25

Preconditioning Discrete Approximations of the Reissner–Mindlin Plate Model

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1 Introduction

The purpose of this paper is to summarize the work of [AFW97]. We consider iterative methods for the solution of indefinite linear systems of equations arising from discretizations of the Reissner–Mindlin plate model.

Like the biharmonic plate model, the Reissner–Mindlin model is a two-dimensional plate model which approximates the behavior of a thin linearly elastic three-dimensional body using unknowns and equations defined only on the middle surface, Ω, of the plate. The basic variables of the model are the transverse displacement ω and the rotation vector $\boldsymbol{\phi}$ which solve the system of partial differential equations

$$\begin{aligned} -\operatorname{\mathbf{div}} \mathcal{C}\mathcal{E}\boldsymbol{\phi} + \lambda t^{-2}(\boldsymbol{\phi} - \operatorname{\mathbf{grad}} \omega) &= 0, \\ \lambda t^{-2}(-\Delta\omega + \operatorname{div} \boldsymbol{\phi}) &= g, \end{aligned} \tag{1}$$

on Ω together with suitable boundary conditions. For the hard clamped plate, which we consider throughout this paper, these are $\omega = 0$, $\boldsymbol{\phi} = 0$. In (1), g is the scaled transverse loading function, t is the plate thickness, $\mathcal{E}\boldsymbol{\phi}$ is the symmetric part of the gradient of $\boldsymbol{\phi}$, and the scalar constant λ and constant tensor $\mathcal{C}$ depend on the material properties of the body.

A variational formulation of this system states that the solution $(\boldsymbol{\phi}, \omega)$ minimizes the total energy of the plate, which is given by

$$E(\boldsymbol{\phi}, \omega) = \frac{1}{2}\int_\Omega \{(\mathcal{C}\mathcal{E}\boldsymbol{\phi}) : (\mathcal{E}\boldsymbol{\phi}) + \lambda t^{-2}|\boldsymbol{\phi} - \operatorname{\mathbf{grad}} \omega|^2\}dx - \int_\Omega g\omega dx \tag{2}$$

over $\boldsymbol{H}_0^1(\Omega) \times H_0^1(\Omega) = \boldsymbol{H}_0^1 \times H_0^1$. Here $H^1 \subset L^2 = L^2(\Omega)$ denotes the Sobolev space of functions with first derivatives in L^2, while H_0^1 is the subspace of functions which vanish on the boundary. Boldface symbols are used to denote 2–vector valued functions and function spaces.

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An advantage of the Reissner–Mindlin model over the biharmonic plate model is that the energy involves only first derivatives of the unknowns and so conforming finite element approximations require the use of merely continuous finite element spaces rather than the C^1 spaces required for the biharmonic model. However, for many choices of finite element spaces, severe difficulties arise due to the presence of the small parameter t. If the finite element subspaces are not properly related, the phenomenon of "locking" occurs, causing a deterioration in the approximation as the plate thickness t approaches zero. A key step in understanding and overcoming locking is passage to a mixed formulation of the Reissner–Mindlin model. The mixed formulation may be derived from the alternative system of differential equations

$$\begin{aligned} -\operatorname{\mathbf{div}} \mathcal{C}\mathcal{E}\boldsymbol{\phi} - \boldsymbol{\zeta} &= 0, \\ -\operatorname{div} \boldsymbol{\zeta} &= g, \\ -\boldsymbol{\phi} + \operatorname{\mathbf{grad}} \omega - \lambda^{-1} t^2 \boldsymbol{\zeta} &= 0, \end{aligned} \tag{3}$$

arising from (1) through the introduction of the shear stress $\boldsymbol{\zeta} = -\lambda t^{-2}(\boldsymbol{\phi} - \operatorname{\mathbf{grad}} \omega)$. This mixed system also makes sense for $t = 0$. In this case the system corresponds to a constrained minimization problem.

