A Funaro-Quarteroni Procedure for Singularly Perturbed Elliptic Boundary Value Problems

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1 Introduction

We analyze the Funaro-Quarteroni alternative procedure for the solution of singular perturbation problems. We show that for an appropriate choice of the domain decomposition, one obtains a fast convergent iterative scheme with *no relaxation* that resolves the boundary layers. The convergence is superlinear with respect to the singular perturbation parameter in the following sense: the amplification factor is $o(\epsilon)$. We give sharp estimates of the interface position and convergence rates for homogeneous domain decomposition in one dimensional space. This analyse can be generalized in a two dimensional space on a disk ([GVC96]). We extend our results to heterogeneous domain decomposition arising in a simplified model of an electromagnetic problem. Our method has been implemented with finite difference approximations and finite element codes (Modulef).

2 Boundary Layers in One-dimensional Space

Homogeneous Domain Decomposition

In this section, we consider a linear second-order singular perturbation problem of the following type:

$$\left\{ \begin{array}{l} L_\epsilon \phi = -\epsilon \phi'' + \phi = F \; in \; \Omega = (0,1); \\ \phi(0) = \alpha_0 \; ; \; \phi(1) = \alpha_1 . \end{array} \right. \tag{2.1}$$

ϵ is a small positive parameter, $\epsilon \in]0, \epsilon_0]$ for some $\epsilon_0 > 0$. Problems of this type exhibit boundary layers usually at both ends of the interval. This trivial one dimensional problem will be used as a motivation for our method. In order to get fast convergence for the Funaro-Quarteroni iterative solver(F.Q) with *no relaxation* parameter, the domain decomposition must be properly designed. We restrict ourselves to the case of a single boundary layer in the neighborhood of 1. According to the

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asymptotic analysis we should split the domain Ω into two subdomains $\Omega_{inner} = (a, 1)$ and $\Omega_{outer} = (0, a)$ where $a > 0$. Ω_{inner} covers the boundary layer at 1 and Ω_{outer} covers the domain of validity of the regular approximation. In order to make it easier to get sharp estimates in the maximum norm, we are going to use the finite difference framework. We keep the mesh in each subdomain regular and adapt the domain decomposition to the boundary layer stiffness. This should be very efficient on a parallel computer. Let us denote h_1 (respectively h_2) the mesh size on Ω_{outer} (respectively Ω_{inner}). Let us denote L^{h_i}, $i = 1, 2$ the discretized operator that corresponds to L_ϵ. We will also restrict ourselves to the case where we have the same asymptotic order of grid points in each subdomain, i.e

$$h_1 \approx \frac{h_2}{1-a} \approx \frac{1}{N},$$

in order to balance the amount of work in each subdomain.

Dirichlet-Neumann Scheme

To solve (2.1), we introduce the following iterative procedure [FQZ88]

$$\left\{ \begin{array}{l} L^{h_1} \phi^p_{outer} = F \ in \ \Omega_{outer}; \\ \phi^p_{outer}(0) = \alpha_0 \ ; \ \phi^p_{outer}(a) = \phi^p_{inner}(a) \\ L^{h_2} \phi^{p+1}_{inner} = F \ in \ \Omega_{inner}; \\ \phi^{p+1}_{inner}(0) = \alpha_1 \ ; \\ \dfrac{\phi^{p+1}_{inner}(a+h_2) - \phi^{p+1}_{inner}(a)}{h_2} = \dfrac{\phi^p_{outer}(a) - \phi^p_{outer}(a-h_1)}{h_1} \end{array} \right. \tag{2.2}$$

To start the scheme, we impose an artificial boundary condition at point a. We use the same finite difference scheme in each subdomain with Dirichlet boundary condition at a in Ω_{outer} and Neumann boundary condition at a in Ω_{inner}.

We will proceed with the analysis of this iterative method in three steps: firstly we define the best interface location between the subdomains, based on a truncation error analysis, secondly we derive from the stability property of the discretized operator the rate of damping of the artificial boundary condition error. Lastly we combine these two results to get an estimate of convergence of the iterative solver to the *exact* solution of the differential problem (2.1).

