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JOSÉ JAVIER (CHICHO) GUADALUPE HERNÁNDEZ
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THE FRATTINI SUBGROUP OF A GROUP

JAVIER OTAL

Abstract. In this paper, we review the behaviour of the Frattini subgroup $\Phi(G)$ and the Frattini factor group $G/\Phi(G)$ of an infinite group G having in mind the most standard results of the finite case.

Este artículo está dedicado con todo cariño a la memoria de Chicho.

Al no formar parte de tu grupo de investigación, no tuve la fortuna de relacionarme contigo por este motivo, pero sí que tuve la oportunidad de participar en algunos de los múltiples seminarios que organizabas. Por la naturaleza de mi especialidad y las características de las personas a los que aquellos se dirigían, siempre me sugerías temas que pudieran ser bien comprendidos por la audiencia, una preocupación que te caracterizó siempre. Yo quería hablar (suavemente, claro) de aspectos de Grupos y de Curvas Elípticas, ante lo que siempre ponías el grito en el cielo. Pero te hice trampas y conseguí hablar delante de ti algo de tales curvas, con la excusa de referir el teorema de Fermat y describir unas parametrizaciones con la circunferencia. Y te gustó, ¿recuerdas? Pero nunca te pude contar nada de mi especialidad ni de sus aplicaciones y es una pena que llevaré siempre conmigo.

Lamento de corazón que ahora ya no me puedas poner trabas.

1. Introduction

In studying the behaviour of the maximal subgroups of a finite group G, Giovanni Frattini [11] introduced what he called the Φ-subgroup of G, the intersection of the maximal subgroups of G. Since then, this subgroup is usually known as *the Frattini subgroup of* G and can be defined as follows. Given a group G, by definition, the Frattini subgroup of G is

$$\Phi(G) = \bigcap \{M \leq G \mid M \text{ maximal in } G\},$$

provided G has maximal subgroups. Otherwise $\Phi(G) = G$.

In order to set up clearly the contents of this survey, we begin with finite groups G. The main result of Frattini is

Theorem A. *If G is finite then $\Phi(G)$ is nilpotent.*

His proof is very simple and is a fairly application of what he knows as *the Frattini argument* and set off the importance of the normality of maximal subgroups of G, a property which characterizes the nilpotency of G. On the other hand, $\Phi(G)$ is known

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to be the set of *non generators* of the group G, that is, the set of elements of G which can be removed in any generator system of G. As an immediate consequence, *the Frattini factor group $G/\Phi(G)$ of G is cyclic if and only if G is cyclic.* Gathering now, among other properties (see [17]), nilpotency of the Frattini subgroup can be characterized by the following ones.

Theorem B. *If G is a finite group then the following are equivalent:*
(1) G is nilpotent.
(2) $G/\Phi(G)$ is nilpotent.
(3) Every maximal subgroup of G is normal in G.
(4) $G' \leq \Phi(G)$.

We mention that these (and other) properties do not characterize nilpotency of infinite groups; see [28]. The main goal of this note is to give an small survey on the current state of these concepts and related ideas. Throughout, our group-theoretic notation is standard and is taken from [28]. For the reader's convenience we briefly recall here some of it.

• If A abelian, the *torsion-free rank* $r_0(A)$ of A is the cardinal of a maximal set of elements of infinite order, and if p is a prime its *p-rank* $r_p(A)$ is the cardinal of a maximal set of non-trivial elements of order a power of p. The *rank* of A is $r(A) = r_0(A) + \sum_p r_0(A)$ (see Fuchs [12]). These concepts have been naturally extended to arbitrary groups. We mention that a group G is said to have *finite abelian section rank* if every abelian section of G is has finite rank, for $p = 0$ or a prime. A soluble group G has finite abelian section rank if an only if there exists a abelian subnormal series $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$ such that each $r_p(G_{i+1}/G_i) < \infty$ $(p \geq 0)$.

