1

Stabilization Techniques for Domain Decomposition Methods with Non-Matching Grids

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1 Introduction

The use of domain decomposition methods with non-matching grids is becoming increasingly popular. In particular, its use is recommended when the splitting into subdomains is dictated by physical and/or geometrical reasons rather than merely by computational ones. Without underestimating the relevance of this latter group of applications (which can be extremely important and even crucial in a number of practical cases), we shall concentrate on the former one. To fix ideas, let us consider a "toy-problem" which will show well enough what we have in mind without using too heavy notation. Suppose therefore that we have a domain $\Omega =]-1,1[\times]0,1[$ split into $\Omega_1 =]-1,0[\times]0,1[$ and $\Omega_2 =]0,1[\times]0,1[$. In order to solve the problem, say,

$$-\Delta u \;=\; f \qquad \text{in } \Omega, \qquad u \;=\; 0 \qquad \text{on } \partial\Omega, \tag{1.1}$$

we decompose separately Ω_1 and Ω_2 by means of two finite element grids $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$ respectively, and we want to approximate (for $i = 1,2$) u^i (restriction of u to Ω_i) by u_h^i, continuous and piecewise linear on the grid $\mathcal{T}_h^i$. Clearly, on the interface $\Gamma = \{0\}\times]0,1[$ we have two 1-d decompositions, induced by $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$, which, in general, do not match. A typical solution to this (as in the mortar method [Mar90]) is to choose one of the two, say $\mathcal{T}_h^2$, and require that $u^1_{h|\Gamma}$ match $u^2_{h|\Gamma}$ only in some weak sense, with the use of suitable Lagrange multipliers. (In the mortar method terminology, the nodes of $\mathcal{T}^2_{h\ |\Gamma}$ will be "masters" and the nodes of $\mathcal{T}^1_{h\ |\Gamma}$, "slaves".)

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However, in certain cases, it can be useful to choose a *third* 1-d decomposition on Γ, (say $\mathcal{T}_h^3(\Gamma)$ or simply $\mathcal{T}^\Gamma$) and have both the $\mathcal{T}^1_{h\,|\Gamma}$ and $\mathcal{T}^2_{h\,|\Gamma}$ nodes as "slaves". An example where this approach can be convenient is when both $\mathcal{T}^1_{h\,|\Gamma}$ and $\mathcal{T}^2_{h\,|\Gamma}$ are non uniform (being dictated by approximation problems that might occur in Ω_1 and Ω_2, or by self-adaptive procedures that have been used in both subdomains), but a uniform grid on Γ is recommended in order to apply a better preconditioner on the final interface problem. This suggests the use of two different Lagrange multipliers, one for matching u_h^1 with u_h^Γ, and the other one for matching u_h^2 with u_h^Γ, where, obviously, we denoted by u_h^Γ the discretization of $u_{|\Gamma}$. As it is well known, this requires suitable inf-sup conditions (see e.g. [GPP96]) to be fulfilled, one on each side of Γ. Recently, an intensive study has been carried out in order to avoid this type of inf-sup conditions by adding of suitable stabilizing terms, thus allowing more freedom in the choice of grids and multipliers (see e.g. [AG93, GG95]). In turn, in different contexts, these techniques have been reinterpreted and/or improved as the addition-elimination of suitable bubble functions to the finite element spaces in use (see e.g. [Pes72, Glo84]).

In this paper, we present a new way for stabilizing Dirichlet problems with Lagrange multipliers for the particular case where u is approximated by a piecewise linear continuous function, and the Lagrange multipliers are approximated by piecewise constant functions on a nonmatching grid. Our stabilization is made by adding suitable bubble functions only on the triangles having an edge on the boundary. It is interesting to note that elimination of the bubbles by static condensation leads to a scheme very similar to that introduced a long time ago by Nitsche [DW95] and recently reproposed and analyzed in [Osw95].

For the sake of simplicity, we shall only discuss a single-domain problem. The extension to many subdomains can then be carried out by means of the usual coupling procedures (Dirichlet-Dirichlet or Neumann-Neumann or something else).

The organization of the paper is the following. In Sect. 2 we present the single-domain problem, where the Dirichlet condition is imposed via Lagrange multipliers. In Sect. 3 we discuss its discretization with nonmatching grids and the bubble stabilization. In Sect. 4 we show that it is possible to eliminate *both* bubbles and Lagrange multipliers, thus obtaining a scheme that is easy to implementation and that strongly resembles the one discussed in [DW95, Osw95]. If needed, the Lagrange multipliers can be recovered by a simple and economical post-processing. This will be useful in a true domain decomposition situation, in order to carry out the iterative procedure.