The technique of demonstration is quite elementary but uses two types of small parameters: first the space steps h_1, second the small singular perturbation parameter ϵ. Our goal is to find the best path in the parameter space (h_1, ϵ) which provides superlinear convergence and optimal uniform approximation.

- **First Step: interface position**

We wish to determine the optimal interface position a, which minimizes the maximum error in both subdomains under the constraint that we have the same asymptotic order of mesh points N inside each subdomain. In this part, we neglect the artificial boundary condition error inherent to the Funaro-Quarteroni alternate $(F.Q)$ procedure. This error will be taken care of later on.

Let ϕ_{outer}(respectively ϕ_{inner}) be the restriction of ϕ to Ω_{outer} (respectively Ω_{inner}).

We define the following errors :

$$e_{outer} = \max_{\Omega_{outer}} |\phi_{outer} - \phi_{outer}^{h_1}|$$
$$e_{inner} = \max_{\Omega_{inner}} |\phi_{inner} - \phi_{inner}^{h_2}|$$

A classical center finite difference scheme applied to $-\epsilon u'' + u = f$ with exact Dirichlet boundary conditions gives

$$e_{outer} \approx \epsilon h_1^2 a^2 \max_{\Omega_{outer}} |\tfrac{d^{(4)}\phi}{dx^4}|$$

The analysis of the inner subdomain approximation with mixed exact boundary conditions gives two truncation errors, which we should consider in addition to the discretisation error of the Neumann boundary condition. We have

$$e_{inner} \approx \epsilon h_2^2 (1-a)^2 \max_{\Omega_{inner}} |\frac{d^{(4)}\phi}{dx^4}| + h_2^2 \frac{R}{1-R} \max_{\Omega_{inner}} |\frac{d^{(2)}\phi}{dx^2}|, \tag{2.3}$$

where $R = 1 + \frac{h_2^2}{2\epsilon} + \frac{h_2}{\sqrt{\epsilon}}(1 + \frac{h_2^2}{2\epsilon})^{\frac{1}{2}}$.

We first notice that the truncation errors defined above depend strongly on the property of the solution that we want to approximate in each subdomain.

Let ϕ_0 be the outer expansion of ϕ and $\Theta(x,\epsilon)$ be the corrector i.e

$$\Theta(x,\epsilon) = \phi(x,\epsilon) - \phi_0(x,\epsilon) \approx exp(-\eta),$$

in the boundary layer with $\eta = \frac{1-x}{\epsilon}$. We show that the truncation error is dominated by the behavior of the corrector as in ([Gar96]).

Secondly we remark that the error in both subdomains is coupled because the Neumann boundary condition for the *inner* domain is only an approximation of a derivative in the *outer* domain. Thys we need to compute directly the error between the exact solution of the continuous problem and the formal limit of (2.2) when $p \to \infty$.

Lemma 1 *Let $\tilde{\phi} = (\tilde{\phi}_{i,j})_{i=0\ldots N, j=1,2}$ be the solution of the following linear system.*

$$\begin{cases} L^{h_1}\tilde{\phi}_{i,1} = F \quad i = 1, \ldots N-1, \\ L^{h_2}\tilde{\phi}_{i,2} = F \quad i = 1, \ldots N-1, \\ \tilde{\phi}_{0,1} = \alpha_0 \ ; \ \tilde{\phi}_{N,1} = \tilde{\phi}_{0,2}; \ \tilde{\phi}_{N,2} = \alpha_1, \\ \dfrac{\tilde{\phi}_{1,2} - \tilde{\phi}_{0,2}}{h_2} = \dfrac{\tilde{\phi}_{N,1} - \tilde{\phi}_{N-1,1}}{h_1}, \end{cases}$$

$$\textit{where} \qquad L^h \phi_i = -\epsilon \tfrac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{h} + \phi_i.$$

Let M be the composite grid $M = M_{outer} \cup M_{inner}$, with

$$\begin{cases} M_{outer} = \{x_{i,1} = i(\frac{a}{N}); i = 0 \ldots N\} \\ M_{inner} = \{x_{i,2} = a + i(\frac{1-a}{N}); i = 0 \ldots N\} \end{cases}$$

Let us suppose that: $N^{-1} \approx \sqrt{\epsilon}\delta$ with $\delta \succ\succ 1$ and $\| \ \|_\infty$ be the maximum norm on the composite grid M.

