30

Hierarchical Basis for the Convection-Diffusion Equation on Unstructured Meshes

Randolph E. Bank and Sabine Gutsch

1 Introduction

The Hierarchical Basis Multigrid Method was originally developed for sequences of refined meshes. Hierarchical basis functions can be constructed in a straightforward fashion on such sequences of nested meshes. The HBMG iteration itself is just a block symmetric Gauß-Seidel iteration applied to the stiffness matrix represented in the hierarchical basis. Because the stiffness matrix is less sparse than when the standard nodal basis functions are used, the iteration is carried out by forming the hierarchical basis stiffness matrix only implicitly. The resulting algorithm is strongly connected to the classical multigrid V-cycle, except that only a subset of the unknowns on each level is smoothed during the relaxation steps [BDY88].

In recent years, we have generalized such bases to completely unstructured meshes, not just those arising from some refinement process. This is done by recognizing the strong connection between the Hierarchical Basis Multigrid Method and an Incomplete LU factorization of the nodal stiffness matrix. This can best be understood in terms of the graph elimination model for Gaussian elimination. This connection is explored in detail in [BX94]. Once this connection is made, it is fairly easy to make a symbolic ILU algorithm for unstructured meshes which mimics the ILU process on a structured mesh leading to the classical hierarchical basis. This symbolic elimination process essentially defines the *supports* of the hierarchical basis functions (or the sparsity structure of the hierarchical basis stiffness matrix).

In the classical case, certain linear combinations of fine grid basis functions are

Ninth International Conference on Domain Decomposition Methods
Editor Petter E. Bjørstad, Magne S. Espedal and David E. Keyes

formed, with the combination coefficients derived from the geometry of the mesh. For these special choices, the linear combinations simplify to coarse grid nodal basis functions. In the case of completely unstructured meshes, the coefficients for the linear combinations can also be specified in a natural way using the geometry of the mesh. However, since there is no coarse grid as such, in general no simplification of the basis functions occurs. Different choices of expansion coefficients typically have no effect on the supports of the basis functions, but do have a profound effect on the shape of the basis functions themselves, and hence on the numerical values appearing in the hierarchical basis stiffness matrix [BX94], [BX96]. One can even choose different coefficients for the trial and test spaces, leading to different hierarchical basis functions, similar to Petrov-Galerkin finite element approximations.

In terms of ILU, the expansion coefficients are just the multipliers in the ILU decomposition. This leads one to consider the possibility of defining these expansion coefficients in a more algebraic fashion. For example, one can choose these coefficients to eliminate certain off diagonal elements of the hierarchical basis stiffness matrix, as is done in the case of classical ILU. While geometry based coefficients seem adequate for isotropic, self-adjoint problems (e.g. $-\Delta u = f$), we have found that our algebraic choices can greatly improve the robustness of the HBMG iteration for other types of equations, notably convection dominated convection-diffusion equations.

In Section 2, we review the connection between HBMG and ILU from a graph theoretical point of view. In Section 3, we analyze this method in the one-dimensional case, where algebraic simplicity allows a fairly complete treatment. In Section 4, we present some algorithms and numerical results for two-dimensional problems.

2 HBMG and ILU

As with typical multigrid methods, classical hierarchical basis methods are usually defined in terms of an underlying refinement structure for a sequence of nested meshes. In many cases, this is no disadvantage, but it limits the applicability of the methods to truly unstructured meshes, which might be highly nonuniform but not derived from some grid refinement process. Here we view the transformation of the stiffness matrix A from nodal basis representation to hierarchical basis representation as a special ILU decomposition. This generalizes the construction of hierarchical bases to unstructured meshes, allowing HBMG and other hierarchical basis methods to be applied. A more complete discussion of this point can be found in [BX94] and [BX96]. See [HS95], [Kor96], [HK94], [KY94], and [CS94] for related algorithms. See [Xu89], [BPX91], and [Zha88] for discussions of non-nested multigrid algorithms.

Graph Theoretical Properties of Hierarchical Bases

We begin by exploring the connection between the HBMG method and ILU decomposition in terms of graph theory. We consider the standard Gaussian elimination and the classical ILU factorization from a graph theoretical point of view, and then develop a simple graph elimination model for classical hierarchical basis methods on sequences of nested meshes. We can interpret this model as a particular ILU decomposition and generalize this graph elimination model to the case of completely unstructured meshes. See Rose [Ros72] and George and Liu [GL81] for a complete discussion of graph theoretical aspects of Gaussian elimination. Corresponding to a sparse $n \times n$ matrix A with symmetric pattern (i.e. $A_{ij} \neq 0$ if and only if $A_{ji} \neq 0$), let $G(X,E)$ be the graph that consists of a set of n ordered vertices $v_i \in X$, $1 \leq i \leq n$, and a set of edges E such that the edge (connecting vertices v_i and v_j) $e_{ij} \in E$ if and only if $a_{ij} \neq 0$, $i \neq j$. The edges in the graph G correspond to the nonzero off diagonal entries of A. If A is the stiffness matrix for the space of continuous piecewise linear polynomials represented in the standard nodal basis, the graph G is just the underlying triangulation of the domain (with minor modifications due to Dirichlet boundary conditions). We define for a vertex v_i the set of adjacent vertices $adj(v_i)$ by

$$adj(v_i) = \{v_j \in X | e_{ij} \in E\}.$$

A clique $C \subseteq X$ is a set of vertices which are all pairwise connected; that is $v_i, v_j \in C, i \neq j \Rightarrow e_{ij} \in E$. With a proper ordering of the vertices, a clique corresponds to a dense submatrix of A. In graph theoretic terms, a single step of Gaussian elimination transforms $G(X,E)$ to a new graph $G'(X',E')$ as follows:

1. Eliminate vertex v_i and all its incident edges from G. Set $X' = X - \{v_i\}$. Denote the resulting set of edges $E_1 \subseteq E$.
2. Create a set F of fillin edges as follows: For each distinct pair $v_j, v_k \in adj(v_i)$ in G, add the edge e_{jk} to F if not already present in E_1. Set $E' = E_1 \cup F$.

Note that the set $adj(v)$ in G becomes a clique in G'. Within this framework, the classical ILU factorization is one in which *no* fillin edges are allowed, i.e. $F \equiv \emptyset$. This forces the matrix A' corresponding to the new graph G' to have the same sparsity structure as the corresponding submatrix of A.

To define HBMG as a generalized ILU procedure, we must first introduce the concept of *vertex parents.* We will begin with the case of two nested meshes where the fine mesh is a uniform refinement of a coarse mesh, generated by pairwise connecting the midpoints of the coarse grid edges in the usual way [BDY88], [Yse86], [Hac85]. Here we can make the direct sum decomposition $X = X_c \oplus X_f$, where X_c is the set of coarse grid vertices and X_f is the set of fine grid vertices (those not in X_c). For each vertex $v_i \in X_f$, there is a unique pair of vertex parents $v_j, v_k \in X_c$ such that v_i is the midpoint of the edge connecting v_j and v_k ($v_i = (v_j + v_k)/2$).

We now view HBMG as an ILU algorithm in which only selected fillin edges are allowed, namely those connecting vertex parents. In this algorithm, we sequentially eliminate the vertices in the set X_f as follows:

1. Eliminate vertex $v_i \in X_f$ and all its incident edges from G. Set $X' = X - \{v_i\}$. Denote the resulting set of edges $E_1 \subseteq E$.
2. Add one fillin edge connecting the vertex parents $v_j, v_k \in X_c$ of v_i. Set $E' = E_1 \cup \{e_{jk}\}$.

Note that the triangulation $\mathcal{T}_f$ is the graph for the original stiffness matrix A represented in the standard nodal basis. After all the vertices in X_f are eliminated, the resulting graph is just the triangulation $\mathcal{T}_c$, i.e. the sparsity pattern of the coarse grid matrix corresponds to the coarse grid triangulation. For completely unstructured meshes, the main problem is to determine reasonable vertex parents for each vertex to be eliminated. Once this is done, the elimination/unrefinement/coarsening is done exactly as in the case of nested meshes. This may lead to graphs that are not necessarily triangulations of the domain, but typically contain polygonal elements of various orders. Even if the graphs remain triangulations, they will generally not be nested. Nonetheless, such a scheme still defines the linear combinations of fine grid basis functions used to create the coarse grid basis functions (but not the values of the coefficients in the linear combinations).

Algorithms for selecting vertex parents are currently an area of active research. The scheme developed in [BX96] is based on the geometry of the triangulation, and seeks to coarsen the grid in a fashion that maintains the shape of the region and the shape regularity of the coarse grid elements to the extent possible. For this scheme, the supports of the resulting hierarchical basis functions grow in a fashion analogous to the classical case. We are also considering more algebraic schemes which rely only on the sparsity structure of the stiffness matrix and the numerical values of its matrix elements; these schemes have much in common with more classical sparse matrix ordering algorithms.

Algebraic HBMG and ILU

In this section, we consider the algebraic aspects of the HBMG method and its relation to Gaussian elimination. Again we will consider the case of only two levels. Let A denote the nodal basis stiffness matrix for the fine grid, and consider the block partitioning

$$A = \begin{pmatrix} A_{cc} & A_{cf} \\ A_{fc} & A_{ff} \end{pmatrix}, \tag{1}$$

where A_{ff} corresponds to the nodal basis functions of the fine grid nodes, A_{cc} corresponds to the (fine grid) nodal basis functions of the coarse grid nodes and A_{cf} and A_{fc} correspond to the coupling between the two sets of basis functions. We consider transformations of the form $A' = S^T A \tilde{S}$ where S and $\tilde{S}$ have the block structure

$$S = \begin{pmatrix} I & 0 \\ R & I \end{pmatrix}, \quad \tilde{S} = \begin{pmatrix} I & 0 \\ \tilde{R} & I \end{pmatrix}. \tag{2}$$

By direct calculation, we obtain

$$S^T A \tilde{S} = \begin{pmatrix} \hat{A}_{cc} & A_{cf} + R^T A_{ff} \\ A_{fc} + A_{ff}\tilde{R} & A_{ff} \end{pmatrix} \tag{3}$$

where

$$\hat{A}_{cc} = A_{cc} + R^T A_{fc} + A_{cf}\tilde{R} + R^T A_{ff}\tilde{R}. \tag{4}$$