2 The Single Domain Problem

In order to introduce our stabilization technique we shall consider a problem on a single domain, thinking of it as one of the subdomains. Always referring for simplicity to the global problem (1.1), at each step of the domain decomposition procedure we have to solve, in each subdomain, a problem of the type

$$-\Delta u \;=\; f \qquad \text{in } \Omega, \qquad u \;=\; g \qquad \text{on } \partial\Omega =: \Gamma, \tag{2.2}$$

where Ω is now the subdomain under consideration (that we assume to be a polygon), and g denotes any continuous function which, eventually, should be the value of the solution of (1.1) on $\partial\Omega \equiv$ interface between subdomains. By enforcing the boundary conditions in (2.1) with Lagrange multipliers [Ben95b], the variational formulation of (2.1) reads

$$\left\{\begin{array}{lll} \text{Find } u \in V,\ \lambda \in M \text{ such that} & & \\ \int_\Omega \underline{\nabla} u \cdot \underline{\nabla} v \, dx - \int_\Gamma \lambda v \, ds & = \int_\Omega f v \, dx & \forall v \in V, \\ \int_\Gamma u\mu \, ds & = \int_\Gamma g\mu \, ds & \forall \mu \in M, \end{array}\right. \tag{2.3}$$

where λ is the multiplier, and V and M are the spaces

$$V \ := \ H^1(\Omega), \qquad M \ := \ H^{-1/2}(\Gamma)$$

with their usual norms (see [Ben95a]). With this choice for V and M, the abstract theory applies (see [GPP96]) so that problem (2.2) has a unique solution (u, λ), verifying

$$\left\{\begin{array}{rcll} -\Delta u & = & f & \text{in } \Omega \\ \lambda & = & \frac{\partial u}{\partial n} & \text{on } \Gamma \\ u & = & g & \text{on } \Gamma. \end{array}\right. \tag{2.4}$$

The usual finite element approximation of (2.2) would be to choose a decomposition $\mathcal{T}^u$ of Ω for discretizing the u variable, and take as a decomposition of Γ for the λ variable the restriction of $\mathcal{T}^u$ to Γ. Next, finite element spaces verifying the Inf-Sup condition can easily be constructed in many ways. This cannot be done in our case. Actually, in order that the discretization of (2.2) mimic the situation occurring in the domain decomposition procedure, we have to assume that the decompositions for u and g are given by $\mathcal{T}^u$ and $\mathcal{T}^g$, which do not match. Consequently, we have to introduce another decomposition of Γ, say $\mathcal{T}^\lambda$, for dealing with the multipliers λ and μ. This decomposition cannot be chosen arbitrarily, since it has to guarantee some Inf-Sup condition between the λ's and the g's, and therefore either has to coincide with $\mathcal{T}^g$ or depend on it strongly. More precisely, $\mathcal{T}^\lambda$ can be chosen finer than $\mathcal{T}^g$ without violating the Inf-Sup condition between the variables μ and the interface variables g, but it can never be coarser. In the next section we shall deal with this problem.

3 Discretization and Stabilization

Let us turn to the discretization of (2.2). Let then $\mathcal{T}_H^u$ be a decomposition of Ω into triangles $\{T\}$, H being the mesh size, and let $\mathcal{T}_h^\lambda$ be a decomposition of Γ into intervals I, h being the mesh size. We define

$$V_H \ = \ \{v \in H^1(\Omega) : \ v_{|T} \in P_1(T) \ \forall T \in \mathcal{T}_H^u\}, \tag{3.5}$$

$$M_h \ = \ \{\mu \in L^2(\Gamma) : \ \mu_{|I} \in P_0(I) \ \forall I \in \mathcal{T}_h^\lambda\}. \tag{3.6}$$

We now look for an approximate solution (u_H, λ_h) of (2.2), with $u_H \in V_H$, and $\lambda_h \in M_h$. As already pointed out, the two decompositions $\mathcal{T}_H^u$ and $\mathcal{T}_h^\lambda$ are not compatible, that is, the decomposition $\mathcal{T}_H^u$ generates a decomposition of Γ which is, in general, different from the decomposition $\mathcal{T}_h^\lambda$ of Γ. Our first step will then be to relate the two decompositions of Γ, the second step will consist in the introduction of the bubble functions, and the final step will be to analyze the stabilized problem.

