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Domain Decompositions of Wave Problems Using a Mixed Finite Element Method

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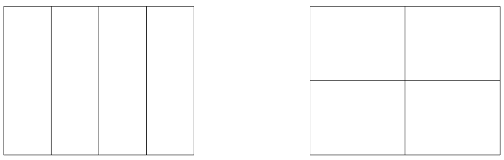
1 Introduction

In this article we discuss the numerical solution of the *wave equation* by *domain decomposition methods.* Such methods, for the numerical solution of partial differential equations, have become very popular in recent years due to the emergence of parallel computers. While most of the emphasis has been on elliptic and parabolic problems, a few authors ([MS87, Far91, DG93, Dup94]) have considered the hyperbolic case. We will discuss the domain decomposition solution of a non-constant coefficient wave equation, with (first order) absorbing boundary conditions, using a mixed finite element formulation. The mixed formulation, in addition to obtaining accurate gradient approximations, will better handle problems with rapidly varying or discontinuous coefficients. The mixed formulation also allows us to treat both striped and box decompositions (Figure 1) in the same manner. This is in contrast to a conforming method where the intersection of the interfaces, in a box decomposition, can present additional complexity. (This difficulty, and a remedy, is discussed in [DG93].) For the mixed method, interface conditions will be treated by a method combining Lagrange multipliers and a conjugate gradient algorithm. The results of numerical experiments will be presented.

Let Ω be a bounded domain of R^d ($d \geq 1$) with boundary Γ. Motivated by wave

Figure 1. A striped decomposition and a box decomposition.



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propagation problems in geophysics, we consider the numerical solution of the following linear wave problem:

$$\rho\, u_{tt} - \nabla\cdot(a\nabla u) = f \qquad \text{in} \quad \Omega\times(0,T), \tag{1.1}$$

with boundary condition:

$$\sqrt{a\rho}\, u_t + a\nabla u\cdot\mathbf{n} = 0 \qquad \text{on} \quad \Gamma\times(0,T), \tag{1.2}$$

and initial conditions:

$$u(0) = u_0,\ u_t(0) = u_1. \tag{1.3}$$

Here $\mathbf{n}$ is the unit outward normal vector on Γ. We will assume that a, ρ are two piecewise continuous functions on $\bar{\Omega}$ satisfying: $a(x)\ge a_0 > 0$, $\rho(x)\ge\rho_0>0$.

If we introduce the new variable

$$\mathbf{p} = a\nabla u, \tag{1.4}$$

then it follows from (1.1) and (1.4) that u and $\mathbf{p}$ satisfy the variational equations:

$$\int_\Omega (\rho\, u_{tt} - \nabla\cdot\mathbf{p} - f)\, v\, dx = 0, \qquad \forall v\in L^2(\Omega), \tag{1.5}$$

and

$$\int_\Omega a^{-1}\,\mathbf{p}\cdot\mathbf{q}\, dx + \int_\Omega u\,\nabla\cdot\mathbf{q}\, dx = \int_\Gamma u\,\mathbf{q}\cdot\mathbf{n}\, d\Gamma, \qquad \forall\mathbf{q}\in H(\Omega, div). \tag{1.6}$$

(Here $H(\Omega, div) = \{\mathbf{q}\in (L^2(\Omega))^d : \nabla\cdot\mathbf{q}\in L^2(\Omega)\}$).

We can accommodate the boundary condition (1.2) by differentiating (1.6) in time, and using (1.2), to get

$$\begin{aligned}\int_\Omega a^{-1}\,\mathbf{p}_t\cdot\mathbf{q}\, dx &+ \int_\Omega u_t\,\nabla\cdot\mathbf{q}\, dx \\ &+ \int_\Gamma (a\rho)^{-\frac{1}{2}}\,(\mathbf{p}\cdot\mathbf{n})\,(\mathbf{q}\cdot\mathbf{n})\, d\Gamma = 0, \qquad \forall\mathbf{q}\in H(\Omega, div).\end{aligned} \tag{1.7}$$