2 Mapping Properties

The system (3) can be written in the form

$$\mathcal{A}_t \begin{pmatrix} \boldsymbol{\phi} \\ \omega \\ \boldsymbol{\zeta} \end{pmatrix} = \begin{pmatrix} 0 \\ g \\ 0 \end{pmatrix}, \tag{4}$$

where the coefficient operator, $\mathcal{A}_t$, is given by

$$\mathcal{A}_t = \begin{pmatrix} -\operatorname{\mathbf{div}} \mathcal{C}\mathcal{E} & 0 & -\boldsymbol{I} \\ 0 & 0 & -\operatorname{div} \\ -\boldsymbol{I} & \operatorname{\mathbf{grad}} & -\lambda^{-1} t^2 \boldsymbol{I} \end{pmatrix}.$$

The mapping properties of the coefficient operator of the continuous system are key to the design of preconditioners for discrete approximations of the system. The indefinite operator $\mathcal{A}_t$ is L^2–symmetric, and, for any $t > 0$ is an isomorphism from $\boldsymbol{H}_0^1 \times H_0^1 \times \boldsymbol{L}^2$ to the L^2–dual $\boldsymbol{H}^{-1} \times H^{-1} \times \boldsymbol{L}^2$. However, in order to obtain bounds on the operator norms which are independent of the thickness t, we are forced to introduce t–dependent norms. Let

$$\boldsymbol{H}_0(\mathrm{rot}) = \{\boldsymbol{\eta} \in \boldsymbol{L}^2 : \mathrm{rot}\, \boldsymbol{\eta} \in \boldsymbol{L}^2, \boldsymbol{\eta} \cdot \boldsymbol{s} = 0 \quad \text{on } \partial\Omega\},$$

with the natural norm. Here $\boldsymbol{s}$ is the unit tangent to $\partial\Omega$ and $\mathrm{rot}\, \boldsymbol{\eta} = \partial \boldsymbol{\eta}_1/\partial y - \partial \boldsymbol{\eta}_2/\partial x$.

It can be shown that the dual space of $\boldsymbol{H}_0(\mathrm{rot})$ with respect to the L^2–inner product is given by

$$\boldsymbol{H}^{-1}(\mathrm{div}) = \{\boldsymbol{\eta} \in \boldsymbol{H}^{-1} : \mathrm{div}\, \boldsymbol{\eta} \in H^{-1}\}.$$

Using sums and intersections of Hilbert spaces (cf. [BL76]) we now define

$$X_t = \boldsymbol{H}_0^1 \times H_0^1 \times [\boldsymbol{H}^{-1}(\mathrm{div}) \cap t \cdot \boldsymbol{L}^2]$$

and its L^2–dual

$$X_t^* = \boldsymbol{H}^{-1} \times H^{-1} \times [\boldsymbol{H}_0(\mathrm{rot}) + t^{-1} \cdot \boldsymbol{L}^2].$$

In particular,

$$X_0 = \boldsymbol{H}_0^1 \times H_0^1 \times \boldsymbol{H}^{-1}(\mathrm{div}) \qquad \text{and} \quad X_0^* = \boldsymbol{H}^{-1} \times H^{-1} \times \boldsymbol{H}_0(\mathrm{rot}).$$

Using these spaces, it is then possible to establish the following result.

Theorem 25.1 *The operator $\mathcal{A}_t$ is an isomorphism from X_t to X_t^*. Furthermore, the associated operator norms $||\mathcal{A}_t||_{\mathcal{L}(X_t,X_t^*)}$ and $||\mathcal{A}_t^{-1}||_{\mathcal{L}(X_t^*,X_t)}$ are independent of t.*

3 Preconditioning

Before turning to the description of discretizations schemes, we will discuss preconditioning for the continuous system (4). Our aim is to replace the system (4) by an equivalent system of the form

$$\mathcal{B}_t\mathcal{A}_t \begin{pmatrix} \boldsymbol{\phi} \\ \omega \\ \boldsymbol{\zeta} \end{pmatrix} = \mathcal{B}_t \begin{pmatrix} 0 \\ g \\ 0 \end{pmatrix}, \tag{5}$$

which is more easily solved by iterative methods. The operator $\mathcal{B}_t$ will be symmetric positive definite and hence the indefinite operator $\mathcal{B}_t\mathcal{A}_t$ will be symmetric with respect to the inner product $(\mathcal{B}_t^{-1}\cdot\,,\,\cdot)$ on X_t.

