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Schwarz Preconditioners for the Spectral Element Stokes and Navier-Stokes Discretizations

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1 Introduction

We consider fast methods of solving the linear system

$$\begin{cases} A\underline{\mathbf{u}} + B^t\underline{p} = \underline{\mathbf{f}} \\ \\ B\underline{\mathbf{u}} = \underline{0}. \end{cases} \tag{1}$$

resulting from the discretization of the Stokes problem by the spectral element method; see (3).

The efficient solution of this and analogous systems, generated by a variety of discretization methods, has been the object of various studies. The Uzawa procedure is a relatively standard technique [GPAR86], and more recently block-diagonal and block-triangular preconditioners have been proposed [Elm94, Kla97]. Global pressure variables are used in [BP89] and [TP95] as Lagrange multipliers to constrain the interface velocities and to guarantee that the divergence free condition holds.

Rønquist has proposed an iterative substructuring method that is based on a decomposition of the domain into interiors of subregions, faces, edges, and vertices. The coarse problem is a Stokes problem approximated by a lower-dimensional pair of discrete spaces on the coarse mesh. Stokes problems are solved within the subregions, while a diagonal scaling using elements of the matrix A is performed on the interface velocity variables. This scheme avoids costly inner iterations, and its built-in parallelism is certainly a very desirable feature. In [Røn95], relatively large problems in three dimensions are solved with modest computer resources. The small iteration count and the excellent approximation properties of the spectral element method for flow problems makes this a very efficient scheme.

Inspired by Rønquist's scheme, we have developed iterative substructuring methods, for which the velocities are restricted to the space of discretely divergence-free functions in the spectral element sense. The PCG method is applied to the resulting

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symmetric, positive definite linear system. The condition number of our algorithms grows at most like

$$\frac{C(1+\log(N))^3}{\beta_N},$$

where β_N is the Babuška-Brezzi constant; see Lemma 2.1. Our approach is also related to the methods of [BP89] and [TP95].

The next section introduces the details of the discretization method. Section 3 presents an important extension operator, while in Section 4, the $Q_2 - Q_0$ pair is used to generate a coarse space for the Stokes problem. The theory carries over without any substantial change to a variety of mixed discretizations using a discontinuous pressure space.

In Sections 5 and 6, we extend the Schwarz theory for indefinite and non-symmetric problems to the Navier-Stokes problem, taking advantage of the Stokes preconditioner developed here. In each step of Newton's method, only the velocity of the previous step is used. The pressure is computed only when required, typically after the velocity has been obtained to the prescribed accuracy. The key point in the success of this method is the construction of an appropriate coarse space.

2 Discretization Method

Let Ω be a domain in $\mathbb{R}^d$, $d = 2$ or 3. We triangulate Ω into non-overlapping *substructures* $\{\Omega_i\}_{i=1}^M$ of diameter H_i. Each Ω_i is the image of the reference substructure $\hat{\Omega} = [-1,+1]^3$ under a mapping $F_i = D_i \circ G_i$ where D_i is an isotropic dilation and G_i a C^∞ mapping such that its Jacobian and the inverse thereof are uniformly bounded by a constant. We assume, e.g., in three dimensions, that the intersection between the closures of two distinct substructures is either empty, a vertex, a whole edge or a whole face.

We define the space $P^N(\hat{\Omega})$ as the space of polynomials of degree at most N in each of the variables separately. The space $P^N(\Omega_i)$ is the space of functions v_N such that $v_N \circ F_i$ belongs to $P^N(\hat{\Omega})$. The conforming discretization space $P_0^N(\Omega) \subset H_0^1(\Omega)$ is the space of continuous functions the restrictions of which to Ω_i belong to $P^N(\Omega_i)$.

Let $\Lambda = [-1,1]$. For each N, the Gauss-Lobatto-Legendre quadrature of order N is denoted by GLL(N) and satisfies: $\forall p \in P^{2N-1}(\Lambda)$, $\int_{-1}^1 p(x)\,dx = \sum_{j=0}^N p(\xi_j)\rho_j$. Here, the quadrature points ξ_j are numbered in increasing order, and are the zeros of $(1-x^2)L_N'(x)$, and $L_N(x)$ is the Legendre polynomial of degree n.