Similarly, we can remove the direct dependence of (1.7) on u_t by differentiating (1.7) in time. By (1.5), and since $\nabla\cdot\mathbf{q}\in L^2(\Omega)$, we get

$$\begin{aligned}\int_\Omega a^{-1}\mathbf{p}_{tt}\cdot\mathbf{q}\, dx &+ \int_\Omega \rho^{-1}\,(\nabla\cdot\mathbf{p} + f)\,\nabla\cdot\mathbf{q}\, dx \\ &+ \int_\Gamma (a\rho)^{-\frac{1}{2}}\,(\mathbf{p}_t\cdot\mathbf{n})\,(\mathbf{q}\cdot\mathbf{n})\, d\Gamma = 0, \qquad \forall\mathbf{q}\in H(\Omega, div).\end{aligned} \tag{1.8}$$

2 Domain Decomposition

To simplify the discussion, we will partition the domain Ω into only two subdomains Ω_1 and Ω_2, with the interface γ between Ω_1 and Ω_2. We let a_i, ρ_i, and f_i denote the restriction of a, ρ, and f to subdomain Ω_i, $i = 1, 2$, respectively. If $\mathbf{p}_i \in H(\Omega_i, div), i = 1, 2$, then for $\mathbf{p}_i$ to be the restriction of $\mathbf{p} \in H(\Omega, div)$ to Ω_i it is necessary that

$$\mathbf{p}_1 \cdot \mathbf{n}_1 + \mathbf{p}_2 \cdot \mathbf{n}_2 = 0 \tag{2.9}$$

on the interface γ. Here $\mathbf{n}_i$ is the unit outward normal vector on γ for subdomain Ω_i. Using Lagrange multiplier theory, we can enforce the constraint (2.9) by finding a multiplier $\lambda \in \Lambda$ satisfying the following domain decomposition formulation of (1.8):

Find $\{\mathbf{p}_1(t), \mathbf{p}_2(t), \lambda(t)\} \in H(\Omega_1, div) \times H(\Omega_2, div) \times \Lambda$ so that

$$\begin{aligned}\sum_{i=1}^{2} \Big[\int_{\Omega_i} a_i^{-1} \mathbf{p}_{itt} \cdot \mathbf{q}_i \, dx \quad &+ \quad \int_{\Omega_i} \rho_i^{-1} (\nabla \cdot \mathbf{p}_i + f_i) \, \nabla \cdot \mathbf{q}_i \, dx \\ &+ \int_{\Gamma \cap \partial\Omega_i} (a_i \rho_i)^{-1/2} \, (\mathbf{p}_{it} \cdot \mathbf{n}_i) \, (\mathbf{q}_i \cdot \mathbf{n}_i) \, d\Gamma \Big] \\ &= \int_\gamma \lambda \, (\mathbf{q}_1 \cdot \mathbf{n}_1 + \mathbf{q}_2 \cdot \mathbf{n}_2) \, d\gamma, \\ &\qquad \forall \{\mathbf{q}_1, \mathbf{q}_2\} \in H(\Omega_1, div) \times H(\Omega_2, div),\end{aligned} \tag{2.10}$$

and

$$\int_\gamma (\mathbf{p}_1 \cdot \mathbf{n}_1 + \mathbf{p}_2 \cdot \mathbf{n}_2) \, \mu \, d\gamma = 0, \quad \forall \mu \in \Lambda, \text{ a.e. on } (0, T). \tag{2.11}$$

Remark 1: The choice of Λ is a delicate matter (involving spaces such as $H_{00}^{1/2}(\gamma)$). We have implicitly assumed, in (2.9), that $\mathbf{p}_1 \cdot \mathbf{n}_1 + \mathbf{p}_2 \cdot \mathbf{n}_2 \in L^2(\gamma)$, implying that we can take $\Lambda = L^2(\gamma)$. There will not be a problem with this choice in finite dimensions.
Remark 2: The Lagrange multiplier plays the role of u.