• If $\mathfrak{X}$ is a class of groups, a *poly-$\mathfrak{X}$-group* is a group G with a finite series between 1 and G with factors $\mathfrak{X}$-groups. Similarly is defined a *poly-$\mathcal{X}$-group* if $\mathcal{X}$ is a theoretical property of groups. For example, a *soluble group* is a poly-abelian group while a *nilpotent group* is a poly-central group. On the other hand, a *minimax group* is a poly-(**Min** or **Max**) group. If we replace *finite series* by *ascending series* we obtain *hyper-* instead of *poly-*.

• A group G is said to be an *$\mathfrak{X}\mathbf{C}$-group* if $G/C_G(x^G) \in \mathfrak{X}$ for all $x \in G$. Best examples of this are FC, PC, CC, and MC-groups $\mathfrak{X}$ = finite, polycyclic-by-finite, Chernikov and minimax groups); see [31], [10], [1], [19].

2. Chief factors and maximal subgroups

In infinite groups when dealing with the Frattini subgroup, the first difficulty which appears is the non existence of maximal subgroups.

Example 1. *Let p be a prime. Then the complex p^n-roots of unity $(n \geq 0)$ form a group $G = \langle x_1, x_2, \ \ldots \mid x_1 = 1, x_i^p = x_{i-1}, \ i > 1\rangle$.*

In this example, G is periodic, non finitely generated, with proper subgroups cyclic and without maximal subgroups. G is called *a Prüfer p-group.*

Example 2. *The wreath product G of a Prüfer p-group and a Prüfer q-group ($p \neq q$) is not locally nilpotent without maximal subgroups.*

The structure of the minimal normal subgroups M of a group G Is known provided some local information is available and we it (see Section B of [18]).

Theorem 2.1. *Let U be a chief factor of a group G.*
(1) If G is locally soluble, then U is abelian.
(2) If G is locally polycyclic, then U is abelian.
(3) If G is locally supersoluble, then U is cyclic of prime order.
(4) If G is locally nilpotent, then U is central of prime order.

Even in the case of locally finite groups, very little it is known about general chief factors. In many cases the maximal subgroups of certain groups G use to have finite index provided the chief factors of G are finite. In 1968, Robinson [26] (Theorems C and D) achieved a result of this kind.

Theorem 2.2. *The chief factors of a hyperabelian group with finite abelian section rank are finite and its maximal subgroups have finite index.*

If G is an FC-group, each normal closure $\langle x \rangle^G$ is finite, and so a similar result holds for these groups. The same is true for PC and CC-groups ([10] and [1]) and more generally for MC-groups ([19]).

Theorem 2.3. *Let G be an MC-group. Then each chief factor of G is finite and each maximal subgroup of G has finite index.*

In this direction, Wehrfritz proved (see [33], Corollary 4.28)

Theorem 2.4. *The chief factors of a soluble-by-finite subgroup of a finitely generated linear group are finite and the maximal subgroups have finite index.*

In what it concerns to the normality of the maximal subgroups, for many years, the best result was due to McLain.

Theorem 2.5. *A maximal subgroup of a locally nilpotent group is normal.*

As a consequence of an influential result of Robinson [27] characterizing the nilpotency of a finitely generated group by that of its images, we have

Theorem 2.6. *A finitely generated hyper-(abelian or finite) group is nilpotent if and only if each of its maximal subgroups is normal.*

Recently, Franciosi and de Giovanni [8] have shown the following.

Theorem 2.7. *Let G be a soluble residually finite minimax group. Then G is finite-by-nilpotent if and only if it has finitely many maximal subgroups which are not normal in G.*

Corollary 2.7.1. *If G is a soluble minimax group with finitely many non-normal maximal subgroups, then G is Chernikov-by-nilpotent.*

3. The nilpotency of the Frattini subgroup

In contrast with the celebrated result of Frattini quoted in the Introduction, just in the beginning of Section 2, we have seen an example of an infinite group G in which $\Phi(G)$ is neither locally nilpotent. Certainly, this group G is neither finitely generated nor has maximal subgroups, and these seem the basic reasons why the nilpotency of $\Phi(G)$ can fail. In his pioneer work on polycyclic groups, Hirsch (see [16]) stated the first fundamental result, which dealt with soluble finitely generated groups only.

