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# Overlapping Schwarz for Parabolic Problems

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## 1 Introduction

The basic ideas underlying waveform relaxation were first suggested in the late 19th century by Picard [Pic93] and Lindelöf [Lin94] to study initial value problems from a theoretical viewpoint. Much recent interest in waveform relaxation as a practical parallel method for the solution of stiff ordinary differential equations (ODEs) has been generated by the publication of a paper by Lelarasmee and coworkers [LRSV82] in the VLSI literature. Recent work in this field includes papers by Miekkala and Nevanlinna [MN87], Nevanlinna [Nev89, Nev90], Bellen and Zennaro [BZ93], Reichelt, White and Allen [RWA95], Jeltsch and Pohl [JP95], Burrage [Bur95] and Lumsdaine, Reichelt, Squyres and White [LRSW96].

There are two classical convergence results for waveform relaxation algorithms for ODEs: (i) for linear systems of ODEs on unbounded time intervals one can show linear convergence of the algorithm under some dissipation assumptions on the splitting; (ii) for nonlinear systems of ODEs (including linear ones) on bounded time intervals one can show superlinear convergence assuming a Lipschitz condition on the splitting function.

For classical relaxation methods (Jacobi, Gauss Seidel, SOR) the above convergence results depend on the discretization parameter if the ODE arises from a partial differential equation (PDE) which is discretized in space. The convergence rates deteriorate as one refines the mesh.

Jeltsch and Pohl propose in [JP95] a multi-splitting algorithm with overlap. They prove results (i) and (ii) for their algorithm, but the convergence rates are mesh-dependent. However they show numerically that increasing the overlap accelerates the convergence of the waveform relaxation algorithm. We quantify their numerical results by formulating the waveform relaxation algorithm at the space-time continuous level using overlapping domain decomposition; this approach was motivated by the work of Bjørhus [Bjø95]. We show linear convergence of this algorithm on unbounded time intervals at a rate depending on the size of the overlap. This is an extension of the first classical convergence result (i) for waveform relaxation from ODEs to PDEs.

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Discretizing the algorithm, the size of the physical overlap corresponds to the overlap of the multi-splitting algorithm analyzed by Jeltsch and Pohl. We show furthermore that the convergence rate is robust with respect to mesh refinement, provided the physical overlap is held constant during the refinement process. The details of the analysis can be found in [GS97].

Independently Giladi and Keller [GK97] studied superlinear convergence of domain decomposition algorithms for the convection-diffusion equation on bounded time intervals, hence generalizing the second classical waveform relaxation result (ii) from ODEs to PDEs.

## 2 Continuous Case

Consider the one-dimensional inhomogeneous heat equation on the interval $[0, L]$,

$$\begin{array}{rcll} \dfrac{\partial u}{\partial t} & = & \dfrac{\partial^2 u}{\partial x^2} + f(x,t) & 0 < x < L,\ t > 0 \\ u(0,t) & = & g_1(t) & t > 0 \\ u(L,t) & = & g_2(t) & t > 0 \\ u(x,0) & = & u_0(x) & 0 < x < L, \end{array} \tag{1}$$

where we assume enough smoothness on the data such that (1) has a unique bounded solution [Can84]. Given any function $f(t) : \mathbb{R}^+ \longrightarrow \mathbb{R}$ we define

$$||f(\cdot)||_\infty := \sup_{t>0} |f(t)|.$$

We decompose the domain $\Omega = [0, L] \times [0, \infty)$ into two overlapping subdomains $\Omega_1 = [0, \beta L] \times [0, \infty)$ and $\Omega_2 = [\alpha L, L] \times [0, \infty)$, where $0 < \alpha < \beta < 1$. The solution $u(x, t)$ of (1) can now be obtained by composing the solutions $v(x, t)$ on $\Omega_1$ and $w(x, t)$ on $\Omega_2$, which satisfy the same inhomogeneous heat equation on the subdomains with the new interior boundary conditions $v(\beta L, t) = w(\beta L, t)$ and $w(\alpha L, t) = v(\alpha L, t)$, respectively. Note that $v(x, t) \equiv w(x, t)$ in the overlap. The system, which is coupled through the boundary, can be solved using an alternating Schwarz iteration, where the new function $v^{k+1}(x, t)$ on $\Omega_1$ is obtained using the previous iterate $w^k(x, t)$ at the interior boundary and similarly on $\Omega_2$. Let $d^k(x, t) := v^k(x, t) - v(x, t)$ and $e^k(x, t) := w^k(x, t) - w(x, t)$ and consider the error equations

