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Decoupling and Modal Synthesis of Vibrating Continuous Systems

Frédéric Bourquin and Rabah Namar

Laboratoire des Matériaux et des Structures du Génie Civil, UMR113 LCPC/CNRS, 2 allée Kepler, 77420 Champs sur Marne, France

1 Introduction

The modal synthesis of a structure that is decomposed into substructures is a Rayleigh-Ritz approximation of the global eigenvalue problem over a space spanned by a few eigenmodes of each substructure in addition to some functions, called *coupling modes* describing the interfacial displacements. Different methods based on intrinsic choices of such *coupling modes* are presented. In particular, extension operators from the boundary of each subdomain to the whole interface are introduced in view of defining, both in a continuous and in a discrete setting, "generalized" Neumann-Neumann preconditioners, the eigenfunctions of which are used as *coupling modes* for approximating the global eigenvalue problem.

Let us consider the model problem of a vibrating three-dimensional body Ω. The family of eigenpairs $\{\lambda_k, \boldsymbol{u}_k\}_{k=1}^{+\infty}$, arranged in nondecreasing order of the eigenvalues λ_k, solve the eigenvalue problem

$$\begin{aligned} -div(\mathcal{A}\boldsymbol{e}(\boldsymbol{u})) &= \lambda\rho\boldsymbol{u} \quad \text{in} \quad \Omega \\ \boldsymbol{u} &= 0 \quad\quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.1}$$

where $\boldsymbol{e}(\boldsymbol{u})$ denotes the linearized strain tensor associated with the displacement field $\boldsymbol{u}$, $\mathcal{A}$ the tensor of elastic moduli, ρ the mass density, and $\partial\Omega$ the boundary of Ω. The structure is assumed to be clamped for the sake of simplicity, but any set of boundary conditions yielding a symmetric variational formulation like traction free, mixed, or third kind boundary conditions may lead to the same conclusions. Other models like membrane or plate models can also be considered.

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The domain Ω is partitioned into p nonoverlapping subdomains Ω_1,..., Ω_p, which are separated by an interface Γ. For modal synthesis with overlap see [CDVM96, CDVM95]. The fixed interface eigenpairs $\left\{\lambda_j^i, \boldsymbol{u}_j^i\right\}_{j=1}^{+\infty}$ of subdomain Ω_i solve the problem

$$\begin{aligned} -div(\mathcal{A}\mathbf{e}(\mathbf{u})) &= \lambda\rho\mathbf{u} \quad \text{in} \quad \Omega_i \\ \mathbf{u} &= 0 \qquad \text{on} \quad \partial\Omega_i, \end{aligned} \tag{1.2}$$

The eigenfunctions $\mathbf{u}_j^i$ are extended by zero outside Ω_i and are thus defined on the whole domain Ω. Therefore, these so-called *fixed interface modes* can be used as trial functions in a Rayleigh-Ritz approximation of problem (1.1.1). Of course, they must be supplemented by a family of trial functions $\mathbf{w}_{\Gamma\ell}$ that do not identically vanish on the interface Γ between the subdomains. We call them *coupling modes*. The basic modal synthesis method thus amounts to a Galerkin approximation of problem (1.1.1) over the space

$$V_N = Span\left\{\bigcup_{i=1}^{p}\left(\mathbf{u}_j^i\right)_{j=1}^{N_i}\bigcup\left(\mathbf{w}_{\Gamma\ell}\right)_{\ell=1}^{N_\Gamma}\right\} \tag{1.3}$$

for some numbers N_i, $1 \leq i \leq p$, and N_Γ. Therefore, in its original version, modal synthesis is not an iterative algorithm, in contrast to those proposed in [Mal92, Mal96, SC96, CL96, Lui96, dV96], but rather a method of approximation. However, the approximate eigenpairs may be enhanced by postprocessing as in [Cha83] or more specifically in [Bal96], and generally speaking they may serve as good starting points for iterative domain decomposition correction algorithms.

