33

A Lagrangian Approach to a DDM for an Optimal Control Problem

Aïcha Bounaïm

1 Introduction

We give a Lagrangian interpretation to a domain decomposition method for an optimal control problem governed by an elliptic partial differential equation. We minimize the cost function and the links between subdomains, the constraints at the interfaces being treated by augmented Lagrangian techniques. An algorithm is proposed and illustrated with numerical computations.

We consider the following distributed optimal control problem governed by an elliptic partial differential equation:

$$J(u) = \inf_{v \in U_{ad}} J(v), \; u \in U_{ad}, \tag{1}$$

where

$$J(v) = \frac{1}{2}\left(\int_\Omega (y(v) - y_d)^2 dx + \nu \int_\Omega v^2 dx\right),$$

with Ω is a bounded smooth open subset of $\mathbb{R}^q$, y_d is the desired state given in $L^2(\Omega)$, U_{ad} is a closed convex subset of $L^2(\Omega)$ (space of admissible controls), ν is a strictly positive real number, $y(v)$ is the solution of the system:

$$\begin{cases} -\Delta y(v) = f + v & \text{in } \Omega, \\ y(v) = 0 & \text{on } \Gamma = \partial\Omega, \end{cases} \tag{2}$$

and $f \in L^2(\Omega)$.

Proposition 1 *The problem* (1) *has a unique solution* u *and the mapping* $u \mapsto y(u)$ *is affine continuous from* U_{ad} *into* $H^1(\Omega)$.

Proof. See [Lio68].

Remark 1 *In the unconstrained case, the control is given by* $u = -\frac{p}{\nu}$ *where* $p = p(u)$ *is the adjoint state, solution of:*

$$\begin{cases} -\Delta p(u) = y(u) - y_d & \text{in } \Omega, \\ p(u) = 0 & \text{on } \Gamma = \partial\Omega. \end{cases} \tag{3}$$

Ninth International Conference on Domain Decomposition Methods
Editor Petter E. Bjørstad, Magne S. Espedal and David E. Keyes

In the general case, the problem (1) *is characterized by the following system called "optimality system":*

- *Direct state* (1).
- *Adjoint state* (3).
- *Optimality condition:* $\int_\Omega (p(u)+\nu u)(v-u)dx \geq 0 \qquad \forall v \in U_{ad}$.

It is also proved in [Lio68] that there exists a unique solution (u, y, p) *of the above optimality system and* u *is the minimizer of* (1).

2 Domain Decomposition for the Optimal Control Problem

We will be interested in a non-overlapping domain decomposition method. Thus, Ω is decomposed into m subdomains and we introduce the following notations:

$$\begin{array}{ll} \Omega = \bigcup_{i=1}^m \Omega_i \cup \Sigma, & \Gamma_i = \partial\Omega \cap \partial\Omega_i, \\ \sigma_{ij} = \partial\Omega_i \cap \partial\Omega_j, & \Sigma = \bigcup_{1\leq i\neq j\leq m} \sigma_{ij}, \\ V_i = \{\ y_i \in H^1(\Omega_i);\ y_i = 0 \ in\ \Gamma_i\ \}, & \end{array}$$

where Ω_i are disjoint open sets in $\mathbb{R}^q$.

The idea of the domain decomposition method proposed is to define the augmented Lagrangian associated with the decomposed optimal control problem. Therefore, the continuity of function value at the interfaces is treated by Lagrangian techniques, whereas the flux continuity is formulated explicitly in the direct *pde* on each subdomain Ω_i:

$$\left\{ \begin{array}{rcll} -\Delta y_i &=& f_i + v_i & \text{in} \quad \Omega_i, \\ y_i &=& 0 & \text{on} \quad \Gamma_i, \\ \frac{\partial y_i}{\partial \eta} &=& \omega_{ij} & \text{on} \quad \sigma_{ij}, j \neq i. \end{array} \right. \tag{4}$$

where $\eta = \eta_{ij} = -\eta_{ji}$ is the unit outward normal to Ω_i on σ_{ij}, for $i < j$ and ω_{ij} is an extra variable introduced to ensure the continuity of the normal derivative and satisfies $\omega_{ij} = \omega_{ji} \in H^{-\frac{1}{2}}(\sigma_{ij})$ for $j \neq i$ and $\sigma_{ij} \neq \emptyset$.

