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The Overlapping Component Mode Synthesis Method: The Shifted Eigenmodes Strategy and the Case of Selfadjoint Operators with Discontinuous Coefficients

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1 Introduction

The Component Mode Synthesis (CMS) method is a domain decomposition strategy for the approximation of eigenmodes of partial differential elliptic operators. It makes use of local functions that are the eigenmodes of the same global operator but restricted over each subdomain. The first local eigenfunctions suitably extended over the whole domain (plus eventually some interface modes, see [CB68] for more details) are then used to span a discrete space that allows one to approximate the global eigenmodes through a Galerkin-type strategy. Whereas the standard method, based on a nonoverlapping domain decomposition, is of low order of accuracy, our variant [CDM96a] relies on an overlapping domain decomposition

$$\Omega = \bigcup_{k=1}^{K} \Omega^k, \tag{1.1}$$

and produces a method of infinite order accuracy (see below for the definition). The analysis and the first results we have presented in [CDM96a] deal with the case of a constant coefficient operator and the computation of the spectrum, starting from the lowest eigenmode. We generalize here the domain of application of the method by first introducing the shifted eigenmode strategy and secondly by considering elliptic operators with discontinuous coefficients.

This paper is divided as follows:

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- In section 2 is discussed the shifted eigenmodes strategy. One and two-dimensional numerical tests are discussed.
- In section 3 are discussed in detail the additional problems arising with the presence of nonconstant coefficients.

2 Computing Eigenmodes of Energy Close to a Given One

For both engineering and mathematical purposes, it is sometimes attractive only to compute a part of the spectrum: low, medium or high frequencies. Conventional finite element or component mode synthesis methods are known not to be so efficient in the approximation of large eigenvalues. Indeed, in order to approximate high frequency eigenmodes, the trial functions of the discrete spaces have to be able to reproduce these modes. This generally requires high dimensional discrete spaces inducing a corresponding algebraic system that quickly becomes very large. Hence, the restriction in spectrum comes from the restriction in computational range.
Provided that all the first local eigenmodes are present in the discrete space, it has been shown in [CDM96a] that the overlapping method provides an infinite order of accuracy in the following sense: suppose that all the eigenmodes of energy less than $\widetilde{\lambda}$ are used; then the convergence is controlled by a constant times $(\frac{\overline{\lambda}}{\widetilde{\lambda}})^p$ for <u>any</u> p for the approximation of all the global eigenmodes of energy less than $\overline{\lambda}$.
We here prove that the use of the only local basis functions with energy close to the expected value is sufficient to capture the expected mode with an infinite order of accuracy.
For the sake of simplicity, we present this "*shifted eigenmodes strategy*" applied to the Laplace operator, but it can be extended to any linear elliptic selfadjoint operator.

Presentation and Numerical Analysis of the Shifted Eigenmodes Strategy

Let $\lambda^\star$ be a given positive real value. We are interested in the following problem: *find a pair* $(\lambda, u) \in R^+ \times H_0^1(\Omega)$ *such that* λ *is close to* $\lambda^\star$ *and*

$$\left\{ \begin{array}{rcl} -\Delta u &=& \lambda\, u \text{ in } \Omega, \\ u &=& 0 \text{ over } \partial\Omega. \end{array} \right. \tag{2.2}$$

On the subdomain Ω^k, we consider the same problem as (2.2) quoted here as $(2.2)^k$ where Ω is replaced by Ω^k. We denote by $\{\lambda_i^k, u_i^k\}_{i=1,\infty}^{k=1,K}$ the corresponding eigenmodes ranged in *increasing* order of eigenvalue. We denote by $\tilde{u}_i^k$ the extension of u_i^k by 0 over Ω.

Definition - Let α be a positive constant. The shifted mode strategy consists in considering a Galerkin method based on the discrete space $X_{\alpha,\lambda^\star}$ spanned by all local eigenmodes $\{\tilde{u}_i^k\}_{i=m_k,M_k}^{k=1,K}$, where m_k and M_k $(m_k < M_k)$ are some integers chosen such that

$$\lambda_{m_k}^k \;<\; \lambda^\star - \alpha \;<\; \lambda^\star + \alpha \;<\; \lambda_{M_k}^k \;\text{ for all integer } k \text{ in } \{1, ..., K\},$$

This problem is defined by: *find a pair* $(\mu, v) \in R^+ \times X_{\alpha,\lambda^\star}$ *such that*

$$\forall w \in X_{\alpha,\lambda^\star}, \int_\Omega \nabla v \nabla w = \mu \int_\Omega vw. \tag{2.3}$$

