Steps in Differential Geometry, Proceedings of the Colloquium on Differential Geometry, 25–30 July, 2000, Debrecen, Hungary

ON TOTALLY UMBILICAL HYPERSURFACE WITH CONHARMONIC CURVATURE TENSOR

FÜSUN ÖZEN AND SEZGIN ALTAY

ABSTRACT. The purpose of this paper is to study conharmonically recurrent Weyl spaces corresponding to the tensor K_{hijk}. In Section II, some relations which are needed in Section III are obtained. In Section III, it is shown that while the totally umbilical hypersurface W_n of the recurrent Weyl space is conharmonically Ricci recurrent, W_n is recurrent. After then, it is proved that conharmonically recurrent Weyl space is also conformally recurrent, but the converse is true if and only if the condition $\dot{\nabla}_l R = \lambda_l R$ holds.

1. INTRODUCTION

The geometrical features of Weyl's theory consists of a space-time manifold W_n on which is defined a symmetric (torsion free) linear connection Γ and, in the first instance, a Lorentz metric g. The manifold W_n and all structures on W_n are assumed smooth. The connection Γ is not assumed to be a metric connection with respect to g or any other metric on W_n. Rather, Γ and g are related in such a way as to recreate Weyl's original idea that parallel transport, with respect to Γ, of a tangent vector k at $p \in W_n$ along a curve c to a point $q \in W_n$ may result in change of the length of k (with respect to g). However the ratio of the lengths of k at p and q, where this makes sense (i.e., if k is non-null), depends only on p, q and c and not on k . Let W_n be a manifold of dimension n ($n > 2$) and let Γ be a symmetric linear connection on W_n. Then Γ is called a Weyl connection if there exists a metric g on W_n such that $\nabla g = g \otimes T$ for some 1-form T on W_n, where ∇ denotes covariant differentiaton with respect to Γ. If W_n admits a Weyl connection, it is called a Weyl manifold.

In local coordinates this reads $\nabla_k g_{ij} = 2T_k g_{ij}$ where in coordinate notation ∇_k denotes the covariant derivative with respect to Γ, and is just means that the tensor g is recurrent with respect to Γ with recurrence 1-form T . With g_{ij}, Γ, and the complementary vector T_k, this is equivalent to the following expression for the connection associated with Γ:

$$\Gamma^h_{kl} = \frac{1}{2} g^{hm}(\partial_l g_{mk} + \partial_k g_{ml} - \partial_m g_{kl}) - (\delta^h_k T_l + \delta^h_l T_k - g^{hm} g_{kl} T_m), \tag{1.1}$$

Key words and phrases. Totally umbilical Weyl hypersurface, conharmonic curvature tensor, conharmonically and conformally recurrent Weyl hypersurface.

Now suppose that Γ is fixed but g and T are changed to $\breve{g} = \lambda^p g$ and $\breve{T} = T + \partial(\ln\lambda)$ where λ is real valued function on W_n. Then $\nabla\breve{g} = \breve{g}\otimes\breve{T}$ still holds as does (1.1) for $\Gamma, \breve{g}$ and $\breve{T}$. Such changes $(g, T) \to (\lambda^p g, T + \partial(\ln\lambda))$ are the gauge transformations introduced by Weyl [1], [2].

Suppose that the metrics of W_n and W_{n+1} are elliptic and that they are given by $g_{ij}du^i du^j$ and $g_{ab}dx^a dx^b$, respectively, which are connected by the relation

$$g_{ij} = g_{ab}x_i^a x_j^b \ (i, j = 1, 2, ..., n; a, b = 1, 2, ..., n+1) \tag{1.2}$$

where x_i^a denotes the covariant derivative of x^a with respect to u^i. On the basis of (1.1) [3] and [4], using T_k as a normalizer Zlatanov introduced in [5] a prolonged covariant differentiation of the satellites A of g_{ij} with weight $\{p\}$ by the law

$$\dot{\nabla}_k A = \nabla_k A - pT_k A. \tag{1.3}$$

One can show that the prolonged covariant derivative of A, relative to W_n and W_{n+1}, is related by

$$\dot{\nabla}_k A = x_k^c \dot{\nabla}_c A. \tag{1.4}$$

By [5] we have $\dot{\nabla}_k g_{ij} = 0$ and $\dot{\nabla}_k g^{ij} = 0$ where g^{ij} is the reciprocal tensor of g_{ij}.