$$\begin{array}{rcll} \dfrac{\partial d^{k+1}}{\partial t} & = & \dfrac{\partial^2 d^{k+1}}{\partial x^2} & 0 < x < \beta L,\ t > 0 \\ d^{k+1}(0,t) & = & 0 & t > 0 \\ d^{k+1}(\beta L,t) & = & e^k(\beta L,t) & t > 0 \\ d^{k+1}(x,0) & = & 0 & 0 < x < \beta L \end{array} \tag{2}$$

and

$$\begin{array}{rcll} \dfrac{\partial e^{k+1}}{\partial t} & = & \dfrac{\partial^2 e^{k+1}}{\partial x^2} & \alpha L < x < L,\ t > 0 \\ e^{k+1}(\alpha L,t) & = & d^k(\alpha L,t) & t > 0 \\ e^{k+1}(L,t) & = & 0 & t > 0 \\ e^{k+1}(x,0) & = & 0 & \alpha L < x < L. \end{array} \tag{3}$$

Given any function $g(x,t) : [a,b] \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ we define

$$||g(\cdot,\cdot)||_{\infty,\infty} := \sup_{a<x<b,t>0} |g(x,t)|.$$

**Theorem 2.1** *The Schwarz iteration for the heat equation with two subdomains converges at a rate depending on the size of the overlap. The error on the two subdomains decays at the rate*

$$\begin{aligned} ||d^{2k+1}(\cdot,\cdot)||_{\infty,\infty} &\leq \left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^k ||e^0(\beta L,\cdot)||_\infty && (4)\\ ||e^{2k+1}(\cdot,\cdot)||_{\infty,\infty} &\leq \left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^k ||d^0(\alpha L,\cdot)||_\infty . && (5)\end{aligned}$$

**Proof** The proof is obtained using the maximum principle of the heat equation and can be found in [GS97]. ■

## 3 Semi-Discrete Case

Consider the heat equation continuous in time, but discretized in space using a centered second order finite difference scheme on a grid with $n$ grid points and $\Delta x = \frac{L}{n+1}$. This gives

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} &= A_{(n)}\boldsymbol{u} + \boldsymbol{f}(t) \qquad t > 0\\ \boldsymbol{u}(0) &= \boldsymbol{u}_0, \end{aligned} \tag{6}$$

where the $n \times n$ matrix $A_{(n)}$ is given by

$$A_{(n)} = \frac{1}{(\Delta x)^2}\begin{bmatrix} -2 & 1 & & 0\\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1\\ 0 & & 1 & -2 \end{bmatrix} \tag{7}$$

and $\boldsymbol{f}(t) = (f(\Delta x,t) + \frac{g_1(t)}{(\Delta x)^2}, f(2\Delta x,t),\ldots,f((n-1)\Delta x,t), f(n\Delta x,t) + \frac{g_2(t)}{(\Delta x)^2})^T$, $\boldsymbol{u}_0 = (u_0(\Delta x),\ldots,u_0(n\Delta x))^T$.

We decompose the domain into two overlapping subdomains $\Omega_1$ and $\Omega_2$. We assume for simplicity that $\alpha L$ falls on the grid point $i = a$ and $\beta L$ on the grid point $i = b$. We therefore have $a\Delta x = \alpha L$ and $b\Delta x = \beta L$. As in the continuous case, the solution $\boldsymbol{u}(t)$ of (6) can be obtained by composing the solutions $\boldsymbol{v}(t)$ on $\Omega_1$ and $\boldsymbol{w}(t)$ on $\Omega_2$, which satisfy the corresponding equations on the subdomains. Applying a Schwarz iteration one obtains the error equations

$$\begin{aligned} \frac{\partial \boldsymbol{d}^{k+1}}{\partial t} &= A_{(b-1)}\boldsymbol{d}^{k+1} + \boldsymbol{f}^{(e^k)} \qquad t > 0\\ \boldsymbol{d}^{k+1}(0) &= \boldsymbol{0} \end{aligned} \tag{8}$$