Then, the cost functional has the new expression:

$$J(v_i, y_i) = \sum_{i=1}^m \frac{1}{2} \left[\int_{\Omega_i} (y_i - y_d^i)^2 dx + \nu \int_{\Omega_i} v_i^2 dx \right].$$

Finally, we obtain a new optimization problem [1]:

$$\left\{ \begin{array}{l} \text{Minimize} \quad J(v_i, y_i) \\ v_i \in U_{ad}^i \\ y_i \ \text{ and } \ v_i \ \text{ linked by (4)} \\ y_i = y_j \ \text{ on } \ \sigma_{ij},\ j \neq i. \end{array} \right. \tag{5}$$

1 We use the notation $(v_i, y_i) = ((v_i)_{1\leq i\leq m}, (y_i)_{1\leq i\leq m})$.

Lagrangian Formulation

Now, we introduce the augmented Lagrangian associated with the above minimization problem. In order to enforce and to decouple the constraint of the interface continuity, we add an extra-variable q_{ij} such that: $q_{ij} = q_{ji} \in H^{\frac{1}{2}}(\sigma_{ij})$, for $j \neq i$, $\sigma_{ij} \neq \emptyset$ and

$$y_i = y_j = q_{ij} \text{ on } \sigma_{ij} \tag{6}$$

(see [GL90] for such consideration).
Then, after making explicit the constraint between v_i and y_i, the augmented Lagrangian L_r is given by [2]:

$$\begin{aligned} L_r(v_i, y_i, p_i, \omega_{ij}, \lambda_{ij}, q_{ij}) &= J(v_i, y_i) - \sum_{i=1}^{m} \left(\int_{\Omega_i} \nabla y_i \nabla p_i dx - \int_{\Omega_i} (f_i + v_i) p_i dx \right) \\ &+ \sum_{1 \leq i < j \leq m} \int_{\sigma_{ij}} \omega_{ij}(p_i - p_j) d\sigma_{ij} + \sum_{i=1}^{m} \sum_{j,\ \sigma_{ij} \neq \emptyset} \int_{\sigma_{ij}} \lambda_{ij}(y_i - q_{ij}) d\sigma_{ij} \\ &+ \frac{r}{2} \sum_{i=1}^{m} \sum_{j,\ \sigma_{ij} \neq \emptyset} \int_{\sigma_{ij}} (y_i - q_{ij})^2 d\sigma_{ij}, \end{aligned}$$

where λ_{ij} are the Lagrange multipliers corresponding to the constraint (6) and r (augmented Lagrangian constant) is a positive real number.

Remark 2 *In the L_r expression, the ω_{ij} can be interpreted as Lagrange multipliers corresponding to the continuity of the adjoint state on the interface.*

Proposition 2 *If $(u_i, y_i, p_i, \omega_{ij}, \lambda_{ij}, q_{ij})$ is a saddle point of L_r, when it exists, then,*

$$u_i = u/\Omega_i, \ y_i = y/\Omega_i, \ p_i = p/\Omega_i. \tag{7}$$

where (u, y, p) is the saddle point of the Lagrangian associated with the problem (1) *and u is the minimizer of* (1).

Proof. In the proof, we explicit characterization of a saddle point of L_r; we denote: $(s.p) = (v_i, y_i, p_i, \omega_{ij}, \lambda_{ij}, q_{ij})$. As L_r is differentiable in each variable, for $i : 1 \leq i \leq m$, j such that $j \neq i$ and $\sigma_{ij} \neq \emptyset$, we have:

$$\left(\frac{\partial L_r}{\partial p_i}(s.p), \phi_i\right) = 0 \quad \forall \phi_i \in V_i \tag{8}$$

$$\left(\frac{\partial L_r}{\partial y_i}(s.p), \phi_i\right) = 0 \quad \forall \phi_i \in V_i \tag{9}$$