Theorem 3.1. *If G is a polycyclic group then $\Phi(G)$ is nilpotent.*

We recall that a polycyclic group G is a soluble group with **Max** and so all subgroups of G are finitely generated. After Hirsch's result, Itô asked whether or not there exist metabelian groups G with $\Phi(G)$ not nilpotent. The answer of this question is negative as P. Hall [15] showed.

Example 3. *If $p \equiv 1 \pmod{q^2}$ and C is a Prüfer p-group, then C has an automorphism α of order q^2. Let $A = \langle\alpha\rangle$, $B = \langle\alpha^q\rangle$ and $G = CA$.*

The proper subgroups C_n of C are cyclic, characteristic and not maximal. Hence $\Phi(G) = CB$ is the unique maximal subgroup of G. Since $\alpha^q - 1$ represents a unit of $\mathrm{End}(C)$, we have $[C_n, B] = C_n$ for all n and then $\Phi(G)$ is not locally nilpotent. G is a non finitely generated metabelian group, but Hall also constructed finitely generated examples ([15], Section 3.2):

Example 4. *Let G be the above group. There exists a finitely generated metabelian group H such that $\Phi(G)$ is isomorphic to a section of $\Phi(H)$.*

In the same paper, Hall found the best answer to Itô's question.

Theorem 3.2. *$\Phi(G)$ is nilpotent if G is finitely generated metanilpotent.*

If $q = 2$ and $p = 5$ or $p = 13$, then $p \equiv 1 \pmod{q^2}$ so that the above $G \leq GL(4, F)$, where F is an algebraic closed field and G is given by $p = 5$ $q = 2$, if $\mathrm{char}\, F \neq 5$ or $p = 13$ $q = 2$ otherwise ([33] 2.3). Thus

Example 5. *There are periodic metabelian linear groups G (in particular locally finite linear groups) such that $\Phi(G)$ is not nilpotent.*

Moreover, O. H. Kegel has been able to construct torsion-free examples.

Example 6. *There are torsion-free metabelian residually finite linear groups whose Frattini subgroup is not nilpotent.*

Indeed, let F be an uncountable field. Then F^* contains a subgroup A isomorphic to the additive group of rational numbers with odd denominators. If α is a homomorphism of A onto a cyclic group of order 4 contained in $GL(4, F)$, let G be the split extension of four copies of A by a further copy of A that acts by permuting the first copies cyclically via α. Then G is residually finite group and its maximal subgroups have finite index. If M is a maximal then G/M_G is a finite 2-group and so $G^2 \leq M$ and $G^2 \leq \Phi(G)$. Since G^2 has a factor isomorphic to the wreath product of

C_∞ by C_2, G^2 and $\Phi(G)$ are not locally nilpotent. By construction $G \le GL(5,F)$. However, Platonov [24] and Wehrfritz [32] proved good results on linear groups.

Theorem 3.3. *If G is a subgroup of a finitely generated linear group, then $\Phi(G)$ is nilpotent and $G' \le \Phi(G)$ if and only if G is nilpotent.*

Note that these G are not necessarily finitely generated. Generalizing Hall's result, Robinson [28] proved results as the following.

Theorem 3.4. *Let G be a finitely generated hyper-(abelian or finite) group. Then $\Phi(G)$ is nilpotent.*

We finally mention two results related to $\mathfrak{X}\mathbf{C}$-groups described in [9].

Theorem 3.5. *Let G be a PC-group. Then $\Phi(G)$ is locally nilpotent.*

Theorem 3.6. *Let G be a soluble-by-finite minimax group with $D(G)=1$. Then $\Phi(G)$ is locally nilpotent.*

Here and elsewhere $D(G)$ denotes the largest periodic divisible abelian subgroup of G. If $D(G)=1$, the group G is said to be *reduced*.

