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k-COHERENCE OF MEASURES WITH NON-CLASSICAL WEIGHTS

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Dedicated to Prof. Dr. José Javier Guadalupe, Chicho.
His memory will always remain with us.

Abstract. The concept of k-coherence of two positive measures μ_1 and μ_2 is useful in the study of the Sobolev orthogonal polynomials. If μ_1 or μ_2 are compactly supported on $\mathbb{R}$ then any 0-coherent pair or symmetrically 1-coherent pair (μ_1, μ_2) must contain a Jacobi measure (up to affine transformation). Here examples of k-coherent pairs ($k \geq 1$) when neither μ_1 nor μ_2 are Jacobi are constructed.

1. Introduction and k-coherence

Given two positive distribution functions (see, e.g., [10]), μ_1 and μ_2, supported on $\mathbb{R}$ and with an infinite number of points of increase, we can define the Sobolev inner product

$$(p, q)_S = \int pq \, d\mu_1 + \int p'q' \, d\mu_2 \,, \tag{1}$$

on the space $\mathbb{P}$ of algebraic polynomials with real coefficients. The sequence Q_n, $\deg Q_n(x) = n$, $n = 0, 1, \dots$ of monic polynomials orthogonal with respect to (1) has been object of interest in the last years. Algebraic and analytic properties of these polynomials (known as Sobolev orthogonal polynomials) have been studied and compared with the corresponding properties of the monic polynomial sequences $P_n(\mu_i, \cdot)$ such that for $i = 1, 2$,

$$\deg P_n(\mu_i, \cdot) = n \,, \quad \langle P_n(\mu_i, x), x^j \rangle_i = 0 \,, \quad j = 0, \dots, n-1 \,,$$

with

$$\langle p, q \rangle_i = \int pq \, d\mu_i \,,$$

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(see [5]–[8]). This study is much simplified if we assume a link between the measures μ_1 and μ_2, restrictive enough to follow the structural properties of the sequence $\{Q_n\}$, but sufficiently loose to get some insight concerning the general situation.

One of these links, appeared in [3], is the concept of the *coherence of measures*; it proved to be a fruitful tool in the study of the Sobolev orthogonal polynomials. All the coherent pairs of measures (μ_1, μ_2) have been obtained in [9]; in particular, it was shown that at least one of them must be a classical weight function (Jacobi, Laguerre or Hermite).

Thus, in order to deal with pairs (μ_1, μ_2) with non-classical weights, we need to extend the concept of coherence. The main goal of this work is to show how to construct such pairs of measures, linked by the so-called k-coherence relation. These measures are in particular suitable for the study of structural and analytic properties of the Sobolev polynomials $\{Q_n\}$ (see, e.g., [7]).

Definition 1 (k-coherence). *Let $\mu_i\,, i = 1, 2\,,$ be two positive measures. Then (μ_1, μ_2) is a k-coherent pair of measures ($k \geq 0$) if for every $n \in \mathbb{N}\,,\ n \geq k+1\,,$ there exist $\sigma_n^{(n)}(\mu_1, \mu_2)\,, \ldots, \sigma_n^{(n-k)}(\mu_1, \mu_2) \in \mathbb{R}\,,$ with $\sigma_n^{(n-k)}(\mu_1, \mu_2) \neq 0$, such that*

$$P_n(\mu_2, x) = \frac{P'_{n+1}(\mu_1, x)}{n+1} - \sum_{j=0}^{k} \sigma_n^{(n-j)}(\mu_1, \mu_2) \frac{P'_{n-j}(\mu_1, x)}{n-j}\,. \tag{2}$$

Remark: Relation (2) can be considered for $k = -1$, but this case is trivial and is only verified by the classical polynomials: Jacobi, Laguerre, Hermite and Bessel (see [10]). If $k = 0$, we obtain the 0-coherence (or just coherence) introduced by Iserles et al. in [3], and $k = 1$ yields the 1-coherence studied by de Bruin and Meijer in [2] as well as by Kwon et al in [4]. A particular case of 1-coherence is the symmetric coherence, also introduced by Iserles et al. in [3], which takes place when both measures are symmetric with respect to the origin.

