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A Note on Domain Decomposition of Singularly Perturbed Elliptic Problems

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1 Introduction

Considerable progress can be observed in the design and analysis of parallelizable domain decomposition methods for singularly perturbed elliptic problems (see references). The purpose of this paper is to report on some results and problems with two domain decomposition methods for the advection–diffusion–reaction model.

Reasonable results are now available for *overlapping* Schwarz methods. In particular, the overlap can be minized in the singularly perturbed case using certain exponential decay of Dirichlet data in overlap regions. In Sec. 3 we consider a modified Schwarz method which is easy to parallelize in case of a simple geometry. We derive error estimates in the continuous case which can be extended to the discrete case.

Non–overlapping methods are better suited for parallel implementation. The problem consists in deriving appropriate interface conditions. We consider in Sec. 4 a method with an adaptive interface condition and discuss recent variants. Strong convergence is proven for the continuous method. We obtained a robust behaviour of both methods for two– and three–dimensional test problems using stabilized Galerkin finite element methods as the basic discretization.

2 Preliminaries

Consider the following Dirichlet problem in a bounded domain $\Omega \subset \mathbf{R}^d,\ d = 2, 3$ with Lipschitzian boundary $\partial\Omega$:

$$L_\epsilon u := -\epsilon\Delta u + (\boldsymbol{a} \cdot \nabla)u + cu = f \quad \text{in } \Omega; \qquad u = 0 \quad \text{on } \partial\Omega \tag{1}$$

Of particular interest are singularly perturbed problems with $\|\boldsymbol{a}\|_\infty \gg \epsilon$ (advection dominated case) or $\|c\|_\infty \gg \max\{\|\boldsymbol{a}\|_\infty; \epsilon\}$ (reaction dominated case). The latter case appears e.g. in an implicit time discretization method. We assume sufficiently smooth

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data of the problem satisfying

(H.1) $\quad c(x) \geq c_0 > 0,\ x \in \Omega; \qquad \nabla \cdot \boldsymbol{a} = 0.$

Remark 2.1. The condition $c(x) \geq c_0 > 0$ can be always guaranteed if all characteristic curves, the solutions of $dx(\tau)/d\tau = \boldsymbol{a}(x(\tau)), x(0) \in \overline{\Omega}$, leave $\overline{\Omega}$ in finite time. The incompressibility condition on $\boldsymbol{a}$ is not essential. ☐

The basic variational problem is: Find $u \in W \equiv H_0^1(\Omega)$, such that for all $v \in W$

$$B^G(u,v) := (\epsilon \nabla u, \nabla v)_\Omega + (\boldsymbol{a} \cdot \nabla u, v)_\Omega + (cu, v)_\Omega = L^G(v) := (f, v)_\Omega \tag{2}$$

Let $\mathcal{T} = \{K\}$ be an admissible triangulation and $W^h \subset W$ a finite element space of piecewise polynomials of degree $k \geq 1$. The (unusual) stabilized Galerkin method [FFLR96] for problem (1) is to find $U \in W^h$ such that

$$\begin{array}{rcl} B^{SG}(U,v) & = & L^{SG}(v) \quad \forall v \in W^h, \\ B^{SG}(u,v) & := & B^G(u,v) - \sum_K \sigma_K (L_\epsilon u, L_\epsilon^* v)_K, \\ L^{SG}(v) & := & L^G(v) - \sum_K \sigma_K (f, L_\epsilon^* v)_K, \end{array} \tag{3}$$

where L_{ϵ^*} denotes the adjoint operator to L_ϵ and σ_K are suitably chosen parameters. For a subdomain $D \subset \Omega$ set $H_{0,\Omega}^1(D) := W \cap H^1(D)$. $B_D^{SG}(\cdot,\cdot)$ and $L_D^{SG}(\cdot,\cdot)$ are the obvious restrictions of $B^{SG}(\cdot,\cdot)$ and $L^{SG}(\cdot,\cdot)$ to D if D consists of finite elements in $\mathcal{T}$. Then define

$$V^h(D) := H^1(D) \cap W^h; \quad V_0^h(D) := H_0^1(D) \cap V^h(D).$$

By ∂D^+, ∂D^- and ∂D^0 we denote the outflow, inflow and characteristic parts of ∂D where the scalar product $\boldsymbol{a} \cdot \boldsymbol{\nu}_D$ with the outer normal $\boldsymbol{\nu}_D$ is positive, negative or zero, respectively.