4. Frattini properties of infinite groups

Let $\mathfrak{X}$ be a class of groups. A group theoretical property $\mathcal{P}$ is said to be a *Frattini property of $\mathfrak{X}$-groups* provided an $\mathfrak{X}$-group G satisfies the following:

$$G/\Phi(G) \text{ has } \mathcal{P} \Longrightarrow G \text{ has } \mathcal{P}.$$

Some authors prefer to say better that the $\mathfrak{X}$-groups which satisfy the property $\mathcal{P}$ form a *saturated* class of groups, although this terminology is almost reserved for *formations* of groups. We recall that a class $\mathcal{F}$ of groups is called *a formation* simply if it is $\mathbf{Q}$ and $\mathbf{R}_0$-closed (see [28]), that is if $\mathcal{F} = \mathbf{Q}\mathcal{F} = \mathbf{R}_0\mathcal{F}$. It is well known the influential rôle of these properties when $\mathfrak{X}$ is the class of finite soluble groups (see [17]). A fundamental result of Gaschütz and Lubeseder [13] stated that saturated formations of finite soluble groups can be locally defined through a family $\{f(p)\}$ of formations of finite soluble groups $f(p)$, one for each prime p. Concretely, Gaschütz and Lubeseder proved that $\mathcal{F}$ is saturated if and only if $\mathcal{F} = \bigcap \mathfrak{S}^*_{p'}\mathfrak{S}^*_p f(p)$, where $\mathfrak{S}^*$ is the class of finite soluble groups.

It turns out that the extension of these ideas to other classes of groups, close to finite groups, is very natural. One of the outstanding results in this direction was established by the paper of Robinson [28] who proved

Theorem 4.1. *Nilpotency is a Frattini property of reduced soluble minimax groups.*

Later on, in a series of papers, Lennox [20], [21], [22], [23] showed others Frattini properties of several groups. Lennox's results are:

Theorem 4.2.
- *Finiteness is a Frattini property of the following groups:*
(1) Finitely generated hyper-(finite or locally soluble) groups.
(2) Finitely generated soluble groups.

(3) Subgroups of finitely generated abelian-by-nilpotent groups.

• *If $\mathfrak{H}$ is the class of all finitely generated hyper-(abelian-by-finite) groups then the following are Frattini properties of $\mathfrak{H}$-groups:*

(i) Supersolubility, (ii) Polycyclicity, (iii) Being polycyclic-by-finite, and (iv) Being finite-by-nilpotent.

Since an $\mathfrak{H}$-group G is polycyclic-by-finite $\iff G \in$**Max**, we have

Corollary 4.2.1. **Max** *is a Frattini property of $\mathfrak{H}$-groups.*

Further, the techniques of Roseblade in his report [29] allow to establish:

Theorem 4.3. *Polycyclicity is a Frattini property of finitely generated metabelian groups.*

Going back to above results, we remark that **Max-n**, the maximal condition for normal subgroups, is not a Frattini property of $\mathfrak{H}$-groups. In fact there exists a finitely generated central-by-metabelian group G which does not satisfy **Max-n** as the mentioned Hall's examples show.

The paper Wehrfritz [32] also shows a Frattini property of linear groups.

Theorem 4.4. *Nilpotency is a Frattini property of subgroups of finitely generated linear groups.*

It follows that Lennox's results may be applied to these groups as follows. By using an idea due to Tits, it is possible to show:

Corollary 4.4.1. *The following properties are Frattini properties of finitely generated linear groups: (i) Polycyclicity, (ii) Being polycyclic-by-finite, and (iii)* **Max**.

Other properties of these groups appear in the papers of Wehrfritz (see [33]). Combining his ideas with those of Huppert, we have.