1^{st} step - *Generation of a new decomposition.*

We create a new decomposition of Γ, say $\widetilde{\mathcal{T}}_h^\lambda$, by merging the two decompositions $\mathcal{T}_H^u$ and $\mathcal{T}_h^\lambda$, i.e., we add to $\mathcal{T}_h^\lambda$ the nodes of $\mathcal{T}_H^u$ belonging to Γ. In doing this, it may occur that some of the nodes of $\widetilde{\mathcal{T}}_h^\lambda$ get too close to each other, thus complicating the analysis of our procedure. To avoid this we may proceed as follows: when the distance between two nodes of $\widetilde{\mathcal{T}}_h^\lambda$ is less than or equal to some tolerance, one of the two nodes is eliminated. This can be easily done by slightly changing either the $\mathcal{T}_H^u$ or the $\mathcal{T}_h^\lambda$ decomposition, so that the two nodes become coincident. In other words, we are making the following assumption: for every triangle T in $\mathcal{T}_H^u$ having an edge E on the boundary, let H_T be the diameter of T, and let h_T be the smallest length of the intervals of $\widetilde{\mathcal{T}}_h^\lambda$ belonging to E. We assume that there exists a constant γ independent of the decompositions, such that

$$h_T \geq \gamma H_T. \tag{3.7}$$

2^{nd}step - *Introduction of the bubbles.*

We add to the discretization of u as many bubble functions as the intervals of $\widetilde{\mathcal{T}}_h^\lambda$. More precisely, we proceed as follows. Let T be a triangle having an edge on Γ. Let T' be such an edge; in general, we will have a situation of the type $T' = \cup I_k,\ I_k \in \widetilde{\mathcal{T}}_h^\lambda$ and, accordingly, $T = \cup T_k$ (see Fig. 1 as an example). We call *bubble* a function $b_k \in H^1(\Omega)$ such that $supp(b_k) \subset T_k$, and $\int_{I_k} b_k\,ds \neq 0$. (See Fig. 2). In order to have uniform estimates, we need however that the bubbles have "similar" shape. For that, let $\hat{T}$ be the reference triangle: $\hat{T} = \{(\xi,\eta):\ 0 \leq \xi \leq 1,\ 0 \leq \eta \leq 1-\xi\}$, and let $\hat{b}$ be a function in $H^1(\hat{T})$, with $\hat{b} = 0$ on the edges $\xi = 0$ and $\eta = 0$, and $\int_{\partial\hat{T}} \hat{b}\,ds \neq 0$. (As a simple example, we can take $\hat{b}(\xi,\eta) = \xi\eta$. Many other choices are possible, and the optimal shape of $\hat{b}$ is still under investigation.) Our bubble b_k will then be given by $b_k(x,y) = \hat{b}(\xi,\eta)$ under the affine mapping $(\xi,\eta) \to (x,y)$ from $\hat{T}$ to T_k which maps the edge $\eta = 1-\xi$ on the boundary edge I_k.

3^{rd}step - *The stabilized problem.*

Let B_h be the space spanned by the bubbles introduced above. We then write the new discrete problem with V_H replaced by

$$\widetilde{V}_H := V_H \oplus B_h, \tag{3.8}$$

and M_h replaced by

$$\widetilde{M}_h \;=\; \{\mu \in L^2(\Gamma):\ \mu_{|I} \in P_0(I)\ \forall I \in \widetilde{\mathcal{T}}_h^\lambda\}. \tag{3.9}$$

Figure 1

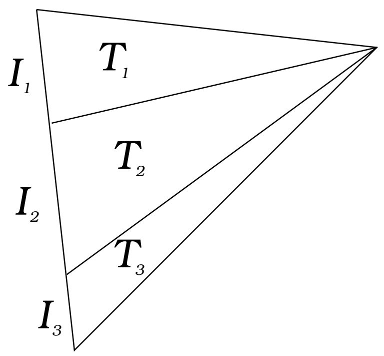
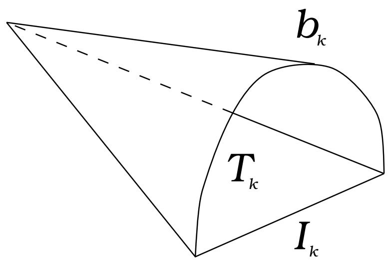