Furthermore we introduce a nonoverlapping admissible partition $\overline{\Omega} = \cup_i \overline{\Omega}_i$ with Lipschitz $\partial\Omega_i$ which aligns with the triangulation $\mathcal{T}$. Finally, define $\Gamma_i := \partial\Omega_i \setminus \partial\Omega$ and $\Gamma_{ik} = \Gamma_{ki} = \partial\Omega_i \cap \partial\Omega_k$.

3 An Overlapping Schwarz Method

Overlapping Schwarz methods for elliptic problems guarantee good convergence properties if the overlap is sufficiently large. On the other hand, they are not easy to implement. We propose a *modified* Schwarz method for singularly perturbed problems which is more appropriate for parallelization and allows minimal overlap [BS91]. A description in the 2D–case (with obvious modifications in 3D) is as follows: Starting from the non–overlapping partition $\overline{\Omega} = \bigcup_i \overline{\Omega}_i$, we introduce small *interface domains* $\mathcal{O}_{ik}$ covering the interface Γ_{ik} between adjacent subdomains with thickness $\Delta \approx kh$ and *crosspoint regions* $\mathcal{C}$ of diameter kh each covering a crosspoint. Starting from an initial guess $U^0 \in V_0^h(\Omega)$, the iteration method for problem (3) reads for $n \in \mathbf{N}$:

1. Solve in parallel on each subdomain Ω_i:

$$B_{\Omega_i}^{SG}(U_i^{n+1}, v) = L_{\Omega_i}^{SG}(v),\ \forall v \in V_0^h(\Omega_i); \qquad U_i^{n+1} - U_i^n \in V_0^h(\Omega_i).$$

2. Solve in parallel (redundantly) on each interface domain $\mathcal{O} = \mathcal{O}_{ik}$:
$$B^{SG}_{\mathcal{O}}(V^{n+1}_{\mathcal{O}}, v) = L^{SG}_{\mathcal{O}}(v),\ \forall v \in V^h_0(\mathcal{O}); \qquad V^{n+1}_{\mathcal{O}} - U^{n+1} \in V^h_0(\mathcal{O}).$$
Then set $U^{n+1} = V^{n+1}_{\mathcal{O}}$ on the interface Γ_{ik} generating $\mathcal{O}$.
3. Solve in parallel (redundantly) on each crosspoint region $\mathcal{C}$
$$B^{SG}_{\mathcal{C}}(W^{n+1}_{\mathcal{C}}, v) = L^{SG}_{\mathcal{C}}(v),\ \forall v \in V^h_0(\mathcal{C}); \qquad W^{n+1}_{\mathcal{C}} - U^{n+1} \in V^h_0(\mathcal{C}).$$
Then set $U^{n+1} := W^{n+1}_{\mathcal{C}}$ on the corresponding interfaces in the crosspoint region $\mathcal{C}$.
4. Set $n \mapsto n+1$ and goto step 1.