Since we only require $u \in L^2(\Omega)$, the restrictions u_i need only to be in $L^2(\Omega_i)$ and satisfy:

Find $\{u_1(t), u_2(t)\} \in L^2(\Omega_1) \times L^2(\Omega_2)$ so that

$$\begin{aligned}\sum_{i=1}^{2} \Big[\int_{\Omega_i} (\rho_i \, u_{itt} - \nabla \cdot \mathbf{p}_i - f_i) \, v_i \, dx \Big] = 0, \\ \forall \{v_1, v_2\} \in L^2(\Omega_1) \times L^2(\Omega_2), \text{ a.e. on } (0, T).\end{aligned} \tag{2.12}$$

3 Space and Time Discretization

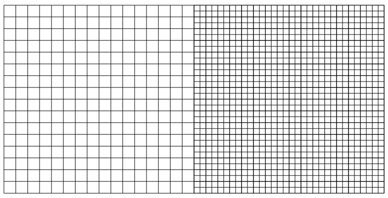
For simplicity we will assume that the spatial dimension $d = 2$ and that the domain Ω, as well as the subdomains $\Omega_i, i = 1, 2$, are rectangles whose boundaries are parallel to the coordinate axes. We will approximate the spaces $H(\Omega_i, div)$ and $L^2(\Omega_i)$ by the lowest order Raviart-Thomas spaces. To this end, we triangulate each rectangle Ω_i into a uniform partition of subrectangles $R_{h_i} = \{K\}$. We will also assume that each rectangle K has edges parallel to the coordinate axes and each rectangle is of uniform size, with h_i the length of the longest side.

We will use the approximation spaces $Q_{h_i} \approx H(\Omega_i, div)$ where

$$Q_{h_i} = \{\mathbf{q}|_K = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} : q_1 = \alpha_K + \beta_K x_1, q_2 = \gamma_K + \delta_K x_2, \forall K \in R_{h_i}\}. \tag{3.13}$$

(Here $x = \binom{x_1}{x_2}$ is a generic point in Ω_i.) We see then that over each rectangle K the vector-valued function $\mathbf{q} \in Q_{h_i}$ will have the first component linear with respect to x_1 and constant with respect to x_2. The situation for the second component of $\mathbf{q}$ is reversed. For the displacement spaces, we will use the approximation $L^2(\Omega_i) \approx V_{h_i} = \{v|_K = \varepsilon_K, \forall K \in R_{h_i}\}$, i.e. the space of piecewise constant functions. We will also assume that the triangulations R_{h_1} and R_{h_2} will be semi-matching at the interface γ as in Figure 2. Finally, the multiplier space Λ is approximated by Λ_h, the space of functions piecewise constant on the edges of the finer triangulation located on γ.

Figure 2. Semi-matching grid.



The time discretization is a domain decomposition implementation of a well known second order *explicit* finite difference scheme for the wave equation. We let $\Delta t (> 0)$ be a time discretization step and let $\mathbf{p}^n_{h_i} \approx \mathbf{p}_i(n\Delta t), u^n_{h_i} \approx u_i(n\Delta t)$, and $\lambda^n_h \approx \lambda(n\Delta t)$, for $i = 1, 2$ and for $n = 0, 1, 2, \ldots$. The full approximate problem to problem (2.10)-(2.12) is:

For $n = 0, 1, 2, \cdots$, find $\{\mathbf{p}^{n+1}_{h_1}, \mathbf{p}^{n+1}_{h_2}, \lambda^n_h\} \in Q_{h_1} \times Q_{h_2} \times \Lambda_h$ so that