Figure 2



The approximate problem now reads

$$\left\{\begin{array}{l} \text{Find } u_H \in \widetilde{V}_H,\ \lambda_h \in \widetilde{M}_h \text{ such that} \\ \quad \int_\Omega \underline{\nabla} u_H \cdot \underline{\nabla} v_H \, dx - \int_\Gamma \lambda_h v_H \, ds \quad = \int_\Omega f v_H \, dx \qquad \forall v_H \in \widetilde{V}_H, \\ \qquad\qquad\qquad\qquad \int_\Gamma \mu\, u_H \, ds \quad = \int_\Gamma g\mu \, ds \qquad \forall \mu \in \widetilde{M}_h. \end{array}\right. \tag{3.10}$$

Existence, uniqueness, and optimal error bounds for the solution of (3.6) will follow if we can prove the following Inf-Sup condition relating $\widetilde{V}_H$ and $\widetilde{M}_h$:

$$\left\{\begin{array}{l} \qquad \exists \beta > 0 \text{ independent of } h \text{ such that:} \\ \dfrac{\int_\Gamma \mu v \, ds}{||\mu||_M ||v||_V} \geq \beta \qquad \forall v \in \widetilde{V}_H,\ \forall \mu \in \widetilde{M}_h. \end{array}\right. \tag{3.11}$$

As the Inf-Sup condition holds for the continuous problem, (3.7) will follow from the general results of [For77], if we prove the following theorem.

Theorem 3.1 *There exists a constant C, and, for every H, a linear continuous operator $\Pi_H : V \longrightarrow \widetilde{V}_H$ such that*

$$\int_\Gamma (\Pi_H v - v)\mu \, ds \;=\; 0 \qquad \forall \mu \in \widetilde{M}_h, \tag{3.12}$$

and

$$||\Pi_H v||_V \leq C||v||_V \qquad \forall v \in V. \tag{3.13}$$

Proof. We start by observing, cf. [GPP94], that it is possible to construct a linear operator $\Pi_H^1 :\ V = H^1(\Omega) \longrightarrow V_H$ with the following properties:

$$\Pi_H^1 v = v \qquad \forall v \in V_H \tag{3.14}$$

$$||\Pi_H^1 v||_V \leq C||v||_V \qquad \forall v \in V, \tag{3.15}$$

$$\forall T' \in (\mathcal{T}_H^u)_{|\Gamma}\ ||\Pi_H^1 v||_{0,E} \leq C||v||_{0,\widetilde{E}} \qquad \forall v \in V, \tag{3.16}$$

where, here and in the following, $\widetilde{E}$ is the union of the boundary edges in $\mathcal{T}_H^u$ having at least one vertex in common with E, $||v||_{0,D}$ is the norm in $L^2(D)$, $||v||_{s,D}$ the norm in $H^s(D)$, and C denotes a constant independent of the mesh size. We want to check that, for every edge E on Γ, we also have

$$||v - \Pi_H^1 v||_{0,E} \le C H_T^{1/2} ||v||_{1/2,\widetilde{E}}. \tag{3.17}$$

For this, using interpolation theory (see [Ben95a, DSW96]) and (3.12), we only need to show that, for all v in $H^1(\widetilde{E})$, we have

$$||v - \Pi_H^1 v||_{0,E} \le C H_T ||v||_{1,\widetilde{E}}, \tag{3.18}$$

which easily follows from (3.12) and (3.10) by the following standard argument:

$$\begin{aligned} ||v - \Pi_H^1 v||_{0,E} &\le \inf_p ||(v-p) - \Pi_H^1(v-p)||_{0,E} \\ &\le C \inf_p ||v-p||_{0,\widetilde{E}} \le C H_T ||v||_{1,\widetilde{E}}, \end{aligned} \tag{3.19}$$

where the infimum is taken over the polynomials p of degree ≤ 1 in $\widetilde{E}$. Then, define another linear continuous operator $\Pi_h^2 : V \longrightarrow B_h$ as

$$\int_\Gamma (\Pi_h^2 v - v)\mu \, ds = 0 \qquad \forall \mu \in \widetilde{M}_h. \tag{3.20}$$

It can be proved that Π_h^2 is uniquely defined by (3.16), and verifies

$$||\Pi_h^2 v||_{0,T} \le C H_T^{1/2} ||v||_{0,E} \qquad \forall T \in \mathcal{T}_H^u, \tag{3.21}$$

$$||\Pi_h^2 v||_{1,T} \le C h_T^{-1} ||\Pi_h^2 v||_{0,T} \qquad \forall T \in \mathcal{T}_H^u. \tag{3.22}$$