Modal synthesis methods have been introduced in aerospace engineering in the sixties in order to save memory storage when analyzing the dynamics of large structures. These methods are now used in particular by nuclear, off-shore, automobile, and aerospace industries. The advantages of such methods are potentially numerous. Of course, they are amenable to parallel implementation. This point will be made clearer in the third and fourth sections. Furthermore, they can include experimental measurements on the substructures. Moreover, parametric studies involving local perturbations in view of sensitivity analysis or reanalysis can be performed cheaply [Tra96]. Finally, the substructuring concept extends to fluid-structure interaction [MO79], soil-structure interaction [Clo93], and buckling [Val82]. There exists a wide variety of methods, depending on the boundary conditions imposed to each substructure, and on the coupling strategy. In particular, many hybrid methods have been proposed and discussed; see, e.g., [MN71, Des89, DO96, J́85, Tra92b, Tra92a]. They correspond to other kinds of boundary conditions for the definition of the local modes. To a large extent, their numerical analysis is open. They fall in the general class of non-conforming methods since, at the continuous level at least, the continuity across the interface cannot be imposed if for example the local modes are associated with boundary conditions of Neumann type along the interface. General expositions on modal synthesis can be found in [Imb79, Mei80, Cra85, J́85, Mas88, Gib88, Tra92b].

This paper aims first at reviewing *a priori* error estimates for such methods and second at exploring new coupling strategies. In the next section, we revisit the pioneering work of [Hur65]. The third section is devoted to more recent methods involving *coupling modes* defined as the eigenfunctions of the Poincaré-Steklov operator [Bou92, Bd92b, Bd92a]. Since the computation of these *coupling modes* may be expensive when the interface is large and complex, cheaper *coupling modes* are introduced in section four. To this end, new operators of Neumann-Neumann type are defined and put in vibration. They are based on special extension operators from the boundary of each subdomain to the whole interface. Numerical tests confirm the accuracy of the resulting modal synthesis method. We close with a few comments.

2 Hurty's method

Let V denote the space of global test functions and V_Γ the space of their restrictions to the interface Γ, namely $V = H_0^1(\Omega)$ and $V_\Gamma = H_{00}^{1/2}(\Gamma)$ if Dirichlet boundary conditions are imposed on $\partial\Omega$ as in (1.1.1). In a continuous setting, Hurty's method (see [Hur65] and [CB68]) would amount to choose the *coupling modes* as the "harmonic" extensions to each subdomain of all elements of a given basis of V_Γ.

Now, if N_i *fixed interface modes* are retained to describe the dynamics of subdomain Ω_i, and if $\lambda_k^{HCB,N}$ denotes the k^{th} eigenvalue resulting from Hurty's procedure, the following error bound is derived in [Bou92]:

$$0 \leq \lambda_k^{HCB,N} - \lambda_k \leq \sum_{i=1}^{p} C_i(1+N_i)^{-1}. \tag{2.1}$$

The constants C_i depend on k but not on N_i. Similar error bounds hold for the eigenfunctions in L^2-norm as well as in energy norm. For two-dimensional elasticity, and for bars under traction [Bou90], the estimate would behave like $N_i^{-3/2}$, and N_i^{-3} respectively. Note that the rate of convergence deteriorates when the dimension of the problem increases.

The constants C_i in (1.2.1) behave as follows: $C_i \sim \left(\frac{\rho_i}{E_i}\right)^{3/2} Vol(\Omega_i)$, if ρ_i and E_i stand for the mass density and Young's modulus of substructure Ω_i, when these quantities are constant on Ω_i. For two-dimensional elasticity, we obtain $C_i \sim \left(\frac{\rho_i}{E_i}\right)^{3/2} (Vol\,(\Omega_i))^{3/2}$. Combining above estimates yields a rational way to choose the number of *fixed interface modes* of each substructure relative to the others.

