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Subspace Correction Multilevel Methods for Elliptic Eigenvalue Problems

Tony F. Chan and Ilya Sharapov

1 Introduction

Domain decomposition and multigrid methods are powerful techniques for solving elliptic linear problems. Unfortunately the straightforward implementation of the methods is limited to linear problems and relatively little work has been done for nonlinear applications. The goal of this paper is to analyze the application of the multiplicative Schwarz methods to the eigenvalue problem without linearization. An important distinction of this approach is that the subspace problem is also a generalized eigenvalue problem which allows to apply the algorithm recursively and formulate a multilevel method of optimal complexity.

Solution of eigenvalue problems by multigrid methods using linearization was discussed by Hackbusch ([Hac84]) and McCormick ([McC92]). The idea to use coordinate relaxation applied directly for a matrix eigenvalue problem goes back to the book by Fadeev and Fadeeva [FF63] (1963) where they applied a technique similar to Gauss-Seidel method for minimizing the Rayleigh quotient. This approach was extended by Kaschiev [Kas88] and Maliassov [Mal92] for PDE-based problems. In this case the resulting method of minimizing the Rayleigh quotient is analogous to the block Gauss-Seidel method for linear problems.

Several domain decomposition-based methods were proposed by Lui [Lui95], in particular the method based on a nonoverlapping partitioning where the interface problem is solved either using a discrete analogue of a Steklov-Poincare operator or using Schur complement-based techniques. The former approach resembles the component mode synthesis method (cf. Bourquin and Hennezel [BH92]) which is an approximation rather than iterative technique for solving eigenproblems. The component mode synthesis was also used by Farhat and Géradin in [FG94]. Stathopoulos, Saad and Fischer [SSF95] considered iterations based on the Schur complement of the block corresponding to the interface variables.

Anther way to implement the domain decomposition technique on eigenvalue

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problems is the divide and conquer method proposed by Dongarra and Sorensen [DS87]. An attempt to link relaxation methods (in particular SOR) to eigenvalue problem was made by Ruhe [Ruh74].

In this work we extend the results of [Mal92] and [Kas88] for the multiplicative Schwarz method by considering the two-level scheme. Convergence is proved for a more suitable class of initial approximations and an asymptotic convergence analysis is given. We also describe the recursive implementation of the method, which results in a multilevel algorithm. Finally we present an alternative variational formulation of the problem, which is equivalent mathematically but more suitable for theoretical considerations.

2 Subspace Correction for Eigenvalue Problems

Let us consider the problem of finding the minimal eigenvalue λ and the corresponding eigenvector u of

$$Lu \equiv -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} a_{i,j} \frac{\partial u}{\partial x_j} + p(x)u = \lambda u \tag{2.1}$$

$$x \in \Omega, \qquad u\mid_{\partial\Omega} = 0 \qquad a_{i,j} > 0 \ ,$$

where Ω is a bounded region in R^2 and $a_{i,j}(x) = a_{j,i}(x)$, $p(x) \geq 0$ are piecewise smooth real functions.

To discretize the problem, we can perform a triangulation of Ω with triangles of quasi-uniform size h and use the standard finite element approach to represent (2.1) as

$$Au = \lambda M u \ , \tag{2.2}$$

where $A = A^T > 0$ and $M = M^T > 0$ are stiffness and mass matrices respectively. The problem of finding the minimal eigenvalue of (2.2) can be viewed as a minimization of the Rayleigh quotient

$$\lambda_1 = \min_u F(u) = \min_u \frac{u^T A u}{u^T M u} \ . \tag{2.3}$$

In order to apply domain decomposition technique to this problem we can represent Ω as a union of overlapping subdomains with Lipschitz boundaries: $\Omega = \cup_{i=1}^{J} \Omega_i$. Let $\{V_i\}_{i=1}^{J}$ be finite element subspaces corresponding to this partition and let P_i^T denote the orthogonal projection into the subspace V_i; its transpose P_i is the prolongation operator from V_i to H_h^2. We also introduce the M-norm of a vector $\|v\|_M = (v^T M v)^{1/2}$.

A scheme analogous to the multiplicative Schwarz algorithm for solving (2.2) was proposed in [Mal92] and [Kas88]:

Algorithm 1 (Multiplicative subspace correction)
Starting with u^0 for $k = 0$ until convergence
for $i = 1 : J$
find $u^{k+i/J}$ such that

$$F(u^{k+i/J}) = \min_{d_i \in V_i} F(u^{k+(i-1)/J} + P_i d_i) \tag{2.4}$$

end
end

To have some control over the norm of the iterates we can M-normalize the approximations either after each subiteration or after a loop over all subdomains is completed.

At each step the algorithm performs a subspace search minimizing the Rayleigh quotient using the correction from the current subspace. We will now show that the minimization (2.4) results in minimizing the Rayleigh quotient for the local $(n_i + 1) \times (n_i + 1)$ problem, where n_i is the dimension of the current subspace.