$$\begin{aligned} \sum_{i=1}^{2} \Big[\int_{\Omega_i} & a_i^{-1} \frac{\mathbf{p}^{n+1}_{h_i} - 2\mathbf{p}^n_{h_i} + \mathbf{p}^{n-1}_{h_i}}{|\Delta t|^2} \cdot \mathbf{q}_{h_i} \, dx \qquad (3.14) \\ &+ \int_{\Omega_i} \rho_i^{-1} (\nabla \cdot \mathbf{p}^n_{h_i} + f_i) \, \nabla \cdot \mathbf{q}_{h_i} \, dx \\ &+ \int_{\Gamma \cap \partial\Omega_i} (a_i \rho_i)^{-1/2} \left(\frac{\mathbf{p}^{n+1}_{h_i} - \mathbf{p}^{n-1}_{h_i}}{\Delta t} \cdot \mathbf{n}_i \right) (\mathbf{q}_{h_i} \cdot \mathbf{n}_i) \, d\Gamma \ \Big] \\ &= \int_\gamma \lambda^n_h \, (\mathbf{q}_{h_1} \cdot \mathbf{n}_1 + \mathbf{q}_{h_2} \cdot \mathbf{n}_2) \, d\gamma, \quad \forall \{\mathbf{q}_{h_1}, \mathbf{q}_{h_2}\} \in Q_{h_1} \times Q_{h_2}, \end{aligned}$$

and

$$\int_\gamma (\mathbf{p}_{h_1}^{n+1} \cdot \mathbf{n}_1 + \mathbf{p}_{h_2}^{n+1} \cdot \mathbf{n}_2)\, \mu_h \, d\gamma = 0, \quad \forall \mu_h \in \Lambda_h, \tag{3.15}$$

with $\mathbf{p}_{h_i}^0 = a_i \nabla u_{0h_i}$ and $\mathbf{p}_{h_i}^1 - \mathbf{p}_{h_i}^{-1} = 2\Delta t \; a_i \nabla u_{1h_i}, \quad i = 1, 2,$

and find $\{u_{h_1}, u_{h_2}\} \in V_{h_1} \times V_{h_2}$ so that

$$\sum_{i=1}^{2} \Big[\int_{\Omega_i} (\rho_i \, \frac{u_{h_i}^{n+1} - 2u_{h_i}^n + u_{h_i}^{n-1}}{|\Delta t|^2} - \nabla \cdot \mathbf{p}_{h_i}^n - f_i)\, v_{h_i}\, dx \Big] = 0, \tag{3.16}$$

$$\forall \{v_{h_1}, v_{h_2}\} \in V_{h_1} \times V_{h_2},$$

with $u_{h_i}^0 = u_{0h_i}$ and $u_{h_i}^1 - u_{h_i}^{-1} = 2\Delta t \, u_{1h_i}, \quad i = 1, 2.$

Notice that (3.14),(3.15) do not depend on the displacement approximation $u_{h_i}^{n+1}$. Hence (3.16) needs to be calculated only if we are interested in approximating the displacements $u(t)$ as well as $\mathbf{p}(t)$. For the applications in which we are interested, the material coefficients a_i and $\rho_i, i = 1, 2$, are assumed to be piecewise constant. If we approximate the forcing term f by a piecewise constant interpolant, then all the integrals in (3.14)-(3.16) can be computed exactly using Simpson's rule.

To find $u_{h_i}^{n+1}$ in (3.16) we need only to solve a diagonal linear system. To find $\mathbf{p}_{h_i}^{n+1}$ and λ_h^n in (3.14),(3.15) we solve, at each time step, a system of linear equations of the form

$$\mathbf{A}\hat{\mathbf{p}} + \mathbf{B}^T \hat{\lambda} = \hat{\mathbf{b}} \tag{3.17}$$

$$\mathbf{B}\hat{\mathbf{p}} = \hat{\mathbf{c}} \tag{3.18}$$

where $\mathbf{A} \in R^{N \times N}$ is symmetric positive definite and $\mathbf{B} \in R^{M \times N}$ $(M << N)$. Using the Schur Complement we can solve for $\hat{\lambda}$ by solving

$$(\mathbf{B}\mathbf{A}^{-1}\mathbf{B}^T)\hat{\lambda} = \mathbf{B}\mathbf{A}^{-1}\hat{\mathbf{b}} - \hat{\mathbf{c}} \tag{3.19}$$

using, for example, the Conjugate Gradient Algorithm in the form given by Glowinski and LeTallec [GL89]:

0) $\hat{\lambda}_0$ is given. $(\hat{\lambda}_0 \equiv \hat{\lambda}^{n-1})$

Solve $\mathbf{A}\hat{\mathbf{p}}_0 = \hat{\mathbf{b}} - \mathbf{B}^T \hat{\lambda}_0$.