The Schwarz method is similarly defined for the continuous problem (1). We consider the convergence of the (*continuous*) Schwarz method, for simplicity, in the following model problems in $\Omega = (0,1)^2$ with a very simple flow field $\boldsymbol{a}$ and substructuring according to

$$\textbf{(H.1)}^* \qquad \begin{cases} A: & a_1(x) \geq A_1 \gg \epsilon > 0, \quad a_2(x) = 0, \qquad\qquad x \in \Omega \\ B: & a_i(x) \geq A_i \gg \epsilon > 0, \quad i = 1,2 \qquad\qquad x \in \Omega \\ C: & a_i(x) = 0, \quad i = 1,2; \quad c(x) \geq c_0 \gg \epsilon > 0 \quad x \in \Omega, \end{cases}$$

$$\textbf{(H.2)} \qquad \begin{cases} \Omega = (0,1)^2 \text{ is split into non–overlapping subdomains } \Omega_{ij} := \\ ((i-1)H_1, iH_1) \times ((j-1)H_2, jH_2),\ i = 1,...,M_1,\ j = \\ 1,...,M_2,\ H_k := \frac{1}{M_k}. \end{cases}$$

The crucial point in (H.1)*, (H.2) is a uniform behaviour of $\mathrm{sgn}(\boldsymbol{a}\cdot\boldsymbol{\nu})$ on the interface between adjacent subdomains. In case B there exist only inflow and outflow parts $\partial\Omega^-_{ij}$ and $\partial\Omega^+_{ij}$. In case A we additionally have characteristic parts $\partial\Omega^0_{ij}$. In case C we obtain the trivial case $\partial\Omega_{ij} = \partial\Omega^0_{ij}$. Assume that the interface regions are generated by narrow strips in $\overline{\Omega}_{ij}$ of thickness Δ^-_{ij}, Δ^+_{ij} or Δ^0_{ij} at the inflow outflow or characteristic part of $\partial\Omega_{ij}$. The intersection of the interface strips generates crosspoint regions $\mathcal{C}$.

Theorem 3.1. *Assume* (H.1)*, (H.2) *and set* $TOL > 0$, $K := \|u - u^0\|_{L^\infty(\Omega)}$. *Furthermore assume that the minimal overlap width of the interface strips satisfies*

$$\Delta^+_{ij}, \Delta^-_{ij} \geq \alpha^{-1}\varepsilon \left|\ln \frac{TOL}{4K}\right|,\ \ \Delta^0_{ij} \geq \beta^{-1}\sqrt{\varepsilon}\left|\ln\frac{TOL}{4K}\right|. \tag{4}$$

Then we obtain after $M+k$ steps of the modified Schwarz method that

$$\|u - u^{M+k}\|_{L^\infty(\Omega)} \leq C(TOL)^{k+1}, \qquad C \sim M, \tag{5}$$

with $M = M_1$, $M = \sum_{i=1}^2 M_i$ and $M = 0$ in case A, B and C, respectively, and appropriate ϵ–independent constants $\alpha, \beta > 0$. □

Outline of the proof: The key of the proof is some exponential decay of presumably wrong Dirichlet data (appearing during the iteration) in overlapping regions leading to artificial layers. The proof is based on the barrier function technique using the following variant of the maximum principle on an arbitrary subdomain $G \subset \Omega$ with Lipschitzian and piecewise C^2–boundary: Suppose that for $T, S \in C^2(G) \cap C(\overline{G})$ holds

$$|(L_\varepsilon T)(x)| \leq (L_\varepsilon S)(x),\ x \in G,\ |T(x)| \leq S(x),\ x \in \partial G.$$

Then we obtain $|T(x)| \leq S(x)$ in $\overline{G}$.

Consider now in particular case A: The manifolds ∂G^-, ∂G^+ and ∂G^0 for $G = (a,b) \times (c,d) \subset \Omega = (0,1)^2$ are located at $\{a\} \times (c,d)$, $\{b\} \times (c,d)$ and $[a,b] \times \{c\} \cup [a,b] \times \{d\}$. Let $T_1, T_2 \in C^2(G) \cap C(\overline{G})$ be solutions of (1). Then $T = T_1 - T_2$ satisfies

$$\begin{aligned}|T(x)| \quad &\leq \quad \|T\|_{L^\infty(\partial G^-)} + \exp\left[\frac{-\alpha}{\varepsilon}\operatorname{dist}(x, \partial G^+)\right] \|T\|_{L^\infty(\partial G^+)} \\ &\quad + (1 + x_1 - a)\exp\left[\frac{-\beta}{\sqrt{\varepsilon}}\operatorname{dist}(x, \partial G^0)\right] \|T\|_{L^\infty(\partial G^0)}.\end{aligned}$$