with $\boldsymbol{f}^{(e^k)} = (0, \ldots, 0, \frac{\boldsymbol{e}^k(b-a,t)}{(\Delta x)^2})^T$ and

$$\begin{aligned} \frac{\partial \boldsymbol{e}^{k+1}}{\partial t} &= A_{(n-a)}\boldsymbol{e}^{k+1} + \boldsymbol{f}^{(d^k)} \qquad t > 0 \\ \boldsymbol{e}^{k+1}(0) &= \boldsymbol{0} \end{aligned} \tag{9}$$

with $\boldsymbol{f}^{(d^k)} = (\frac{\boldsymbol{d}^k(a,t)}{(\Delta x)^2}, 0, \ldots, 0)^T$.

Given any vector valued function $\boldsymbol{h}(t) : \mathbb{R}^+ \longrightarrow \mathbb{R}^n$ we define

$$||\boldsymbol{h}(\cdot,\cdot)||_{\infty,\infty} := \max_{1<j<n} \sup_{t>0} |\boldsymbol{h}(j,t)|,$$

where $\boldsymbol{h}(j,t)$ denotes the $j$-th component of the vector $\boldsymbol{h}(t)$.

**Theorem 3.1** *The Schwarz iteration for the semi-discrete heat equation with two subdomains converges at a rate depending on the size of the overlap. The error on the two subdomains decays at the rate*

$$\begin{aligned} ||\boldsymbol{d}^{2k+1}(\cdot,\cdot)||_{\infty,\infty} &\leq \left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^k ||\boldsymbol{e}^0(b-a,\cdot)||_\infty \\ ||\boldsymbol{e}^{2k+1}(\cdot,\cdot)||_{\infty,\infty} &\leq \left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^k ||\boldsymbol{d}^0(a,\cdot)||_\infty. \end{aligned}$$

**Proof** The proof uses the discrete maximum principle and follows as in the continuous case [GS97]. ■

The results shown for two subdomains can be generalized to an arbitrary number of subdomains, although the analysis is more involved. The theorems corresponding to Theorem 2.1 and 3.1, and their proofs, can be found in [GS97].

## 4 The Algorithm in the Framework of Waveform Relaxation

For a linear initial value problem

$$\frac{d\boldsymbol{u}(t)}{dt} = A\boldsymbol{u}(t) + \boldsymbol{f}(t), \quad \boldsymbol{u}(0) = \boldsymbol{u}_0$$

the standard waveform relaxation algorithm is based on a splitting of the matrix $A$ into $A = M + N$, which yields

$$\frac{d\boldsymbol{u}(t)}{dt} = M\boldsymbol{u}(t) + N\boldsymbol{u}(t) + \boldsymbol{f}(t), \quad \boldsymbol{u}(0) = \boldsymbol{u}_0.$$

This system of ODEs is solved using an iteration of the form

$$\frac{d\boldsymbol{v}^{k+1}}{dt} = M\boldsymbol{v}^{k+1} + N\boldsymbol{v}^k + \boldsymbol{f}, \quad \boldsymbol{v}^{k+1}(0) = \boldsymbol{u}_0, \tag{10}$$

where the starting function $\boldsymbol{v}^0(t)$ is usually chosen to be constant. In the case of Block-Jacobi the matrix $M$ is chosen to be block diagonal, for example for two subblocks

$$M = \begin{bmatrix} D_1 & \\ & D_2 \end{bmatrix}, \tag{11}$$

and $N$ contains the remaining off diagonal blocks. This allows for solution of the subsystems $D_i$, $i = 1,2$ in equation (10) in parallel. In the case where $A$ equals $A_{(n)}$ from the semi-discrete heat equation (6), the waveform relaxation algorithm with Block-Jacobi splitting computes the same iterates as the Schwarz domain decomposition algorithm presented in subsection 3 with overlap $\Delta x$ (i.e. one grid point only). This result can be generalized to an arbitrary number of subdomains, as shown in [GS97].