When the measures μ_1 and μ_2 are linked by k-coherence, a connection between the families $\{Q_n\}$ and $\{P_n(\mu_1, \cdot)\}$ can be established. Indeed, denote $k_n(\mu_i) = \langle P_n(\mu_i, \cdot), P_n(\mu_i, \cdot)\rangle_i\,,\ i = 1, 2\,,$ and $\widetilde{k}_n = (Q_n, Q_n)_S$. In order to simplify notations we omit the explicit reference to the pair (μ_1, μ_2) in the coefficients $\sigma_n^{(n-j)}$, whenever it cannot lead to confusion.

We have the following relation between the polynomials $P_n(\mu_1, x)$ and $Q_n(x)$:

Proposition 1. *Let (μ_1, μ_2) be a k-coherent pair of measures satisfying (2). Then, for $n \geq k+1\,,$ there exist $\alpha_n^{(n-j)} \in \mathbb{R}\,,\ j = 0, \ldots, k\,,$ with $\alpha_n^{(n-k)} \neq 0$, such that*

$$P_{n+1}(\mu_1, x) - \sum_{j=0}^{k} \sigma_n^{(n-j)} \frac{n+1}{n-j} P_{n-j}(\mu_1, x) = Q_{n+1}(x) + \sum_{j=0}^{k} \alpha_n^{(n-j)} Q_{n-j}(x)\,.$$

Proof. Consider the 1-parametric family of Sobolev inner products, given by

$$(p, q)_S = \int pq\, d\mu_1 + \lambda \int p'q'\, d\mu_2\,, \quad \lambda \geq 0\,. \tag{3}$$

Standard arguments allow to express the polynomial $Q_n(x; \lambda)$, orthogonal with respect to (3), as a ratio of two determinants. It can be observed that $Q_n(x; 1) =$

$Q_n(x)$, and that the coefficients of $Q_n(x;\lambda)$ are rational functions in λ such that for each n a non-trivial limit polynomial

$$R_n(x) = \lim_{\lambda\to\infty} Q_n(x;\lambda)$$

exists. This monic polynomial has degree n and satisfies (see [8]):

$$\langle R_n, 1\rangle_1 = 0\,, \quad n \geq 1\,, \tag{4}$$

$$\langle R_n', x^m\rangle_2 = 0\,, \quad n \geq 2\,, \quad 0 \leq m \leq n-2\,. \tag{5}$$

From (5) we get

$$R_{n+1}'(x) = (n+1)P_n(\mu_2, x)\,, \quad n \geq 1\,. \tag{6}$$

For the time being, we have not used the concept of k-coherence. Now, by (2), relation (6) can be rewritten as

$$R_{n+1}'(x) = P_{n+1}'(\mu_1, x) - (n+1)\sum_{j=0}^{k} \sigma_n^{(n-j)}(\mu_1,\mu_2)\frac{P_{n-j}'(\mu_1,x)}{n-j}\,, \quad n \geq k+1\,.$$

Taking into account (4),

$$R_{n+1}(x) = P_{n+1}(\mu_1, x) - \sum_{j=0}^{k} \sigma_n^{(n-j)}\frac{n+1}{n-j}P_{n-j}(\mu_1,x)\,, \quad n \geq k+1\,. \tag{7}$$

On the other hand, expanding R_{n+1} in the basis $\{Q_0, Q_1, \ldots, Q_{n+1}\}$, and using (5) and (7) we get

$$R_{n+1}(x) = Q_{n+1}(x) + \sum_{j=0}^{k} \alpha_n^{(n-j)} Q_{n-j}(x)\,, \tag{8}$$

where, using (6) and (7),

$$\alpha_n^{(n-j)} = \frac{(R_{n+1}, Q_{n-j})_S}{\widetilde{k}_{n-j}} = \frac{\langle R_{n+1}, Q_{n-j}\rangle_1}{\widetilde{k}_{n-j}}\,, \quad j = 0, \ldots, k\,.$$

Moreover, we have, for $j = k$,

$$\alpha_n^{(n-k)} = -\sigma_n^{(n-k)}\frac{n+1}{n-k}\frac{k_{n-k}(\mu_1)}{\widetilde{k}_{n-k}} \neq 0\,.$$

It remains to gather (7) and (8). □

2. k-COHERENT PAIRS

Given a positive and integrable function f on an open bounded interval $\Delta \subset \mathbb{R}$, $f(x)\,dx$ stands for an absolutely continuous measure, supported on Δ, whose Radon-Nikodym derivative with respect to the Lebesgue measure dx on Δ is $f(x)$. Furthermore, for $\xi \in \mathbb{R}$, $\delta(\xi)$ will be the Dirac delta (mass point) at ξ.