In three dimensions, the discrete $L^2(\Omega)$-inner product is defined by

$$(u,v)_N = \sum_{i=1}^M \sum_{j,k,l=0}^N (u \circ F_i)\cdot(v \circ F_i)\cdot|J_i|(\xi_j,\xi_k,\xi_l)\cdot\rho_j\rho_k\rho_l, \qquad (2)$$

where $|J_i|$ is the Jacobian determinant of F_i.

We next consider the variational form of the Stokes equation in the velocity-pressure formulation, discretized by the spectral elements. While the velocities are taken to be continuous functions, the pressures can be discontinuous across substructure boundaries. The restriction of the pressure space $\bar{P}^{N-2}(\Omega)$ to each Ω_i is $P^{N-2}(\Omega_i)$. We note that $\bar{P}^{N-2}(\Omega) \subset L^2(\Omega)$, but $\bar{P}^{N-2}(\Omega) \not\subset H^1(\Omega)$.

The discrete problem is given by:
Find $(\mathbf{u}, p) \in (P_0^N(\Omega))^d \times \bar{P}^{N-2}(\Omega) \cap L_0^2(\Omega)$ such that:

$$\left\{ \begin{array}{rcl} a_Q(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) & = & (\mathbf{f}, \mathbf{v})_N \quad \forall \mathbf{v} \in (P_0^N(\Omega))^d, \\ \\ b(\mathbf{u}, q) & = & 0 \quad \forall q \in \bar{P}^{N-2}(\Omega) \cap L_0^2(\Omega). \end{array} \right. \tag{3}$$

Here, $a_Q(\cdot,\cdot)$ is given by $a_Q(\mathbf{u},\mathbf{v}) = \sum_{i,j=1}^{d} \nu \left(\frac{\partial \mathbf{u}_i}{\partial x_j}, \frac{\partial \mathbf{v}_i}{\partial x_j} \right)_N$. We assume, for simplicity, that $b(\mathbf{v}, q) = -\int_\Omega q \nabla \cdot \mathbf{v} \, dx$; see [MPR92]. The right-hand side is assumed to be in $L^2(\Omega)$. Our analysis also applies to the more general non-homogeneous problem, and also to mixed Dirichlet and Neumann boundary conditions, with only minor changes.

For the velocities, we choose standard nodal basis functions $\phi_j^N \in (P_0^N(\Omega))^d$. We number the GLL($N$) nodes ξ *within* the subregions Ω_i by an index r, and define a basis for $\bar{P}^{N-2}(\Omega)$ by $\beta_{r_1}(\xi_{r_2}) = \delta_{r_1 r_2}$, for all r_1, r_2, where δ is the Kronecker symbol. We note that any function of $\bar{P}^{N-2}(\Omega)$ is uniquely represented by its values at the interior GLL(N) nodes ξ_r. By writing the system (3) in terms of these two bases, we arrive, in a standard way, at the system (1). To each component of the velocity, there corresponds a diagonal block of A which is equal to the standard scalar spectral element stiffness matrix K_N. The entries of B are given by $B_{jr} = b(\phi_j^N, \beta_r)$, and $\underline{\mathbf{f}}$ is a vector with components $\underline{\mathbf{f}}_j = (\mathbf{f}, \phi_j^N)$.

The next lemma is the key point in the error analysis of this discretization; see [MPR92].

Lemma 2.1 *For each N, there exists a $\beta_N > 0$ such that*

$$\inf_{q \in \bar{P}^{N-2}(\Omega) \cap L_0^2(\Omega)} \quad \sup_{\mathbf{v} \in (P_0^N(\Omega))^d} \frac{b(\mathbf{v}, q)}{||\mathbf{v}||_{H^1(\Omega)} ||q||_{L^2(\Omega)}} \geq \beta_N.$$

If the geometry is rectilinear, i.e. the F_i are affine mappings, then there exists a constant β, independent of N, and such that $\beta_N \geq \beta N^{\frac{1-d}{2}}$, for $d = 1$, 2, or 3.