Theorem 4.5. *Let $G \leq GL(n, F)$.*

(1) If $\operatorname{char} F > 0$*, then G is parasoluble $\iff G/\Phi(G)$ is hypercyclic.*

(2) If G is finitely generated, then $G/\Phi(G)$ hypercyclic $\Rightarrow G$ supersoluble.

At the time of writing these notes, we do not know whether or not hypercyclycity is a Frattini property of arbitrary linear groups.

Extending Robinson's result mentioned above, Franciosi and de Giovanni [8] showed another interesting result:

Theorem 4.6. *Nilpotency-by-finiteness is a Frattini property of the following types of groups:*

(1) Soluble residually finite minimax groups.

(2) Reduced soluble-by-finite minimax groups.

In general, if G is an abelian group and $G/\Phi(G)$ is finite, we note that G is divisible-by-finite and could be infinite. It follows that there are properties of $G/\Phi(G)$ which are not inherited for G. Therefore it is natural to look for classes of groups which are *good* under this point of view. One of these is the class of

FC-groups. For example, if G is an FC-group and $G/\Phi(G)$ is hypercentral then G is hypercentral too. Since $\Phi(G)$ is hypercentral when G is an FC-group, Franciosi and de Giovanni [9] considered Frattini properties for an extension of FC-groups, namely PC-groups. More precisely, they realized the connection between the properties of the factor-group $G/\Phi(G)$ and those of $G/Z(G)$ in a PC-group G as they were able to show the following result

Theorem 4.7. *Let G be a PC-group. Then the statement*

$$G/\Phi(G) \text{ has } \mathcal{P} \Longrightarrow G/Z(G) \text{ has } \mathcal{P}$$

holds provided the property $\mathcal{P}$ is one of the following:
(i) Being finite,
(ii) Being polycyclic-by-finite,
(iii) Being soluble-by-finite minimax,
(iv) Having finite abelian section rank,
(v) Having finite torsion-free rank, and
(vi) Being periodic.

In the same paper they achieved some further Frattini properties:

Theorem 4.8. *The following are Frattini properties of PC-groups:*
(i) Being (locally nilpotent)-by-finite,
(ii) Being finite-by-(locally nilpotent), and
(iii) Being locally supersoluble.

Similar questions could be asked for the other fairly generalization of the class of FC-groups, namely the class of CC-groups. In a private communication, Beidleman [6] has shown to me an unpublished proof of the following results

Theorem 4.9. *Let G be a CC-group with $G/Z(G)$ periodic. If $G/\Phi(G)$ is a minimax group then $G/Z(G)$ is a Chernikov group.*

Corollary 4.9.1. *Being a Chernikov group is a Frattini property of CC-groups with trivial centre.*

Beidleman also claims for other Frattini properties of CC-groups. However in this case we think that the relation between $G/\Phi(G)$ and $G/Z(G)$ does not exist, as the following example constructed in [14] shows.

Example 7. *If p is a prime, each element of Q can be written in the form (not unique!) $\frac{a}{bp^m}$, where $(p,b) = 1$ and $m \geq 0$. If $Z(p^\infty) = \langle x_i \rangle$ is a copy of the Prüfer p-group, a central extension C of $Z(p^\infty)$ by $Q \oplus Q$ is given by $[\frac{a}{bp^m}, \frac{c}{dp^n}] = (\frac{1}{bd})(acx_{m+n+1})$. Since each element of $Z(p^\infty)$ has a unique k^{th}-root if $(p,k) = 1$, our construction can be carried out.*

Therefore, if Z is a copy of the Prüfer p-group and A and B are copies of the full rational group then C is a split extension of $Z \times A$ by B and $[Z,B] = [Z,A] = 1$, $[A,B] = Z$. It follows that $C' = Z = Z(C)$ and so C is a CC-group. Since $C/Z \cong Q \oplus Q$, C is radicable and hence $\Phi(C) = C$.

Thus the direct product $G = \times_{n\in N} C_n$ of a countably infinite number of copies of C is a divisible CC-group so that $G = \Phi(G)$ but $G/Z(G)$ is a torsion-free divisible abelian group of countable rank.