We now give the following result:

Proposition [**error estimate**] **-** *There exists a positive constant* $C(p)$, $p \in N$ *such that for any solution* (u, λ) *of (2.2) with*

$$\lambda^\star - \alpha \;<\; \lambda \;<\; \lambda^\star + \alpha,$$

the following error estimate holds for one of the eigenmode (μ, v) *of problem (2.3):*

$$\|u - v\|_{H_0^1(\Omega)} \leq \; C(p) \left[\left(\frac{\lambda_{m_k - 1}}{\lambda}\right)^p + \left(\frac{\lambda}{\lambda_{M_k+1}}\right)^p \right] \; \textit{for any } p \in N. \tag{2.4}$$

The above estimate expresses that the method is of infinite order of accuracy since it is better than any fixed order.

Sketch of the proof. - From the standard abstract results on the numerical analysis of the Galerkin approximation, it is well-known (see Châtelin [Cha83] for example) that the proof of the previous proposition reduces itself into the evaluation of the distance (in the H_0^1-norm)

$$\operatorname{dist}\left(u, \; \operatorname{span}(u_i^k)_{i=m_k,M_k}^{k=1,K}\right).$$

In order to evaluate this distance, we follow the same strategy as in [CDM96a]. A regular partition of unity $\{\chi_k\}_{k=1,K}$ is first associated to the domain decomposition of Ω. We are now looking for the ability of the functions $(u_i^k)_{i=m_k,M_k}$ to approximate the function $\chi_k u$. Because the set of functions $\{u_i^k\}_{i=1,+\infty}$ spans the space $H_0^1(\Omega^k)$, there exists a ℓ_2-summable family of coefficients $\{\alpha_i^k\}_i$ such that

$$\chi_k u \;=\; \sum_{i=1}^{+\infty} \alpha_i^k \; \tilde{u}_i^k.$$

We then approximate $(\chi_k u)$ by a truncated series

$$\chi_k u \;\approx\; \tilde{u}^k \;\stackrel{\text{def}}{=}\; \sum_{i=m_k}^{M_k} \alpha_i^k \; \tilde{u}_i^k,$$

so that the term $\sum_{k=1}^K \sum_{i=m_k}^{M_k} \alpha_i^k \, \tilde{u}_i^k$ will be a candidate for bounding the distance from above. We are left to estimate in the H_0^1-norm the quantities

$$\chi_k u \;-\; \tilde{u}^k \;=\; \left\{ \sum_{i=1}^{m_k-1} \alpha_i^k \; \tilde{u}_i^k \;+\; \sum_{i=M_k+1}^{+\infty} \alpha_i^k \; \tilde{u}_i^k \right\}.$$

Two residual terms are present. The second one was already considered in the analysis of the initial version of the method (see [CDM96a]) and decays exponentially fast:

$$\left\| \sum_{i=M_k+1}^{+\infty} \alpha_i^k \ \tilde{u}_i^k \right\|_{L^2(\Omega)} \leq \ c(p) \left(\frac{\lambda}{\lambda_{M_k+1}} \right)^p \quad \forall p \in N. \tag{2.5}$$

Recalling that u is an eigenvalue of the original problem, we write

$$\begin{aligned} \alpha_i^k &= \int_{\Omega^k} (\chi_k u) \ u_i^k \\ &= \int_{\Omega} u \ (\chi_k \ \tilde{u}_i^k) \\ &= \frac{1}{\lambda} \int_{\Omega} -\Delta u \ (\chi_k \ \tilde{u}_i^k) \\ &= \frac{1}{\lambda} \int_{\Omega} u \left[-\Delta(\chi_k \ \tilde{u}_i^k) \right]. \end{aligned}$$

Iterating this argument p times leads to

$$\alpha_i^k = \frac{1}{\lambda^p} \int_{\Omega} u \ \ [-\Delta]^p \, (\chi_k \ \tilde{u}_i^k) \tag{2.6}$$

From $(2.2)^k$, there exists a positive constant $C(p)$ such that

$$\| \, [-\Delta]^p \, (\chi_k \ u_i^k) \|_{L^2(\Omega)} \leq C(p) \ \ (\lambda_{m_k-1})^p \, , \tag{2.7}$$

and the result thus follows from (2.5), (2.6) and (2.7).