Under the previous hypothesis concerning the discretization and approximation of the operators in each subdomain, $\| \phi - \tilde{\phi} \|_\infty$ is asymptotically minimum when $1 - a \sim \sqrt{\epsilon}\log(\epsilon^{-2})$

• **second step: Damping of artificial boundary errors**

The convergence of the method depends essentially on the way an error which is introduced at the artificial interface propagates inside the subdomain.

In [Gar96] it is shown that we may have fast convergence with relatively small overlap even if we apply the straightforward Schwarz alternate procedure with Dirichlet boundary conditions. We will prove that the F.Q procedure may also have fast convergence and that the choice of boundary conditions is critical.

Let us consider the F.Q iterative procedure applied to the following homogeneous problem:

$$\left\{\begin{array}{c} L_1^h e_{i,1}^p = 0 \ ; \\ e_{0,1}^p = 0 \ ; \ e_{N,1}^p = e_{0,2}^{p-1} \ ; \\ L_2^h e_{N,2}^p = 0 \\ \frac{e_{1,2}^p - e_{0,2}^p}{h_2} = \frac{e_{N,1}^p - e_{N-1,1}^p}{h_1} \ ; e_{N,2}^p = 0 \ ; \end{array}\right.$$

with a domain decomposition given by (Lemma 1), i.e $b = 1 - a \approx \sqrt{\epsilon}\log\epsilon^{-1}$. The discretized operator satisfies a maximum principle and we can show that:

$$|e_{0,2}^p| = |e_{N,1}^{p+1}| \geq max_{i=0,\ldots N-1}|e_{i,1}^{p+1}|$$

and

$$|e_{0,2}^p| \geq max_{i=0,\ldots,N}|e_{i,2}^p|.$$

We will call *damping factor* a real ξ such that: $|e_{0,2}^{p+1}| \leq \xi|e_{0,2}^p|$, $\forall p$.

Lemma 2 *Let ϕ_{outer}^p and ϕ_{inner}^p defined by the iterative scheme:*

$$\left\{\begin{array}{l} L^{h_1}\phi_{outer}^p = F \ in \ \Omega_{outer}; \\ \phi_{outer}^p(0) = \alpha_0 \ ; \ \phi_{outer}^p(a) = \phi_{inner}^p(a) \\ L^{h_2}\phi_{inner}^{p+1} = F \ in \ \Omega_{inner}; \\ \phi_{inner}^{p+1}(0) = \alpha_1 \ ; \\ \dfrac{\phi_{inner}^{p+1}(a+h_2) - \phi_{inner}^{p+1}(a)}{h_2} = \dfrac{\phi_{outer}^p(a) - \phi_{outer}^p(a-h_1)}{h_1}. \end{array}\right.$$

Let a be the interface position between the subdomains such that $1 - a \approx \epsilon^{1/2}\log\epsilon^{-1}$. Suppose that $N^{-1} \approx \epsilon^{\frac{1}{2}}\delta$, with $\delta \succ\succ 1$. Then the amplification factor of the iterative scheme is: $\xi \approx \delta^{-1}$.