Let n^a be the contravariant components of vector field in W_{n+1} normal to W_n and let it normalized by the condition $g_{ab}n^a n^b = 1$. The moving frame $\left\{x_a^i, n_a\right\}$ in W_n, reciprocal to moving frame $\{x_i^a, n^a\}$ is defined by the relations [2]

$$n^a n_a = 1, \ n_a x_i^a = 0, \ n^a x_a^i = 0, \ x_i^a x_a^j = \delta_i^j. \tag{1.5}$$

Differentiating covariantly of each side of $(1.5)_4$ with respect to u^k and remembering that the weight of x_i^a is $\{0\}$, the following form

$$\dot{\nabla}_k x_i^a = \nabla_k x_i^a = w_{ik} n^a \tag{1.6}$$

holds.

The curvature tensor of the hypersurface R_{ijk}^h is given by

$$R_{ijk}^h = \frac{\partial\Gamma_{ik}^h}{\partial x^j} - \frac{\partial\Gamma_{ij}^h}{\partial x^k} + \Gamma_{mj}^h\Gamma_{ik}^m - \Gamma_{mk}^h\Gamma_{ij}^m \tag{1.7}$$

where $R_{ijk}^h = g^{hm}R_{mijk}$.

2. Totally umbilical hypersurface immersed in a recurrent Weyl space

If W_n admits of a tensor field $T_{...}$ such that

$$\dot{\nabla}_k T... = \lambda_k T_{...} \tag{2.1}$$

where λ_k is a non-zero vector field of W_n, then W_n is called a T-recurrent Weyl space and is denoted by $T_n - W$.

We note that, since the prolonged covariant derivative preserves the weight, ϕ_s is a satellite of g_{ij} with weight $\{0\}$.

A hypersurface of a Weyl space is called totally umbilical if $w_{ij} = \rho g_{ij}$ where ρ is a satellite of g_{ij} with weight $\{-1\}$. From this definition it follows that $\rho = \frac{M}{n}$ where M is the mean curvature of the hypersurface defined by $M = w_{ij} g^{ij}$. A hypersurface of a Weyl space is called totally geodesic if $w_{ij} = 0$.

The generalization of Gauss and Mainardi-Codazzi equations have the following forms [6]

$$R_{hijk} = \Omega_{hijk} + \bar{R}_{abcd} x^a_h x^b_i x^c_j x^d_k \tag{2.2}$$

$$\dot{\nabla}_k w_{ij} - \dot{\nabla}_j w_{ik} + \bar{R}_{abcd} x^b_i x^c_j x^d_k n^a = 0 \tag{2.3}$$

where $\bar{R}_{abcd}$ is the covariant curvature tensor of W_{n+1} and Ω_{hijk} is the Sylvestrian of w_{ij} defined by $\Omega_{hijk} = w_{hj} w_{ik} - w_{hk} w_{ij}$. These formulae have also been obtained in [7].

Let W_n be a hypersurface of recurrent Weyl space W_{n+1} with recurrence vector ϕ_a which is not orthogonal to the hypersurface W_n. If we denote the tangential component of ϕ_a by ϕ_r, then we have

$$\phi_k = x^a_k \phi_a. \tag{2.4}$$

Since W_{n+1} is recurrent-Weyl space, we can write

$$\lambda_r \bar{R}_{abcd} = \dot{\nabla}_r \bar{R}_{abcd} = x^e_r \dot{\nabla}_e \bar{R}_{abcd}. \tag{2.5}$$

Using (2.2), we get

$$\dot{\nabla}_r R_{hijk} = \dot{\nabla}_r \Omega_{hijk} + \dot{\nabla}_r (\bar{R}_{abcd} x^a_h x^b_i x^c_j x^d_k). \tag{2.6}$$

With the help of the equations (1.6) and (2.5), the formula (2.6) can be brought in the following form [6]

$$\begin{aligned} \dot{\nabla}_r R_{hijk} = {} & \dot{\nabla}_r \Omega_{hijk} + \phi_e \bar{R}_{abcd} x^a_h x^b_i x^c_j x^d_k x^e_r + \bar{R}_{abcd} x^b_i x^c_j x^d_k w_{hr} n^a + \\ & + \bar{R}_{abcd} x^a_h x^c_j x^d_k w_{ir} n^b + \bar{R}_{abcd} x^a_h x^b_i x^d_k w_{jr} n^c + \\ & + \bar{R}_{abcd} x^a_h x^b_i x^c_j w_{kr} n^d \ . \end{aligned} \tag{2.7}$$