One-dimensional Numerical Tests

In this section we consider the following eigenvalue problem with constant coefficients: *find all pairs* $(\lambda, u) \in R^+ \times H_0^1(]0,1[)$ *such that*

$$\begin{cases} -\ u''(x) \ = \ \lambda \ u(x) \ \ \textit{for all } x \textit{ in }]0,1[, \\ u(0) = u(1) = 0. \end{cases}$$

Let us assume that $]0,1[$ is split into K overlapping sets

$$]0,1[\ = \bigcup_{k=1}^{K} \ ]a_k, b_k[.$$

Here, the exact local eigenvalues are known: $\lambda_i^k = \left[\frac{i\pi}{b_k - a_k} \right]^2$. This will help us in the numerical experiments for verifying the infinite order of convergence. On each subdomain $\Omega^k =]a_k, b_k[$, we denote by N_k the number of consecutive local eigenmodes between λ_{m_k} and λ_{M_k} of the partial differential problem set on Ω^k, surrounding the eigenvalue $\lambda^\star$ we want to identify.

Table 1 Identification of the spectrum $\{\lambda_{min}, .., \lambda_{max}\}$ with precision of order 10^{-6} for decompositions D_1 and D_2, and with precision 10^{-4} for D_3.

	D_1	D_2	D_3
N	60	52	51
$\lambda^\star$=100	$\{88, .., 120\}$	$\{86, .., 121\}$	$\{91, .., 115\}$
$\lambda^\star$=200	$\{185, .., 220\}$	$\{183, .., 221\}$	$\{189, .., 221\}$
$\lambda^\star$=400	$\{385, .., 420\}$	$\{383, .., 421\}$	$\{386, .., 421\}$
$\lambda^\star$=510	$\{495, .., 531\}$	$\{493, .., 531\}$	$\{496, .., 531\}$

¿From the previous numerical analysis (see (2.4)), it is natural to tune the different pairs $(N_k, \lambda_{m_k})_{k=1,..,K}$ such that

$$\frac{N_1}{\mid \Omega^1 \mid} \approx \frac{N_2}{\mid \Omega^2 \mid} \approx \cdots \frac{N_K}{\mid \Omega^K \mid}$$

and

$$\frac{\lambda_{m_k}}{\lambda^\star} \approx \frac{\lambda^\star}{\lambda_{M_k}} \text{ for } k = 1, \ldots, K.$$

It thus appears that the accuracy of the discretization only depends on the parameter $n = \frac{N_1}{|\Omega^1|}$ since all the others are then deduced from the relations

$$\frac{N_k}{\mid \Omega^k \mid} = n, \quad \frac{\lambda_{m_k}}{\lambda^\star} = \frac{\lambda^\star}{\lambda_{m_k} + N_k\,\pi^2} \quad \text{for} \quad k = 1, .., K. \tag{2.8}$$

Numerical experiments - We use our overlapping CMS method on the three following interval decompositions of Ω to highlight its accuracy:

$$\begin{cases} D_1 & : \emptyset =]0, 0.4[\cup]0.3, 0.7[\cup]0.6, 1[\ , \quad K = 3, \\ D_2 & : \emptyset =]0, 0.75[\cup]0.5, 1[\ , \quad K = 2, \\ D_3 & : \emptyset =]0, 0.6[\cup]0.6, 1[\cup]0.3, 0.7[\ , \quad K = 3. \end{cases}$$

Table 1 indicates the part of the spectrum surrounding the eigenvalue $\lambda^\star$ that has been identified with a relative error of order 10^{-6} or 10^{-4}.

Remark [unexpected discrete eigenvalues] - Particular unexpected eigenvalues may appear in the approximate spectrum. The abstract results on the approximation of eigenmodes (see Châtelin [Cha83]) indicate that (in all good cases) there exists a sequence of discrete eigenvalues that converges towards each exact eigenvalue. This does not prevent that, at some fixed discretization, there may exist spurious eigenmodes that eventually will converge towards an exact one that may be still far away. The problem is more present in the Shifted method than in the non shifted one since the convergence is not monotonic. We are indeed not certain that the discrete eigenvalues are larger than the exact ones.

In order to cure the problem, we propose two different solutions.

1. Let A be the (discrete) stiffness matrix built from *all* the first local eigenmodes. For each approximate eigenpair $(\tilde\lambda, \tilde u)$, we compare the vector $(A\,\tilde u - \tilde\lambda\,\tilde u)$ to zero (using the euclidian ℓ_2-norm for example); if the difference is small, then $(\tilde{\lambda}, \tilde{\mu})$ is a good approximation of an exact eigensolution. But this technique is expensive since it requires to build up the complete (large) matrix that we expected to avoid. Nevertheless, it is not so expensive as computing the eigenmodes of the total matrix A.