Different algorithms can be characterized by different choices of R and $\tilde{R}$. For example, in the classical block Gaussian elimination we have $R = -A_{ff}^{-T} A_{cf}^T$ and $\tilde{R} = -A_{ff}^{-1} A_{fc}$, and $\hat{A}_{cc} = A_{cc} - A_{cf} A_{ff}^{-1} A_{fc}$ is the Schur complement. In this case, the off diagonal blocks are reduced to zero, but at the cost of having fairly dense matrices R, $\tilde{R}$ and $\hat{A}_{cc}$. In the case of HBMG, the matrices R and $\tilde{R}$ are sparse and contain information about changing from the nodal to the hierarchical basis. The sparsity patterns of R and $\tilde{R}$ are the same, and both are determined by the vertex parent relationship described above. In particular, each row of R and $\tilde{R}$ is zero except for the two entries which correspond to the (coarse grid) vertex parents for the given fine grid vertex. In the classical case, where each fine grid vertex is the midpoint of the edge connecting its vertex parents, $R = \tilde{R}$ and both nonzero entries in a given row are equal to $1/2$. In the generalized HBMG, these values are replaced by $\theta_i, \tilde{\theta}_i, \nu_i$ and $\tilde{\nu}_i$. Often one has $\tilde{\theta}_i = 1 - \theta_i$ and $\tilde{\nu}_i = 1 - \nu_i$. Choosing $R \neq \tilde{R}$ corresponds to choosing a test space different from the trial space for the coarser grids, as in a Petrov-Galerkin method. Several alternatives for choosing $\theta_i, \tilde{\theta}_i, \nu_i$ and $\tilde{\nu}_i$ are discussed in Section 4.

3 Analysis of a One-Dimensional Model Problem

The Two-Level HBMG Iteration

In this section, we will analyze the case of a constant coefficient two two-point boundary value problem, giving rise to a constant coefficient tridiagonal stiffness matrix when discretized using some finite element approximation on a uniform mesh. See [Hac84] and [BB91] for other analyses of multilevel methods for one-dimensional model problems. Let $n > 2$ be an integer and set $h = 1/(2n)$. The uniform fine mesh $\mathcal{T}_h$ of size h has $2n + 1$ grid points $x_k = kh, 0 \leq k \leq 2n$. The coarse mesh has $n + 1$ grid points $x_{2k}, 0 \leq k \leq n$. We will refer to the set of coarse grid points as level 1 vertices, and $x_{2k+1}, 0 \leq k \leq n - 1$, as level 2 vertices.

Let $\mathcal{P}_h$ be a $2n - 1$ dimensional trial space of functions associated with the fine mesh $\mathcal{T}_h$ satisfying the boundary conditions $v(x_0) = v(x_{2n}) = 0$ for all $v \in \mathcal{P}_h$. Let $\hat{\phi}_k, 1 \leq k \leq 2n - 1$, denote the nodal basis for the trial space $\mathcal{P}_h$. We assume that $support\{\hat{\phi}_k\} = (x_{k-1}, x_{k+1})$ and that $\hat{\phi}_k(x_j) = \delta_{kj}$. Similarly, we define the $2n - 1$ dimensional test space $\mathcal{S}_h$ with nodal basis $\hat{\psi}_k, 1 \leq k \leq 2n - 1$.

We will use a discretization on the mesh $\mathcal{T}_h$ given by a bilinear form $b(\cdot,\cdot) : \mathcal{P}_h \times \mathcal{S}_h \to \mathbb{R}$. The details of the bilinear form $b(\cdot,\cdot)$ are arbitrary for the moment. The discrete system of equations to be solved is: Find $u_h \in \mathcal{P}_h$ such that

$$b(u_h, v) = rhs(v)$$

for all $v \in \mathcal{S}_h$, where $rhs(\cdot)$ is an appropriate linear functional.

Suppose

$$b(\hat{\phi}_k, \hat{\psi}_j) = \begin{cases} a & j = k - 1 \\ b & j = k + 1 \\ c & j = k \\ 0 & |k - j| > 1 \end{cases} \tag{5}$$

where a, b, and c are constants.

The resulting nodal basis stiffness matrix in natural vertex order is the constant coefficient tridiagonal matrix

$$A^{NB} = \begin{pmatrix} c & b & & & \\ a & c & b & & \\ & \ddots & \ddots & \ddots & \\ & & a & c & b \\ & & & a & c \end{pmatrix}. \tag{6}$$