Rewrite (2.4) as

$$\rho(u^{k+i/J}) = \min_{\tilde{d}_i} \rho(\tilde{P}_i \tilde{d}_i) = \min_{\tilde{d}_i} \frac{(\tilde{P}_i \tilde{d}_i)^T A (\tilde{P}_i \tilde{d}_i)}{(\tilde{P}_i \tilde{d}_i)^T M (\tilde{P}_i \tilde{d}_i)} = \min_{\tilde{d}_i} \frac{\tilde{d}_i^{\,T} \tilde{A} \tilde{d}_i}{\tilde{d}_i^{\,T} \tilde{M} \tilde{d}_i} \ ,$$

where

$$\tilde{d}_i = \begin{pmatrix} d_i \\ 1 \end{pmatrix}, \qquad \tilde{P}_i = \begin{pmatrix} P_i & u^{k+(i-1)/J} \end{pmatrix}$$

and

$$\tilde{A} = \tilde{P}_i^{\,T} A \tilde{P}_i, \qquad \tilde{M} = \tilde{P}_i^{\,T} M \tilde{P}_i \ . \tag{2.5}$$

Thus the subspace problem is an eigenvalue problem with $\tilde{A}$ and $\tilde{M}$.

The matrices $\tilde{A}$ and $\tilde{M}$ preserve the sparsity of the original matrices A and M except for the last row and column, therefore the minimization subproblem can be efficiently solved.

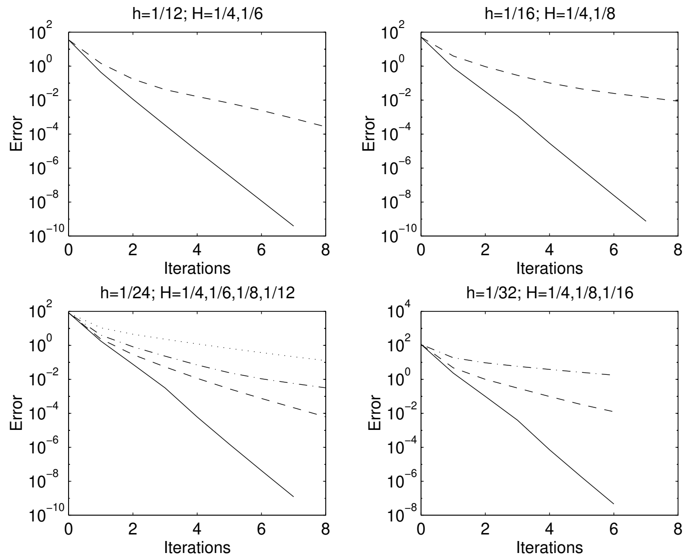
Convergence for **Algorithm 1** was proven in [Mal92] and [Kas88] with the assumption that the initial guess u^0 satisfies

$$\lambda_1 < F(u^0) < \lambda_2 \ . \tag{2.6}$$

Lui [Lui96] pointed out that the algorithm can break down in certain degenerate cases and proposed a modified procedure proving convergence for the case of two subdomains under condition (2.6).

This condition is difficult to control unless we use the method as some refinement procedure using it after a good approximation to the lowest eigenmode was produced by some other method. We can formulate a stronger result that **Algorithm 1** converges to the lowest eigenpair of (2.2) in the case of any number of subdomains and with the more practical assumption that all the components of the initial approximation u^0 are of the same sign.

Figure 1 Error reduction for the model problem without the coarse grid correction. Higher curves show slow convergence for the large number of subdomains.



Theorem 1 *Vectors u^k and the corresponding Rayleigh quotients $\rho(u^k)$ produced by* **Algorithm 1** *converge to the lowest eigenmode of discretized problem (2.2) if all the components of the initial approximation u^0 satisfy $u_i^0 > 0$.*

The proof (to appear in a full version of this paper) relies on a natural assumption that the eigenvector we are looking for is not contained in any of the subspaces V_i:

Assumption 1 *For any subspace V_i there is a constant $C_i > \lambda_1$ such that*

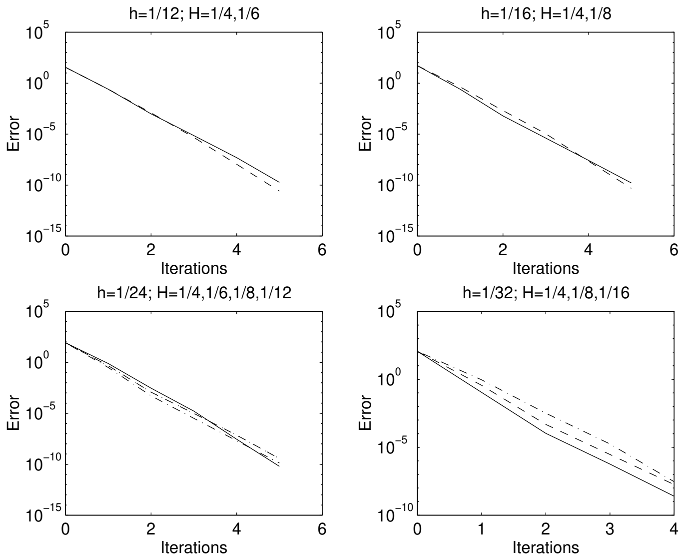
$$v^T A v \geq C_i v^T M v \qquad \text{forany} \qquad v \in V_i \ .$$

3 Coarse Grid Correction and Multilevel Method

In this work we modify **Algorithm 1** by adding a coarse grid correction after a loop over the subdomains is completed. By doing so in the case of a linear elliptic problem with sufficient subdomain overlap, the convergence rate becomes independent of both the meshsize and the number of subdomains [BPWX91, Xu92].

