32

Spectral Elements on Infinite Domains

Kelly Black

1 Introduction

The use of infinite computational domains for the approximation of partial differential equations (PDEs) can arise in many applications. For example, in the approximation of external flows, or the approximation of some geophysical flows, the flow field may be extended to an infinite domain to simplify the treatment of boundaries. Also, electromagnetic fields may have an influence over an extremely large domain, or because of the relatively strong forces on small intervals they may be best approximated by extending the domain to an infinite interval [Bel94, Lan60, Sec95, Wil93, Wil94]. This may be particularly true when the treatment of boundary conditions can lead to artificial reflections that may pollute the approximation near the origin.

Two methods to approximate PDEs on infinite domains are examined here. Both methods are spectral element techniques and rely on high-order polynomials to construct the local approximation within the individual subdomains. To compare the two methods, a simple one-dimensional Helmholtz equation is examined:

$$\begin{aligned} u_{xx} + \lambda u &= f(x), \qquad -\infty < x < \infty \qquad (1) \\ \lim_{x\to\infty} u(\pm x) &= 0. \end{aligned}$$

For both methods, the computational domain is divided into non-overlapping subdomains. Near the origin, finite subdomains are employed, while far from the origin, semi-infinite subdomains are employed. Both methods rely on local spectral approximations and are not collocation methods.

The first method examined is based on a Laguerre polynomial expansion while the second method is based on a mapping of the semi-infinite interval to a finite interval proposed by Boyd [Boy87] for a single domain approximation. The local approximation is found as a linear combination of basis elements that are constructed from the Legendre polynomials. These basis functions are based on those proposed by Shen [She94]. Unlike the method proposed by Karageorghis and Phillips [Kar87],

Ninth International Conference on Domain Decomposition Methods
Editor Petter E. Bjørstad, Magne S. Espedal and David E. Keyes

a variational method is constructed in which the basis functions are divided between polynomials which are zero on the boundaries and polynomials which are not zero on the boundary.

In this presentation, the basis functions are defined first. Once done, two methods for extending the method to accommodate semi-infinite domains are given. The first employs Laguerre polynomials while the second employs the use of the mapping method proposed by Boyd [Boy87]. A direct comparison of the two methods is given. In the comparisons, equations are examined in which the true solution decays to zero exponentially and a true solution that decays only algebraicly.

2 Introduction to Spectral Elements with Local Spectral Basis

To introduce the method, a simple example is given. A one dimensional Helmholtz equation is examined on a finite domain:

$$\begin{aligned} u_{xx} + \lambda u &= f(x), \qquad -1 < x < 1 \\ u(\pm 1) &= 0. \end{aligned} \tag{2}$$

To construct an approximation for this example, the domain, $-1 \leq x \leq 1$, is divided into two subdomains, [-1,0] and [0,1].

The global approximation is found from the space of piecewise polynomials. For example, in a given subdomain the local approximation is a polynomial up to a given degree, N. This approximation is found as a linear combination of polynomials which are divided into those that are zero on the boundary and those that are not. The choice for these test functions is motivated by the results of Shen [She94]:

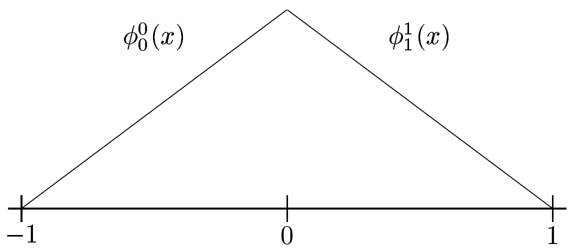
$$\phi_i(x) = \begin{cases} \frac{1+x}{2} & i = 0, \\ \frac{1-x}{2} & i = 1, \\ L_i(x) - L_{i-2}(x) & i > 1, \end{cases} \tag{3}$$

for $i = 0 \dots N$, and $L_i(x)$ is the Legendre polynomial of degree i. The span of the test functions is the span of polynomials up to degree N, and each of the ϕ_i's are linearly independent. The primary difference from the work of Shen [She94] is the addition of ϕ_0 and ϕ_1 and implementing it as a spectral element method. The two new linear basis functions are used to form a hat function whose support includes adjacent subdomains (see Figure 1).

The linear basis functions, $\phi_0(x)$ and $\phi_1(x)$, allow for a straightforward method to handle boundary conditions. Unlike collocation methods such as the one proposed by Karageorghis and Phillips [Kar87], the method is closer in spirit to the methods originally proposed by Patera [Pat84], and the discretization requires that only C^0 continuity at the subdomain interfaces be enforced.