with $0 < \alpha \neq \alpha(\varepsilon)$, $0 < \beta \neq \beta(\varepsilon)$. The exponential terms of the barrier function mimic artificial layers of width $0(\epsilon \log 1/\epsilon)$ or $0(\sqrt{\epsilon} \log 1/\epsilon)$ at ∂G^+ and ∂G^0, respectively. They are small in subdomains of G with appropriate distance to ∂G^+ and ∂G^0. A corresponding result holds in case B and C.

The idea in case A and B is a *downwind correction* of the solution from subdomain to subdomain. The chosen thickness of the interface and crosspoint regions guarantees the *exponential decay* of the influence of wrong Dirichlet data in upwind and crosswind directions. The first iteration cycle 1–4 yields in particular in case A an error of $0(TOL)$ in subdomains $\Omega_{1j}, j = 1, ..., M_2$. The next iteration gives an error of $0(TOL)^2$ there and of $0(TOL)$ in subdomains $\Omega_{2j}, j = 1, ..., M_2$. The desired result follows by induction. The idea is the same in case B. In case C we have an *isotropic propagation* of information. After the first iteration cycle the error is of order TOL in all subdomains Ω_{ij}. The result follows then again by induction. □

Remark 3.1. It is possible to extend the result of Theorem 3.1 to 3D and to more general domains and macro partitions, in particular of singularly perturbed diffusion–reaction problems (cf. case C). The case of nonsymmetric singularly perturbed problems is more involved due to the possibly complicated behaviour of the characteristics at the interface $\Sigma := \cup \Gamma_{ik}$.

Furthermore, a result corresponding to Theorem 3.1 can be derived for the energy norm $|||\cdot||| := \sqrt{B^{SG}(\cdot,\cdot)}$. The proof of similar results for the *discrete* Schwarz method depends strictly on the discretization method. Some ideas and technical details for the streamline upwind method which is closely related to (3) can be found in [RZ95], [Zho95]. In particular, a discrete maximum principle is not available. □

4 An Adaptive Non–overlapping Method

Consider again a non–overlapping partition $\overline{\Omega} = \bigcup_i \overline{\Omega}_i$. The results of Sec. 3 indicate that a transition to a non–overlapping method should be possible with appropriate interface conditions at $\Sigma := \cup_{i,k} \Gamma_{ik}$. A first insight is given with the *fictitious overlapping method* [LeT94]. Consider in the (continuous) overlapping method of Sec. 3 with overlap width Δ_{ik} at Γ_{ik} a first order Taylor expansion of the solution at Γ_{ik}. This leads to the non–overlapping Schwarz Method proposed by P. L. Lions [Lio89]. Set $\rho_{ik} = \frac{\epsilon}{\Delta_{ik}}$. Starting from an initial guess u^0, the iterative procedure reads: Solve (in parallel) on Ω_i

$$L_\epsilon u_i^{n+1} = f \quad \text{in } \Omega_i; \qquad u_i^{n+1} = 0 \quad \text{on } \partial\Omega_i \cap \partial\Omega, \tag{6}$$

with interface condition

$$\epsilon \frac{\partial u_i^{n+1}}{\partial \nu_i} + \rho_{ik} u_i^{n+1} = -\epsilon \frac{\partial u_k^n}{\partial \nu_k} + \rho_{ik} u_k^n \text{ on } \Gamma_{ik}. \tag{7}$$

A first convergence result was given by P.L. Lions [Lio89] for the fictitious overlapping method. Assume (H.1) and that no crosspoints occur (strip partitions of Ω):

$$u^n \to u \text{ strongly in } L_2(G),\ G \subset\subset \Omega_i; \quad u^n \to u \text{ weakly in } L_2(\partial\Omega_i) \tag{8}$$