$$\left(\frac{\partial L_r}{\partial v_i}(s.p), v_i - u_i\right) \geq 0 \quad \forall v_i \in U_{ad}^i \tag{10}$$

$$\left(\frac{\partial L_r}{\partial \lambda_{ij}}(s.p), d\lambda\right) = 0 \quad \forall\, d\lambda \in H^{-\frac{1}{2}}(\sigma_{ij}),\ i \neq j \tag{11}$$

$$\left(\frac{\partial L_r}{\partial \omega_{ij}}(s.p), d\omega\right) = 0 \quad \forall\, d\omega \in H^{-\frac{1}{2}}(\sigma_{ij}),\ i < j \tag{12}$$

$$\left(\frac{\partial L_r}{\partial q_{ij}}(s.p), dq\right) = 0 \quad \forall\, dq \in H^{\frac{1}{2}}(\sigma_{ij}),\ i < j \tag{13}$$

---

2 We use short expression of the L_r variables: $(v_i, y_i, p_i, \omega_{ij}, \lambda_{ij}, q_{ij})$ instead of $((v_i)_{1 \leq i \leq m}, (y_i)_{1 \leq i \leq m}, (p_i)_{1 \leq i \leq m}, (\omega_{ij})_{1 \leq i < j \leq m}, (\lambda_{ij})_{1 \leq i \neq j \leq m}, (q_{ij})_{1 \leq i \neq j \leq m})$.

Let (u, y, p) be the saddle point of the Lagrangian associated with the problem (1). From (8) and (11), it follows that: $y_i = y_j$ and $\frac{\partial y_i}{\partial \eta_{ij}} = \frac{\partial y_i}{\partial \eta_{ij}}$ on σ_{ij}. So, y_i is exactly the restriction of y on Ω_i.

As for p_i, from (9), (12) and (13), we obtain both the continuity and the flux continuity of the adjoint state. Thus, $p_i = p/\Omega_i$.

Finally, using the optimality of u_i (10) and the equation for p_i, we deduce $(p_i + \nu u_i, v_i - u_i)_{L^2(\Omega_i)} \geq 0, \ \forall v_i \in U^i_{ad}$ and get the optimality condition in the global domain.

Solution Algorithm

Due to the above proposition, we have just to search a saddle point of L_r. So, we propose a modification of the algorithm ALG3 in [GL89b]:

Algorithm

Step 1. Initialization: u_i^1 given in $L^2(\Omega_i)$, $(\omega_{ij}^1)_{i<j}$ and $(\lambda_{ij}^1)_{i\neq j}$ given in $H^{-\frac{1}{2}}(\sigma_{ij})$ and $(q_{ij}^0)_{i<j}$ given in $H^{\frac{1}{2}}(\sigma_{ij})$.

Step 2. Iteration: For $n = 1, 2, \ldots$, compute y_i^n and p_i^n such that $y_i^n \in V_i$, $p_i^n \in V_i$ and

$$\left\{ \begin{array}{rcll} -\Delta y_i^n & = & f_i + u_i^n & in \quad \Omega_i, \\ \frac{\partial y_i^n}{\partial \eta} & = & \omega_{ij}^n & on \quad \sigma_{ij}, \ j \neq i \end{array} \right.$$

$$\left\{ \begin{array}{rcll} -\Delta p_i^n & = & y_i^n - y_d^i & in \quad \Omega_i \\ \frac{\partial p_i^n}{\eta_{ij}} & = & \lambda_{ij}^n + r(y_i^n - q_{ij}^{n-1}) & on \quad \sigma_{ij}, \ j \neq i \end{array} \right.$$