The optimality of the error bounds is highlighted by the numerical experiments presented in [Bd92b]: the true error behaves like $N^{-3/2}$ for the two-dimensional membrane problem which has the same properties as plane elasticity from the viewpoint of modal synthesis. This optimality is also suggested by the proof of the error bound which is obtained as the rest of a series expansion that converges no faster than indicated. The key ingredient of the error analysis is real interpolation theory in Sobolev spaces [LM68]. In order to explain in a simple way where the exponents come from, let us consider the Fourier series expansion on the basis $(\sin j\pi x)_{j=1}^{+\infty}$ of the function 1 over [0,1]: $1 = \sum_{j=1}^{+\infty} \alpha_j \sin j\pi x$. Of course $\sum_{j=1}^{+\infty} (\alpha_j)^2 < \infty$. However, if $\nu_j = j^2\pi^2$, then $\sum_{j=1}^{+\infty} \nu_j\,(\alpha_j)^2 = +\infty$ otherwise the function 1 would vanish at both ends of the interval, but $\sum_{j=1}^{+\infty} (\nu_j)^{1/2-\varepsilon} (\alpha_j)^2 < \infty \ \forall \ \varepsilon > 0$. This property extends *mutatis mutandis* to arbitrary n-dimensional domains as a consequence of [LM68] interpolation theory. Weyl's formula proves also useful to derive the error bound. Working directly at the level of the finite element discretization is possible, but does not yield optimal error bounds because the underlying PDE is hidden. Notice that different error bounds can be derived for plates [Bd92a].

In order to compute the final generalized mass and stiffness matrices, all the modes are usually discretized with finite elements. If h denotes the discretization parameter, and $\lambda_k^{HCB,N,h}$ the k^{th} eigenvalue computed with Hurty's method, then the error $\lambda_k^{HCB,N,h} - \lambda_k$ can be estimated by the sum of the right-hand side of (1.2.1), and a discretization error Ch^β, for some constant $C(N)$, and some $\beta > 0$ which depends on the finite element method and on the smoothness of the *fixed interface modes* [Bou92]. A variant of this method enables one to use incompatible meshes on the different subdomains [FG94]. See [RTG96] for a comparison in the frequency domain of the continuous version of the modal synthesis method with the associated discrete version.

In general Hurty's method combines F.E. discretization and local mode truncation in a way that prevents one from using it when the interface contains many degrees of freedom, because the resulting mass and stiffness matrices are still quite large and dense, therefore difficult to handle. Even forming these matrices proves time-consuming since the Schur complement matrix has to be computed. A possible solution is to avoid computing these matrices and to use an iterative solver within the eigenvalue solver for the final generalized eigenvalue problem. An other strategy consists of further reducing the size of this final eigenvalue problem. In this direction, *coupling modes* can be defined at both the continuous level and the discrete level such that a prescribed accuracy of modal synthesis is achieved by means of a given number of them that does not depend on the mesh size [Bou89]. We define them in the next section.

3 A Poincaré-Steklov Operator-based Method

Let $\mathbf{u}$ and $\mathbf{v}$ denote any displacement fields in V_Γ, resulting in "harmonic" (in the sense of elasticity) extensions $\tilde{\mathbf{u}}^i$ and $\tilde{\mathbf{v}}^i$ on every subdomain Ω_i. The bilinear form $b(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^p \int_{\Omega_i} \sigma(\tilde{\mathbf{u}}^i) : e(\tilde{\mathbf{v}}^i)$ defines a scalar product over V_Γ and the associated isomorphism T satisfies $T\mathbf{u} = \sum_{i=1}^p \left(\left(\mathcal{A}_i e\left(\tilde{\mathbf{u}}^i\right)\right) .\boldsymbol{n}^i \right)_{|\Gamma\cap\partial\Omega_i}$, if $\boldsymbol{n}^i$ denotes the unit outer normal vector to Ω_i along $\partial\Omega_i$, and $\mathcal{A}_i = \mathcal{A}_{|\Omega_i}$. This well-known Poincaré-Steklov operator is of course the continuous counterpart of the Schur complement matrix.

From standard spectral theory, the problem *find* $(\lambda, \mathbf{u}) \in \mathbb{R} \times V_\Gamma$ *such that*

$$b(\mathbf{u}, \mathbf{v}) = \lambda \int_\Gamma \mathbf{u}\mathbf{v} \quad \forall \mathbf{v} \in V_\Gamma \tag{3.1}$$

is well-posed and admits a family of solutions $\{\lambda_{\Gamma\ell}, \mathbf{u}_{\Gamma\ell}\}_{\ell=1}^{+\infty} \in \mathbb{R} \times V_\Gamma$ such that $(\mathbf{u}_{\Gamma\ell})_{\ell=1}^{+\infty}$ forms a basis of V_Γ. The "harmonic" extension of $\mathbf{u}_{\Gamma\ell}$ to each component Ω_i is continuous across the interface Γ and is thus defined on the whole domain