To extend this analogy to arbitrary overlaps, the concept of *multi-splittings* is needed, which was first introduced by O'Leary and White in [OW85] for solving large systems of linear equations on a parallel computer. Jeltsch and Pohl generalized multi-splittings to linear systems of ODEs and waveform relaxation in [JP95].

Let $A$, $M_i$, $N_i$ and $E_i$, $i = 1,2$ be real $n \times n$ matrices. The set of ordered triples $(M_i, N_i, E_i)$ for $i = 1,2$ is called a *multi-splitting* of $A$ if

1. $A = M_i - N_i \quad$ for $i = 1,2$.
2. The matrices $E_l$ are nonnegative diagonal matrices and satisfy

$$E_1 + E_2 = I. \tag{12}$$

Using the waveform relaxation algorithm, we get two new approximations $\boldsymbol{v}_1^{k+1}$ and $\boldsymbol{v}_2^{k+1}$ at each step according to

$$\frac{d\boldsymbol{v}_i^{k+1}}{dt} = M_i \boldsymbol{v}_i^{k+1}(t) + N_i \boldsymbol{v}_i^k + \boldsymbol{f}_i, \quad \boldsymbol{v}_i^{k+1}(0) = \boldsymbol{u}_0,\ i = 1,2 \tag{13}$$

which are combined using the matrices $E_i$ to form a new approximation $\boldsymbol{v}^{k+1}$ by $\boldsymbol{v}^{k+1} = E_1 \boldsymbol{v}_1^{k+1} + E_2 \boldsymbol{v}_2^{k+1}$. Note that the two equations in (13) can be solved in parallel and in addition, components of $\boldsymbol{v}_i^{k+1}$ where $E_i$ has a zero on the diagonal do not have to be computed at all provided they do not couple to other components of $\boldsymbol{v}_i^{k+1}$ where $E_i$ has a non zero diagonal entry. Jeltsch and Pohl prove in [JP95] that the multi-splitting algorithm converges superlinearly on a finite time interval for all splittings and matrices $A$, and linearly on an infinite time interval if $A$ is an M-matrix and the splitting is an M-splitting. However in the case of the semi-discrete heat equation, the rate of convergence in their analysis depends on $\Delta x$ since their level of generality includes the Schwarz method with one grid point overlap and spectral radius $1 - O(\Delta x^2)$ - the block Jacobi algorithm (11). Jeltsch and Pohl also observe, on the basis of numerical experiments, that increasing the overlap accelerates the convergence rate of the algorithm. Our analysis substantiates and quantifies this observation in the specific case of the heat equation, since the $E_i$ can be chosen in such a way that the domain decomposition algorithm described in the previous section is recovered. Choose the two splittings of $A$ according to the two subdomains of the domain decomposition and let $E_i$ have the value one on the diagonal in the interior of the corresponding subdomain $\Omega_i$, including the first point of the overlap, some arbitrary distribution in the overlap satisfying (12) and zero in the interior of the other subdomain. Then the intermediate solutions $\boldsymbol{v}_i^{k+1}$ computed by the multi-splitting algorithm for the heat equation are identical to the solutions computed by the domain decomposition algorithm described in the previous section. Thus, in this case, multi-splitting gives a $\Delta x$ independent rate of convergence.

Note that one could save half of the computation time by computing only even iterates on $\Omega_1$ and odd iterates on $\Omega_2$ or vice versa, since these two solution sequences are independent of one another. In the terminology of Domain Decomposition this would correspond to the multiplicative Schwarz algorithm with red-black ordering whereas the multi-splitting algorithm corresponds to the additive Schwarz algorithm.

The important point here is that our algorithm converges linearly, independent of the mesh size, on unbounded time intervals. Thus for certain PDEs the analysis of Jeltsch and Pohl can be refined to give $\Delta x$ independent rates of convergence if sufficient overlap is used.