- Compute the gradient g_i^n of $L_r(v_i, \ldots)$, $g_i^n = p_i^n + \nu u_i^n$
- $u_i^{n+1} = u_i^n + t^n d_i^n$, $t^n > 0$ minimization along the descent direction d_i^n computed from g_i^n by BFGS formula.
- Update the Lagrange multipliers and q_{ij}^n: $\rho^n > 0$, $\rho_\omega^n > 0$

$$\begin{array}{rcl} \omega_{ij}^{n+1} & = & \omega_{ij}^n + \rho_\omega^n (p_i^n - p_j^n) \\ \lambda_{ij}^{n+\frac{1}{2}} & = & \lambda_{ij}^n + \rho^n (y_i^n - q_{ij}^{n-1}) \\ 2rq_{ij}^n & = & (\lambda_{ij}^{n+\frac{1}{2}} + \lambda_{ji}^{n+\frac{1}{2}}) + r(y_i^n + y_j^n) \\ \lambda_{ij}^{n+1} & = & \lambda_{ij}^{n+\frac{1}{2}} + \rho^n (y_i^n - q_{ij}^n) \end{array}$$

Remark 3 *In the above algorithm, eliminating λ_{ij}^n and q_{ij}^n for the case where $\rho^n = \rho_\omega^n = r$, the p_i-boundary conditions on σ_{ij} are transformed into*

$$-\frac{\partial p_i^{n+1}}{\partial \eta_{ij}} + r y_i^{n+1} = \frac{\partial p_j^n}{\partial \eta_{ji}} + r y_j^n \ on \ \sigma_{ij}.$$

These are precisely the conditions of algorithm Alg2 in [Ben94], where transmission conditions of [Lio90] are applied to the optimal control problem. The y_i-boundary conditions on σ_{ij} are written as

$$-\frac{\partial y_i^{n+1}}{\partial \eta_{ij}} + r p_i^n = \frac{\partial y_j^n}{\partial \eta_{ji}} + r p_j^n \ on \ \sigma_{ij}$$

which are close (but not identical) to those given in Alg2.

3 Numerical Results

As a test example, we consider a boundary optimal control problem given in [BGL73] defined by its direct state equation:

$$\left\{ \begin{array}{rcll} -\Delta y(v) & = & f & in \quad \Omega, \\ y(v) & = & 0 & on \quad \Gamma_3 \cup \Gamma_4, \\ \frac{\partial y(v)}{\partial \eta} & = & v & on \quad \Gamma_1 \cup \Gamma_2. \end{array} \right. \tag{14}$$

and we want to minimize over $U_{ad} = L^2(\Gamma_1 \cup \Gamma_2)$, the function

$$J(v) = \frac{1}{2}\left(\int_{\Omega} (y(v) - y_d)^2 dx + \nu \int_{\Gamma_1 \cup \Gamma_2} v^2 d\sigma\right).$$

In the computations, we take $\Omega =]0,4[\times]0,1[$, $f(x,y) = 2(-x^2 - y^2 + 4x + y)$ and $y_d(x,y) = (y - y^2)(x^2 - 4x) - 8\nu$, for $(x,y) \in \Omega$.

We split the domain Ω into m subdomains (see Fig. 1). A finite difference scheme is used to discretize both the direct and the adjoint state systems. We run the proposed algorithm with $\rho^n = \rho^n_\omega = r$. The descent direction is computed from the gradient vector of the Lagrangian L_r with respect to the control, the line search process and BFGS formula are carried out similarly to $M1QN3$ algorithm [GL89a]. The iterations of the algorithm are stopped when the norm of the gradient is small enough. For $\nu = 1.25, r = 0.85$ and for different values of h (discretization step) and m (subdomain number), the relative L^2-error on the direct state and the interface error average are worked out. In Table 1, we define

$$err_y = \left(\frac{\sum_{i=1}^{m} \|y_i^n - ye_i\|^2_{L^2(\Omega_i)}}{\sum_{i=1}^{m} \|ye_i\|^2_{L^2(\Omega_i)}} \right)^{\frac{1}{2}}, \quad err_{yij} = \frac{1}{m} \sum_{i=1}^{m} \sum_{j>i,\ \sigma_{ij} \neq \emptyset} \|y_i^n - y_j^n\|_{L^2(\sigma_{ij})}$$

where y_i^n is the computed solution on Ω_i at the nth iteration when the stopping criterion is attained and $ye_i = ye/\Omega_i$ with $ye(x,y) = (y^2 - y)(x^2 - 4x),\ for (x,y) \in \Omega$, is the analytic solution to the global problem. For $m = 1$, the minimization of J is done by the $M1QN3$ algorithm. Table 1 shows that the domain splitting into two subdomains does not affect much the convergence results of our algorithm and the y_i connect very well on the interface because of the symmetry of this problem. The convergence is attained in few iterations and depends on the '*quasi-optimal*' choice of r.