Despite these complications, we can describe the structure of a these groups with the above restrictions on its Frattini factor-group, the main goal of the paper [19].

Theorem 4.10. *If G is an MC-group with $G/\Phi(G)$ minimax, then there is $X \subseteq G$ finite with $H = \langle X\rangle^G$ is minimax and G/H is radicable nilpotent of class ≤ 2.*

Corollary 4.10.1. *If G is a CC-group with $G/\Phi(G)$ minimax, then there is an $H = \langle X\rangle^G$* **Min**-*by*-**Max** *with G/H radicable nilpotent of class ≤ 2.*

Theorem 4.11. *If G is a reduced MC-group with $G/\Phi(G)$ minimax, then there is $X \subseteq G$ finite with $H = \langle X\rangle^G$ reduced minimax and G/H divisible abelian.*

Corollary 4.11.1. *Let G be a group such that $G/\Phi(G)$ is minimax.*

(1) If G is a PC-group, then G contains a polycyclic-by-finite normal subgroup H such that G/H is a divisible abelian group.

(2) If G is an FC-group, then G contains a finitely generated abelian-by-finite normal subgroup H such that G/H is a divisible abelian group.

Theorem 4.12. *Let G be a group such that $G/\Phi(G)$ is periodic.*

(1) If G is an MC-group, then G is a CC-group.

(2) If G is a reduced MC-group, then G is an FC-group.

Corollary 4.12.1. *Let G be a group such that $G/\Phi(G)$ has finite 0-rank.*

(1) If G is an MC-group, then G includes a minimax normal subgroup H such that G/H is a CC-group.

(2) If G is a reduced MC-group, then G includes a reduced minimax normal subgroup H such that G/H is an FC-group.

Theorem 4.13. *Let G be a PC-group. If $G/\Phi(G)$ has finite abelian section rank, then G has an ascending series of normal subgroups*

$$1 = A_0 \leq A_1 \leq \cdots \leq A_\omega = \bigcup_{n\geq 1} A_n \leq A_{\omega+1} \leq A_{\omega+2} = G \textit{ such that}:$$

(1) $A_{\omega+1} \leq Z(G)$.

(2) $Z(G)/A_{\omega+1}$ is a periodic group with finite Sylow p-subgroups. Therefore $Z(G)/A_{\omega+1}$ is a residually finite group of finite abelian section rank.

(3) A_1 is a periodic divisible abelian group.

(4) If $i > 1$, A_{i+1}/A_i is a torsion-free abelian group of rank 1.

(5) Given a prime p, there exists an integer $\ell(p)$ such that A_{i+1}/A_i is p-divisible for every $i \geq \ell(p)$. In particular $A_\omega/A_{\ell(p)}$ is p-divisible.

(6) $A_{\omega+1}/A_\omega$ is divisible.

Theorem 4.14. *Let G be a CC-group. If $G/\Phi(G)$ has finite abelian section rank, then G has an ascending series of normal subgroups*

$$1 = A_0 \leq A_1 \leq \cdots \leq A_\omega = \bigcup_{n\geq 1} A_n \leq A_{\omega+1} \leq A_{\omega+2} = G \textit{ such that}:$$

(1) $A_1 = D(G)$.
(2) $A_{\omega+1}/A_1 \leq Z(G/A_1)$.
(3) $G/A_{\omega+1}$ *is a periodic group with finite Sylow p-subgroups.*
(4) If $i > 1$, A_{i+1}/A_i *is a torsion-free abelian group of rank* 1.
(5) Given a prime p, there exists an integer $\ell(p)$ *such that* A_{i+1}/A_i *is p-divisible for all* $i \geq \ell(p)$. *In particular* $A_\omega/A_{\ell(p)}$ *is p-divisible.*
(6) $A_{\omega+1}/A_\omega$ *is divisible.*

It follows that locally nilpotency is a Frattini property of reduced MC-groups.