2. Another cheaper strategy is to define a set of test cases with slight variations of parameters (number of local basis functions, size of subdomains) and compare the different results. It is reasonable to think that an unexpected approximate eigenvalue is strongly dependent on the parameters of a computation and then will appear or disappear for slight variations of the parameters. This is a manner to localize these spurious modes. We have experimented this approach and it has given good results.

The Shifted Eigenmodes Strategy for Two-dimensional Problems

We extend here the shifted eigenmodes strategy to the multidimensional case. As before, it consists in choosing appropriate local modes according to the eigenvalue we want to approximate.

For the numerical experiments, we present the method using the Laplace operator defined on the unit square. The eigenvalues are analytically known:

$$\lambda_{kl} = (k^2 + \ell^2)\,\pi^2, \quad k, \ell \in N.$$

The unit square is splitted up into three overlapping subdomains

$$\cup_{k=1}^{3}\,\Omega^k \;\; =]0, 0.6[\times]0, 1[\;\cup\;]0.4, 1[\times]0, 0.7[\;\cup\;]0.25, 1[\times]0.4, 1[.$$

We denote by λ_{m_k} and λ_{M_k} two eigenvalues of the problem

$$-\,\Delta\,u \;=\; \lambda\,u \text{ on domain } \Omega^k, \tag{2.9}$$

$$u_{|\partial\Omega^k} \;=\; 0, \tag{2.10}$$

such that, as suggested by (2.4),

$$\frac{\lambda_{m_k}}{\lambda^\star} \simeq \frac{\lambda^\star}{\lambda_{M_k}}, \tag{2.11}$$

where $\lambda^\star$ is the eigenvalue we are interested in. Since the closed form of the global solution is known, the relation between the number of local modes N_k and the other parameters is roughly

$$N_k \;\approx\; \frac{\lambda_{M_k} - \lambda_{m_k}}{\mid \Omega^k \mid}. \tag{2.12}$$

As in the one-dimensional case, a coherent choice of the number of local eigenvalues determines the accuracy of the CMS method.

In order to identify a significant part of the spectrum around the global eigenvalue $\lambda^\star = 600\,\pi^2$, we first select 331 local modes of problem $(2.2)^1$ in Ω^1 dispatched around the local eigenvalue $\lambda^\star$ with respect to the relations (2.11), (2.12): those give

the number of modes on each other subdomain: 231 on Ω^2 and 247 on Ω^3. With these 331+231+247=809 local modes we compute the eigenvalues around $\lambda^\star$ using the shifted CMS method. A similar process is used to select a local basis made of 1355 local modes for the identification of the global eigenvalue surrounding $1000\,\pi^2$.
The following Figures (1 and 2) represent all the couples $(k,\ell) \in N^2$ that define the exact eigenvalues of the Laplace operator for the problem set in the unit square. Each square point (k,ℓ) indicates that there exists a computed eigenvalue close to the corresponding exact eigenvalue $((k^2+\ell^2)\pi^2)$: a square point is plotted as soon as the relative error on the eigenvalue is less or equal than 2.10^{-3}. The black squared points show the couples (k,ℓ) such that $(k^2+\ell^2)$ is close to 600, 1000 or 1500. We have indicated in Figure 4 the accuracy of each identified eigenvalue.

Figure 1 Identified indices (k,l) of eigenvalue $\lambda_{kl} = (k^2+\ell^2)\,\pi^2$ around $600\,\pi^2$

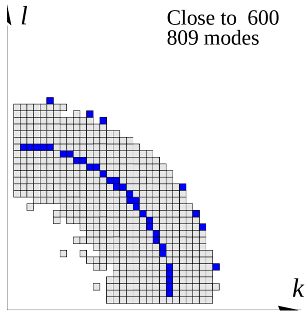
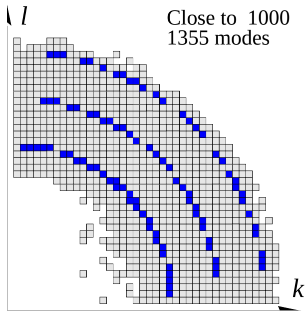


Figure 2 Identified indices (k,l) of eigenvalue $\lambda_{kl} = (k^2+\ell^2)\,\pi^2$ around $1000\,\pi^2$



For five global eigenvalues around $\lambda^\star = 600\,\pi^2$, Figure 5 shows the accuracy of the computation. It is drawn the logarithm of the error between the exact eigenvalue and its approximate computed value with respect to the total number of local modes. This confirms the exponential convergence of the method.