• **third step: Convergence to the solution of the ODE problem and uniform approximation**

Theorem 1 *Let ϕ be the solution of the Dirichlet problem*

$$L[\phi] = -\epsilon\phi'' + \phi = F \ ; \ \phi(0) = \alpha_0, \ \phi(1) = \alpha_1$$

$$Let \qquad \phi^p = \left\{\begin{array}{ll} \phi_{outer}^p & on \quad M_{outer} \\ \phi_{inner}^p & on \quad M_{inner} \end{array}\right.$$

Let $\| \ \|_\infty$ be the maximum norm on the composite grid $M_{outer} \cup M_{inner}$.
Let us suppose that: $N^{-1} \approx \sqrt{\epsilon}\delta, \ ; \ b \sim \sqrt{\epsilon}\log(\epsilon^{-2})$ *with* $\delta \succ\succ 1$
Then

$$\|\phi - \phi^p\|_\infty \leq C(\xi^p + \epsilon\delta)),$$

$$with \qquad \xi \sim \delta^{-1}.$$

Neumann-Dirichlet Scheme

We are going to show that the choice of the boundary conditions at the artificial interface is critical. Let us consider now the F.Q method with Neumann boundary condition at a in Ω_{outer} and the Dirichlet boundary condition at a in Ω_{inner}. The scheme gives:

$$\left\{ \begin{array}{l} L^{h_1}\phi^p_{outer} = F \quad \text{on} \quad \Omega_{outer}, \\ \dfrac{\phi^{p+1}_{outer}(a) - \phi^{p+1}_{outer}(a-h_1)}{h_1} = \dfrac{\phi^p_{inner}(a+h_2) - \phi^p_{inner}(a)}{h_2} \quad ; \quad \phi^p_{outer}(0) = \alpha_0 \ , \\ L^{h_2}\phi^p_{inner} = F \quad \text{on} \quad \Omega_{inner}, \\ \phi^p_{inner}(1) = \alpha_1 \quad ; \quad \phi^p_{inner}(a) = \phi^p_{outer}(a). \end{array} \right. \tag{2.4}$$

To start the scheme, we impose an artificial boundary condition at point a. We show that the best choice for the interface position in terms of accuracy is $1-a \approx \sqrt{\epsilon}\log\epsilon^{-1}$ since the formal limit of (2.4) is identical to the formal limit of (2.2). However, this procedure is then highly unstable:

Theorem 2 *Let us assume that $h_1 \succ\succ h_2$ and $h_1 \succ\succ \sqrt{\epsilon}$. Then the amplification factor of the iterative procedure satisfies $\xi \sim \frac{h_1}{h_2}$ and the F.Q procedure with no relaxation is highly unstable.*

With the same principle as below, we proved that we obtain a fast convergence with a good approximation on problems having different operators for each subdomain with the Neumann-Dirichlet scheme. Therefore we found the F.Q. algorithm very interesting for singularly perturbed transmission problems for which the overlapping domain decomposition technique does not be used.

Heterogeneous Domain Decomposition
In this section, we consider a linear second-order transmission problem of the following type:

$$\left\{ \begin{array}{l} L_1\phi = -\epsilon\phi'' + \phi = F \ in \ \Omega_1 = (0, A); \\ L_2\psi = \psi'' = G \ in \ \Omega_2 = (A, 1); \\ \phi(A) = \psi(A); \ \phi'(A) = \psi'(A); \\ \phi'(0) = 0 \ ; \ \psi(1) = 0. \end{array} \right. \tag{2.5}$$

ϵ is a small positive parameter, $\epsilon \in]0, \epsilon_0]$ for some $\epsilon_0 > 0$. In addition we assume the compatibility condition that all derivatives of F vanish in 0. This very simple model is introduced to study the convergence of an heterogeneous domain decomposition based on the F.Q method. We observe that the domain decomposition is dictated by the definition of the transmission problem and that there is no overlap of the subdomains on A.

Asymptotic Analysis

We studied the boundary layer of (2.5) in ([DLTO$^+$96]) and observed that it is a singular perturbation problem with a weak layer of $\sqrt{\epsilon}$ thickness located to the left of A.

First Numerical Procedure

The asymptotic analysis suggests that the computation domain should be split into three subdomains $\Omega_1 = (O, B)$, $\Omega_2 = (B, A)$ and $\Omega_3 = (A, 1)$ where the intermediate subdomain is used to resolve the boundary layer. We assume that F vanishes in the neighbourhood of 0 and that the space step h_i for each subdomain satisfies the asymptotic relation $h_1 \approx \frac{h_2}{A-B} \approx h_3 \approx \frac{1}{N}$. with $b = A - B \prec\prec 1$.