Finally, define Π_H as

$$\Pi_H v := \Pi_H^1 v + \Pi_h^2 (v - \Pi_H^1 v) \qquad v \in V. \tag{3.23}$$

It is immediate to check that Π_H is linear and verifies (3.8), since, from (3.19), (3.16) we have

$$\int_\Gamma (v - \Pi_H v)\mu \, ds = \int_\Gamma \Big((v - \Pi_H^1 v) - \Pi_h^2 (v - \Pi_H^1 v) \Big) \mu \, ds = 0 \qquad \forall \mu \in \widetilde{M}_h. \tag{3.24}$$

It remains to prove that Π_H verifies (3.9). We first remark that $\Pi_H v = \Pi_H^1 v$ in all triangles T that do not have edges belonging to Γ. For the remaining triangles, using (3.18)-(3.17), and (3.13) gives

$$\begin{aligned} ||\Pi_h^2 (v - \Pi_H^1 v)||_{1,T} &\le C h_T^{-1} H_T^{1/2} ||v - \Pi_H^1 v||_{0,E} \\ &\le C h_T^{-1} H_T ||v||_{1/2,\widetilde{E}}, \end{aligned} \tag{3.25}$$

so that, from the definition (3.19), using (3.11) and (3.21) we have

$$\begin{aligned} ||\Pi_H v||_V &\le C \, ||\Pi_H^1 v||_V + \Big(\textstyle\sum_E ||\Pi_h^2 (v - \Pi_H^1 v)||_{1,T}^2 \Big)^{1/2} \\ &\le C \, \Big(||v||_V + (\textstyle\sum_E h_T^{-2} H_T^2 ||v||_{1/2,\widetilde{E}}^2)^{1/2} \Big) \\ &\le C ||v||_V, \end{aligned} \tag{3.26}$$

where, in the last inequality, we used (3.3) and the fact that

$$\sum_E ||v||^2_{1/2,\widetilde{E}} \leq 3||v||^2_{1/2,\Gamma} \leq C||v||^2_V. \tag{3.27}$$

4 Interpretation of the Scheme

We will show in this section that the approximation (3.6) is directly related to Nitsche's scheme recently analyzed in Stenberg [Osw95]. For that, we rewrite (3.6) using the splitting (3.4) for trial and test functions in $\widetilde{V}_H$

$$u_H = u + \beta,\ v_H = v + b,\ u, v \in V_H,\ \beta, b \in B_h, \tag{4.28}$$

and we obtain

$$\left\{\begin{array}{lcll} \text{Find } u \in V_H,\ \beta \in B_h,\ \lambda_h \in \widetilde{M}_h \text{ such that} & & & \\ \int_\Omega (\underline{\nabla} u + \underline{\nabla}\beta)\cdot\underline{\nabla}\, v\, dx - \int_\Gamma \lambda_h v\, ds & = & \int_\Omega f v\, dx & \forall v \in V_H, \\ \int_\Omega (\underline{\nabla} u + \underline{\nabla}\beta)\cdot\underline{\nabla} b\, dx - \int_\Gamma \lambda_h b\, ds & = & \int_\Omega f b\, dx & \forall b \in B_h, \\ \int_\Gamma \mu\,(u+\beta)\, ds & = & \int_\Gamma g\mu\, ds & \forall \mu \in \widetilde{M}_h. \end{array}\right. \tag{4.29}$$

Let us point out that, by construction, B_h and $\widetilde{M}_h$ have the same dimension, say NB. As a basis in B_h it is natural to use the functions $\{b_k\}$ defined in the previous Section (2^{nd} *step*), while a natural basis in $\widetilde{M}_h$ will be given by the functions $\mu_k =$ *the characteristic function of* I_k, for $k = 1, .., NB$. Then, we can write

$$\beta = \sum_k \beta_k b_k, \qquad\qquad \lambda_h = \sum_k \lambda_k \mu_k. \tag{4.30}$$

From the third equation of (4.2) we can derive the coefficients β_k in terms of the linear unknown u. Taking $\mu = \mu_k$ we have

$$\beta_k = \int_{I_k} (g-u)\, ds \Big/ \int_{I_k} b_k\, ds \qquad \forall k. \tag{4.31}$$