Remark [parallel implementation] - In order to identify a large number of eigenvalues, there exist two solutions. The usual one consists in choosing larger local basis sets. This rapidly becomes too onerous in terms of memory and CPU time.
The second solution, suggested by the shifted eigenmodes strategy, consists in splitting up the expected part of the spectrum into several overlapping frequency bands. We then carry out as many independent calculations as there exists some bands. On

Figure 3 Identified eigenvalues (k, l) (no shift)

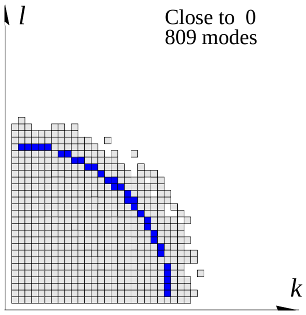


Figure 4 Accuracy of the method

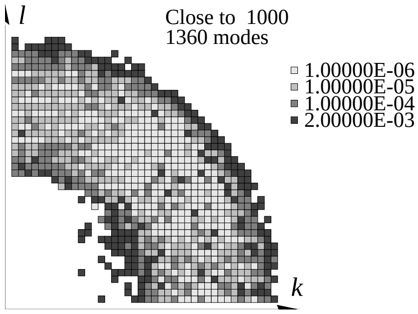
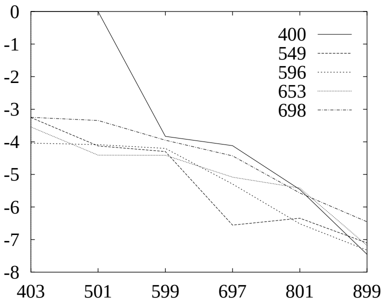


Figure 5 Accuracy of the method



each band, we apply the shifted eigenmodes strategy. Now, the involved matrices are of smaller size and the global algorithm complexity is less than the one of the computation of the whole large band.

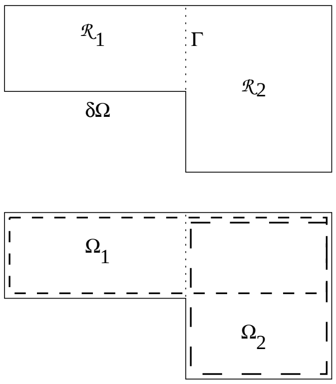
As an illustration, we have considered the test using 809 unshifted local modes. It appears from the Figures 3, 1 and 2 that this parallel approach is a viable tool for computing the whole spectrum from the smallest eigenvalue up to those close to $1500\,\pi^2$.
It is then quite easy to consider parallel algorithms to carry out computations that are completely independent.

Conclusion - The Shifted Component Modes Synthesis (SCMS) method allows for computing eigenmodes corresponding to large eigenvalues without requiring the approximation of all the smaller eigenvalues and without loosing the infinite order of accuracy. It also provides features adapted to the parallelization of the computation of the spectrum. It is interesting to note that the problem of the mass matrix bad conditioning already reported in previous papers (see [CDM96b],[CDM96a]) is weakened by the use of this technique.

3 The Overlapping CMS Method for Operations with Discontinuous Coefficients

As it is often the case in industrial problems, the related partial differential equations are generally set on a quite complex domain and the coefficients involved in the operator are often nonconstant. They may even present some discontinuities. Let us consider here for example the problem of the vibration of a membrane made of two rectangular membranes $\mathcal{R}_1$ and $\mathcal{R}_2$ as it is drawn below. Each rectangular membrane

Figure 6 Overlapping domain decomposition on a "L-shaped" structure involving two materials.



$(\mathcal{R}_i)$, $i = 1, 2$, has its own characteristic of vibration expressed in terms of a constant a_i, $i = 1, 2$. We denote by Γ the boundary between $\mathcal{R}_1$ and $\mathcal{R}_2$. The membrane is fixed at the domain boundary $\partial\Omega$. In order to approximate the eigenmodes on this

membrane, we decide to use a strategy of subdomains decomposition with overlapping (Ω_1, Ω_2) as proposed in [CDM96a]. The overlapping strategy avoids the problem of the definition of local interface basis functions. Otherwise, the decomposition leads to an operator T_k, $k = 1, 2$ defined on Ω^k with discontinuous coefficients at the interface Γ. As before, the analysis of the convergence rate can be achieved.
To solve the problem, one can use an overlapping subdomain with either constant or nonconstant coefficients. The convergence of the CMS method is thus of finite order of accuracy for the first choice and of infinite order for the second one. Of course, in the present situation, a camembert-shaped domain has to be added as suggested in [CDM96a] to solve the corner singularity.
Setting of the problem - Let $\Omega =]0, x_a[\times]0, y_b[$, $x_a, y_b > 0$. Let α be a constant, $0 < \alpha < 1$. We are interested in the solutions of the following eigenvalue problem: *find a pair* $(\lambda, u) \in R^+ \times H_0^1(\Omega)$ *such that*