We are going to study the heterogeneous F.Q procedure for such problem. According to the previous analysis, we adopted the F.Q procedure with the D-N boundary conditions to resolve the layer and with the N-D boundary conditions to resolve the transmission problem. The iteration procedure is as follows:

$$\left\{\begin{array}{l} L_1^{h_1}\phi_1^p = F \ in \ \Omega_1; \\ \phi_1^p(0) = \phi_1^p(h_1) \ ; \ \phi_1^p(B) = \phi_2^p(B); \\ L_2^{h_3}\psi^p = G \ in \ \Omega_3; \\ \psi^p(A) = \phi_2^p(A) \ ; \ \psi^p(1) = 0; \\ L_1^{h_2}\phi_2^{p+1} = F \ in \ \Omega_2; \\ \dfrac{\phi_2^{p+1}(B+h_2) - \phi_2^{p+1}(B)}{h_2} = \dfrac{\phi_1^p(B) - \phi_1^p(B-h_1)}{h_1}; \\ \dfrac{\phi_2^{p+1}(A) - \phi_2^{p+1}(A-h_2)}{h_2} = \dfrac{\psi^p(A+h_3) - \psi^p(A)}{h_3}. \end{array}\right. \quad (2.6)$$

The proof of convergence of this scheme is very similar to that of the previous section. It can be proved that:

Lemma 3 *Let $(\tilde\phi, \tilde\psi)$ with $\tilde\phi = (\tilde\phi_{i,j})_{i=0\ldots N, j=1,2}$ and $\tilde\psi = (\tilde\psi_i)_{i=0\ldots N}$ be the solution of the linear system that is the formal limit of (2.6) when $p \to \infty$. Let M be the composite grid $M = M_1 \cup M_2 \cup M_3$, with*

$$\left\{\begin{array}{l} M_1 = \{x_{i,1} = i(\frac{B}{N}); i = 0\ldots N\} \\ M_2 = \{x_{i,2} = B + i(\frac{A-B}{N}); i = 0\ldots N\} \\ M_3 = \{x_{i,2} = A + i(\frac{1-A}{N}); i = 0\ldots N\} \end{array}\right.$$

Let us suppose that $N^{-1} \approx \sqrt{\epsilon}\delta$, with $\delta \succ\succ 1$. Let $\| \, \|_\infty$ be the maximum norm on the composite grid M.

Under the previous hypothesis concerning the discretization and approximation of the operators in each subdomain, $\max(\| \phi - \tilde\phi \|_\infty, \| \psi - \tilde\psi \|_\infty)$ *is asymptotically minimum when* $b = A - B \approx \sqrt{\epsilon}\log(\epsilon^{-1})$

Proof: see ([DLTO+96])
We have then the following convergence property of the iterative scheme (2.6),

Lemma 4 *Let B be the interface position defined as in Lemma 3. Suppose that $N^{-1} \approx \epsilon^{\frac{1}{2}}\delta$, with $\delta \succ\succ 1$. Then the amplification factor of the iterative scheme is:*

$$\xi \approx \delta^{-1}.$$

Proof: We only need to look at the following homogeneous problem,

$$\left\{\begin{array}{l} L_1^{h_1} e_1^p = 0 \ in \ \Omega_1; \\ e_{1,1}^p = e_{0,1}^p \ ; \ e_{N,1}^p = e_{0,2}^p; \\ L_2^{h_3} e_3^p = 0 \ in \ \Omega_3; \\ e_{0,3}^p = e_{N,2}^p \ ; \ e_{N,3}^p = 0; \\ L_1^{h_2} e_2^{p+1} = 0 \ in \ \Omega_2; \\ \dfrac{e_{1,2}^{p+1} - e_{0,2}^{p+1}}{h_2} = \dfrac{e_{N,1}^p - e_{N-1,1}^p}{h_1}; \\ \dfrac{e_{N,2}^{p+1} - e_{N-1,2}^{p+1}}{h_2} = \dfrac{e_{1,3}^p - e_{0,3}^p}{h_3}. \end{array}\right. \tag{2.7}$$