Now we introduce the generalized hierarchical bases for the spaces $\mathcal{P}_h$ and $\mathcal{S}_h$. Define functions

$$\tilde{\phi}_{2k} = \theta\hat{\phi}_{2k-1} + \hat{\phi}_{2k} + \tilde{\theta}\hat{\phi}_{2k+1} \tag{7}$$

for $1 \leq k \leq n-1$, and $\theta, \tilde{\theta} \in \mathbb{R}$. The generalized hierarchical basis for $\mathcal{P}_h$ consists of the union of the functions $\tilde{\phi}_{2k}$ for $1 \leq k \leq n-1$, and the basis functions for the level 2 nodes, $\hat{\phi}_{2k+1}$, $0 \leq k \leq n-1$. This basis will be denoted by ϕ_k, $1 \leq k \leq 2n-1$. The generalized hierarchical basis introduces a natural direct sum decomposition of the space $\mathcal{P}_h$. If $u \in \mathcal{P}_h$, then we have the unique decomposition $u = v + w$, where $v \in \mathcal{V} = span\{\hat{\phi}_{2k+1}\}_{k=0}^{n-1}$ and $w \in \mathcal{W} = span\{\tilde{\phi}_{2k}\}_{k=1}^{n-1}$. The generalized hierarchical basis $\psi_k, 1 \leq k \leq 2n-1$, of $\mathcal{S}_h$ is defined similarly, but with constants ν and $\tilde{\nu}$ instead of θ and $\tilde{\theta}$.

Using (5) and (7), we obtain

$$b(\phi_{2k}, \psi_j) = \begin{cases} \hat{a} = \theta\tilde{\nu}c + (\tilde{\nu} + \theta)a & j = 2k-2 \\ p = \theta c + a & j = 2k-1 \\ \hat{c} = (1 + \tilde{\theta}\tilde{\nu} + \theta\nu)c + (\theta + \tilde{\nu})b + (\nu + \tilde{\theta})a & j = 2k \\ q = \tilde{\theta}c + b & j = 2k+1 \\ \hat{b} = \nu\tilde{\theta}c + (\nu + \tilde{\theta})b & j = 2k+2 \end{cases} \tag{8}$$

for $1 \leq k \leq n-1$, and

$$b(\phi_{2k+1}, \psi_j) = \begin{cases} r = \tilde{\nu}c + a & j = 2k \\ c & j = 2k+1 \\ s = \nu c + b & j = 2k+2 \end{cases} \tag{9}$$

for $0 \leq k \leq n-1$.

The stiffness matrix A^{HB} corresponding to the hierarchical basis is a pentadiagonal matrix given by

$$A^{HB} = \begin{pmatrix} c & s & 0 & & & & & \\ p & \hat{c} & q & \hat{b} & & & & \\ 0 & r & c & s & 0 & & & \\ & \hat{a} & p & \hat{c} & q & \hat{b} & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & 0 & r & c & s & 0 \\ & & & & \hat{a} & p & \hat{c} & q \\ & & & & & 0 & r & c \end{pmatrix}.$$

Now we apply a simple permutation to the matrix A^{HB}, in which the basis functions associated with the coarse grid points are ordered first, and those associated with the fine grid points are ordered last. If we denote the relevant permutation matrix by P, the permuted matrix is block 2×2 of the form given in (3)-(4):

$$\bar{A}^{HB} = PA^{HB}P^T = \begin{pmatrix} A_{cc} & A_{cf} \\ A_{fc} & A_{ff} \end{pmatrix},$$

where

$$A_{cc} = \begin{pmatrix} \hat{c} & \hat{b} & & & \\ \hat{a} & \hat{c} & \hat{b} & & \\ & \ddots & \ddots & \ddots & \\ & & \hat{a} & \hat{c} & \hat{b} \\ & & & \hat{a} & \hat{c} \end{pmatrix}_{n-1\times n-1}, \quad A_{cf} = \begin{pmatrix} p & q & & & \\ & p & q & & \\ & & \ddots & \ddots & \\ & & & p & q \end{pmatrix}_{n-1\times n},$$

$$A_{fc} = \begin{pmatrix} r & & & \\ s & r & & \\ & s & \ddots & \\ & & \ddots & r \\ & & & s \end{pmatrix}_{n\times n-1}, \qquad A_{ff} = \begin{pmatrix} c & & & \\ & c & & \\ & & \ddots & \\ & & & c \end{pmatrix}_{n\times n}.$$

The 2-level generalized hierarchical basis multigrid method is the block symmetric Gauß-Seidel iteration applied to the linear system $\bar{A}^{HB}\bar{u} = \bar{f}$. If $\bar{A}^{HB}$ is block upper (or lower) triangular, we will obtain the exact solution in one step. To make $\bar{A}^{HB}$ block diagonal, we set $p = q = r = s = 0$, giving

$$\theta = \tilde{\nu} = -\frac{a}{c} \quad \text{and} \quad \tilde{\theta} = \nu = -\frac{b}{c}. \tag{10}$$

For this choice of interpolation coefficients, the transformation of the stiffness matrix $\bar{A}^{NB}$ to the matrix $\bar{A}^{HB} = S^T\bar{A}^{NB}\tilde{S}$ is the classical block Gaussian elimination.

Examples

For our first example, we consider the self adjoint problem $-u'' = f$ with Dirichlet boundary conditions. Discretizing using the standard nodal basis for the space of continuous piecewise linear polynomials leads to the tridiagonal stiffness matrix

$$A^{NB} = \frac{1}{h}\begin{pmatrix} 2 & -1 & \\ -1 & \ddots & \ddots \\ & \ddots & \end{pmatrix}.$$

If we use the standard interpolation constants,

$$\theta = \tilde{\theta} = \nu = \tilde{\nu} = \frac{1}{2},$$

leading to the standard piecewise linear nodal basis functions for the course grid, the permuted hierarchical basis stiffness matrix is of the form

$$\bar{A}^{HB} = \frac{1}{2h}\left(\begin{array}{cccc|ccc} 2 & -1 & & & & & \\ -1 & \ddots & \ddots & & & & \\ & \ddots & & & & & \\ \hline & & & & 4 & & \\ & & & & & & \ddots \end{array}\right).$$

It is well known that the HBMG method is a direct method for this special case [Yse86].