$$\begin{cases} -\operatorname{div}(A\,\nabla u) \;=\; \lambda\, u \text{ in } \Omega, \\ u_{|\partial\Omega} \;=\; 0, \end{cases} \tag{3.13}$$

where $A(x, y) = a_1 > 0$ if $x > \alpha\, x_a$, $A(x, y) = a_2 > 0$ otherwise. For the sake of simplicity, we set $\mathcal{R}_1 =]0, \alpha\, x_a[\times]0, y_b[$ and $\mathcal{R}_2 =]\alpha\, x_a, x_a[\times]0, y_b[$. The solutions cannot be explicitly written in a fully closed form, but they can be approximated as close as we want to the exact solutions. More precisely, we have proved :

Proposition 1 [eigenvalues] - *There exists an infinite countable set of eigenvalues of problem (3.13). These eigenvalues* $\{\lambda_{kl}\}_{l=1,+\infty}$ *(k is a positive integer) are characterized by the relations*

1. *If* $\lambda_{kl} \;\geq\; \max(a_1\,(k/y)^2\,\pi^2,\, a_2\,(k/y)^2\,\pi^2)$*, then* λ_{kl} *are solutions of the equation*

$$a_1\; g_1(\lambda_{kl}) \tan\left(g_2(\lambda_{kl})\,(1-\alpha)\, x_a\right) \;+\; a_2\; g_2(\lambda_{kl}) \tan\left(g_1(\lambda_{kl})\, \alpha\, x_a\right) \;=\; 0, \tag{3.14}$$

with $g_i(\lambda) \stackrel{def}{=} \sqrt{\frac{\lambda - a_i\,(k/y_b)^2\,\pi^2}{a_i}}$, $i = 1, 2$.

2. *If* λ_{kl} *is such that* $\min(a_1\,(k/y)^2\,\pi^2,\, a_2\,(k/y)^2\,\pi^2) \;\leq\; \lambda_{kl} \;\leq\; \max(a_1\,(k/y)^2\,\pi^2,\, a_2\,(k/y)^2\,\pi^2)$*, then it is solution of the equation*

$$a_1\; h_1(\lambda_{kl}) \tanh\left(h_2(\lambda_{kl})\,(1-\alpha)\, x_a\right) \;+\; a_2\; h_2(\lambda_{kl}) \tan\left(h_1(\lambda_{kl})\, \alpha\, x_a\right) \;=\; 0, \tag{3.15}$$

where $h_i(\lambda) \stackrel{def}{=} \sqrt{\frac{|\lambda - a_i\,(k/y_b)^2\,\pi^2|}{a_i}}$, $i = 1, 2$.

When the determination of each eigenvalue is achieved, through the solution of equation (3.14) or (3.15) , the eigenmodes can be obtained in closed form. It is easy to prove that the solutions restricted to each subdomain $\mathcal{R}_1$ or $\mathcal{R}_2$ are either sin or sinh functions. The constants of integration are determined via the boundary conditions and continuity conditions for both u_{kl} and $A(.)\nabla u_{kl}$ at $x = \alpha\, x_a$:

Proposition 2 [eigenmodes] - *The eigenvalues* u_{kl} *of problem (3.13) are given by*

1. *If* $\lambda_{kl} \geq \max(a_1\,(k/y)^2\,\pi^2,\ a_2\,(k/y)^2\,\pi^2)$, *then*

$$\left\{\begin{array}{rcl} u_{kl}(x,y) & = & B_{12}(\lambda)\ \ \sin(g_1(\lambda_{kl})\,x)\sin(k\,\pi\,y) \\ & & \qquad\qquad\qquad \textit{if } x \leq \alpha x_a, \\ u_{kl}(x,y) & = & \sin(g_2(\lambda_{kl})\,(x_a - x))\sin(k\,\pi\,y) \\ & & \qquad\qquad\qquad \textit{otherwise,} \end{array}\right. \tag{3.16}$$

with $B_{12}(\lambda) = \sin\left(g_2(\lambda)\,x_a\,(1-\alpha)\right) / \sin\left(g_1(\lambda)\,x_a\,\alpha\right)$.