The choice of basis functions also yield a symmetric tridiagonal mass matrix and a diagonal stiffness matrix. As pointed out by Shen [She94], a linear combination of the

Figure 1 Example of two subdomains, [-1,0] and [0,1], on the interval [-1,1]. Trial functions $\phi_0^0(x)$ and $\phi_1^1(x)$ combine on adjacent subdomains to assemble a "hat" function on adjacent subdomains.



Legendre polynomials satisfies the following identity:

$$
\begin{aligned}
-\int_{-1}^{1} & \left(L_i(x) - L_{i-2}(x)\right)' \left(L_i(x) - L_{i-2}(x)\right)' dx \qquad (4) \\
&= -\int_{-1}^{1} (2i-1)\, L_{i-1}(x)\, (2j-1)\, L_{j-1}(x) dx, \\
&= -\,(2i-1)\,(2j-1) \int_{-1}^{1} L_{i-1}(x) L_{j-1}(x) dx, \\
&= -2\,(2i-1)\,\delta_{ij}.
\end{aligned}
$$

The final equality follows from the orthogonality of the Legendre polynomials. The result is a nearly diagonal stiffness matrix for the one dimensional problem (see Figure 2). The mass matrix is tridiagonal, and since a tensor product is used for higher dimensions, the result is a sparse linear system.

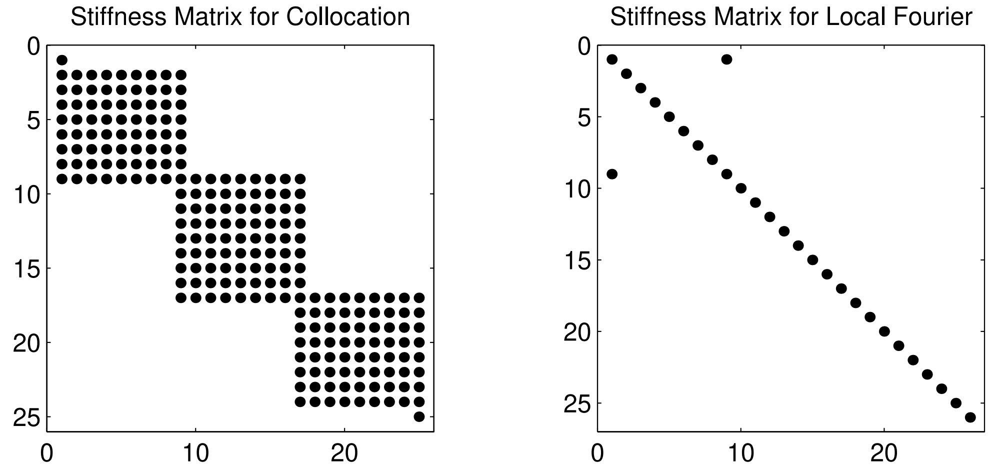
3 Laguerre Polynomials

The Laguerre polynomials are orthogonal on a semi-infinite interval, and the family of polynomials represents a natural candidate to construct an approximation on a semi-infinite domain. The Laguerre polynomials, denoted $L_i^{(0)}(x)$ for the Laguerre polynomial of degree i, are orthogonal on the semi-infinite domain with respect to the weight function e^{-x} [Fun92]:

$$
\int_0^\infty L_i^{(0)}(x) L_j^{(0)}(x) e^{-x} dx \quad = \quad \delta_{ij}. \qquad (5)
$$

The orthogonality relationship has made the Laguerre polynomials the focus of a great deal of attention. In particular, both Funaro [Cou90] and Maday [Mad85] have shown many theoretical results on the accuracy of methods utilizing Laguerre polynomials. However, in practice there are also many difficulties associated with Laguerre polynomials [Mav89]. First, the Laguerre polynomials scale badly for the larger degree polynomials [Fun89]. Moreover, the Laguerre polynomials suffer in the

Figure 2 Comparison of the stiffness matrix for the collocation and the local spectral schemes.



small range of boundaries that can be accommodated at infinity [Mav89]. Finally, the Laguerre polynomials experience spectral convergence only for solutions that decay to zero exponentially [Boy87, Kar87].