We remark that very good convergence properties are predicted by the theory and have been extensively verified in practice; see, e.g., [FR94].

3 An Extension Operator

For a subregion Ω_i, we define an extension operator $E_i^{S,N} : (P^N(\partial\Omega_i))^3 \longrightarrow (P^N(\Omega_i))^3$, where $\mathbf{u}_i = E_i^{S,N}(\mathbf{g}_i)$ is the velocity component of the solution to the following Stokes problem:
Find $(\mathbf{u}_i, p_i) \in ((P^N(\Omega_i))^d, P^{N-2}(\Omega_i) \cap L_0^2(\Omega_i))$, such that:

$$\left\{ \begin{array}{l} a_Q(\mathbf{u}_i, \mathbf{v}_i) + b_{\Omega_i}(\mathbf{v}_i, p_i) = 0 \quad \forall \mathbf{v}_i \in (P_0^N(\Omega_i))^d, \\ b_{\Omega_i}(\mathbf{u}_i, q_i) = 0 \quad \forall q \in P^{N-2}(\Omega_i) \cap L_0^2(\Omega_i), \\ \mathbf{u}_i|_{\partial\Omega_i} = \mathbf{g}_i. \end{array} \right. \tag{4}$$

The subscript Ω_i indicates that the integration or quadrature is taken on Ω_i only. In other words, $\mathbf{u}_i$ is the solution of a homogeneous Stokes problem with $\mathbf{g}_i$ as boundary data, and zero right-hand side within Ω_i. We remark that $\mathbf{u}_i$ always exists, even if the outward fluxes $\int_{\partial\Omega_i} \mathbf{g}_i \cdot \mathbf{n}\, dS$ are not equal to zero, since the pressure test space does not include the constant function, and the Babuška-Brezzi condition is satisfied for the problems restricted to each subregion.

We remark that if $\mathbf{u}_i = E_i^{S,N}(\mathbf{g}_i)$, then

$$a_{Q,\Omega_i}(\mathbf{u}_i, \mathbf{u}_i) = \min_{\mathbf{v}_i|_{\partial\Omega_i} = \mathbf{g}_i} a_{Q,\Omega_i}(\mathbf{v}_i, \mathbf{v}_i) \quad \forall \mathbf{v}_i \in P_\nabla^N(\Omega_i), \tag{5}$$

where $P_\nabla^N(\Omega_i) = \{\mathbf{v}_i \in (P^N(\Omega_i))^d \mid \quad b_{\Omega_i}(\mathbf{v}_i, q_i) = 0 \quad \forall q_i \in P^{N-2}(\Omega_i) \cap L_0^2(\Omega_i)\}$.

Let $P_{0,\nabla}^N(\Omega)$ be the space of discretely divergence-free functions i.e. functions that satisfy the second equation of (3). For $\mathbf{v} \in (P_0^N(\Omega))^d$, let $\tilde{\mathbf{v}}$ be defined by $\tilde{\mathbf{v}}|_{\Omega_i} = E_i^{S,N}(\mathbf{v}|_{\partial\Omega_i})$. It is easy to see that if $\int_{\partial\Omega_i} \mathbf{v} \cdot \mathbf{n}\, dS = 0$ $\forall i$, then $\tilde{\mathbf{v}} \in P_{0,\nabla}^N(\Omega)$.

4 A Domain Decomposition Preconditioner

We describe the construction in detail for two dimensions. The three-dimensional case is analogous; see [GPAR86], Section II.3.1, and Remark 4.1 below. For a reference square $\hat{\Omega} = [-1, 1]^2$, let

$$V_{\mathbf{n}}^H(\hat{\Omega}) = (Q_1(\hat{\Omega}))^2 \oplus \text{span}\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\},$$

where $\mathbf{p}_i \in (Q_2(\hat{\Omega}))^2$ vanishes on the edges $\mathcal{E}_j$ for $j \neq i$, and is normal to $\mathcal{E}_i$. For example, for the edge $\mathcal{E}_1$ given by $x = 1$, $\mathbf{p}_1 = ((1 + x)(1 - y^2), 0)$.