Remark 4.1. A modified approach leading to (6),(7) is the *three–field formulation* of [BBM92] which consists of finding $u = \{u_i\}$ with $u_i \in H^1_{0,\Omega}(\Omega_i)$ and Lagrange multipliers for the Neumann and Dirichlet data on the interface. After using an augmented Lagrangian technique [LeT94] this formulation can be decoupled iteratively ending up with the method (6),(7). □

The result of [Lio89] gives no indication for the *design* of ρ_{ik} in the interface condition (7). Generalizing an idea of [Nat96] we propose the following modification of (7)

$$\rho_{ik} := \rho_i^- := -\frac{1}{2}(\boldsymbol{a} \cdot \boldsymbol{\nu}_i - Z_i|_{\Gamma_{ik}}) = \rho_k^+ := \tfrac{1}{2}(\boldsymbol{a} \cdot \boldsymbol{\nu}_k + Z_k|_{\Gamma_{ik}}) \tag{9}$$

with Z_i a strictly positive function on Γ_i. We analyze the *convergence* of the adaptive method (6),(7), (9) applied to (1).

Theorem 4.1 *Assume* (H.1) *(or* $c - \frac{1}{2}\nabla \cdot \boldsymbol{a} \geq 0$ *and* $\partial\Omega_i \cap \partial\Omega \neq \emptyset\ \forall i$*) and* $\frac{\partial u}{\partial \boldsymbol{\nu}_i}|_{\Gamma_{ik}} \in L^2(\Gamma_{ik})$ *for the solution of (1). Then the sequence* $u^n = \{u_i^n\}$ *generated by algorithm (6),(7),(9) converges for* $n \to \infty$ *with* $u_i^n \to u$ strongly in $H^1(\Omega_i)$.

Outline of the proof: First of all we observe that algorithm (6), (7), (9) is well defined provided that $\frac{\partial u_i^0}{\partial \boldsymbol{\nu}_i} \in L^2(\Gamma_i)$. This implies $\frac{\partial u_i^n}{\partial \boldsymbol{\nu}_i} \in L^2(\Gamma_i)$ for all $n \in \mathbf{N_0}$.
The key step is a modification of Lemma 4.4 in [Nat96]: A function $u \in H^1_{0,\Omega}(\Omega_i)$ with $L_\epsilon u = 0$ in $L^2(\Omega_i)$ and $\frac{\partial u}{\partial \boldsymbol{\nu}_i} \in L^2(\Gamma_i)$ satisfies

$$|||u|||_i^2 + \int_{\Gamma_i} \frac{1}{2Z_i} \left[\left(\epsilon \frac{\partial}{\partial \boldsymbol{\nu}_i} - \rho_i^+ \right) u \right]^2 ds = \int_{\Gamma_i} \frac{1}{2Z_i} \left[\left(\epsilon \frac{\partial}{\partial \boldsymbol{\nu}_i} + \rho_i^- \right) u \right]^2 ds \tag{10}$$

with the outer unit normal ν_i on Γ_i and

$$|||u|||_i^2 := \epsilon \|\nabla u\|^2_{L^2(\Omega_i)} + \|\sqrt{\mu} u\|^2_{L^2(\Omega_i)},\ \mu := c - \frac{1}{2}\nabla \cdot \boldsymbol{a},\ Z_i \geq z_0 > 0 \text{ a.e. on } \Gamma_i.$$

The error $e^{n+1} := u_i - u_i^{n+1}$ satisfies $L_\epsilon e^{n+1} = 0$ in Ω_i, hence

$$|||e_i^{n+1}|||^2_{\Omega_i} + \int_{\Gamma_i} \frac{1}{2Z_i} \left[\left(\epsilon \frac{\partial}{\partial \boldsymbol{\nu}_i} - \rho_i^+ \right) e_i^{n+1} \right]^2 ds = \int_{\Gamma_i} \frac{1}{2Z_i} \left[\left(\epsilon \frac{\partial}{\partial \boldsymbol{\nu}_i} + \rho_i^- \right) e_i^{n+1} \right]^2 ds$$