Despite these drawbacks, Laguerre polynomials offer a simple and elegant implementation for the approximation of solutions which decay to zero fast enough [Mad85]. For example, for a spectral element method, semi-infinite domains can be accommodated through the following basis functions:

$$\phi_0(x) = L_0^{(0)}(x)e^{-x/2}, \tag{6}$$

$$\phi_i(x) = \left(L_n^{(0)}(x) - L_{n-1}^{(0)}(x)\right) e^{-x/2}, \quad i > 0. \tag{7}$$

A variational approximation can be constructed resulting in symmetric, tridiagonal mass and stiffness matrices. This process is made easier since, as defined, the basis functions satisfy the following identities,

$$\phi_0(0) = 1, \tag{8}$$

$$\phi_i(0) = 0, \qquad i > 0, \tag{9}$$

$$L_0^{(0)}(x) = 1, \tag{10}$$

$$\frac{d}{dx}\left(L_i^{(0)}(x) - L_{i-1}^{(0)}(x)\right) = -L_{i-1}^{(0)}(x), \qquad i > 0. \tag{11}$$

The basis functions satisfy similar boundary conditions as the the basis functions for the finite subdomains. Continuity at the interface is handled through the first basis function, $\phi^0(x)$, in the same manner as for the finite subdomains.

4 Mapping to a Finite Subdomain

Another common method for building an approximation on a semi-infinite interval is to make use of an algebraic mapping. By mapping the semi-infinite interval, $[0,\infty)$, to a finite interval, $[-1,1]$. Orthogonal polynomial methods can be utilized to construct an approximation on the resulting finite interval.

We propose to adapt the mapping proposed by Boyd [Boy87]:

$$y = M\frac{1+x}{1-x}, \tag{12}$$

$$x = \frac{y-M}{y+M}. \tag{13}$$

While the method has been employed for the single domain case, we propose to extend its use to the multidomain approach in a manner similar to that proposed by Karageorghis and Phillips [Kar87].

For example, given a simple 1D Helmholtz equation, the area near the origin can be approximated through the use of finite subdomains. Away from the origin, semi-infinite subdomains can be employed. For the semi-infinite subdomains, the following functions are defined (the notation introduced by Boyd [Boy87] is employed here),

$$\begin{aligned} LM_n(y) &= L_n\left(\frac{y-M}{y+M}\right), \\ &= L_n(x), \end{aligned} \tag{14}$$

where $L_n(x)$ is the n^{th} Legendre polynomial. To take advantage of the orthogonality of the Legendre polynomials, a weight function is required,

$$\begin{aligned} \int_0^\infty LM_n(y)LM_m(y)\frac{2M^2}{(y+M)^2}dy &= \frac{M}{2}\int_{-1}^{1} L_n(x)L_m(x)dx, \\ &= \frac{M}{2n+1}\delta_{nm}. \end{aligned} \tag{15}$$

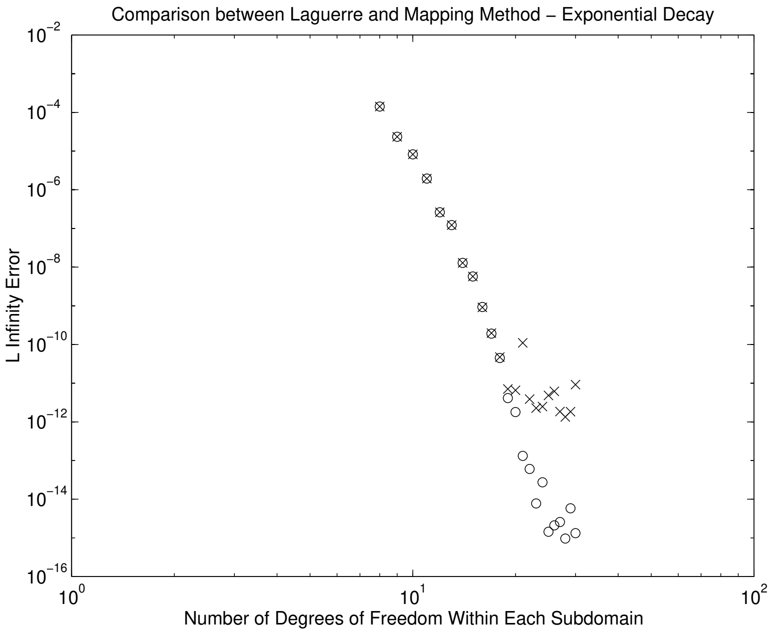
The test functions are chosen to remain consistent with those found in Section 2:

$$\phi M_j(y) = \begin{cases} \frac{1+x}{2} & j=0, \\ \frac{1-x}{2} & j=1, \\ L_j(x)-L_{j-2}(x) & j>1. \end{cases} \tag{16}$$

The continuity conditions remain identical to those found for the finite subdomain, and the choice of basis functions leads to a mass matrix that is identical to that found for a finite subdomain. The stiffness matrix is not as elegant, though. The stiffness matrix, while not diagonal, is a banded matrix, with band width 5. While the new stiffness matrix is not symmetric, the new approximation can be employed for a much wider collection of boundary conditions at infinity. In fact, the boundaries are enforced in exactly the same way as is done for a finite domain.