If we use the equations (2.2),(2.3),(2.4),(2.7) and remembering that $w_{ij} = \frac{M}{n} g_{ij}$ and M is scalar invariant, then we find

$$\begin{aligned} \dot{\nabla}_r R_{hijk} = {} & \phi_r R_{hijk} + \frac{M}{n^2}[(\dot{\nabla}_j M) G_{hirk} + (\dot{\nabla}_k M) G_{hijr} + (\dot{\nabla}_i M) G_{kjrh} \\ & + (\dot{\nabla}_h M) G_{kjir}] + \frac{2M}{n^2}(\dot{\nabla}_r M) G_{hijk} - \frac{M^2}{n^2} \phi_r G_{hijk} \end{aligned} \tag{2.8}$$

where $G_{hijk} = g_{hj} g_{ik} - g_{hk} g_{ij}$. Multiplying (2.8) by g^{hk} and g^{ij}, we obtain, respectively

$$\begin{aligned} \dot{\nabla}_r R_{ij} = {} & \phi_r R_{ij} + \frac{M}{n^2}[(2-n)(\dot{\nabla}_j M) g_{ir} - 2n(\dot{\nabla}_r M) g_{ij} + \\ & + (2-n)(\dot{\nabla}_i M) g_{jr}] + \frac{M^2}{n^2}(n-1)\phi_r g_{ij} \end{aligned} \tag{2.9}$$

$$\dot{\nabla}_r R = \phi_r R + \frac{2M}{n^2}(\dot{\nabla}_r M)(-n^2 - n + 2) + \frac{M^2}{n}(n-1)\phi_r. \tag{2.10}$$

3. Conharmonic curvature tensor of a Weyl space

Let $W_n(g_{ij}, T_k)$ and $\bar{W}_n(\bar{g}_{ij}, \bar{T}_k)$ be two Weyl spaces with connections ∇_k and $\bar{\nabla}_k$, respectively, and let the map $\tau : W_n \to \bar{W}_n$ be a conformal mapping. As a special case, let the transformed expressions of the fundamental metric tensor g_{ij} and the coefficients of Weyl connection Γ^i_{kl} be the following forms [8]

$$\bar{g}_{ij} = g_{ij}\ ,\ \bar{g}^{ij} = g^{ij}, \tag{3.1}$$

$$\bar{\Gamma}^i_{kl} = \Gamma^i_{kl} + \delta^i_k P_l + \delta^i_l P_k - g_{kl} g^{im} P_m, \tag{3.2}$$

where the vector P_k is called the vector of conformal mapping such as

$$P_k = T_k - \bar{T}_k. \tag{3.3}$$

Let us seek the differentiable harmonic function A with weight $\{p\}$ defined by [9]

$$\bar{A} = e^{c\int P_j du^j} A,\ \ c = \frac{2-n-2p}{2}. \tag{3.4}$$

and then, we have the following expression

$$g^{kl}\nabla_k P_l + \frac{1}{2}(n-2)P^k P_k = 0. \tag{3.5}$$

Since a conformal transformation with P_k satisfying (3.5) transforms a harmonic function into a harmonic one in above sense: (3.4), we call it conharmonic transformation.

The conharmonic curvature tensor is in the following form [10]

$$\begin{aligned} K^h_{ijk} = R^h_{ijk} &- \frac{1}{n}(\delta^h_k R_{[ij]} - \delta^h_j R_{[ik]} + g_{ij} g^{hm} R_{[mk]} - g_{ik} g^{hm} R_{[mj]} + \\ &+ 2\delta^h_i R_{[kj]}) - \frac{1}{(n-2)}(\delta^h_k R_{(ij)} - \delta^h_j R_{(ik)} + g_{ij} g^{hm} R_{(mk)} - \\ &- g_{ik} g^{hm} R_{(mj)})\ . \end{aligned} \tag{3.6}$$

The conharmonic curvature tensor K^h_{ijk} of a Weyl space satisfies the following condition [10]

$$K_{ij} = \frac{1}{2-n} g_{ij} R \tag{3.7}$$

where K_{ij} is conharmonic Ricci tensor.