The complete classification of the coherent and symmetrically coherent pairs was given by Meijer in [9]. In particular, he proved that if $\operatorname{supp}(\mu_1) = [-1,1]$ then all the 0-coherent pairs (μ_1, μ_2) are those described in the following table:

TABLE 1

Case	μ_1	μ_2
1	$(1-x)^{\alpha-1}(1+x)^{\beta-1}dx$	$\frac{1}{\lvert x-\xi_2\rvert}(1-x)^{\alpha}(1+x)^{\beta}dx + M\delta(\xi_2)$
2	$\lvert x-\xi_1\rvert(1-x)^{\alpha-1}(1+x)^{\beta-1}dx$	$(1-x)^{\alpha}(1+x)^{\beta}dx$
3	$(1+x)^{\beta-1}dx + M\delta(1)$	$(1+x)^{\beta}dx$
4	$(1-x)^{\alpha-1}dx + M\delta(-1)$	$(1-x)^{\alpha}dx$

with $\alpha, \beta > 0$, $|\xi_1| > 1$, $|\xi_2| \geq 1$, and $M \geq 0$.

In a similar way, all the symmetrically coherent pairs (μ_1, μ_2) were classified as follows:

TABLE 2

Case	μ_1	μ_2	
5	$(1-x^2)^{\alpha-1}dx$	$\frac{(1-x^2)^{\alpha}}{x^2+\xi_2^2}dx$	$\xi_2 \neq 0$
6	$(1-x^2)^{\alpha-1}dx$	$\frac{(1-x^2)^{\alpha}}{\xi_2^2-x^2}dx + M[\delta(-\xi_2)+\delta(\xi_2)]$	$\lvert\xi_2\rvert \geq 1$
7	$(x^2+\xi_1^2)(1-x^2)^{\alpha-1}dx$	$(1-x^2)^{\alpha}dx$	$\xi_1 \neq 0$
8	$(\xi_1^2-x^2)(1-x^2)^{\alpha-1}dx$	$(1-x^2)^{\alpha}dx$	$\lvert\xi_1\rvert > 1$
9	$dx + M[\delta(1)+\delta(-1)]$	dx	

with $\alpha > 0$ and $M \geq 0$. Notice that in all the cases at least one of the measures is the Jacobi weight.

The main goal of this work is to give some non-trivial examples of k-coherent pairs (μ_1, μ_2), $k \geq 1$, for which neither μ_1 nor μ_2 are Jacobi, combining coherent pairs from Tables 1 and 2:

Proposition 2.

(i) *The pair*

Case 1	μ_1	μ_2
	$\lvert x-\xi_1\rvert(1-x)^{\alpha-1}(1+x)^{\beta-1}dx$	$\frac{1}{\lvert x-\xi_2\rvert}(1-x)^{\alpha}(1+x)^{\beta}dx + M\delta(\xi_2)$

with $\alpha, \beta > 0$, $|\xi_1| > 1$, $|\xi_2| \geq 1$ *and* $M \geq 0$, *is* 1*-coherent.*

(ii) *Pairs*

Case 2	μ_1	μ_2	
2a	$(x^2+\xi_1^2)(1-x^2)^{\alpha-1}dx$	$\frac{1}{\lvert x-\xi_2\rvert}(1-x^2)^{\alpha}dx + M\delta(\xi_2)$	$\xi_1 \neq 0,\ \lvert\xi_2\rvert \geq 1$
2b	$(\xi_1^2-x^2)(1-x^2)^{\alpha-1}dx$	$\frac{1}{\lvert x-\xi_2\rvert}(1-x^2)^{\alpha}dx + M\delta(\xi_2)$	$\lvert\xi_1\rvert > 1,\ \lvert\xi_2\rvert \geq 1$