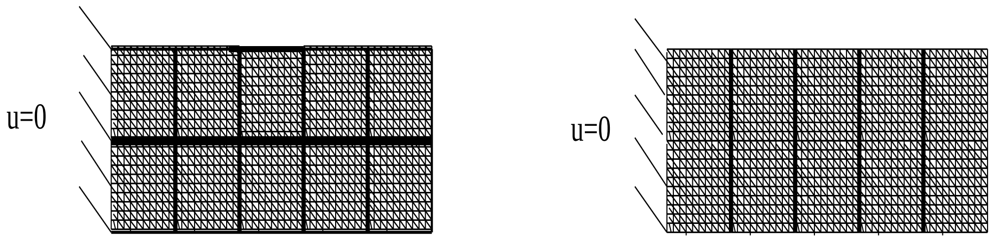

Figure 1 A two-dimensional elastic beam, clamped on the left side and free everywhere else, decomposed in two ways.

as a trial function $\tilde{\mathbf{u}}_{\Gamma\ell} \in V$ which we take as *coupling mode.* The Rayleigh-Ritz approximation of the global eigenvalue problem based on N_i *fixed interface modes* from each substructure as in Hurty's method and N_Γ such *coupling modes* yield approximate eigenvalues $\lambda_k^{PS1,N}$. For three-dimensional elasticity, the error bound

$$0 \leq \lambda_k^{PS1,N} - \lambda_k \leq \sum_{i=1}^{p} C_i {N_i}^{-1} + C_\Gamma {N_\Gamma}^{-\alpha} \tag{3.2}$$

has been derived in [Bou89, Bou91]. In (1.3.2), α denotes the exponent of the most severe vertex or edge singularity of the local source problems with homogeneous Dirichlet conditions on the interface. For two-dimensional elasticity, we get the exponent 2α instead of α. See also [Bd92a] for plates. That the rate of convergence does not depend directly on the smoothness of the eigenmode is an interesting feature. For the approximation of *coupling modes*, we refer to [Bou92], and also to [BVPA94]. The discrete *coupling modes* can be computed by a Lanczos method combined with the Neumann-Neumann algorithm of [BGLT88], as in [Bd92b] without forming the Schur complement matrix, or as in [CL96] where the descent directions of the inner loop are stored in view of solving more efficiently the source problem $T\mathbf{u} = \mathbf{g}$ with successive right-hand sides.

This basic Poincaré-Steklov operator based modal synthesis method proves very accurate because the first few global eigenpairs are correctly described by means of a small number of *fixed interface* and *coupling modes* [Bd92b]. Moreover, it can be parallelized in the same way as the Neumann-Neumann algorithm applied to source problems.

Although taking advantage of the identity matrix along Γ instead of the consistent mass matrix associated with the scalar product $\int_\Gamma \mathbf{uv}$ apparently does not deteriorate too much the accuracy of the whole procedure, one may get rid of the possibly non standard implementation of this mass matrix by using a non-local inertia along Γ [BN97a].

4 Extended Neumann-Neumann Preconditioners

Now, can one speed up the method by defining new *coupling modes* that would be much cheaper to compute, and that would yield comparable accuracy? A natural idea is to replace the Poincaré-Steklov operator T by a preconditioner, like the Neumann-Neumann preconditioner introduced in [BGLT88]. This operator S is defined as

$$S : V'_\Gamma \longrightarrow V_\Gamma, \quad \mathbf{v} \longrightarrow S\mathbf{v} = \sum_{i=1}^{p} P_i S_i R_i \mathbf{v}, \tag{4.1}$$

where $P_i : W_i = tr_{|\Gamma\cap\partial\Omega_i}(V) \longrightarrow V_\Gamma$ is a continuous extension operator, $R_i : V'_\Gamma \longrightarrow W'_i$ is a continuous restriction operator and $S_i : W'_i \longrightarrow W_i$ is the Neumann-to-Dirichlet operator associated with the subdomain Ω_i and its boundary, that is to say $S_i\mathbf{u} = tr_{|\Gamma\cap\partial\Omega_i}(\tilde{\mathbf{u}}^i)$, where