As our second example, we consider the one-dimensional convection diffusion equation $-(u' + \beta u)' = f$ on an interval I, with Dirichlet boundary conditions and constant β. Here we will use the well known Scharfetter-Gummel discretization on a uniform mesh. There are several standard interpretations of this discretization. One which is especially useful here is to view the discretization as a Petrov-Galerkin method using the standard bilinear form

$$b(u, v) = \int_I (u' + \beta u)v' \, dx.$$

For the test space we use the standard continuous piecewise linear polynomials, while the trial space is composed of piecewise functions of the form $\alpha e^{-\beta x} + \gamma$, since these are the fundamental solutions of the homogeneous equation. The nodal basis function $\phi_i(x)$ is given by

$$\phi_i(x) = \begin{cases} (e^{-\beta(x-x_i)} - e^{-\beta h})/(1 - e^{-\beta h}) & x_i \le x \le x_{i+1} \\ (e^{-\beta(x-x_i)} - e^{\beta h})/(1 - e^{\beta h}) & x_{i-1} \le x \le x_i \\ 0 & \text{elsewhere} \end{cases}$$

The resulting tridiagonal nodal basis stiffness matrix has entries

$$\begin{aligned} a &= -\frac{\mathcal{B}(-\beta h)}{h}, \\ b &= -\frac{\mathcal{B}(\beta h)}{h}, \\ c &= \frac{\mathcal{B}(\beta h) + \mathcal{B}(-\beta h)}{h}, \end{aligned}$$

where $\mathcal{B}(\cdot)$ denotes the Bernoulli function

$$\mathcal{B}(x) = \frac{x}{e^x - 1}.$$

To make the off-diagonal blocks zero, we choose interpolation coefficients

$$\begin{aligned} \theta &= \tilde{\nu} = \frac{\mathcal{B}(-\beta h)}{\mathcal{B}(-\beta h) + \mathcal{B}(\beta h)} = \frac{1 - e^{-\beta h}}{1 - e^{-2\beta h}}, \\ \tilde{\theta} &= \nu = 1 - \theta. \end{aligned}$$

The entries of the resulting coarse grid matrix are given by

$$\begin{aligned}\hat{a} &= -\frac{\mathcal{B}(-2\beta h)}{2h},\\ \hat{b} &= -\frac{\mathcal{B}(2\beta h)}{2h},\\ \hat{c} &= \frac{\mathcal{B}(2\beta h)+\mathcal{B}(-2\beta h)}{2h},\end{aligned}$$

i.e., with this choice of interpolation coefficients the Scharfetter-Gummel discretization on the coarse grid is reproduced. This is the same result we get calculating θ from $u(x+h/2) = \theta u(x) + (1-\theta)u(x+h)$ and making the interpolation exact for functions of the form $u(x) = \alpha e^{-\beta x} + \gamma$.

For our third example, we consider the case of the L^2-projection into the space of continuous piecewise linear finite elements. The mass matrix for the space of continuous piecewise linear polynomials on a uniform mesh is given by

$$M = \frac{h}{6}\begin{pmatrix} 4 & 1 & \\ 1 & \ddots & \ddots \\ & \ddots & \end{pmatrix}.$$

In order to make the off-diagonal blocks zero, we have to choose

$$\theta = \tilde{\theta} = \nu = \tilde{\nu} = -\frac{1}{4}.$$

Here the coefficients are negative and $\theta + \tilde{\theta} \neq 1$. Note that this leads to oscillatory wavelet like basis functions in the k-level recursion.

4 Some Interpolation Algorithms for Two Dimensions

In this section, we consider the two-dimensional convection-diffusion equation

$$-\nabla(\nabla u + \beta u) = f \quad \text{in } \Omega \tag{11}$$

with boundary conditions

$$u = 0 \quad \text{on } \partial\Omega, \tag{12}$$

where Ω is a polygonal domain and β is constant. We will apply the Scharfetter Gummel discretization on an unstructured triangular mesh. We will discuss several alternatives for choosing interpolation coefficients $\theta, \tilde{\theta}, \nu$ and $\tilde{\nu}$ which determine the generalized hierarchical basis functions. Unfortunately, in 2-D it is not possible to make the off-diagonal blocks equal to zero. Most alternatives are motivated by the one-dimensional case examined in Section 3.

Classical HBMG (Linear Interpolation)

In the classical HBMG method, the interpolation coefficients are chosen to exactly interpolate one-dimensional linear polynomials along element edges. The hierarchical basis stiffness matrix is of the form $A^{HB} = S^T A^{NB} S$ as given in (3)-(4). As usual, the matrices R and $\tilde{R}$ of (2) contain the interpolation coefficients, which in the case of a regular uniform refinement satisfy $\theta = \tilde{\theta} = \nu = \tilde{\nu} = 1/2$. If a fine grid point v_i is not the midpoint of its vertex parents v_j, v_k, we take the corresponding fractions $\theta = \nu = dist(v_i, v_k)/dist(v_j, v_k)$ and $\tilde{\theta} = \tilde{\nu} = 1 - \theta$.