Let $\boldsymbol{D}_t$ denote the operator

$$\boldsymbol{D}_t = \boldsymbol{I} + (1-t^2)\,\mathbf{curl}(I - t^2\Delta)^{-1}\,\mathrm{rot},$$

where

$$\mathbf{curl} = \begin{pmatrix} -\partial/\partial y \\ \partial/\partial x \end{pmatrix}.$$

When $t = 0$, this is a differential operator which is an isomorphism from $\boldsymbol{H}_0(\mathrm{rot})$ into $\boldsymbol{H}^{-1}(\mathrm{div})$. In general, it can be shown that that $\boldsymbol{D}_t$ is an isomorphism from $\boldsymbol{H}_0(\mathrm{rot}) + t^{-1} \cdot \boldsymbol{L}^2$ to $\boldsymbol{H}^{-1}(\mathrm{div}) \cap t \cdot \boldsymbol{L}^2$, with the operator norms of $\boldsymbol{D}_t$ and $\boldsymbol{D}_t^{-1}$ independent of t. An immediate consequence of this is that the block diagonal operator

$$\mathcal{B}_t = \begin{pmatrix} -\boldsymbol{\Delta}^{-1} & 0 & 0 \\ 0 & -\Delta^{-1} & 0 \\ 0 & 0 & \boldsymbol{D}_t \end{pmatrix}$$

is an isomorphism mapping X_t^* to X_t with the norms $||\mathcal{B}_t||_{\mathcal{L}(X_t^*,X_t)}$ and $||\mathcal{B}_t^{-1}||_{\mathcal{L}(X_t,X_t^*)}$ independent of t. From Theorem 25.1, we conclude that the block diagonal positive definite operator $\mathcal{B}_t$ has the same mapping property as $\mathcal{A}_t^{-1}$. Hence, the composition $\mathcal{B}_t\mathcal{A}_t$,

$$X_t \xrightarrow{\mathcal{A}_t} X_t^* \xrightarrow{\mathcal{B}_t} X_t,$$

is an isomorphism from X_t to X_t with operator norms

$$||\mathcal{B}_t\mathcal{A}_t||_{\mathcal{L}(X_t,X_t)} \quad \text{and} \quad ||(\mathcal{B}_t\mathcal{A}_t)^{-1}||_{\mathcal{L}(X_t,X_t)}$$

independent of t. Therefore, $\mathcal{B}_t\mathcal{A}_t$ is a bounded operator on X_t with bounded inverse, and as a consequence, the spectral condition number

$$\kappa(\mathcal{B}_t\mathcal{A}_t) = \frac{\sup|\sigma(\mathcal{B}_t\mathcal{A}_t)|}{\inf|\sigma(\mathcal{B}_t\mathcal{A}_t)|}$$

is finite and independent of t.

A preconditioned differential system of the form (5) can be solved by a Krylov space method like MINRES or CGN (conjugate gradients applied to the normal equations). These methods are well defined as long as the coefficient operator $\mathcal{B}_t\mathcal{A}_t$ maps X_t into itself, and convergence in the norm of X_t is guaranteed as long as the spectral condition number of $\mathcal{B}_t\mathcal{A}_t$ is finite. Therefore, we obtain the following result.

Theorem 25.2 *Assume that MINRES or CGN is applied to the preconditioned system* (5)*. Then the sequence of approximations converges to the solution in X_t, with a convergence rate independent of t.*

4 Stable Discretizations

The continuous theory presented above may serve as a guideline for the problem of real interest, i.e., how to construct effective preconditioners for the discrete systems arising from finite element approximations of the differential system (3). Here, we shall just give a brief outline of the discrete theory. For full details and proofs we refer to the original paper [AFW97].

A finite element approximation of the system (3) (or (4)) will typically give rise to a discrete system of the form

$$\mathcal{A}_{t,h}\begin{pmatrix}\phi_h\\ \omega_h\\ \zeta_h\end{pmatrix} = \begin{pmatrix}0\\ g_h\\ 0\end{pmatrix},$$

where $\mathcal{A}_{t,h}$ is an indefinite, L^2–symmetric operator mapping a finite dimensional space $X_h = \boldsymbol{V}_h \times W_h \times \boldsymbol{\Gamma}_h$ into itself. Here $h > 0$ is a discretization parameter. Examples of stable and locking free finite element discretizations have been proposed by Arnold and Falk [AF89], Brezzi, Fortin and Stenberg [BFS91], Duran and Liberman [DL92], and others. The main purpose of this work is to construct preconditioners $\mathcal{B}_{t,h}$ for these systems such that the spectral condition number of $\mathcal{B}_{t,h}\mathcal{A}_{t,h}$ is independent of the thickness t and the discretization parameter h. The construction of $\boldsymbol{D}_{t,h}$ is analogous to that of $\boldsymbol{D}_t$ in the continuous case.

