedure based on directed graphs to partition and order the unknowns of the Gauss-Seidel process. This is performed by inspection of the coefficient matrix. Nevertheless, this algorithm is expensive for nonlinear problems, like those coming from the Navier-Stokes or Boussinesq equations, when the coefficients are solution-dependent and require the reconstruction of the directed graph several times. The penalty for such a choice is proportional to the number of mesh points. Here the FDHVI scheme is applied to the Boussinesq flow problem.

3 The Boussinesq Flow Problem

The problem specifically considered here is that of the two-dimensional flow of a Boussinesq fluid of Prandtl number $Pr = 0.71$ (i.e., air) in an upright square cavity (see [BWD90], [Dav83]). The walls are non-slip and impermeable. The horizontal walls are adiabatic and the vertical sides are at fixed temperatures. In addition to the Navier-Stokes equations we have one further equation for the temperature T. By Ra we denote the Rayleigh number. The Boussinesq flow problem in vorticity-streamfunction formulation reads as follows:

$$
\begin{aligned}
4\Delta\psi + \omega &= 0 \text{ in } \Omega = (-1,1)^2, && (3.4)\\
-2Pr\Delta\omega + \frac{\partial}{\partial x}(v_1\omega) + \frac{\partial}{\partial y}(v_2\omega) &= RaPr\frac{\partial T}{\partial x} \text{ in } \Omega, && (3.5)\\
-2\Delta T + \frac{\partial}{\partial x}(v_1 T) + \frac{\partial}{\partial y}(v_2 T) &= 0 \text{ in } \Omega. && (3.6)
\end{aligned}
$$

As usual $(v_1, v_2)^t$ denotes the velocity. ψ fulfills homogeneous Dirichlet boundary conditions and T fulfills mixed Dirichlet/Neumann boundary conditions. The homogeneous Neumann boundary conditions correspond to the fact that the horizontal walls are adiabatic.

Now the equations (4)–(6) are linearized by a Quasi-Newton method, where the velocity from the previous iteration is employed. The linearized system is then approximately solved by a spectral multigrid (SMG) method (see [Hei88a], [Hei88b], [Hei92], [Hei93]). In the spectral scheme the solution is approximated by polynomials in $\mathbf{P}_N$, $N \in \mathbf{N}$ where $\mathbf{P}_N$ denotes the space of polynomials of degree $\leq N$. The discretization is performed by a pseudo spectral (or collocation) method in the Chebyshev-Gauss-Lobatto nodes. In the SMG method we use the same components as already introduced. We employ the FDHVI iteration for preconditioning. In order to handle the complex eigenvalues of the preconditioned spectral operator we employ nonsymmetric matrix iterations. Here we choose the GMRES iteration.

By using these components we numerically calculated for various Rayleigh numbers and mesh sizes the following quantities:

$|\psi|_{mid}$: absolute value of the streamfunction at the midpoint of the cavity,
$|\psi|_{max}$: maximum absolute value of the streamfunction,
$v_{1,max}$: maximum horizontal velocity on the vertical mid-plane of the cavity,
$v_{2,max}$: maximum horizontal velocity on the horizontal mid-plane of the cavity.

The local heat flux in a horizontal direction at any point in the cavity is given by

$$Q \ := \ v_1 T \ - \ 2\frac{\partial T}{\partial x}.$$

Let us further introduce the following Nusselt numbers:

$\overline{Nu} \ := \ \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1} Q(x,y)dxdy$: average Nusselt number throughout the cavity,
$Nu_{\frac{1}{2}} \ := \ \frac{1}{2}\int_{-1}^{1} Q(0,y)dy$: average Nusselt number on the vertical mid-plane,
$Nu_0 \ := \ \frac{1}{2}\int_{-1}^{1} Q(-1,y)dy$: average Nusselt number on the vertical boundary.

The above integrals in the definition of $\overline{Nu}$, $Nu_{\frac{1}{2}}$ and Nu_0 are evaluated by the Clenshaw-Curtis quadrature. In Table 2, we present the numerical results for the Rayleigh numbers $Ra = 10^5$. The numerical results are in good accordance with the results obtained in [Dav83].

Table 2. Results for $Ra = 10^5$.

N	$\lvert\psi\rvert_{mid}$	$\lvert\psi\rvert_{max}$	$v_{1,max}$	$v_{2,max}$	$\overline{Nu}$	$Nu_{\frac{1}{2}}$	Nu_0
8	14.3409	18.8519	37.8844	40.2643	4.4140	4.7345	4.7590
16	11.3720	12.3330	36.3420	61.3420	4.5030	4.5061	4.5313
24	9.1600	9.6530	34.6320	67.9120	4.5100	4.5120	4.5231

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