The effect of the coarse grid correction for a model problem of the 2D Laplacian is shown in the following figures, where h and H are fine and coarse meshsizes respectively. We see that without the coarse grid the convergence rate is dependent on both the meshsize and the number of subdomains, whereas after the inclusion of the coarse grid correction the convergence rate becomes independent of both h and H.

Figure 2 Error reduction for the the model problem with the coarse grid correction. Convergence is independent of the number of subdomains.



Since the subspace problem is of the same type as the original one, i.e. a generalized eigenvalue problem, we can make the algorithm more efficient by applying it recursively. Instead of solving the eigenvalue subproblem over a subdomain by some other method we can apply several iterations of the same algorithm. Applying that recursion to the multiplicative method with coarse grid correction we can view the resulting scheme as a multilevel method and the iterations performed on each level as the smoothing of the solution (in the spirit of the multigrid method). The recursion can be stopped once the subproblems reach some sufficiently small fixed size.

Though the algorithms as presented are sequential we can add some degree of parallelism using multicoloring techniques (see, e.g., [CM94]).

4 Alternative Formulation

A different variational formulation for the symmetric positive definite eigenvalue problem (2.2) was recently proposed by Mathew and Reddy (94) [MR94]. They pointed out that the minimal eigenpair (u_1, λ_1) can be characterized as:

$$J(u_1) = \min_v J(v) \equiv \min_v \left[v^T A v + \mu (1 - v^T M v)^2 \right] \tag{4.7}$$

with

$$\lambda_1 = 2\mu - \sqrt{4\mu^2 - 4\mu J(u_1)}$$

and

$$\|u_1\|_M^2 = 1 - \frac{\lambda_1}{2\mu}$$

for any

$$\mu > \lambda_1/2\ . \tag{4.8}$$

Unlike the Rayleigh quotient minimization, formulation (4.7) is unconstrained. The μ-term in $J(v)$ serves as a barrier to pull the solution u away from the trivial solution.

The subspace problem for (4.7) is again of the same form as the original one with dimension $n_i + 1$, i.e. an eigenvalue problem of this size. For

$$\tilde{P}_i = \begin{pmatrix} P_i & u^{k+(i-1)/J} \end{pmatrix}, \qquad \tilde{d}_i = \begin{pmatrix} d_i \\ \alpha \end{pmatrix},$$

we can write the minimization step of the algorithm as

$$J(u^{k+i/J}) = \min_{d_i \in V_i, \alpha} J(\tilde{P}_i \tilde{d}_i)$$

$$= \min_{\tilde{d}_i} \left[(\tilde{P}_i \tilde{d}_i)^T A (\tilde{P}_i \tilde{d}_i) + \mu (1 - (\tilde{P}_i \tilde{d}_i)^T M (\tilde{P}_i \tilde{d}_i))^2 \right]$$

$$= \min_{\tilde{d}_i} \left[\tilde{d}_i^{\,T} \tilde{A} \tilde{d}_i + \mu (1 - \tilde{d}_i^{\,T} \tilde{M} \tilde{d}_i)^2 \right] \ ,$$

where

$$\tilde{A} = \tilde{P}_i^{\,T} A \tilde{P}_i, \qquad \tilde{M} = \tilde{P}_i^{\,T} M \tilde{P}_i$$

Therefore, we can see that for any choice of μ satisfying (4.8), one subspace correction step for formulations (2.3) and (4.7) results in the same reduced generalized eigenvalue problem with matrices (2.5). The application of the multiplicative Schwarz algorithm to both formulations results in the same approximations to the lowest eigenvalue and the approximations to the eigenvector are the same up to scaling.

The objective function in (4.7) is convex near the solution so for the local analysis we can use apply the theory of multiplicative Schwarz methods for minimization problems [TE96]. Equivalence of the formulations gives the asymptotic result:

Theorem 2 *The iterates produced by* **Algorithm 1** *with coarse grid correction applied to formulations (2.3) and (4.7) (in SPD case) satisfy for k large enough*

$$\|u_1 - u^{k+1}\| \leq (1 - \delta) \|u_1 - u^k\| \ ,$$

where (u_1, λ_1) is the minimal eigenpair of (2.2) with $\|u_1\|_M = 1$ for (2.3) and $\|u_1\|_M = 1 - \frac{\lambda_1}{2\mu}$ for (4.7) and the value of $\delta > 0$ is independent of the meshsize h and the number of subdomains J.

The proof of this theorem will be given in a larger version of this paper.

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