## 5 Numerical Experiments

We perform numerical experiments to measure the actual convergence rate of the algorithm. We consider first the linear example problem

$$\begin{array}{rcll} \dfrac{\partial u}{\partial t} & = & \dfrac{\partial^2 u}{\partial x^2} + 5e^{-(t-2)^2-(x-\frac{1}{4})^2} & 0<x<1,\ 0<t<3 \\ u(0,t) & = & 0 & 0<t<3 \\ u(1,t) & = & e^{-t} & 0<t<3 \\ u(x,0) & = & x^2 & 0<x<1. \end{array} \tag{14}$$

To solve the semi-discrete heat equation (6), (7), we use the backward Euler method in time. The experiment is done splitting the domain $\Omega = [0,1] \times [0,3)$ into the two subdomains $\Omega_1 = [0,\alpha] \times [0,3)$ and $\Omega_2 = [\beta,1] \times [0,3)$ for three pairs of values $(\alpha,\beta) \in \{(0.4,0.6),(0.45,0.55),(0.48,0.52)\}$. As initial guess for the iteration we use the constant value 1. Figure 1 shows the convergence of the algorithm at the grid point $b$ for $\Delta x = 0.01$ and $\Delta t = 0.01$. The solid line is the predicted bound on the convergence rate according to Theorem 3.1 and the dashed line is the measured one. The measured error displayed is the difference between the numerical solution on the whole domain and the solution obtained from the domain decomposition algorithm. We also checked the robustness of the method by refining the time step and obtained similar results.

Now consider the nonlinear example problem

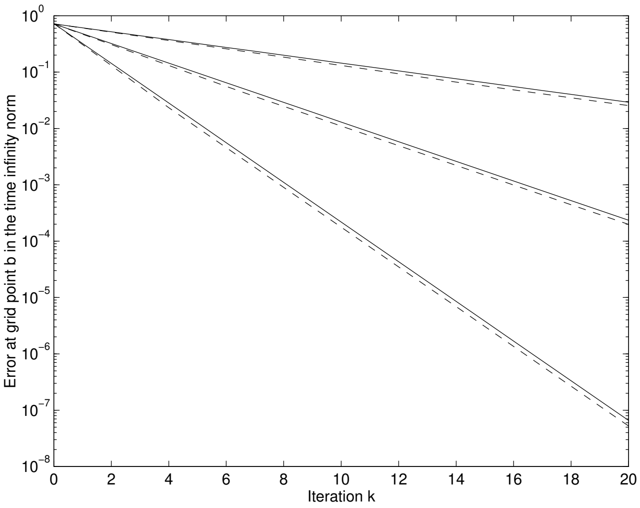
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 5(u-u^3) \qquad 0<x<1,\ 0<t<3 \tag{15}$$

with the same initial and boundary conditions as in the linear case. We discretize in space as before and use the backward Euler method in time for the Laplacian, keeping the nonlinear part explicit. Figure 2 shows the convergence of the algorithm at the grid point $b$ for $\Delta x = 0.01$ and $\Delta t = 0.01$ using the same overlaps as in the linear case.

## 6 Conclusion

Although the analysis presented is restricted to the one-dimensional heat equation, the underlying ideas are more general. As suggested by the nonlinear example, the

**Figure 1** Theoretical and measured decay rate of the error for two subdomains and three different sizes of the overlap for the linear example problem




analysis can be generalized to nonlinear problems, convection-diffusion equations, variable coefficients, and higher dimensions; this is the subject of ongoing research.

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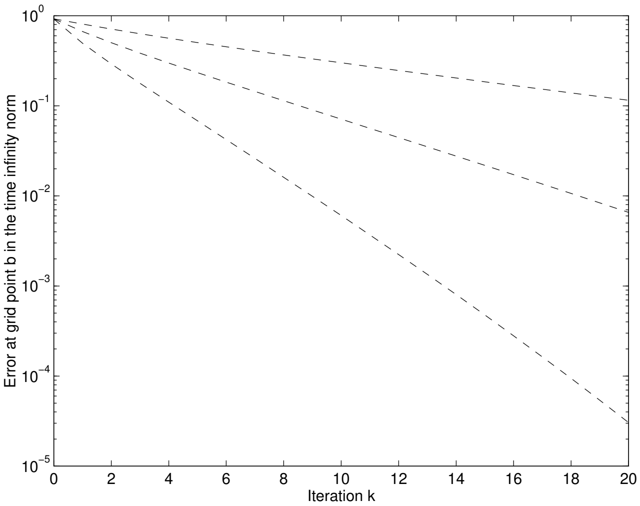
**Figure 2** Measured decay rate of the error for two subdomains and three different sizes of the overlap for the nonlinear example problem





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