If a Weyl hypersurface W_n immersed in a recurrent Weyl space W_{n+1} is totally geodesic, then the hypersurface is recurrent Weyl with recurrence vector λ_r [6].

A totally geodesic hypersurface W_n immersed in a recurrent Weyl space W_{n+1} is conharmonically recurrent $(n > 2)$.

Proposition 3.1. If W_n is conharmonically Ricci-recurrent (may not be Ricci-recurrent), then the expression $\phi_r - 2T_r$ is locally gradient $(n > 2)$.

Proof. From (3.7) and (2.1), we get $\dot{\nabla}_r R = \phi_r R$. Thus, remembering that the scalar curvature R is scalar invariant with weight $\{-2\}$, using (1.3), we have

$$\phi_s - 2T_s = \frac{\nabla_s R}{R} \ (R = c_1 \bar{R};\ c_1 \neq 0,\ \text{const.})$$

where $\bar{R}$ is the scalar curvature of Weyl space W_{n+1}. Then, we say that $\phi_s - 2T_s$ is locally gradient.

Theorem 3.1. If a totally umbilical Weyl hypersurface W_n immersed in a recurrent Weyl space W_{n+1} is a conharmonically Ricci-recurrent, then W_n is a conharmonically recurrent Weyl space $(n > 2)$.

Proof. Let W_n be a totally umbilical hypersurface of a recurrent Weyl space W_{n+1}. Let W_n be also conharmonically Ricci-recurrent. Multiplying (2.10) by g_{ij}, we get

$$\dot{\nabla}_r(g_{ij}R) = \phi_r[Rg_{ij} + \frac{M^2}{n}(n-1)g_{ij}] + \frac{2M}{n^2}g_{ij}(\dot{\nabla}_r M)(-n^2 - n + 2). \tag{3.8}$$

Using the equation (3.7) in the form (3.8), we find

$$\frac{1}{n^2}g_{ij}(n-1)[nM^2\phi_r - 2(n+2)M(\dot{\nabla}_r M)] = 0, \tag{3.9}$$

then, we obtain

$$\frac{\dot{\nabla}_r M}{M} = \frac{n}{2(n+2)}\phi_r. \tag{3.10}$$

On the other hand, from the equation (2.9),

$$\dot{\nabla}_r R_{[jk]} = \phi_r R_{[jk]} \tag{3.11}$$

and

$$\begin{aligned} \dot{\nabla}_r R_{(jk)} &= \phi_r R_{(jk)} + \frac{M^2}{n^2}(n-1)\phi_r g_{jk} \\ &+ \frac{M}{n^2}[(2-n)(\dot{\nabla}_k M)g_{jr} - 2n(\dot{\nabla}_r M)g_{jk} + (2-n)(\dot{\nabla}_j M)g_{kr}]. \end{aligned} \tag{3.12}$$

Taking the prolonged covariant derivative of (3.6) with respect to u^r and putting the equations (3.11) and (3.12) in this expression, then we get

$$\begin{aligned} \dot{\nabla}_r K_{hijk} &= \dot{\nabla}_r R_{hijk} + \phi_r(K_{hijk} - R_{hijk}) - \\ &- \frac{2M^2}{n^2(n-2)}(n-1)\phi_r G_{ihjk} - \frac{M}{n^2(n-2)}[(2-n)(\dot{\nabla}_i M)G_{kjhr} + \\ &+ (2-n)(\dot{\nabla}_j M)G_{hikr} - 4n(\dot{\nabla}_r M)G_{hikj} + \\ &+ (2-n)(\dot{\nabla}_h M)G_{irjk} - (2-n)(\dot{\nabla}_k M)G_{hijr}]\ . \end{aligned} \tag{3.13}$$

Using (2.8) in (3.13), we obtain

$$\dot{\nabla}_r K_{hijk} = \phi_r K_{hijk} - \frac{1}{n(n-2)} G_{hijk} M[(\dot{\nabla}_r M)\frac{2(n+2)}{n} - \phi_r M]. \tag{3.14}$$

Using the expression (3.10), we get

$$\dot{\nabla}_r K_{hijk} = \phi_r K_{hijk}. \tag{3.15}$$

Corollary 3.1. If a totally umbilical Weyl hypersurface W_n immersed in a recurrent Weyl space W_{n+1} is a conharmonically Ricci-recurrent, then W_n is a recurrent Weyl space $(n > 2)$.