For the locking free methods, it is possible to establish a discrete version of Theorem 25.1 above, i.e., to prove that the operator norms

$$||\mathcal{A}_{t,h}||_{\mathcal{L}(X_{t,h},X^*_{t,h})} \quad \text{and} \quad ||\mathcal{A}^{-1}_{t,h}||_{\mathcal{L}(X^*_{t,h},X_{t,h})}$$

are bounded uniformly in t and h. Here the spaces $X_{t,h}$ and $X^*_{t,h}$ coincide with X_h as a set, but are endowed with norms which resemble the norms in X_t and X^*_t. As in the continuous case, this property of $\mathcal{A}_{t,h}$ suggests a symmetric, positive definite and block diagonal preconditioner such that

$$||\mathcal{B}_t||_{\mathcal{L}(X^*_{t,h},X_{t,h})} \quad \text{and} \quad ||\mathcal{B}^{-1}_t||_{\mathcal{L}(X_{t,h},X^*_{t,h})}$$

are independent of t and h. As a consequence, the spectral condition number of $\mathcal{B}_{t,h}\mathcal{A}_{t,h}$ is independent of t and h.

The preconditioner $\mathcal{B}_{t,h}$ will be of the form

$$\mathcal{B}_t = \begin{pmatrix} \boldsymbol{L}_h & 0 & 0 \\ 0 & M_h & 0 \\ 0 & 0 & \boldsymbol{N}_{t,h} \end{pmatrix},$$

where $\boldsymbol{L}_h$ and M_h are chosen spectrally equivalent to approximations of the inverse of the negative Laplace operator on $\boldsymbol{V}_h$ and W_h, respectively, while $\boldsymbol{N}_{t,h}$ is a discrete analog of $\boldsymbol{D}_t$.

5 Numerical Examples

In the examples presented below, the domain Ω is taken to be the unit square. A triangulation of Ω is obtained by first dividing Ω into squares of size $h \times h$, and then dividing each square into two triangles using the positively sloped diagonal. All the computations are done with the method of Arnold and Falk [AF89]. Hence, the space $\boldsymbol{V}_h$ consists of continuous piecewise linear functions plus cubic bubbles on each triangle, W_h is the nonconforming piecewise linear space, with continuity requirements only at the midpoint of each edge, and $\boldsymbol{\Gamma}_h$ is the space of piecewise constants with respect to the triangulation.

The operators $\boldsymbol{L}_h$ and M_h are essentially constructed from a standard V–cycle multigrid operator with a Gauss–Seidel smoother. These operators are fixed throughout all the experiments. For the method considered here, the proper discrete analog of the operator $\boldsymbol{D}_t$ is of the form

$$\boldsymbol{I} + (1-t^2)\,\mathbf{curl}_h(I - t^2\Delta_h)^{-1}\,\mathrm{rot}_h$$

mapping $\boldsymbol{\Gamma}_h$ into itself. If $Q_h \subset H^1$ is the space of continuous piecewise linear functions with respect to the triangulation, then $\mathbf{curl}_h : Q_h \mapsto \boldsymbol{\Gamma}_h$ is defined by restricting the ordinary **curl**–operator to Q_h. Furthermore, $\mathrm{rot}_h : \boldsymbol{\Gamma}_h \mapsto Q_h$ is the adjoint operator, while $\Delta_h : Q_h \mapsto Q_h$ is the standard finite element approximation of the Laplace operator generated by the space Q_h.

By replacing $(I - t^2\Delta_h)^{-1}$ by a spectrally equivalent (with respect to t and h) preconditioner $\Phi_{t,h}$, again derived from a V–cycle multigrid algorithm, we obtain a computational feasible operator

$$\boldsymbol{D}_{t,h} = \boldsymbol{I} + (1-t^2)\,\mathbf{curl}_h\,\Phi_{t,h}\,\mathrm{rot}_h\,.$$

We observe that the operator $\boldsymbol{D}_{t,h}$ simplifies when $t = 0$, by taking $\Phi_{0,h} = I$, and for $t = 1$, since $\boldsymbol{D}_{1,h} = \boldsymbol{I}$.