with $\alpha > 0$, and $M \geq 0$, and

Case 3	μ_1	μ_2	
3a	$\lvert x-\xi_1\rvert(1-x^2)^{\alpha-1}dx$	$\frac{(1-x^2)^\alpha}{x^2+\xi_2^2}dx$	$\lvert\xi_1\rvert > 1,\ \xi_2 \neq 0$
3b	$\lvert x-\xi_1\rvert(1-x^2)^{\alpha-1}dx$	$\frac{(1-x^2)^\alpha}{\xi_2^2-x^2}dx + M[\delta(-\xi_2)+\delta(\xi_2)]$	$\lvert\xi_1\rvert > 1,\ \lvert\xi_2\rvert \geq 1$

with $\alpha > 0$, are 2-coherent.

(iii) *Pairs*

Case 4	μ_1	μ_2	
4a	$(x^2+\xi_1^2)(1-x^2)^{\alpha-1}dx$	$\frac{(1-x^2)^\alpha}{x^2+\xi_2^2}dx$	$\xi_1 \neq 0,\ \xi_2 \neq 0$
4b	$(x^2+\xi_1^2)(1-x^2)^{\alpha-1}dx$	$\frac{(1-x^2)^\alpha}{\xi_2^2-x^2}dx + M[\delta(-\xi_2)+\delta(\xi_2)]$	$\xi_1 \neq 0,\ \lvert\xi_2\rvert \geq 1$
4c	$(\xi_1^2-x^2)(1-x^2)^{\alpha-1}dx$	$\frac{(1-x^2)^\alpha}{x^2+\xi_2^2}dx$	$\lvert\xi_1\rvert > 1,\ \xi_2 \neq 0$
4d	$(\xi_1^2-x^2)(1-x^2)^{\alpha-1}dx$	$\frac{(1-x^2)^\alpha}{\xi_2^2-x^2}dx + M[\delta(-\xi_2)+\delta(\xi_2)]$	$\lvert\xi_1\rvert > 1,\ \lvert\xi_2\rvert \geq 1$

with $\alpha > 0$, and $M \geq 0$, are 3-coherent.

In all the cases, the absolutely continuous parts of μ_1 and μ_2 are supported on $[-1,1]$.

Proof. Fix $\alpha, \beta > 0$ corresponding to μ_2 and denote the Jacobi measures

$$\nu_1 = (1-x)^{\alpha-1}(1+x)^{\beta-1}dx\,, \quad \nu_2 = (1-x)^\alpha(1+x)^\beta dx\,, \tag{9}$$

with the following convention: if β does not appear explicitly in the expression of μ_2 (Cases 2–4), then assume

$$\alpha = \beta\,. \tag{10}$$

We use the fact that the derivative of a Jacobi polynomial is still a Jacobi polynomial (see [10]):

$$P_n'(\nu_1, x) = nP_{n-1}(\nu_2, x)\,. \tag{11}$$

Consider Case 1. Take

$$\mu_1 = |x-\xi_1|(1-x)^{\alpha-1}(1+x)^{\beta-1}dx, \quad \mu_2 = \frac{1}{|x-\xi_2|}(1-x)^\alpha(1+x)^\beta dx + M\delta(\xi_2)\,.$$

Since μ_2 is 0-coherent with ν_1 (see Case 1 of Table 1), we have

$$P_n(\mu_2, x) = \frac{P_{n+1}'(\nu_1, x)}{n+1} - \sigma_n^{(n)}(\nu_1, \mu_2)\frac{P_n'(\nu_1, x)}{n}\,,$$

so that using (11),

$$P_n(\mu_2, x) = P_n(\nu_2, x) - \sigma_n^{(n)}(\nu_1, \mu_2)P_{n-1}(\nu_2, x)\,.$$

But ν_2 is 0-coherent with μ_1 (see Case 2, Table 1). Then,

$$P_n(\nu_2, x) = \frac{P_{n+1}'(\mu_1, x)}{n+1} - \sigma_n^{(n)}(\mu_1, \nu_2)\frac{P_n'(\mu_1, x)}{n}\,,$$

so that,

$$P_n(\mu_2,x)=\frac{P'_{n+1}(\mu_1,x)}{n+1}-\left(\sigma_n^{(n)}(\mu_1,\nu_2)+\sigma_n^{(n)}(\nu_1,\mu_2)\right)\frac{P'_n(\mu_1,x)}{n}$$
$$+\,\sigma_n^{(n)}(\nu_1,\mu_2)\sigma_{n-1}^{(n-1)}(\mu_1,\nu_2)\frac{P'_{n-1}(\mu_1,x)}{n-1}\,,\qquad n\geq 2\,.$$