Using (7),(9), we obtain

$$|||e_i^{n+1}|||^2_{\Omega_i} + \sum_{k(\neq i)} |[e_i^{n+1}]|^2_{\Gamma_{ik}} = \sum_{k(\neq i)} |[e_k^n]|^2_{\Gamma_{ik}}$$

where

$$|[e_i^n]|^2_{\Gamma_{ik}} := \int_{\Gamma_{ik}} \frac{1}{2Z_i} \left[\left(\epsilon \frac{\partial}{\partial \boldsymbol{\nu}_i} - \rho_i^+ \right) e_i^n \right]^2 \; ds.$$

Summing over the subdomains yields

$$\sum_{i=1}^{N} |||e_i^{n+1}|||^2_{\Omega_i} + |[e^{n+1}]|^2_\Sigma = |[e^n]|^2_\Sigma, \qquad |[e^n]|^2_\Sigma := \sum_{i=1}^{N} \sum_{k(\neq i)} |[e_i^n]|^2_{\Gamma_{ik}}$$

where $\Sigma := \cup_{i=1}^N \Gamma_i$. Summation over n implies strong H^1-convergence of $\{e^n\}$ to zero due to the equivalence of norms. □

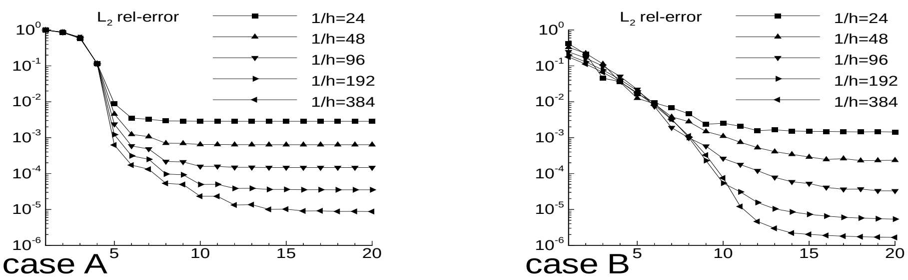
Remark 4.2. Possible choices of Z_i: (i) The choice $Z_i = \sqrt{(\boldsymbol{a} \cdot \boldsymbol{\nu}_i)^2 + 4\epsilon c}$ was derived in [Nat96] from an zeroth order approximate factorization of the elliptic operator L_ϵ. But this choice failed in case $c = 0$ if the flow field is parallel to some Γ_{ik}. In the original proof ([Nat96], Th. 4.1) Robin-type boundary conditions on $\partial\Omega$ were assumed.
(ii) We suggest $Z_i = \sqrt{(\boldsymbol{a} \cdot \boldsymbol{\nu}_i)^2 + \lambda\epsilon}$ with some arbitrary positive parameter λ. Then even if $c = 0$ no restriction to the flow field is necessary. Thus the Theorem is also applicable to the Poisson problem and improves the result of [Lio89].
(iii) With $Z_i = | \boldsymbol{a} \cdot \boldsymbol{\nu}_i |$ one obtains the so-called *adaptive Robin-Neumann algorithm* [CQ95], [Tro96], [GGQ]. Then Theorem 4.1 is applicable only if $| \boldsymbol{a} \cdot \boldsymbol{\nu}_i | > 0$. In case of two subdomains [GGQ] prove weak H^1-convergence under the more general assumption that $| \boldsymbol{a} \cdot \boldsymbol{\nu}_i |$ can vanish in a finite number of points. □