2. *If* $\min(a_1\,(k/y)^2\,\pi^2,\ a_2\,(k/y)^2\,\pi^2)\ \leq\ \lambda_{kl}\ \leq\ \max(a_1\,(k/y)^2\,\pi^2,\ a_2\,(k/y)^2\,\pi^2)$, *then (in case* $a_1 < a_2$*)*

$$\left\{\begin{array}{l} u_{kl}(x,y) = \sin(h_1(\lambda_{kl})\,x)\sin(k\,\pi\,y) \\ \qquad\qquad\qquad\qquad\qquad \textit{if } x \leq\ \alpha x_a, \\ u_{kl}(x,y) = B_{21}(\lambda)\ \ \sinh(h_2(\lambda_{kl})(x_a - x))\ \sin(k\,\pi\,y) \\ \qquad\qquad\qquad\qquad\qquad \textit{otherwise,} \end{array}\right. \tag{3.17}$$

with $B_{21}(\lambda) = \sin\left(h_1(\lambda)\,x_a\,\alpha\right) / \sinh\left(h_2(\lambda)\,x_a\,(1-\alpha)\right)$. *(the case where* $a_1 > a_2$ *is completely symmetric).*

Remark [lack of regularity of the solutions] - The closed forms of the eigenmodes on the rectangular domain show that the solutions are not generally smooth. Indeed, whenever $a_1 \neq a_2$, the normal derivative of $A(.)\nabla u_{kl}$ is continuous at the interface $x = \alpha$, but it is not generally the case for the function ∇u_{kl}. Thus $u_{kl} \notin \mathrm{H}^2(\Omega)$. The singularities are then localized on the lines of discontinuity of the function $A(.)$. This limitation of regularity of solutions would drastically limit the order of accuracy of a standard method of approximation. We have considered the following test: $a_1 = 1$, $a_2 = 5$, $\alpha = 1/2$, $x_a = 1/2$ and $y_b = 1$.
A dichotomy algorithm has been implemented in order to find the roots of the two equations (3.14),(3.15) up to the precision of the computer.
On table 2 we display the 30 first computed eigenvalues (arranged in increasing order). One can notice that, for the modes number 7 and 29, the eigenvalues are exactly $40\,\pi^2$ and $160\,\pi^2$ respectively. For example, mode 7 is the function $u_7(x,y) = \sin(6\pi x)\cos(2\pi y)\mathbb{1}_{\,0\leq x\leq 1/4} + \sin(2\pi x)\cos(2\pi y)\mathbb{1}_{\,1/4\leq x\leq 1/2}$. At the interface $x = 1/4$, both functions u_7 and ∇u_7 are continuous (since ∇u_7 vanishes) hence all the successive derivatives of u_7 are continuous too.
Let us also observe that some eigenvalues can be very close together. It is the case for modes number 19 and 20, and also 29 and 30 (see table 2). It is interesting to check that the method correctly captures these approximate eigenvalues and eigenmodes. We plot below the eigenmodes associated to eigenvalues number 19 and 20. The corresponding modes are different.

4 Numerical Results for Operators with Discontinuous Coefficients

Our purpose is now to illustrate numerically the accuracy of different CMS methods in case of discontinuous coefficients. On the unit square domain Ω, we decide to

Table 2 Exact eigenvalue for Test Case 1 (up to the computer precision)

eig. nb	eigenvalue / π^2	eig. nb	eigenvalue / π^2
1	11.381858321428	16	86.561289001402
2	15.874044219346	17	93.126493282476
3	22.045698340432	18	96.014731853507
4	29.771455839467	19	107.36015972229
5	30.811703131273	20	107.65137053876
6	39.222849777720	21	115.11237794574
7	40.0	22	123.19446535873
8	50.522457182429	23	131.67906445402
9	53.599749966838	24	134.06922677463
10	63.734486614315	25	137.31868568331
11	67.387028033897	26	140.80059050878
12	68.390060393702	27	146.50622200202
13	74.340847129365	28	157.47087545240
14	78.892410930881	29	160.0
15	80.106572892321	30	160.26355418007