$$\begin{aligned} -div\mathcal{A}_i e(\tilde{\mathbf{u}}^i) + d\tilde{\mathbf{u}}^i &= 0 \text{ in } \Omega_i, \\ \mathcal{A}_i e(\tilde{\mathbf{u}}^i).\boldsymbol{n}_i &= \mathbf{u} \text{ on } \Gamma\cap\partial\Omega_i, \\ \tilde{\mathbf{u}}_i &= 0 \text{ on } \partial\Omega_i\cap\partial\Omega, \end{aligned} \tag{4.2}$$

for some $d > 0$. In order to ensure the symmetry of S, we shall always choose $R_i = P_i^t$. We have the following

Proposition: *let us set* $E_i = P_i P_i^t$, *and* $E = \sum_{i=1}^p E_i$. *Then*

i) $\exists C>0$, *such that* ${}_{V_\Gamma}\langle Sv, v\rangle_{V'_\Gamma} \le C\|v\|^2_{V'_\Gamma}$, $\forall v \in V'_\Gamma$.

ii) $Ker(S) = Ker(E)$ *and* $Ker(E) = \bigcap_{i=1}^p Ker(E_i)$.

iii) The direct sum of the eigenspaces of S associated with strictly positive eigenvalues and of $Ker(E)$ *is dense in* V'_Γ *and in* $L^2(\Gamma)$.

Proof: i) is a direct consequence of the continuity of all operators involved in definition (1.4.1). On the other hand, the coerciveness of each operator S_i which is due to the positivity of d, leads to the inequality ${}_{V_\Gamma}\langle Sv, v\rangle_{V'_\Gamma} \ge \alpha\sum_{i=1}^p \|P_i^t v\|^2_{W'_i}$, for some positive α. Since ${}_{V_\Gamma}\langle Ev, v\rangle_{V'_\Gamma} = \sum_{i=1}^p \|P_i^t v\|^2_{W'_i}$, we also have the opposite inequality. Moreover, each operator E_i is non-negative, hence ii). Finally iii) follows from the symmetry and compactness of S on $L^2(\Gamma)$ and from ii).

Interface without Cross-points

Consider a domain decomposed in p slices separated by edges Γ_j that of course do not intersect. In this case, the space V_Γ coincides with $\prod_{j=1}^{p-1} tr_{|\Gamma_j}(V)$ and the extension operators $P_i : W_i \longrightarrow V_\Gamma$, defined by

$$P_i\mathbf{u} = \begin{cases} \mathbf{u} & \text{on} \quad \partial\Omega_i\cap\Gamma, \\ 0 & \text{on} \quad \Gamma - (\partial\Omega_i\cap\Gamma), \end{cases} \tag{4.3}$$

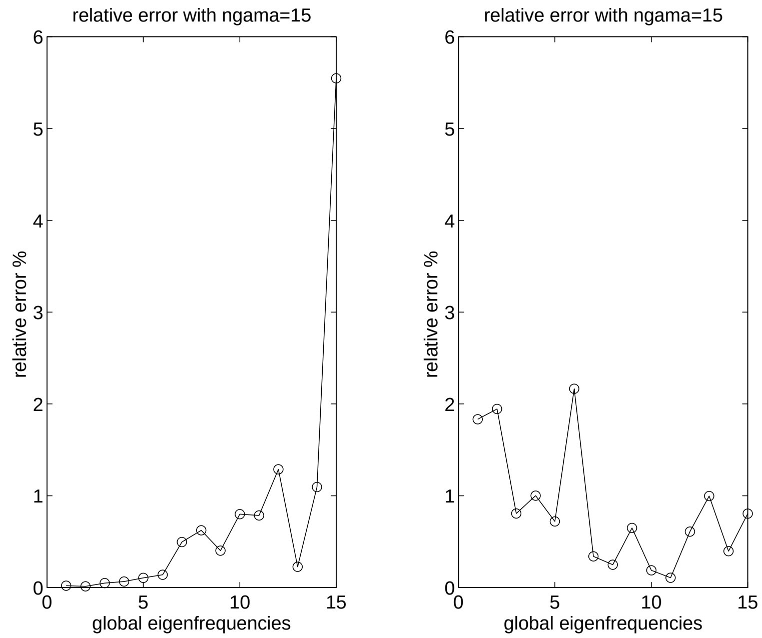

Figure 2 Interface *without* cross-points (5 slices): Relative accuracy on the 15 smallest global eigenfrequencies for the standard Poincaré-Steklov based method (left) with $N_\Gamma = 15$ and for the Neumann-Neumann based method (right) with $N_\Gamma = 15$