The space $V_{\mathbf{n}}^H(\Omega) \subset (H_0^1(\Omega))^2$ is the space whose restrictions to each Ω_i is the image of $V_{\mathbf{n}}^H(\hat{\Omega})$ under the mapping F_i, which is here taken to be isoparametric with respect to the space $(Q_1(\hat{\Omega}))^2$; see [GPAR86], Section A.2. There are 12 degrees of freedom per element, namely the nodal values at each vertex and the fluxes across each of the edges.

Let $Q_0^H(\Omega)$ be the space of functions of zero mean on Ω that are constant within each substructure Ω_i. It is well-known that for the discretization of the Stokes problem on the coarse mesh, the pair $V_{\mathbf{n}}^H - Q_0^H$ yields a stable discretization in the Babuška-Brezzi sense, with a stability constant bounded away from zero independently of H.

Let $V_{\mathbf{n},\nabla_H}^H(\Omega)$ be defined by: $V_{\mathbf{n},\nabla_H}^H(\Omega) = \{\mathbf{u} \in V_{\mathbf{n}}^H \mid \int_{\Omega_i} \nabla \cdot \mathbf{u} dx = \int_{\partial\Omega_i} \mathbf{u} \cdot \mathbf{n} dS = 0\}$. This space plays the role of our coarse space, but it is clearly not contained in $P_{0,\nabla}^N(\Omega)$, since a function $\mathbf{u} \in V_{\mathbf{n},\nabla_H}^H(\Omega)$ in general fails to have a divergence orthogonal to the space $\bar{P}^{N-2}(\Omega)$ in $L^2(\Omega)$. We therefore define a transfer operator $I_H^h : V_{\mathbf{n},\nabla_H}^H(\Omega) \rightarrow P_{0,\nabla}^N(\Omega)$ by:

$$\begin{cases} I_H^h(\mathbf{u}_H)|_{\partial\Omega_i} = \mathbf{u}_H|_{\partial\Omega_i} \\ \\ I_H^h(\mathbf{u}_H)|_{\Omega_i} = E_i^{S,N}(\mathbf{u}_H|_{\partial\Omega_i}). \end{cases} \tag{6}$$

This operator satisfies the usual H^1-stability and L^2-approximation properties used in the Schwarz theory.

For $\mathbf{u}, \mathbf{v} \in H^1(\Omega)$, we define the bilinear form $a(\mathbf{u}, \mathbf{v}) := \int_\Omega \nabla\mathbf{u} \cdot \nabla\mathbf{v}\, dx$. The coarse solver T_h^H is given by

$$a(T_h^H \mathbf{u}, \mathbf{w}) = a_Q(\mathbf{u}, I_H^h \mathbf{w}) \quad \forall \mathbf{w} \in V_{\mathbf{n},\nabla_H}^H(\Omega).$$

For each edge $\mathcal{E}_k$ shared by two subregions Ω_i and Ω_j, let Ω_{ij} be the union of Ω_i, Ω_j, and $\mathcal{E}_k$. The local space $V_{\mathcal{E}_k} \subset P_{0,\nabla}^N(\Omega)$ consists of functions $\mathbf{u}_{\mathcal{E}_k}$ with support in $\bar{\Omega}_{ij}$, and whose values in the interior of Ω_i and Ω_j are given by $E_i^{S,N}$ and $E_j^{S,N}$, respectively. This definition implies that $\forall \mathbf{u}_{\mathcal{E}_k} \in V_{\mathcal{E}_k}$, $\int_{\mathcal{E}_k} \mathbf{u}_{\mathcal{E}_k} \cdot \mathbf{n}\, dS = 0$. The bilinear form associated with $V_{\mathcal{E}_k}$ is $a_Q(\cdot,\cdot)$.