4 Conclusion

Using domain decomposition, the original optimal control problem is transformed into a saddle point problem. We have used augmented Lagrangian techniques studied by Fortin and Glowinski[FG82], Glowinski and Le Tallec[GL89b]. We have combined a descent method with a multipliers one. The proposed algorithm gives different ways of dealing with constraints on interfaces. Moreover, it is well suited to parallel processors.

Figure 1 The decomposed domain of the test example ($m = 8$).

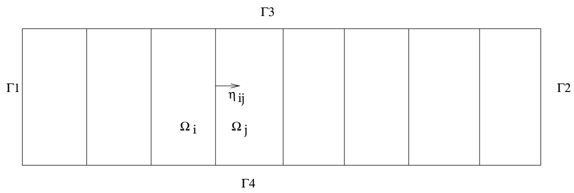

Table 1 Direct State Results. h: step discretization, m: subdomain number.

h^{-1}	m	err_y	err_{yij}
16	1	1.2226217904E-06	- - -
	2	3.5817536178E-05	1.5060315952E-06
	4	1.4511406491E-02	1.5495620667E-02
	8	8.7904304272E-02	8.6599163711E-02
64	1	1.0268570911E-06	- - -
	2	1.3455085083E-05	1.3616281608E-06
	4	1.1562970614E-02	1.5400098264E-02
	8	7.9211773115E-02	8.0130822639E-02

Acknowledgement

This work was carried out within the IDOPT project joint to CNRS, INRIA, Joseph Fourier University (Grenoble I) and Polytechnic National Institute of Grenoble and the computations were done on the parallel machine SP1 of IMAG. The author is grateful to Prof. J. Blum and B. Luong for many valuable discussions and suggestions related to this work and wishes to thank J. C. Gilbert and C. Lemaréchal for the use of their efficient minimization algorithm ($M1QN3$ routine, INRIA MODULOPT library).

REFERENCES

[Ben94] Benamou J.-D. (April 1994) Domain decomposition methods with coupled transmission conditions for the control of systems governed by elliptic partial differential equations. Technical Report 2246, INRIA.

[BGL73] Bensoussan A., Glowinski R., and Lions J.-L. (May 1973) Méthode de décomposition appliquée au contrôle optimal de systèmes distribués. *Lecture Notes in Computer Science* 5. (In the fifth IFIP Conference on optimization techniques).

[FG82] Fortin M. and Glowinski R. (1982) *Méthodes de Lagrangien augmenté, Application aux problèmes aux limites.* Bordas.

[GL89a] Gilbert J.-C. and Lemaréchal C. (1989) Some numerical experiments with variable storage quasi-newton algorithms. *Mathematical Programming* 45: 407–435.

[GL89b] Glowinski R. and Le Tallec P. (1989) *Augmented Lagrangian and Operator Splitting in Nonlinear Mechanics.* SIAM.

[GL90] Glowinski R. and Le Tallec P. (1990) Augmented Lagrangian interpretation of the nonoverlapping Schwarz alternating method. In Chan T. F., Glowinski R., Périaux J., and Widlund O. B. (eds) *Proc. 3rd Conference on Domain Decomposition Methods.*, pages 224–231. SIAM, Philadelphia.

[Lio68] Lions J.-L. (1968) *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles.* Dunod, Paris.

[Lio90] Lions P.-L. (1990) On the Schwarz alternating method III: A variant for nonoverlapping subdomains. In Chan T. F., Glowinski R., Périaux J., and Widlund O. B. (eds) *Proc. 3rd Conference on Domain Decomposition Methods.*, pages 202–223. SIAM, Philadelphia.