are continuous. The adjoint operators R_i coincide with the usual restrictions to $\partial\Omega_i \cap \Gamma$. The operator S is well defined at the continuous level. It has no kernel and possesses a countable family of eigenpairs $\left(\lambda^{\Gamma\ell}, \mathbf{u}^{\Gamma\ell}\right)_{\ell=1}^{\infty}$ arranged such that the eigenvalues decrease towards zero. The "harmonic" extensions to each component of the eigenfunctions $\mathbf{u}^{\Gamma\ell}$ can be chosen as *coupling modes*. This idea is close to the concept underlying [BKP95]. The resulting modal synthesis method proves less accurate than the original one, but the new *coupling modes* $\mathbf{u}^{\Gamma\ell}$ can be computed quite rapidly through a Lanczos method (see e.g. [CL96] and included references), because now each step mainly consists of computing $S\mathbf{g}$ for a given $\mathbf{g}$. There are no more internal loops for solving any source problem on Γ. The accuracy of the resulting modal synthesis method is very encouraging as shown on fig 2 since it yields a similar accuracy as the Poincaré-Steklov based method at a much smaller cost.

When the interface exhibits a more complex geometry than before, extension by zero is no more possible if $H^{1/2}(\Gamma)$ regularity is to be preserved for the *coupling modes*. However, an extension of the method can be designed.

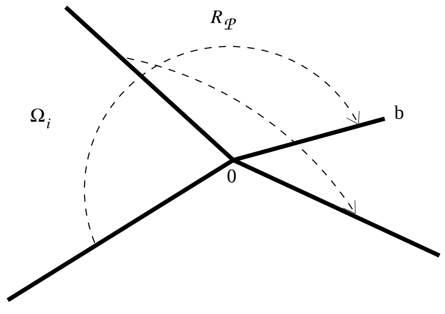

Figure 3 How the extension operator works

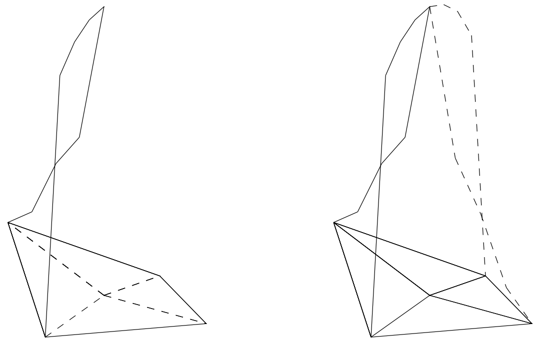

Figure 4 A function defined on the boundary of a subdomain and its extension to the whole interface

Interface with cross-points

Some preliminaries are needed: let f denote a continuous function on the interval $[a, 0]$, $a < 0$, to be extended to some interval $[0, b]$, $b > 0$, then define the function

$$({}^R\mathcal{P} f)(x) = \left\{ \begin{array}{ll} f(x) & \text{on } [a, 0], \\ f(\frac{a}{b}x)\text{'}(x) & \text{on } [0, b], \end{array} \right. \tag{4.4}$$

where ' denotes a smooth cut-off function vanishing in the vicinity of b. The operator ${}^R\mathcal{P}$ enjoys $H^{1/2}$ continuity from interpolation theory. This generic reflection operator can be put to work in order to define the extension operators ${}^RP_i$, as follows: if $\mathbf{u}$ stands for a function defined over $\Gamma \cap \partial\Omega_i$, we want to extend it in a continuous way to the adjacent edges. Define a curvilinear abscissa over the edges of Γ sharing a given vertex which is a cross-point. For such an edge not belonging to $\partial\Omega_i$, parametrized by $s \in [0, b]$, choose such an edge belonging to $\partial\Omega_i$, parametrized by $s \in [a, 0]$, and apply the operator ${}^R\mathcal{P}$ defined above. Repeat this for all edges adjacent to $\Gamma \cap \partial\Omega_i$ (see Figures 3 and 4). Of course, the choice of the edge parametrized by $s \in [a, 0]$