For each interior vertex v_n, let $\mathcal{E}(v_n)$ be the collection of all edges having v_n as an endpoint. We define $\phi_{v_n,x} \in P_{0,\nabla}^N(\Omega)$ by assigning values at the interface nodes, and using the $E_i^{S,N}$ to extend these values to the interior of the substructures. We let $\phi_{v_n,x}(v_n) = (1,0)$, let $\phi_{v_n,x}$ be equal to zero at all the interface nodes not adjacent to v_n, and $\forall \mathcal{E}_k \in \mathcal{E}(v_n)$, we let $\phi_{v_n,x}$ be equal to a constant vector at the node v_n' next to v_n on the edge $\mathcal{E}_k$. This constant vector is taken to be normal to the interface at v_n', and so that $\int_{\mathcal{E}_k} \phi_{v_n,x} \cdot \mathbf{n}\, dS = 0$. We define $\phi_{v_n,y}$ analogously. The one-dimensional vertex spaces are given by:

$$V_{v_n,x} = \mathrm{span}\{\phi_{v_n,x}\} \text{ and } V_{v_n,y} = \mathrm{span}\{\phi_{v_n,y}\}.$$

The bilinear form associated with the vertex spaces is $a_Q(\cdot,\cdot)$.

The interior spaces are $V_{\Omega_i} = P_{0,\nabla}^N(\Omega_i)$, and the bilinear form associated with all of them is $a_Q(\cdot,\cdot)$.

The preconditioned operator is now

$$T_{\mathbf{n}} = I_H^h T_h^H + \sum_{v_n} (T_{v_n,x} + T_{v_n,y}) + \sum_{\mathcal{E}_k} T_{\mathcal{E}_k} + \sum_{i=1}^{M} T_{\Omega_i}. \tag{7}$$

This operator does not exactly fit the Schwarz framework, but an analyisis similar to the proof of that result, together with a decomposition lemma involving the local spaces and bilinear forms just described, yield the following theorem. For the proof, see [Cas96]; cf. [Bre94, Cai95].

Theorem 4.1 *The condition number of $T_{\mathbf{n}}$ satisfies:*

$$\kappa(T_{\mathbf{n}}) \le \frac{C(1+\log(N))^3}{\beta_N}.$$

Remark 4.1 *In three dimensions, edge and face functions play the role of the vertex and edge functions of the two dimensional version, respectively. For each edge, the edge function is the analogue of the ϕ_{v_n} above; it is nonzero for the interface nodes adjacent to the edge, and have zero flux across all the faces of the subregions. The condition number estimate is the same as in Theorem 4.1, where β_N is now the Babuška-Brezzi constant for the three-dimensional discretization.*

5 Schwarz Methods for the Stationary Navier-Stokes Equations

Following [Røn95], we consider a Galerkin spectral element discretization of the velocity-pressure formulation of the Navier-Stokes equations, given by:

Find $(\tilde{\mathbf{u}}_N, \tilde{p}_{N-2}) \in P_0^N(\Omega) \times (\bar{P}^{N-2}(\Omega) \cap L_0^2(\Omega))$ such that

$$\begin{cases} a(\tilde{\mathbf{u}}_N, \mathbf{v}_N) + c(\tilde{\mathbf{u}}_N; \tilde{\mathbf{u}}_N, \mathbf{v}_N) + b(\mathbf{v}_N, \tilde{p}_{N-2}) = \int_\Omega \mathbf{f} \cdot \mathbf{v}_N \, dx \quad \forall \mathbf{v}_N \in P_0^N(\Omega), \\ b(\tilde{\mathbf{u}}_N, q_{N-2}) = 0 \quad \forall q_{N-2} \in \bar{P}^{N-2}(\Omega) \cap L_0^2(\Omega). \end{cases} \tag{8}$$

For $\mathbf{u}, \mathbf{v}$, and $\mathbf{w} \in H^1(\Omega)$, the trilinear form $c(\cdot; \cdot, \cdot)$ is given by:

$$c(\mathbf{u}; \mathbf{v}, \mathbf{w}) := \sum_{i,j=1}^{d} \int_\Omega \mathbf{u}_j \left(\frac{\partial \mathbf{v}_i}{\partial x_j}\right) \mathbf{w}_i \, dx.$$

Numerical computations show that $\tilde{\mathbf{u}}_N$ is a good approximation for $\mathbf{u}$, the exact solution of the Navier-Stokes equations, at least for Reynolds number $Re = 1/\nu$ on the order of 50; see [Røn95].