5. Other Frattini-like subgroups

The non-existence of maximal subgroups of an infinite group G is the main reason why the Frattini subgroup $\Phi(G)$ and the Frattini factor-group $G/\Phi(G)$ are not well-behaved. In the literature there are several attempts to keep the idea involved for that subgroup. Mostly of those attempts slightly modify the definition of the Frattini subgroup in such a way a new subgroup appears and this subgroup retains some original feature of the Frattini subgroup. The purpose of this section is to review a couple of them as well as to indicate the frame in which these new subgroups have been considered.

• *The near Frattini of an infinite group.* This was introduced first by Riles [25] in 1969 as follows. If G is a group, an element $x \in G$ is said to be a *near generator* of G if there is a subset S of G such that $|G : \langle S\rangle| = \infty$ but $|G : \langle x, S\rangle| < \infty$. Hence an element x of G is a *non-near generator* of G precisely when $S \subseteq G$ and $|G : \langle x, S\rangle| < \infty$ always imply that $|G : \langle S\rangle| < \infty$. The set of all non-near generators of a group G forms a characteristic subgroup called the *lower near Frattini subgroup* of G, denoted by $\lambda(G)$.

An $M \leq G$ is *nearly maximal* in G if $|G : M| = \infty$ but $|G : N| < \infty$, whenever $M < N \leq G$. That is, M is maximal with respect to being of infinite index in G. The intersection of all nearly maximal subgroups of G is another characteristic subgroup called the *upper near Frattini subgroup* of G, denoted by $\nu(G)$ (if there are no nearly maximal subgroups, then $\nu(G) = G$). In general $\lambda(G) \leq \nu(G)$ in any group G. However

Example 8. *Let G be the standard wreath product of a Prüfer p-group by a cyclic q-group, where p and q are primes. Then* $\lambda(G) = 1 < G = \nu(G)$.

If $\lambda(G) = \nu(G)$, then their common value is called the *near Frattini subgroup* of G, denoted by $\Psi(G)$. Remark that we do nothing if G is finite since $\Psi(G) = G$. Even in infinite groups the case $\Psi(G) = G$ is very ample. Indeed it can occur if G is divisible abelian or G is soluble with **Min**. When G is subgroup of a free product with or without amalgamations, some authors have been able to extend many of the results proved for $\Phi(G)$ to $\lambda(G)$. $\nu(G)$ and $\Psi(G)$ by different methods (see the report [3] and its references or [2] or [34]), although the difficulties of dealing with non-near generator elements or nearly maximal subgroups still hold. Also, natural definitions of $\Psi(G)$*-free* or λ*-free* groups have been considered as well as the *nearly splitting* of groups and the relation between these concepts.

• *The subgroup* $\mu(G)$ introduced by Tomkinson [30] in 1975 by considering a class of proper subgroups of a a group G, the *major subgroups*. Let U be a subgroup of a group G and consider the properly ascending chains

$$U = U_0 < U_1 < \ldots < U_\alpha = G$$

from U to G. Define $m(U)$ to be the least upper bound of the types α of all such chains. Clearly $m(U) = 1$ if and only if U is a maximal subgroup of G. A proper subgroup M of G is said to be a *major subgroup* of G if $m(U) = m(M)$ whenever $M \leq U < G$. The intersection of all major subgroups of G is denoted by $\mu(G)$. In his former paper on major subgroups, Tomkinson [30] showed that this intersection coincides with $\phi(G)$, if G is finitely generated, and that every proper subgroup is contained in a major subgroup of G so that $\mu(G)$ is always a proper subgroup of G. Tomkinson also proved some properties of $\mu(G)$ for several types of groups G.