is somewhat arbitrary and a weighted average of operators ${}^R\mathcal{P}$ corresponding to different edges of $\Gamma \cap \partial\Omega_i$ can also be used. In any case, we end up with an operator ${}^RP_i$ for every subdomain, and operators ${}^RS$ and ${}^RE$ as in (1.4.1) and in the proposition. They are defined in a continuous setting as well as in a discrete setting. We omit the details of the discretization and of the implementation here. Although the perfect locality of the original Neumann-Neumann preconditioner is lost, some locality is preserved because of the cut-off function '. The kernel of such operators are not reduced to $\{0\}$ in general, but the proposition suggests to choose as *coupling modes* the first few eigenfunctions of ${}^RS$ associated with its largest eigenvalues in addition to $Ker({}^RE)$. At the discrete level, this kernel is easy to compute since the matrices ${}^RE_i ={}^R P_i {}^RP_i^t$ only depend on the mesh and on the geometry of the interface. No subdomain solve is required. From ii) of the proposition, the kernel can be computed as the intersection of the local kernels $Ker({}^RE_i)$, that will be computed in parallel with minimal data exchange between subdomains. Then the Rayleigh-Ritz approximation of the eigenvalue problem ${}^REx = \mu x$ over $\bigcup_{i=1}^p Ker({}^RE_i)$ is performed. The rank of the corresponding Rayleigh matrix may be maximal. In this case, $Ker({}^RE) = \{0\}$. However, the smallest eigenvalue of this matrix can be very small. Therefore, a variant of this modal synthesis method consists of computing several of the smallest eigenvalues of ${}^RE$ and corresponding eigenvectors that will supplement the set of eigenvectors of ${}^RS$. In particular, it may be interesting to keep as *coupling modes* an independent set of vectors in $\bigcup_{i=1}^p Ker({}^RE_i)$, in view of parallel implementation. On the other hand, $Ker({}^RE)$ may be very large, therefore a strategy to filter out unwanted, highly oscillating functions of this kernel should be put to work in this case.

This method, referred to as the R-method, has been tested on the reference structure and the relative error on the eigenfrequencies is reported on Figure 5. It compares favourably with the standard Poincaré-Steklov based method since a similar accuracy is obtained at a smaller cost.

A conceptually simpler variant of above extension operators is based on the linear extension operator defined as

$$({}^L\mathcal{P}f)(x) = \begin{cases} f(x) & \text{on } [a,0], \\ f(0)(1-\frac{x}{b}) & \text{on } [0,b], \end{cases} \tag{4.5}$$

that is well-defined at the discrete level but not at the continuous level because functions in $H^{1/2}$ do not have traces. In this case, we noticed that the associated operators ${}^LE_i$ have no kernel. Results of poorer quality than with the R-method are obtained for this variant, referred to as L-method, as shown on Figure 6.

5 Concluding Remarks

i) According to recent numerical experience of the authors, the Poincaré-Steklov based methods are the most accurate at least at very low frequency. The R-method proves slightly less accurate but a better cost effectiveness is expected. It turns out that the L-method which is not defined at the continuous level is less accurate than the other