The Scharfetter-Gummel method (Exponential Interpolation)

An exponential interpolation scheme can be derived from the Scharfetter-Gummel formula. We begin by noting that, analogous to the one-dimensional case, fundamental solutions of (11) are given by

$$u(x) = \alpha + \gamma e^{-\langle \beta, x\rangle} = \alpha + \gamma e^{-\beta_1 x_1 - \beta_2 x_2}$$

for constants $\alpha, \gamma \in \mathbb{R}$. Suppose the values $u_1 = u(v_1)$ and $u_2 = u(v_2)$ are known and $u_m \equiv u(v_m) = u(\theta v_1 + (1-\theta) v_2)$ is to be approximated. If we require an exact interpolation of the fundamental solutions on the one-dimensional edge between v_1 and v_2, we can obtain by a straightforward calculation $u_m = \nu u_1 + \tilde{\nu} u_2$, where

$$\begin{aligned} \nu &= \frac{\theta \mathcal{B}(\langle \beta, v_2 - v_1\rangle)}{\mathcal{B}(\theta\langle \beta, v_2 - v_1\rangle)} = \frac{e^{\theta\langle \beta, v_2 - v_1\rangle} - 1}{e^{\langle \beta, v_2 - v_1\rangle} - 1}, \\ \tilde{\nu} &= 1 - \nu. \end{aligned}$$

Here $\mathcal{B}(x)$ denotes the Bernoulli function. When $\beta = 0$, this method reduces to the classical HBMG algorithm. Note that the interpolation coefficients lie in $(0, 1)$ and sum up to 1: $\nu + \tilde{\nu} = 1$. We note that problems arise for multilevel methods where the algebraic coarse grid matrices do not correspond to a discretized convection-diffusion equation anymore. In some very special cases, with a slight variation of the SG-coefficients, one can force A^{HB}_{cc} to be the coarse grid Scharfetter-Gummel discretization matrix. However, this leads to poor numerical results.

An Algebraic ILU Method

In this method, we choose interpolation coefficients to create zeroes in the off-diagonal blocks whenever possible, in a fashion analogous to Gaussian elimination. This leads to coefficients $\theta^i_k = -a_{ik}/a_{ii}$, where vertex i is the fine grid vertex to be eliminated and vertex k is one of its vertex parents. If vertex i is the only one to be eliminated, this leads to a minimization of the $\|\cdot\|_1$ and $\|\cdot\|_2$ norms of the vectors $A_{cf} + R^T a_{ff}$ and $A_{fc} + a_{ff}\tilde{R}$. In the general case, the corresponding norms of the affected row and column vectors are locally minimized at each elimination step. The interpolation coefficients can have either sign, and generally $\theta + \tilde{\theta} \neq 1$. A related possibility is to choose $\theta^i_j = a_{ij}/(a_{ij} + a_{ji})$, for which the coefficients will sum to one. Interestingly, for the case of a uniform mesh of isosceles right triangles with regular refinement, the coefficients in $x-$ and $y-$direction are the Scharfetter-Gummel coefficients, but the "diagonal" interpolation coefficients are indeterminate and could for example be taken as zero.

Minimizing the Frobenius Norm

In this method the interpolation coefficients are chosen such that the Frobenius norm of the off-diagonal blocks is minimized. Intuitively it appears that making off-diagonal blocks smaller should increase the rate of convergence, since these are the blocks which are lagged in the iteration. Thus minimizing some norm of the off-diagonal blocks might be good. On the other hand, in this case the minimization leads to small sets of linear equations to be solved for the interpolation coefficients. These linear systems are awkward to assemble if the stiffness matrix is represented in some of the standard sparse matrix formats, and the solution process adds to the cost of the method. In our experience with the method, we noted no significant improvement in the convergence properties of the resulting hierarchical basis iteration. Thus, at this time we cannot recommend this approach.

Hybrid Methods

We have also considered combinations of the above methods for the trial and test space, effectively making a Petrov-Galerkin like method for the coarse grid approximation. Note from (3) that $A_{fc}^{HB} = A_{fc}^{NB} + A_{ff}^{NB}\tilde{R}$ is influenced only by $\tilde{R}$ and $A_{cf}^{HB} = A_{cf}^{NB} + R^T A_{ff}^{NB}$ is influenced only by R. The motivation is that one needs A_{cf}^{HB} *or* A_{fc}^{HB} to be small in order to get good convergence for the symmetric Gauß-Seidel iteration. Thus, one could use interpolation coefficients for one space to make the corresponding off-diagonal block small, and choose those for the other space to influence A_{cc}^{HB} in order to get favorable recurrence relations for several levels. Note that this can lead to a non-symmetric matrix A^{HB} even if A^{NB} is symmetric. One possibility we have found to be effective is to use linear interpolation coefficients for the trial space and algebraic ILU coefficients for the test space.