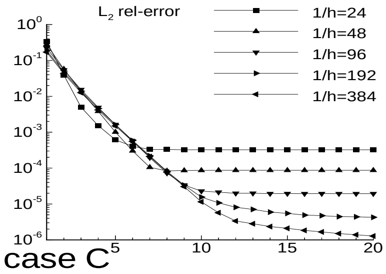
The result of Theorem 4.1 means that the corresponding Poincare–Steklov operator is strictly non–expansive and gives no information on the convergence rate.
On the other hand, with our choice for Z_i very reasonable results are obtained for a *discrete version* of the adaptive non–overlapping method: Instead of (6),(7),(9) one has to solve for $n \in \mathbf{N}_0$ (in parallel for $i = 1, ..., I$)

$$B^{SG}_{\Omega_i}(U_i^{n+1}, v) + \sum_{k(\neq i)} \left(\rho_i^+ U_i^{n+1} - \Lambda^n_{ik}, v \right)_{\Gamma_{ik}} = L^{SG}_{\Omega_i}(v), \quad \forall v \in V^h_{\Omega_i}, \tag{11}$$

$$\Lambda^{n+1}_{ik} := \left(\rho_i^+ + \rho_i^- \right) U_k^{n+1} - \Lambda^n_{ki} = Z_{ik} U_k^{n+1} - \Lambda^n_{ki}. \tag{12}$$

Here we present some *numerical results* for 2D–problems satisfying (H.1)*, (H.2).
Examples: We choose $\epsilon = 10^{-4}$, a $M_1 \times M_2$-partition of Ω and a sequence of different values of h. The (continuous) solution is always $w = \sin \pi x_1 \left(\exp\left(x_2^2 - x_2\right) - 1 \right)$, hence $f = L_\epsilon w$. More precisely, we consider the advection dominated case with (B) and without (A) characteristic interfaces and the reaction dominated case (C) (cf. also Th. 3.1 for the weakly overlapping method).





Here we show the convergence history of the relative discrete L^2–norm vs. the iteration number for the test cases A, B and C for different values of h. For the parameter λ in our formula for Z_i we have chosen 100, 100 and 10 for case A,B and C respectively.

Case A: $\boldsymbol{a} = \frac{1}{\sqrt{5}}(2,1)^T$, $c = 0$ and $M_1 = M_2 = 3$
Case B: $\boldsymbol{a} = (0,1)^T$, $c = 0$ and $M_1 = 4,\ M_2 = 1$
Case C: $\boldsymbol{a} = (0,0)^T$, $c = 1$ and $M_1 = M_2 = 3$

The convergence history is similarly as predicted by Theorem 3.1 for the weakly overlapping Schwarz method: In case A, we observe an initial phase of downwind propagation of information, then we obtain reasonable linear convergence until the discretization error level is reached. In cases B and C, no initial advective propagation of information appears. Reasonable linear convergence is again obtained. The algorithm is only slightly sensitive w. r. t. λ.
A heuristic explanation of the favourable convergence properties of the discrete algorithm can be given by means of singular perturbation arguments. In cases A and C, wrong interface data cause artificial layers of exponential (and essentially 1D-) type for the flux $\epsilon\nabla u\cdot\nu$. The analysis is more involved in case B. Wrong interface data cause artificial layers which are of parabolic type for the flux. A higher order factorization of the operator L_ϵ would better represent the advective transport along the interface [Nat96]. Fortunately, the diffusive interface transport involved in the transmission condition (7), (9) is obviously sufficient to guarantee convergence. Nevertheless, further theoretical foundation of the convergence properties of the discrete algorithm (6), (7) is neccesary.

5 Summary and Open Problems

Two parallelizable methods are considered for singularly perturbed elliptic problems. We obtain linear convergence for the overlapping method in the continuous case. The overlap width can be minimized for advection and/ or reaction dominated problems. The non–overlapping method with properly problem adapted interface condition of Robin type gives strong H^1–convergence in the continuous case. The method applied to the discrete problem provides reasonable performance in the range from diffusion to advection (or reaction) dominated problems. Nevertheless, there are still open problems concerning the convergence analysis even for scalar elliptic problems. The application to incompressible flow problems is considered in a forthcoming paper.

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