We will develop Schwarz preconditioners for the system representing the k^{th} step of the Newton iteration used to solve (8). We fix k, and to simplify notations, set $\mathbf{u}_N := \mathbf{u}_N^k$, $\mathbf{w} := \mathbf{u}_N^{k-1}$, and $\mathbf{g} := \mathbf{f}^k$. Then, $\mathbf{u}_N$ is the solution of the following problem:

Find $\mathbf{u}_N \in P_{0,\nabla}^N(\Omega)$ such that

$$B_{\mathbf{w}}(\mathbf{u}_N, \mathbf{v}_N) = (\mathbf{g}, \mathbf{v}_N) \quad \mathbf{v}_N \in P_{0,\nabla}^N(\Omega), \tag{9}$$

where

$$B_{\mathbf{w}}(\mathbf{u}_N, \mathbf{v}_N) = a(\mathbf{u}_N, \mathbf{v}_N) + c(\mathbf{w}; \mathbf{u}_N, \mathbf{v}_N) + c(\mathbf{u}_N; \mathbf{w}, \mathbf{v}_N). \tag{10}$$

We assume that $\tilde{\mathbf{u}}_N$ is a solution of (8) which is non-singular i.e. (9) is uniquely solvable if we let $\mathbf{w} = \tilde{\mathbf{u}}_N$. If the Reynolds number $\mathrm{Re} = 1/\nu$ is small enough, this can be proved by classical arguments (see [GPAR86], Theorem IV.2.4); our analysis does not assume Re is small enough, although the iteration count of the method may deteriorate when that parameter increases.

6 A Schwarz Preconditioner with a New Coarse Space

We propose a Schwarz preconditioner for $B(\cdot, \cdot)$, by viewing $B(\cdot, \cdot)$ restricted to $P_{0,\nabla}^N(\Omega)$ as a perturbation of the symmetric bilinear form $a(\cdot, \cdot)$. We assume that the coarse triangulation $\tau_H = \cup_{i=1}^M \Omega_i$ is a shape regular triangulation, not necessarily quasi-uniform, and set $H = \max_i H_i$, where H_i is the diameter of Ω_i.

We start the definition of our coarse space by first defining an extension operator $\tilde{I}_H^h$, similar to the operator I_H^h, defined in (6). Let $\tilde{I}_H^h : V_{\mathbf{n},\nabla_H}^H(\Omega) \longrightarrow P_{0,\nabla}^N(\Omega)$, and let $\tilde{\mathbf{u}}_H = \tilde{I}_H^h(\mathbf{u}_H)$ for $\mathbf{u}_H \in V_{\mathbf{n},\nabla_H}^H(\Omega)$. The restriction of $\tilde{\mathbf{u}}_H$ to a subregion Ω_i is the solution of the following non-homogeneous Stokes problem:

Find $\tilde{\mathbf{u}}_H \in P_\nabla^N(\Omega_i)$, with $\tilde{\mathbf{u}}_H = \mathbf{u}_H$ on $\partial\Omega_i$, and $\tilde{p}_H \in P^{N-2}(\Omega_i) \cap L_0^2(\Omega_i)$ such that

$$\begin{cases} a(\tilde{\mathbf{u}}_H, \mathbf{v}_N) + b(\mathbf{v}_N, \tilde{p}_H) = a(\mathbf{u}_H, \mathbf{v}_N) \quad \forall \mathbf{v}_N \in P_0^N(\Omega_i), \\ b(\tilde{\mathbf{u}}_H, q_{N-2}) = 0 \quad \forall q_{N-2} \in P^{N-2}(\Omega_i) \cap L_0^2(\Omega_i). \end{cases} \tag{11}$$