We obtain for the first subdomain: $e_{i,1}^p = e_{N,1}^p \frac{R^{i-1}+R^{-i}}{R^{N-1}+R^{-N}}, \ \forall i,$
with $R = 1 + \frac{h_1^2}{2\epsilon} + \frac{h_1}{\sqrt{\epsilon}}\sqrt{1+\frac{h_1^2}{2\epsilon}} \approx \delta^2$. We have then

$$e_{i,1}^p \prec\prec e_{N,1}^p, \ \forall i < N.$$

For the second subdomain, we have $e_{i,3}^p = \frac{N-i}{N} e_{0,3}^p$, and then $e_{i,N}^p \le e_{0,3}$.
And for the third subdomain:

$$e_{i,2}^p = h_2 \frac{e_{0,2}^p}{h_1}\left(\frac{R_*^{i-N+1}}{(R_*-1)(R_*^{-N+1}-R_*^{N-1})} + \frac{R_*^{N-1-i}}{(R_*^{-1}-1)(R_*^{N-1}-R_*^{-N+1})}\right)$$

$$-h_2 e_{0,3}^p \left(\frac{R_*^i}{(R_*-1)(R_*^{-N+1}-R_*^{N-1})} + \frac{R_*^{-i}}{(R_*^{-1}-1)(R_*^{N-1}-R_*^{-N+1})}\right),$$

where $R_* = 1 + \frac{h_2^2}{2\epsilon} + \frac{h_2}{\sqrt{\epsilon}}\sqrt{1+\frac{h_2^2}{2\epsilon}}$.

Using $R_* - 1 \approx \frac{h_2}{\sqrt{\epsilon}}$, we obtain then

$$e_{0,2}^{p+1} \approx \delta^{-1} e_{0,2}^p + 2\sqrt{\epsilon}\exp\left(-\tfrac{b}{\sqrt{\epsilon}}\right) e_{N,2}^p,$$
$$e_{N,2}^{p+1} \approx \sqrt{\epsilon} e_{N,2}^p + 2\sqrt{\epsilon}\exp\left(-\tfrac{b}{\sqrt{\epsilon}}\right) \frac{e_{0,2}^p}{h_1}.$$

We conclude that the amplification factor of the method is then asymptotically δ^{-1}.
Combining Lemma 3 and Lemma 4 we have finally,

Theorem 3 *With the notations defined above, we have:*

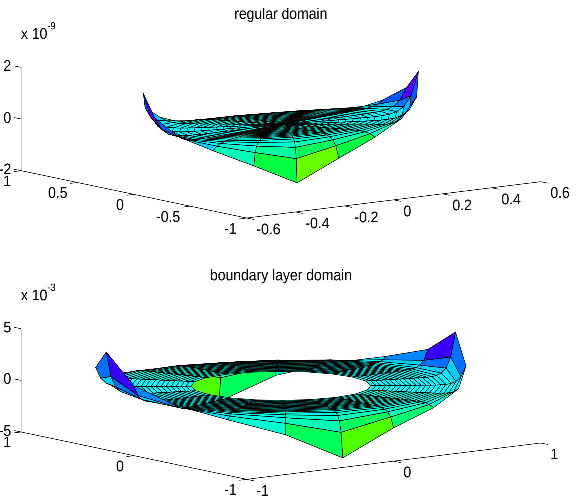
$$\max(\|\phi - \phi^p\|_\infty, \|\psi - \psi^p\|_\infty) \le C(\xi^p + \epsilon\delta^2), \quad with \quad \xi \sim \delta^{-1}.$$

Proof: The proof is a straightforward application of Lemma 3 and Lemma 4.

Composite Method: Schwarz and F.Q.