From the second equation of (4.2), taking $b = b_k$ we can express the λ_k's in terms of u and β_k

$$\begin{array}{rcl} \lambda_k & = & (\int_{T_k} \underline{\nabla}\, u \cdot \underline{\nabla} b_k\, dx + \beta_k \int_{T_k} |\underline{\nabla} b_k|^2\, dx - \int_{T_k} f b_k\, dx)/\int_{I_k} b_k\, ds \\ & = & (\int_{I_k} b_k u_{/n}\, ds + \beta_k \int_{T_k} |\underline{\nabla} b_k|^2\, dx - \int_{T_k} f b_k\, dx)/\int_{I_k} b_k\, ds \qquad \forall k \end{array} \tag{4.32}$$

where we have integrated the first integral by parts, and where $u_{/n}$ denotes the outward normal derivative of u. Using (4.3), the first equation of (4.2) becomes

$$\int_\Omega \underline{\nabla}\, u \cdot \underline{\nabla}\, v\, dx + \sum_k \beta_k \int_{I_k} b_k v_{/n}\, ds - \sum_k \lambda_k \int_{I_k} v\, ds \;=\; \int_\Omega f v\, dx \qquad \forall v \in V_H, \tag{4.33}$$

where again we have integrated the second integral by parts. From (4.4) and $v_{/n} =$ *constant* on I_k we have

$$\sum_k \beta_k \int_{I_k} b_k v_{/n}\, ds = \sum_k \int_{I_k} (g-u) v_{/n}\, ds = \int_\Gamma (g-u) v_{/n}\, ds. \tag{4.34}$$

Setting

$$C_k = \int_{T_k} |\underline{\nabla} b_k|^2\, dx \Big/ \left(\int_{I_k} b_k\, ds \right)^2, \tag{4.35}$$

we deduce from (4.5)

$$\sum_k \lambda_k \int_{I_k} v\, ds = \sum_k \int_{I_k} v u_{/n}\, ds + \sum_k C_k \left(\int_{I_k} (g-u)\, ds \right) \int_{I_k} v\, ds - F(v), \tag{4.36}$$

where, for the sake of simplicity, we set

$$F(v) \;=\; \sum_k \Big(\int_{T_k} f b_k\, dx \Big) \Big(\int_{I_k} v\, ds \Big) \Big/ \Big(\int_{I_k} b_k\, ds \Big). \tag{4.37}$$

The second integral in the right-hand side of (4.9) can be rewritten by using the mean value $\overline{v}$ of v on I_k, leading to

$$\sum_k C_k h_k \int_{I_k} (g-u)\overline{v}\, ds \;=\; \sum_k C_k h_k \int_{I_k} (\overline{g} - \overline{u})\overline{v}\, ds, \tag{4.38}$$

where, obviously, h_k is the length of I_k. To simplify the notation, we can also set

$$B_\Gamma(u,v) \;=\; \sum_k C_k h_k \int_{I_k} \overline{u}\,\overline{v}\, ds. \tag{4.39}$$

Substituting (4.7) and (4.9) into (4.6), and using (4.10), (4.12) we finally obtain

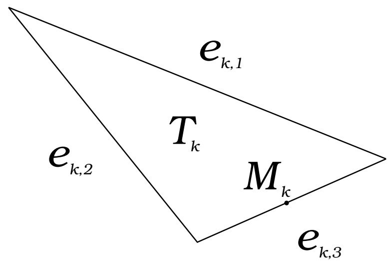
$$\begin{cases} \text{Find } u \in V_H \text{ such that :} \\ \int_\Omega \underline{\nabla} u \cdot \underline{\nabla} v\, dx - \int_\Gamma v u_{/n}\, ds - \int_\Gamma u v_{/n}\, ds + B_\Gamma(u,v) = \\ \int_\Omega f v\, dx - \int_\Gamma g v_{/n}\, ds + B_\Gamma(g,v) - F(v) \qquad \forall v \in V_H. \end{cases} \tag{4.40}$$