The stiffness matrix for the semi-infinite subdomain is found in the same manner

Figure 3 Spectral element approximation of a solution with exponential decay using Laguerre polynomials and algebraic mappings on the outer subdomains. The errors for the Laguerre polynomials are denoted by $\times$, while the errors for the mapping to the finite interval are denoted by $\circ$.

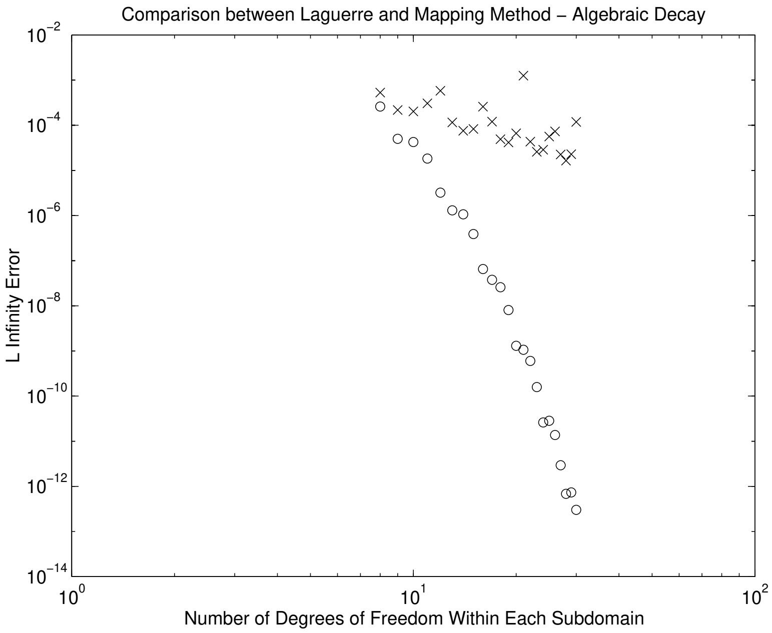


as that for a finite subdomain,

$$\begin{aligned}(\mathcal{S}_N)_{mj} &= -\int_0^\infty \frac{d}{dy}\left(\phi M_j(y)\right)\frac{d}{dy}\left(\phi M_m(y)\frac{2M^2}{(y+M)^2}\right)dy, \qquad (17)\\ &= -\frac{M}{2}\int_{-1}^{1}\phi_j'(x)\frac{(1-x)^2}{2M}\phi_m'(x)\frac{(1-x)^2}{2M}dx,\\ &+\frac{M}{2}\int_{-1}^{1}\phi_j'(x)\phi_m(x)\frac{(1-x)^3}{2M^2}dx.\end{aligned}$$

Note that the weight function is similar to that proposed by Boyd [Boy87] but differs slightly. The weight function includes the square of the mapping parameter in the numerator to insure that the weight function is continuous across the subdomain interface. This choice for the weight function insures that only C^0 continuity need be enforced across the subdomain interface in the resulting variational formulation.

Figure 4 Spectral element approximation of a solution with algebraic decay using Laguerre polynomials and algebraic mappings on the outer subdomains. The errors for the Laguerre polynomials are denoted by $\times$, while the errors for the mapping to the finite interval are denoted by $\circ$.



Comparison of the Two Methods

A comparison of the two methods for two situations is examined. In both of these examples, a 1D Helmholtz equation (Equation 1) is examined (with zero boundaries), and in both cases $\lambda = 2$. In the first example the true solution is e^{-x^2}, and in the second example the true solution is $\frac{1}{1+x^2}$.

The interval $[-5, 5]$ is divided into four equal finite subdomains, and the remaining two subdomains are $(-\infty, 5]$ and $[5, \infty)$. In this situation, the basis functions on subdomain n, $\phi_i^n(x)$, only have support on subdomain n for $i = 2 \ldots N$. The functions $\phi_0^n(x)$ and $\phi_1^n(x)$ are used to construct basis functions whose support only includes adjacent subdomains.