Figure 7 Exact mode number 19 of the test case. $\lambda_{19} = 107.36$

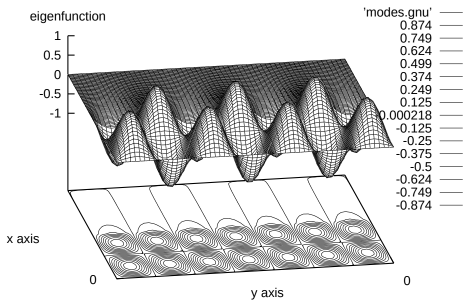


Figure 8 Exact mode number 20 of the test case. $\lambda_{20} = 107.65$

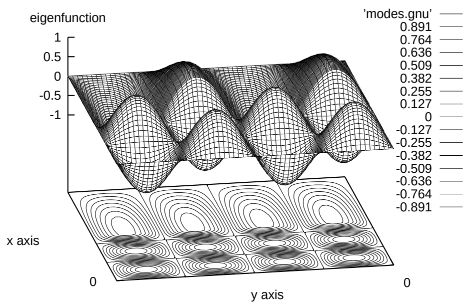
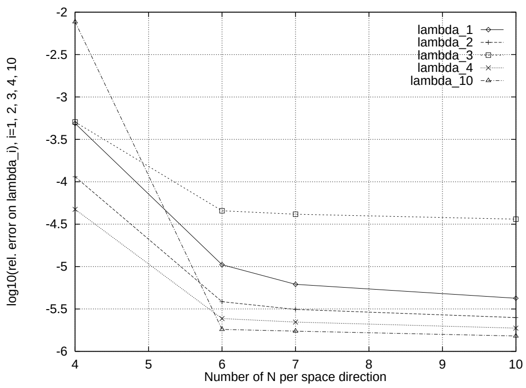


Figure 9 Relative errors on different eigenvalues for different numbers of local modes on the overlapping subdomain



approximate the first eigenvalues of problem (3.13). As in the previous example, we consider a domain decomposition using two nonoverlapping rectangular subdomains with interface localized along the discontinuity line of coefficients and a third overlapping subdomain that covers the interface. We choose

$$\Omega_1 = (0, 0.5) \times (0, 1) \quad (a_1 = 5), \tag{4.18}$$

$$\Omega_2 = (0.5, 1) \times (0, 1) \quad (a_2 = 1), \tag{4.19}$$

$$\Omega_3 = (0.25, 0.75) \times (0, 1) \quad \text{(overlapping subdomain)}. \tag{4.20}$$

For the numerical experiments, we consider:

1. an overlapping subdomain subdomain with a constant coefficient $a(x, y) = a_3 = 1$;
2. an overlapping subdomain with nonconstant coefficients:

$$a(x, y) = 5 \;\; \forall (x, y) \in (0.25, 0.5), \tag{4.21}$$

$$a(x, y) = 1 \;\; \forall (x, y) \in (0.5, 0.75). \tag{4.22}$$

On each subdomain Ω_i $(i = 1, 2)$, we consider 144 local modes that correspond to

$$\sin(2k\pi x)\sin(l\pi y) \quad for \;\; (k, l) \in \{1, .., 12\}^2.$$

For the first test, we consider an overlapping subdomain with constant coefficient. In Table (3) we give the computed eigenvalues with this approach and we plot the relative errors computed with respect to the accurate eigenvalues obtained by dichotomy (see below) in Figure 9. We immediately observe that for the overlapping domain with constant coefficient, the relative errors are bounded by 10^{-4}. But the curves ("constant 6×6", "constant 10×10") are quite similar. That confirms that the

Table 3 Approximate eigenvalues for Test Case 1

eig. nb	eigenvalue / π^2	eig. nb	eigenvalue / π^2
1	11.3819115	13	74.3415565
2	15.8741089	14	78.8925446
3	22.0457681	15	80.1086039
4	29.7715310	16	86.5643435
5	30.8124850	17	93.1282303
6	39.2229341	18	96.0148882
7	40.0015024	19	107.361750
8	50.5225546	20	107.661054
9	53.6021736	21	115.112560
10	63.7346006	22	123.196000
11	67.3894861	23	131.683773
12	68.3902544	24	134.086982

use of overlapping subdomain with constant coefficients has induced a finite order of convergence for the CMS method. In order to raise the rate of convergence, we naturally propose to use an overlapping subdomain with nonconstant coefficients. So in a first step we compute by the dichotomy method the first local modes (of Ω_3) made of sine functions and hyperbolic sine functions.

The preliminary numerical results are in good agreement with the improved accuracy of the method and we shall report in a future work the results of the numerical simulation for this problem.

In this future work, we shall also deal with another important example concerning the capture of singularities not only due to the presence of discontinuous coefficients, but also due to corners at the boundaries. Two ingredients have to be used then: the non constant coefficients on the overlapping domain and the presence of “camembert shaped” subdomains surrounding the corner singularities as it has been explained in [CDM96a].

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