Let us use now the *Schwarz alternate procedure* to solve the layer. We keep the F.Q scheme with N - D boundary conditions, solving for the transmission condition in A. We restrict ourselves to an overlap minimum i.e one cell of step h, between $\Omega_1 = [0, a]$ and $\Omega_2 = [b, 1]$ $(with \quad 0 < b < a < 1)$.
Furthermore, to simplify the demonstration, we impose that the grids of the subdomains Ω_1 and Ω_2 coincide at the boundary points.
It can be proved that $\max(\| \phi - \tilde{\phi}\|_\infty, \| \psi - \tilde{\psi}\|_\infty)$ is asymptotically minimum when

Figure 1 solution in metal domain



$$A - b \approx \sqrt{\epsilon}\log(\epsilon^{-1})$$

We have then the following convergence property of the iterative scheme.

Lemma 5 *Let B the interface position defined above. Suppose that* $N^{-1} \approx \epsilon^{\frac{1}{2}}\delta$, *with* $\delta \succ\succ 1$. *Then the amplification factor of the iterative scheme is:*

$$\xi \approx \sqrt{\epsilon}\log(\epsilon^{-1})\delta$$

Combining these results, we have finally.

Theorem 4 *With the notations defined above, applying the F.Q and Schwarz mixed method, we have:*
$\max(\|\phi - \phi^p\|_\infty, \|\psi - \psi^p\|_\infty) \leq C(\xi^p + \epsilon\delta^2), \qquad with \qquad \xi \sim \delta\epsilon\log\epsilon^{-1}$

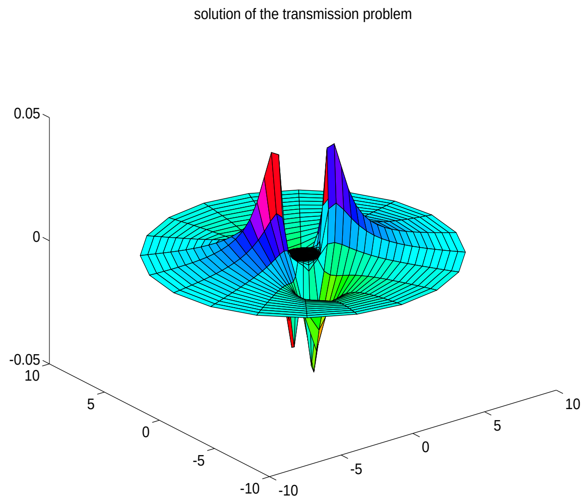
3 Boundary Layers in Two-dimensional Space

Applying some comparison lemmas, we can extend all previous results obtained in a one dimensional space to a two dimensional space with strip domain decomposition. We referred to preprint ([GVC96]) for the detailed of the analysis. We have also applied them to a two dimensional singular perturbed transmission problem that arises in electromagnetic theory [Cou92]. The model is as follows:

$$\begin{cases} -\epsilon\Delta u + a(r,\theta)u = 0 \ on\ \Omega_1 =]0, A[\times[0, 2\pi[, \\ -\Delta u = j(r,\theta)\ on\ \Omega_2 =]A, 1[\times[0, 2\pi[, \\ u_-(A) = u_+(A); u'_-(A) = u'_+(A); \\ u(R_\infty, \theta) = 0\ for\ \theta \in (0, 2\pi), \end{cases}$$

where Ω_1 is a disk of radius one, Ω_2 is a ring for $r \in (1, R_\infty)$; typically j represents the current density in the inductor, Ω_1 is the domain of the liquid metal, Ω_2 is the

Figure 2 global solution



domain with no conduction, and the boundary layer in Ω_1 corresponds to the well known skin effect. This problem has then been numerically efficiently solved using three subdomains with regular finite difference meshes inside each subdomain, and a very large aspect ratio of the mesh width between the subdomains according to our a priori analysis. The method can then be parallelized at various levels but it is still useless since our test case is a small academic problem. Figure1 shows the solution in domain Ω_1 with the domain decomposition that corresponds to the regular part and the boundary layer. Figure2 shows the global solution. We have also tested our method with finite element discretization and unstructured grids using *modulef* [BPA88]; we keep the radius of the elements per subdomain asymptotically equivalent to the space grid used on the finite difference scheme and find good agreement.

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