Theorem 5.1. *Let $\mathfrak{H}$ be the class of groups in which proper subgroups have proper normal closures. If G is a hypercentral $\mathfrak{H}$-group then $\mu(G) \geq G'$.*

Theorem 5.2. *Let $\mathfrak{U}$ be the class of locally finite groups G in which the Sylow subgroups of any subgroup H of G are conjugate in H. Then:*
(1) $\mu(G)$ is locally nilpotent.
(2) $G/\mu(G)$ is locally nilpotent $\Longleftrightarrow$ G is locally nilpotent.

Theorem 5.3. *Let G be a nilpotent-by-finite group. Then:*
(1) $\mu(G)$ is hypercentral.
(2) $G/\mu(G)$ is hypercentral $\Longleftrightarrow$ G is hypercentral.
(3) $G' \leq \mu(G) \Longleftrightarrow G$ is a hypercentral $\mathfrak{B}$-group.

Theorem 5.4. *Let G be a soluble group with finite abelian section rank. Then:*
(1) $\mu(G)$ is hypercentral.
(2) $G/\mu(G)$ is hypercentral $\Longleftrightarrow$ G is hypercentral.

Theorem 5.5. *Let G be group having a finite normal series in which the factors are abelian groups of finite rank and with Chernikov torsion subgroups (see [28]). Then $G' \leq \mu(G) \Longleftrightarrow G$ is nilpotent.*

Theorem 5.6. *Let G be a soluble group with* **Min-n**. *Then*
(1) $G/\mu(G)$ is hypercentral $\Longleftrightarrow$ G is hypercentral.
(2) $G' \leq \mu(G) \Longleftrightarrow G$ is nilpotent.

In a very serious attempt to extend to certain infinite groups some of the results from the theory of saturated formations of finite soluble groups, Ballester-Bolinches and Camp-Mora [4] characterized a saturated formation via $\mu(G)$ instead of $\Phi(G)$. Certainly those results had been extended to various classes of infinite groups (see [7]), but the definition of saturated formation was made locally as indicated above. Concretely, let $c\bar{\mathcal{L}}$ be the class of all countable locally finite and soluble groups satisfying $min{-}p$ for all primes p. We recall that a group G is said to be *primitive* if it has a maximal subgroup with trivial core (Schunck) and G is said to be *semiprimitive* if it is a semidirect product $G = [D]M$, where M is a finite soluble group with trivial core and D is a divisible abelian group such that every proper M-invariant subgroup of D is finite (Hartley). If G is a $c\bar{\mathcal{L}}$-group and M is a major subgroup of G. Then G/M_G is a finite primitive soluble group or a semiprimitive group depending whether or not M is a maximal subgroup of G. In [4] the following extension of the Gaschütz-Lubeseder is proved:

Theorem 5.7. *Let $\mathcal{F}$ be a $c\bar{\mathcal{L}}$-formation. Then the following are equivalent:*
(1) $\mathcal{F}$ is a saturated $c\bar{\mathcal{L}}$-formation.
(2) $\mathcal{F}$ satisfies the two following properties:
(i) A $c\bar{\mathcal{L}}$-group G is in $\mathcal{F}$ if and only if $G/\mu(G)$ is in $\mathcal{F}$.
(ii) A semiprimitive group G is an $\mathcal{F}$-group if and only if it is the union of an ascending chain $\{G_n \mid n \geq 0\}$ of finite $\mathcal{F}$-subgroups.

Having in mind the same idea, Ballester-Bolinches and T. Pedraza-Aguilera [5] obtained several nice extensions of standard results, including those original of Frattini mentioned in the introduction.

Theorem 5.8. *If G is a $c\bar{\mathcal{L}}$-group, then $\mu(G) \in \mathfrak{B}$ and the following are equivalent:*
(1) G is a $\mathfrak{B}$-group,
(2) $G/\mu(G)$ is a $\mathfrak{B}$-group,
(3) $G' \leq \mu(G)$, and
(3) every major subgroup of G is normal in G.

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Departamento de Matemáticas, Universidad de Zaragoza, Edificio de Matemáticas, Ciudad Universitaria s/n, 50009 Zaragoza, Spain

E-mail address: otal@posta.unizar.es