In the examples below, the preconditioned system is solved either by MINRES or CGN. The work estimate for one iteration of CGN corresponds roughly to two MINRES iterations. We therefore compare the number of iterations for MINRES (N_{MR}) with twice the number of iterations for CGN (N_{CGN}). The condition number $\kappa(\mathcal{B}_{t,h}\mathcal{A}_{t,h})$, which is estimated from the conjugate gradient iteration using a standard

Matlab routine, will also be given. The iterations are terminated when the error, measured in the norm associated with the inner product $(\mathcal{B}_{t,h}^{-1}\cdot,\cdot)$, is reduced by a factor of at least $5\cdot 10^4$.

The two extreme cases $t=0$ and $t=1$ are considered in Tables 1 and 2.

Table 1 $t=0$, $\boldsymbol{N}_{t,h}=\boldsymbol{D}_{t,h}=\boldsymbol{D}_{0,h}$

h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
N_{MR}	41	41	35	29	24
N_{CGN}	48	50	48	40	34
$\kappa(\mathcal{B}_{t,h}\mathcal{A}_{t,h})$	8.17	10.7	11.1	10.6	9.62

Table 2 $t=1$, $\boldsymbol{N}_{t,h}=\boldsymbol{D}_{t,h}=\boldsymbol{I}$

h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
N_{MR}	22	22	20	20	20
N_{CGN}	102	104	106	104	102
$\kappa(\mathcal{B}_{t,h}\mathcal{A}_{t,h})$	17.5	18.4	19.0	19.0	18.9

Table 3 $\kappa(\mathcal{B}_{t,h}\mathcal{A}_{t,h})$ for $t=0.01$

h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
$\boldsymbol{N}_{t,h}=\boldsymbol{D}_{0,h}$	8.15	10.7	11.4	32.9	113
$\boldsymbol{N}_{t,h}=\boldsymbol{D}_{t,h}$	8.15	10.7	11.2	11.1	9.68

The results clearly seem to confirm the boundedness of $\kappa(\mathcal{B}_{t,h}\mathcal{A}_{t,h})$ with respect to h for these values of t. Observe also the substantial difference in the behavior of MINRES and CGN in the case $t=1$.

Our theory predicts that if $\boldsymbol{N}_{t,h}$ is chosen to be $\boldsymbol{D}_{t,h}$, then the condition numbers $\kappa(\mathcal{B}_{t,h}\mathcal{A}_{t,h})$ (and hence N_{MR} and N_{CGN}) are bounded independently of t and h. Furthermore, if t is sufficiently small compared to h ($t=O(h)$) then the simpler choice $\boldsymbol{N}_{t,h}=\boldsymbol{D}_{0,h}$ is a good one as well. In Table 3, we compare the condition numbers $\kappa(\mathcal{B}_{t,h}\mathcal{A}_{t,h})$ for $t=0.01$ and $\boldsymbol{N}_{t,h}=\boldsymbol{D}_{t,h}$ or $\boldsymbol{N}_{t,h}=\boldsymbol{D}_{0,h}$. As expected, the choice $\boldsymbol{N}_{t,h}=\boldsymbol{D}_{0,h}$ works well when h is large, but deteriorates when h becomes too small. In contrast, the condition numbers for the choice $\boldsymbol{N}_{t,h}=\boldsymbol{D}_{t,h}$ appear to be bounded uniformly with respect to h.

For any fixed $t>0$ the choice $\boldsymbol{N}_{t,h}=\boldsymbol{D}_{1,h}=\boldsymbol{I}$ leads to condition numbers $\kappa(\mathcal{B}_{t,h}\mathcal{A}_{t,h})$ which are independent of h, but which may increase with decreasing values of t. The effect of this is illustrated in Table 4. These results confirm the uniformity with respect to h for the simple choice $\boldsymbol{N}_{t,h}=\boldsymbol{I}$, but the experiments also clearly indicate that this is not a good choice for t sufficiently small.

Table 4 $\kappa(\mathcal{B}_{t,h}\mathcal{A}_{t,h})$ for $t = 0.1$

h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
$\boldsymbol{N}_{t,h} = \boldsymbol{I}$	90.2	78.5	72.7	70.1	70.7
$\boldsymbol{N}_{t,h} = \boldsymbol{D}_{t,h}$	8.64	10.8	11.1	11.2	11.1

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