Numerical Results

In this section, we present numerical illustrations for some of the interpolation schemes applied to the model convection-diffusion equation (11)-(12), with $f \equiv 1$ and $\Omega \equiv (0,1) \times (0,1)$. The problem is discretized using the Scharfetter-Gummel method. The level 1 mesh is a uniform 5×5 mesh shown in Figure 1 (note boundary vertices do not correspond to unknowns). This mesh is uniformly refined by dividing each triangle into four congruent triangles using regular refinement. The refinement is continued until we reach level 5 (6) with 4225 (16641) vertices. We used uniform grids with structured refinement rather than the symbolic elimination for unstructured grids as described in Section 2 in order to treat all test cases in a more standardized setting. To show the performance of the methods on unstructured grids, we adaptively refined the level 5 grid until we reached 10000 vertices. This results in 7 levels of refinement.

We illustrate the dependence of the convergence rate on the direction and magnitude of β. The results for the different methods are shown in Table 1 for structured grids and in Table 2 for unstructured grids. We accelerated the iteration with the bicg

Table 1 Average convergence rates for various methods on structured grids and for several values of $\beta = (\beta_1, \beta_2)^T$. "Lin", "SG" and "ILU" denote linear, exponential, and ILU interpolation coefficients, respectively. For example, "ILU, Lin" means ILU interpolation coefficients were used for the trial space, and linear interpolation was used for the test space.

$N = 4225$	Lin, Lin	SG, SG	SG, Lin	ILU, ILU	ILU, Lin
$(0, 0)$	0.45	0.45	0.45	0.73	0.42
$(0, 1000)$	fails	0.88	0.73	0.49	0.50
$(0, 5000)$	fails	0.85	0.87	0.67	0.61
$(707, 707)$	fails	0.87	0.61	0.70	0.48
$(3536, 3536)$	fails	fails	0.61	0.70	0.48
$(-707, 707)$	fails	0.85	0.75	0.67	0.54
$(-3536, 3536)$	fails	0.85	0.75	0.67	0.54
$N = 16641$	Lin, Lin	SG, SG	SG, Lin	ILU, ILU	ILU, Lin
$(0, 0)$	0.47	0.47	0.47	0.85	0.52
$(0, 1000)$	fails	0.93	0.84	0.76	0.50
$(0, 5000)$	fails	0.94	0.92	0.71	0.61
$(707, 707)$	fails	fails	0.71	0.94	0.61
$(3536, 3536)$	fails	fails	0.71	0.92	0.60
$(-707, 707)$	fails	0.94	0.84	0.97	0.68
$(-3536, 3536)$	fails	0.94	0.88	0.92	0.68

Table 2 Average convergence rates for various methods on unstructured grids and for several values of $\beta = (\beta_1, \beta_2)^T$. "Lin", "SG" and "ILU" denote linear, exponential, and ILU interpolation coefficients, respectively. For example, "ILU, Lin" means ILU interpolation coefficients were used for the trial space, and linear interpolation was used for the test space.

$N = 10000$	Lin, Lin	SG, SG	SG, Lin	ILU, ILU	ILU, Lin
$(0, 0)$	0.44	0.44	0.44	0.84	0.34
$(0, 1000)$	fails	0.44	0.78	0.69	0.56
$(0, 5000)$	fails	0.92	0.90	0.76	0.71
$(707, 707)$	fails	0.95	0.70	0.73	0.53
$(3536, 3536)$	fails	0.94	0.70	0.75	0.57
$(-707, 707)$	fails	0.93	0.76	0.80	0.61
$(-3536, 3536)$	fails	0.92	0.74	0.82	0.60

method. We record average rates of convergence after $k = \min\{100, \bar{k}\}$ iterations, where the residual is reduced by 10^{-2} in $\bar{k}$ steps. The average rate of convergence is given by $\gamma = (\|r_k\|/\|r_0\|)^{\frac{1}{k}}$, where r_i denotes the residual after i steps. We choose $x_0 = (0, 0, \cdots, 0)^T$ as starting vector for the purpose of standardization. All calculations were done in double precision on a Sparc10/41.

Figure 1 The initial mesh (left) and convergence histories for N=16641, ILU/ILU, $\beta = (-3536, 3536)^T$ (middle) and ILU/Lin, $\beta = (-3536, 3536)^T$ (right).

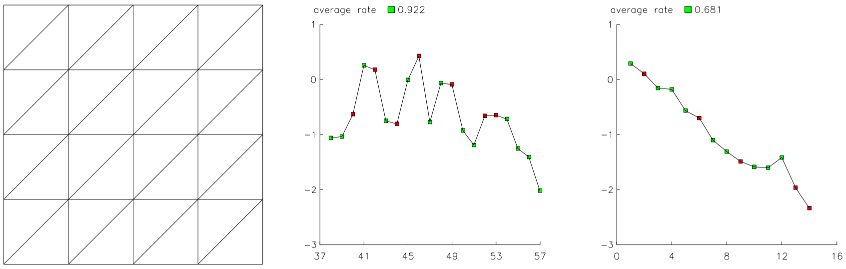
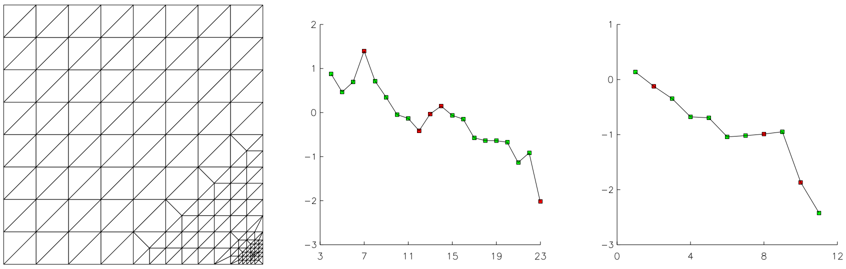