Compute $\hat{\mathbf{g}}_0 = \hat{\mathbf{c}} - \mathbf{B}\hat{\mathbf{p}}_0$.

Set $\hat{\mathbf{w}}_0 = \hat{\mathbf{g}}_0$.

1) For $k = 0, 1, 2, \ldots$ until convergence:

1.1) Solve $\mathbf{A}\hat{\mathbf{z}}_k = \mathbf{B}^T \hat{\mathbf{w}}_k$.

1.2) $\rho_k = \frac{|\hat{\mathbf{g}}_k|^2}{(\mathbf{B}\hat{\mathbf{z}}_k, \hat{\mathbf{w}}_k)}$.

1.3) $\hat{\lambda}_{k+1} = \hat{\lambda}_k - \rho_k \hat{\mathbf{w}}_k$.

1.4) $\hat{\mathbf{p}}_{k+1} = \hat{\mathbf{p}}_k + \rho_k \hat{\mathbf{z}}_k$.

1.5) $\hat{\mathbf{g}}_{k+1} = \hat{\mathbf{g}}_k - \rho_k \mathbf{B}\hat{\mathbf{z}}_k$.

1.6) $\gamma_k = \frac{|\hat{\mathbf{g}}_{k+1}|^2}{|\hat{\mathbf{g}}_k|^2}$.

1.7) $\hat{\mathbf{w}}_{k+1} = \hat{\mathbf{g}}_{k+1} + \gamma_k \hat{\mathbf{w}}_k$.

We note that the matrix $\mathbf{A}$ is block diagonal with symmetric positive definite tridiagonal blocks so the linear system in steps 0) and 1.1) can be solved very efficiently. We also mention that the entire iterative algorithm is very efficient, usually requiring only 1 or 2 iterations to get a substantial reduction in the relative size of the gradient $\hat{\mathbf{g}}_k$.

4 Numerical Experiments

The experiments discussed here are motivated by applications in geophysics and are related to the numerical simulation of an explosion. To this end, we have taken the forcing term f to be the Ricker pulse (see [BT91]):

$$f(\mathbf{x}, t) = \begin{cases} d(t)s(r), & \text{if } 0 \le r \le R \text{ and } 0 \le t \le \frac{2}{f_0}, \\ 0, & \text{otherwise,} \end{cases} \tag{4.20}$$

where $s(r) = \left[\frac{3}{\pi}\left(\frac{r^2 - R^2}{R^3}\right)^2\right]$, $r^2 = (x_1 - x_1^0)^2 + (x_2 - x_2^0)^2$, $d(t) = A(1 - 2\tau^2)e^{-\tau^2}$, and $\tau = \pi(f_0 t - 1)$. In (4.20) $s(r)$ is meant to approximate the Dirac measure centered at the point (x_1^0, x_2^0). Here R, A, and f_0 are the radius, amplitude, and frequency parameters for the pulse. We also used the initial conditions $u_0 = u_1 = 0$.

In Figure 3 we see the evolution of the wave over four subdomains arranged in a 2×2 partition of Ω ($\Omega_1 = (0, 0.5) \times (0, 0.5), \Omega_2 = (0, 0.5) \times (0.5, 1), \Omega_3 = (0.5, 1) \times (0, 0.5), \Omega_4 = (0.5, 1) \times (0.5, 1)$.) The discretization was identical in all four subdomains ($h_1 = h_2 = h_3 = h_4$) with matching grids at the interfaces. We notice that there is no deformation as the wave front passes through the interfaces. We should also note that no special arrangements have to be made at the crossing point $(0.5, 0.5)$ since the Lagrange multipliers are discontinuous there. This is in contrast to the conforming method presented in [DG93].