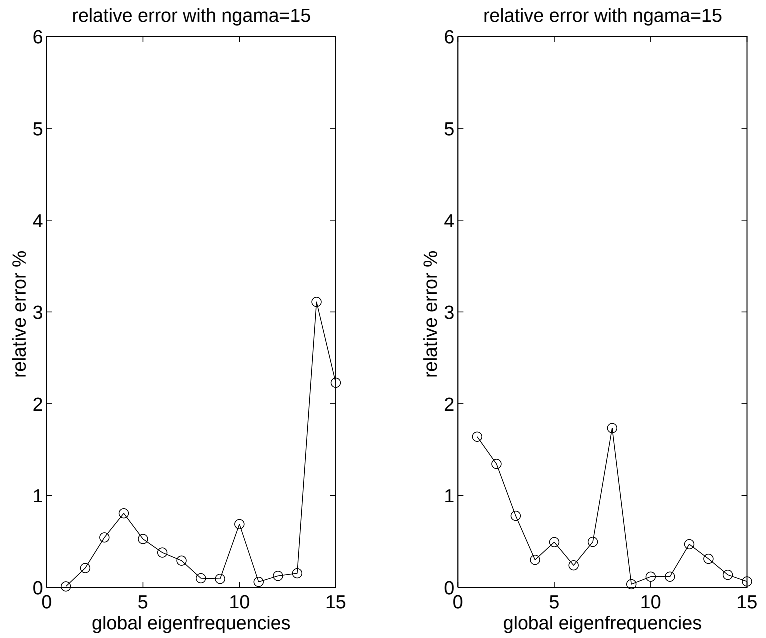

Figure 5 Interface with cross-points (10 subdomains): relative accuracy on the 15 smallest global eigenfrequencies for the standard Poincaré-Steklov based method (left) for $N_\Gamma = 15$ and for the R-method (right) with $N_\Gamma = 15$, and with $1/5$ of the eigenvectors of ${}^R E$ over $\bigcup_{i=1}^{p} Ker({}^R E_i)$

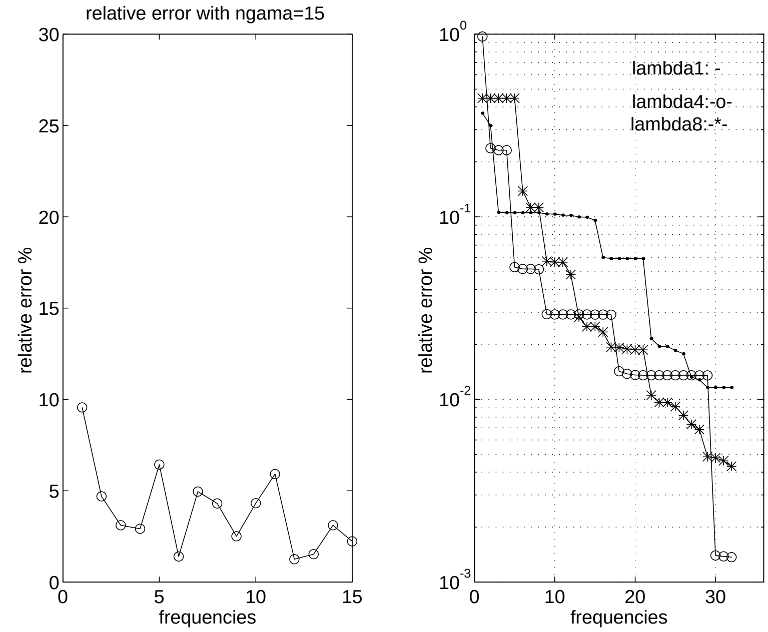

Figure 6 Interface with cross-points (10 subdomains), L-method. Relative accuracy on the 15 smallest global eigenfrequencies (left) for $N_\Gamma = 15$, and decay (right) w.r.t. N_Γ of the relative error on the global eigenfrequencies 1, 4 and 8.

ones. All of them are amenable to parallel computing. The R- and L-methods involve slightly more communication efforts between subdomains.

ii) Variants of the R- or L-method can be designed. In particular, parametrized families of extension operators can be introduced in view of optimizing the accuracy of the method. A detailed version of section 1.4 on the design and use of extended Neumann-Neumann preconditioners will be available in [BN97a].

iii) Extension to three-dimensional problems seems conceptually possible. However, from the practical point of view, defining the extension operator from the boundary of each subdomain to the whole set of interfacial faces may appeal to geometrical data that are not directly available from the numerical description of each subdomain.

iv) On the other hand, the case of plate-like problems can be treated by means of similar but more complicated techniques. Here, extension operators that preserve $H^{3/2} \times H^{1/2}$ continuity along the interface must be designed [BN97b].

v) It would be interesting to combine the Poincaré-Steklov based method with the R-method which starts proving very accurate at frequencies where the former starts deteriorating.

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