Figure 2 An unstructured mesh (left) and convergence histories for N=10000, ILU/ILU, $\beta = (-3536, 3536)^T$ (middle) and ILU/Lin, $\beta = (-3536, 3536)^T$ (right).



Typical convergence histories are shown in Figure 1 for a structured grid and in Figure 2 for an adaptively refined grid, where $\log \|r_i\|/\|r_0\|$ is plotted as a function of the iteration index i. Here we observe the non-monotonic behavior of the residual typical of the bicg method for strongly nonsymmetric problems. Iterations which failed did not reduce the initial residual in 100 iterations, but might have succeeded with more iterations. The coarse grid in these experiments is extremely coarse for such problems. As is standard with multilevel methods, one can overcome convergence failure and/or improve the rates of convergence by making the coarse grid finer. However, our intent here is only to illustrate that our alternative interpolation schemes improve the robustness of the HBMG iteration, often allowing rapid convergence even under very adverse conditions.

Acknowledgement

The work of the first author was supported by the U. S. Office of Naval Research under contract N00014-89J-1440. The second author was supported by a DAAD-fellowship HSPII from the German Federal Ministry for Education, Science, Research and Technology.

REFERENCES

[BB91] Bank R. E. and Benbourenane M. (1991) A Fourier analysis of the two level hierarchical basis multigrid method for convection diffusion equations. In *Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations (R. Glowinski, Y. Kuznetsov, G. Meurant, J. Périaux, and O. Widlund, eds.)*, pages 178–184. SIAM, Philadelphia.

[BDY88] Bank R. E., Dupont T. F., and Yserentant H. (1988) The hierarchical basis multigrid method. *Numer. Math.* 52: 427–458.

[BPX91] Bramble J., Pasciak J., and Xu J. (1991) The analysis of multigrid algorithms with non-imbedded spaces or non-inherited quadratic forms. *Math. Comp.* 56: 1–43.

[BX94] Bank R. E. and Xu J. (1994) The hierarchical basis multigrid method and incomplete LU decomposition. In *Seventh International Symposium on Domain Decomposition Methods for Partial Differential Equations (D. Keyes and J. Xu, eds.)*, pages 163–173. AMS, Providence, Rhode Island.

[BX96] Bank R. E. and Xu J. (1996) An algorithm for coarsening unstructured meshes. *Numerische Mathematik* 73: 1–36.

[CS94] Chan T. F. and Smith B. F. (1994) Domain decomposition and multigrid algorithms for elliptic problems on unstructured meshes. In *Proceedings of Seventh International Conference on Domain Decomposition. (ed. D. Keyes and J. Xu)*, pages 175–189. AMS, Providence, Rhode Island.

[GL81] George A. and Liu J. (1981) *Computer Solution of Large Sparse Positive Definite Systems.* Prentice Hall, Englewood Cliffs, NJ.

[Hac84] Hackbusch W. (1984) Multigrid convergence for a singular perturbation problem. *Lin. Alge. Appl.* 58: 125–145.

[Hac85] Hackbusch W. (1985) *Multigrid Methods and Applications.* Springer-Verlag, Berlin.

[HK94] Hoppe R. H. W. and Kornhuber R. (1994) Adaptive multilevel methods for obstacle problems. *SIAM J. Numer. Anal.* 31: 301–323.

[HS95] Hackbusch W. and Sauter S. A. (1995) A new finite element space for the approximation of pdes on domains with complicated microstructure. Technical report, Universität Kiel.

[Kor96] Kornhuber R. (1996) Monotone multigrid methods for variational inequalities I. *Numer. Math.* 72: 49–60.

[KY94] Kornhuber R. and Yserentant H. (1994) Multilevel methods for elliptic problems of domains not resolved by the coarse grid. In *Seventh International Symposium on Domain Decomposition Methods for Partial Differential Equations (D. Keyes and J. Xu, eds.)*, pages 49–60. AMS, Providence, Rhode Island.

[Ros72] Rose D. J. (1972) A graph theoretic study of the numeric solution of sparse positive definite systems. In *Graph Theory and Computing.* Academic Press, New York.

[Xu89] Xu J. (1989) *Theory of Multilevel Methods.* PhD thesis, Cornell University. Report AM-48, Penn State.

[Yse86] Yserentant H. (1986) On the multi-level splitting of finite element spaces. *Numer. Math.* 49: 379–412.

[Zha88] Zhang S. (1988) *Multilevel Iterative Techniques.* PhD thesis, Pennsylvania

State University, Department of Mathematics Report 88020.