This proves that (μ_1,μ_2) is 1-coherent with coherence parameters

$$\sigma_n^{(n)}(\mu_1,\mu_2)=\sigma_n^{(n)}(\mu_1,\nu_2)+\sigma_n^{(n)}(\nu_1,\mu_2)\,,$$
$$\sigma_n^{(n-1)}(\mu_1,\mu_2)=-\sigma_n^{(n)}(\nu_1,\mu_2)\sigma_{n-1}^{(n-1)}(\mu_1,\nu_2)\,.$$

In the same way, if we combine coherent pairs of Table 1 or Table 2 we get all the pairs that appear in Proposition 2. We omit the details and summarize the coherence parameters $\sigma_n^{(n-j)}(\mu_1,\mu_2)$ using the notation of (9) and (10):

Case	$\sigma_n^{(n)}(\mu_1,\mu_2)$	$\sigma_n^{(n-1)}(\mu_1,\mu_2)$
1	$\sigma_n^{(n)}(\mu_1,\nu_2)+\sigma_n^{(n)}(\nu_1,\mu_2)$	$-\sigma_{n-1}^{(n-1)}(\mu_1,\nu_2)\sigma_n^{(n)}(\nu_1,\mu_2)$
2	$\sigma_n^{(n)}(\nu_1,\mu_2)$	$\sigma_n^{(n-1)}(\mu_1,\nu_2)$
3	$\sigma_n^{(n)}(\mu_1,\nu_2)$	$\sigma_n^{(n-1)}(\nu_1,\mu_2)$
4	0	$\sigma_n^{(n-1)}(\mu_1,\nu_2)+\sigma_n^{(n-1)}(\nu_1,\mu_2)$

Case	$\sigma_n^{(n-2)}(\mu_1,\mu_2)$	$\sigma_n^{(n-3)}(\mu_1,\mu_2)$
1	0	0
2	$-\sigma_{n-1}^{(n-2)}(\mu_1,\nu_2)\sigma_n^{(n)}(\nu_1,\mu_2)$	0
3	$-\sigma_{n-2}^{(n-2)}(\mu_1,\nu_2)\sigma_n^{(n-1)}(\nu_1,\mu_2)$	0
4	0	$-\sigma_{n-2}^{(n-3)}(\mu_1,\nu_2)\sigma_n^{(n-1)}(\nu_1,\mu_2)$

□

From the proof of Proposition 2 it can be seen that in all the cases the coherence parameters $\sigma_n^{(n-j)}(\mu_1,\mu_2)$ are sums or products of the parameters involved in the coherence or symmetrically coherence relations, which are convergent as $n\to\infty$ (see [5]–[8]). Hence, $\sigma_n^{(n-j)}(\mu_1,\mu_2)$ have finite limits when $n\to\infty$, fact which can be used in the study of the asymptotics of $\{Q_n\}$. Actually, following the scheme proposed in [8] we can prove

Theorem 1. *Let (μ_1,μ_2) be a pair as in Proposition 2. Then,*

$$\lim_{n\to\infty}\frac{Q_n(x)}{P_n(\mu_2,x)}=\frac{1}{\Phi'(x)}$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus [-1,1]$ *where* $\Phi(x) = (x + \sqrt{x^2-1})/2$ *with* $\sqrt{x^2-1} > 0$ *when* $x > 1$.

Open questions. In this paper some non-trivial k-coherent pairs of measures are given; still the problem of the complete classification of all k-coherent pairs (with $k \geq 1$) remains open and has an independent interest. In this sense, in [4] the authors have obtained all the 1-coherent pairs when one of the measures is classic (i.e., Jacobi, Laguerre or Hermite).

Furthermore, the approach described in this work can be applied to obtain k-coherent pairs with unbounded support. Asymptotic properties for coherent and symmetrically coherent pairs with unbounded support have been obtained in [6] and [1], respectively, but not much more in this direction is known.

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