It is interesting to compare (4.13) with Nitsche's method that, as studied in [Osw95], reads

$$\begin{cases} \text{Find } u \in V_H \text{ such that :} \\ \int_\Omega \underline{\nabla} u \cdot \underline{\nabla} v\, dx - \int_\Gamma v u_{/n}\, ds - \int_\Gamma u v_{/n}\, ds + \alpha \int_\Gamma u v\, ds = \\ \int_\Omega f v\, dx - \int_\Gamma g v_{/n}\, ds + \alpha \int_\Gamma g v\, ds \qquad \forall v \in V_H, \end{cases} \tag{4.41}$$

where α is a positive parameter to be adjusted, typically, to be of the order of the inverse of the mesh size. As we can see, the only differences between (4.13) and (4.14) are: i) the use of $B_\Gamma(u,v)$ (defined in (4.12)) instead of $\alpha \int_\Gamma uv\, ds$, and ii) the addition of the term $F(v)$ to the right-hand side. In what follows, we will indicate a simple way

Figure 3



for computing $B_\Gamma(u,v)$ and $F(v)$ when using quadratic bubbles, thus producing an estimate of their order of magnitude.

Let then T be a boundary triangle, and let T_k be a subtriangle as in Fig. 1. We denote by $e_{k,i},\ i=1,2,3$ the edges of T_k, and assume $e_{k,3}$ to be the boundary edge; M_k is the midpoint of $e_{k,3}$, and the $\lambda' s$ are the usual barycentric coordinates of T_k (see Fig. 3.) With this notation, the bubble is $b_k(x,y)=\lambda_1(x,y)\lambda_2(x,y)$. With usual techniques we find

$$\int_{I_k} b_k\,ds = |e_{k,3}|/6, \qquad \int_{T_k} |\underline{\nabla} b_k|^2\,dx = \frac{(\sum_{i=1}^3 |e_{k,i}|^2)}{48|T_k|}, \tag{4.42}$$

so that (4.8) becomes

$$C_k = \frac{3(\sum_{i=1}^3 |e_{k,i}|^2)}{4|T_k||e_{k,3}|^2}. \tag{4.43}$$

Since u and v are linear on $e_{k,3}$, combining (4.12) and (4.16), and noting that in this case $h_k=|e_{k,3}|$, we obtain the following expression for $B_\Gamma(u,v)$

$$B_\Gamma(u,v) = \frac{3}{4}\sum_{k=1}^{NB} \frac{(\sum_{i=1}^3 |e_{k,i}|^2)}{|T_k|} u(M_k)v(M_k). \tag{4.44}$$

Notice that, when g is used instead of u, the value $u(M_k)$ has to be replaced by the mean value of g in I_k. We also point out that, comparing (4.17) with (4.14), we see that our method corresponds to choosing, in each I_k, a value of α of the order of H_T/h_k^2.

We now turn to the computation of the term $F(v)$, assuming that f is constant in T_k and v is a basis function in V_H. Clearly, from (4.10) we have $F(v)=0$ if v is associated with an internal vertex of $\mathcal{T}_H^u$. Otherwise, a simple computation shows that

$$F(v) = \sum_{k=1}^{NB} f_k \frac{|T_k|}{2|e_{k,3}|}\int_{I_k} v\,ds. \tag{4.45}$$

In addition, it can easily be checked that

$$\frac{|T_k|}{2|e_{k,3}|}\int_{I_k} v\,ds = \frac{|T_k|v(M_k)}{2} = \frac{3}{4}\int_{T_k} v\,dx. \tag{4,46}$$

Hence,

$$F(v) = \frac{3}{4}\sum_{k=1}^{NB}\int_{T_k} fv\,dx. \tag{4.47}$$

Finally, we point out that, in domain decomposition procedures, the explicit knowledge of the Lagrange multiplier λ_h in (3.6) is needed in order to update the interface unknown g during an iterative solution. With our approach, once u has been computed out of (4.13), the value of λ_h in each I_k can be easily recovered from (4.5), which gives

$$\lambda_k = (u_{/n})_{|I_k} + C_k\int_{I_k}(g-u)\,ds - f_k|T_k|/(2|e_{k,3}|). \tag{4.48}$$

5 Conclusions

The single-domain Dirichlet problem for a linear elliptic operator can be solved by the Lagrange multipliers technique, which is well suited when the boundary condition is given on a grid which does not match with the one used within the domain. If the problem with Lagrange multipliers is stabilized by boundary bubbles, it is possible (with "paper and pencil") to eliminate *a priori* both bubbles *and* Lagrange multipliers. The resulting scheme, which is quite simple to implement, results in a variant of the Nitsche's method [DW95]. As needed in domain decomposition procedures, the Lagrange multipliers can then be computed afterwards, in each subdomain, by an easy and economical post-processing.

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