In Figure 4 we compare a global calculation of the wave propagation with the domain decomposition method. In the domain decomposition calculation we have partitioned Ω into two subdomains ($\Omega_1 = (0, 1) \times (0, 1), \Omega_2 = (1, 2) \times (0, 1)$) where the discretization parameters satisfy $h_2 = h_1/2$ and the grids are semi-matching at the interface as in Figure 2. The material constants are equal in both subdomains. The pulse was centered at the point $(0.5, 0.5)$ and the figure shows the remnants of the wave fronts at a time when the front has passed the interface γ. We notice that the wave fronts are almost identical.

In Figure 5 we have the same domain decomposition described for Figure 4. The material constants in this case satisfy $a_1 = 4a_2$ and $\rho_1 = \rho_2$, so the wave is propagating twice as fast in Ω_1 as in Ω_2. After 100 time steps, we see the wave front about to intersect the interface γ. After 200 time steps, we see the original wave front still developing in Ω_2, with a reflection propagating in the opposite direction.

Figure 3. Four subdomains.

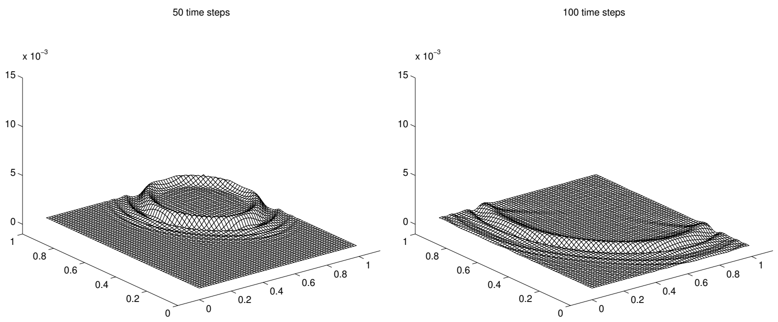


Figure 4. Global versus domain decomposition solutions.

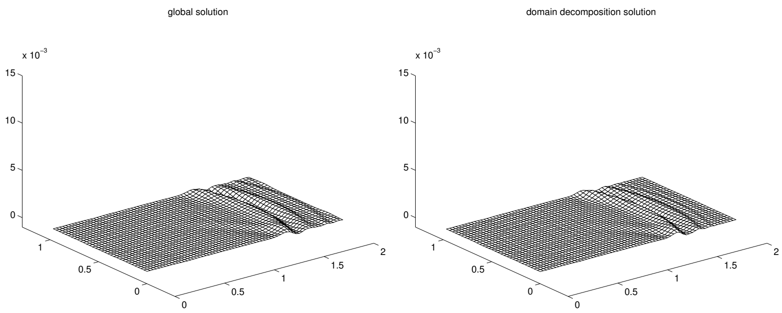
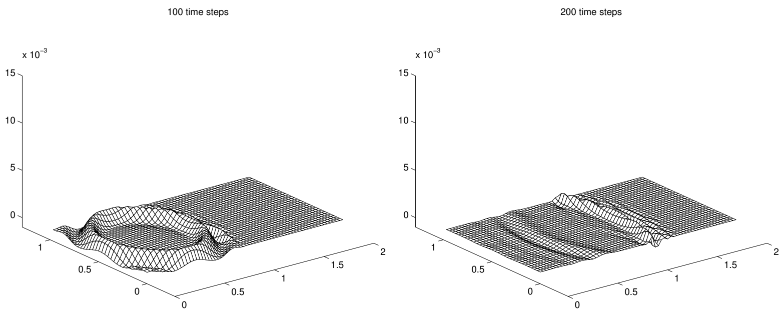


Figure 5. Subdomains with different material constants.



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