The L^∞ errors for the two trials are shown in Figures 3 and 4. The approximation that utilizes Laguerre polynomials does exhibit fast convergence for the equation whose solution decays to zero exponentially (Figure 3). However, this is not the case for the approximation to the equation whose solution does not decay as fast (Figure 4). The method using the proposed mapping, though, does exhibit similar convergence properties for both situations.

To test the mapping method in a two dimensional case, a Poisson equation is

$N_x = N_y$	L^∞ Error
8	5.206259e-04
10	2.438615e-04
12	1.365491e-04
14	7.567740e-05
16	4.658750e-05

Table 1 Maximum Errors for the approximation of a Poisson equation on an infinite domain. For each approximation 16 subdomains are employed, and the polynomial degree is given for each trial. For each trial the polynomial degree for both the x and y direction are equal.

examined,

$$\begin{aligned} \triangle u &= \frac{4x^2+4y^2-4}{(1+x^2+y^2)^3}, \\ \lim_{x\to\infty} u(\pm x, y) &= 0, \\ \lim_{y\to\infty} u(x, \pm y) &= 0. \end{aligned} \tag{18}$$

The solution to this equation, $\frac{1}{1+x^2+y^2}$, decays to zero algebraically. In this test case 16 subdomains are employed. Four finite subdomains are employed on the unit square, $[-1,1]\times[-1,1]$. Away from the unit square semi-infinite subdomains are employed to construct an approximation. The errors are shown in Table 1, and the errors are the maximum errors found on the abscissa of the Legendre-Gauss quadrature within each subdomain.

REFERENCES

[Bel94] Bellan, P.M. (November 1994) Alfen 'Resonance' Reconsidered: Exact equations for wave propogation across a cold inhomogeneous plasma. *Physics of Plasmas* 1(11): 3523–3541.

[Boy87] Boyd, John (1987) Orthogonal Rational Functions on a Semi-Infinite Interval. *Journal of Computational Physics* 70: 63–68.

[Cou90] Coulaud, O., D. Funaro, and O. Kavian (1990) Laguerre Spectral Approximations of Elliptic Problems in Exterior Domains. *Computer Methods in Applied Mechanics and Engineering* 80(1-3): 451–458.

[Fun89] Funaro, D. (October 1989) Computational Aspects of Pseudospectral Laguerre Approximations. Technical Report NAS1-18605, ICASE, NASA Langley Research Center, Hampton VA 23665-5225.

[Fun92] Funaro, D. (1992) *Polynomial Approximation of Differential Equations.* Springer-Verlag, Berlin.

[Kar87] Karageorghis, A., and T.N. Phillips (1987) Spectral Collocation Methods for Stokes Flow in Contraction Geometries and Unbounded Domains. *Journal of Computational Physics* 80: 314–330.

[Lan60] Landau, L. D., and E. M. Lifshitz (1960) *Electrodynamics of Continuous Media*, volume 8 of *Course of Theoretical Physics*. Pergamon Press, New York.

[Mad85] Maday, Y., B. Pernaud-Thomas, and H. Vandeven (1985) Reappraisal of Laguerre Type Spectral Methods. *La Recherche Aerospatial (English Addition)* 6: 13–35.

[Mav89] Mavriplis, C. (1989) Laguerre Polynomials for Infinite-Domain Spectral Elements. *Journal of Computational Physics* 80: 480–488.

[Pat84] Patera, Anthony T. (1984) A Spectral Element Method for Fluid Dynamics: Laminar Flow in a Channel Expansion. *Journal of Computational Physics* 54: 468–488.

[Sec95] Secchi, Paolo (1995) Well-Posedness for a mixed problem for the equations of ideal Magneto-Hydrodynamics. *Archivel der Mathematik* 64: 237–245.

[She94] Shen, J. (November 1994) Efficient Spectral-Galerkin Method I. Direct Solvers of Second and Fourth-Order Equations Using Legendre Polynomials. *SIAM Journal of Scientific Computing* 15(6): 1489–1505.

[Wil93] Wilson, S.K. (1993) The Effect of a Uniform Magnetic Field on the Onset of Marangoni Convection in a Layer of Conducting Fluid. *Quarterly Journal of Mechanics and Applied Mathematics* 46(2): 211–248.

[Wil94] Wilson, S.K. (November 1994) The effect of a uniform magnetic field on the onset of steady Marangoni convection in a layer of conducting fluid with a prescribed heat flux on its lower boundary. *Physics of Fluids* 6(11): 3591–3600.