By restricting the test function $\mathbf{v}_N$ to have zero discrete divergence, i.e. $\mathbf{v}_N \in P^N_{0,\nabla}(\Omega_i)$, $\tilde{\mathbf{u}}_H$ can also be determined by:
Find $\tilde{\mathbf{u}}_H \in P^N_{\nabla}(\Omega_i)$, $\tilde{\mathbf{u}}_H = \mathbf{u}_H$ on $\partial\Omega_i$, and such that

$$a(\tilde{\mathbf{u}}_H, \mathbf{v}_N) = a(\mathbf{u}_H, \mathbf{v}_N) \quad \forall \mathbf{v}_N \in P^N_{0,\nabla}(\Omega_i). \tag{12}$$

The new coarse space is defined by:

$$\tilde{V}^H_{\mathbf{n},\nabla_H}(\Omega) = \tilde{I}^h_H(V^H_{\mathbf{n},\nabla_H}(\Omega)).$$

An easy argument using Green's formula shows that $\tilde{V}^H_{\mathbf{n},\nabla_H}(\Omega) \subset P^N_{0,\nabla}(\Omega)$; it is also easy to see that $\tilde{\mathbf{u}}_H$ is the function of $P^N_{0,\nabla}(\Omega)$ which coincides with $\mathbf{u}_H$ on Γ, and which is the best approximation of $\mathbf{u}_H$ in the $a(\cdot,\cdot)$-semi-norm (and in the H^1-semi-norm, since they differ only by a fixed factor ν).

The operator $Q_H : P^N_{0,\nabla}(\Omega) \rightarrow \tilde{V}^H_{\mathbf{n},\nabla_H}(\Omega)$ is defined by

$$B(Q_H\mathbf{u}, \mathbf{v}_H) = B(\mathbf{u}, \mathbf{v}_H) \quad \forall \mathbf{v}_H \in \tilde{V}^H_{\mathbf{n},\nabla_H}(\Omega). \tag{13}$$

We remark that although $B(\cdot,\cdot)$ is not necessarily positive definite, (13) is guaranteed to have solutions for sufficiently small values of H; see property **P3** below.

Let V_s, $s \geq 1$ be the local spaces used to define the operator $T_{\mathbf{n}}$; see (7). In three dimensions, there is one local space associated with the interior of each Ω_i, one space related to each face, and one for each edge. For $s \geq 1$, the operator $P_s : P^N_{0,\nabla}(\Omega) \rightarrow V_s$ is defined by

$$a(P_s\mathbf{u}, \mathbf{v}_s) = B(\mathbf{u}, v_s) \quad \forall \mathbf{v}_s \in V_s. \tag{14}$$

Theorem 6.1 *There exists a positive constant H_0, depending only on the domain Ω and on the solution $\tilde{\mathbf{u}}_N$, and positive constants $c(H_0)$, and $C(H_0)$ such that the operator*

$$Q_a = Q_H + \sum_{s\geq 1} P_s$$

satisfies, $\forall \mathbf{u} \in P^N_{0,\nabla}(\Omega)$, and for $H \leq H_0$,

$$a(Q_a\mathbf{u}, Q_a\mathbf{u}) \leq C(H_0)a(\mathbf{u},\mathbf{u}),$$

and

$$c(H_0)C_0^{-2}a(\mathbf{u},\mathbf{u}) \leq a(Q_a\mathbf{u},\mathbf{u}).$$

The proof of this result is given in [Cas96].

This estimate immediately implies an upper bound on the iteration count of the GMRES method applied to the preconditioned system

$$Q_{\mathrm{a}}\, \underline{\mathbf{u}}_N = \underline{b},$$

where $\underline{b}$ is chosen so that $\underline{\mathbf{u}}_N$ is the vector of nodal values of $\mathbf{u}_N$. This result is an extension to the Navier-Stokes equation of the Schwarz method for scalar second-order non-symmetric problems studied in [CW92].

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