Proof. Multiplying (2.8) by g^{hr} and g^{ik} and using the equation (3.10), we obtain

$$g^{hr} g^{ik} (\dot{\nabla}_r R_{hijk} - \phi_r R_{hijk}) = \frac{(n-1)(n-2)}{2n^2} M^2 \phi_j. \tag{3.16}$$

If we multiply the expression (3.13) by g^{hr} and g^{ik} and use the equations (3.10), (3.15) and (3.16), we get $M^2\phi_j = 0$. From this, since $\phi_j \neq 0$, $(n > 2)$, we find $M = 0$. In this case, using $M = 0$ and the expression (3.13), the proof is completed.

Corollary 3.2. If a totally umbilical Weyl hypersurface W_n immersed in a recurrent Weyl space W_{n+1} is a conharmonically recurrent, then W_n is a recurrent Weyl space $(n > 2)$.

Proof. Conharmonically recurrent Weyl space is also conharmonically Ricci recurrent. From Corollary 3.1, the result is clear.

Corollary 3.3. If a totally umbilical Weyl hypersurface W_n immersed in a recurrent Weyl space W_{n+1} is a Ricci recurrent, then W_n is a recurrent Weyl space $(n > 2)$.

Proof. Since Ricci recurrent Weyl space is also conharmonically Ricci recurrent, from Corollary 3.1, the proof is clear.

Theorem 3.2. A conharmonically recurrent Weyl space is also a conformally recurrent Weyl space. Conversely, a conformally recurrent Weyl space with its recurrence vector field ϕ_r is conharmonically recurrent if its scalar curvature satisfies $\dot{\nabla}_r R = \lambda_r R$.

Proof. Suppose that W_n be a conharmonically recurrent Weyl space. The so-called conformal curvature tensor introduced by F. Özen and S.A. Uysal [12], is in the following form

$$C_{hijk} = R_{hijk} + \frac{2}{n(n-2)}[g_{hk}R_{[ij]} - g_{hj}R_{[ik]} + g_{ij}R_{[hk]} - g_{ik}R_{[hj]} - (n-2)g_{hi}R_{[kj]}]$$
$$- \frac{1}{n-2}(g_{hk}R_{ij} - g_{hj}R_{ik} + g_{ij}R_{hk} - g_{ik}R_{hj})$$
$$+ \frac{R}{(n-1)(n-2)}(g_{hk}g_{ij} - g_{hj}g_{ik}).$$

The conformal tensor C_{hijk} and conharmonic tensor K_{hijk} are related by the following condition [12]

$$C_{hijk} = K_{hijk} - \frac{R}{(n-1)(2-n)}(g_{hk}g_{ij} - g_{hj}g_{ik}). \tag{3.17}$$

Transvecting (3.17) with g^{hk} and g^{ij} and using (3.7), we have $\dot{\nabla}_r R = \phi_r R$. Consequently, from (3.17), we find

$$\dot{\nabla}_r C_{hijk} = \phi_r C_{hijk}. \tag{3.18}$$

Hence, every conharmonically recurrent Weyl space is conformally recurrent.

Conversely, let W_n be a conformally recurrent Weyl space with the recurrence vector ϕ_r. In this case, the equation (3.18) holds. Thus from (3.17), we get

$$\dot{\nabla}_r C_{hijk} - \phi_r C_{hijk} = \dot{\nabla}_r K_{hijk} - \phi_r K_{hijk} - \frac{(g_{hk}g_{ij} - g_{hj}g_{ik}}{(n-1)(2-n)}(\dot{\nabla}_r R - \phi_r R).$$

Hence, $\dot{\nabla}_r K_{hijk} = \phi_r K_{hijk}$ if $\dot{\nabla}_r R = \phi_r R$ is satisfied.

References

[1] G.S. Hall: Weyl manifolds and connections, J. Math. Phys. **33**, No.7, (1992), 2633-2638.
[2] A. Norden: Affinely Connected Spaces, GRMFL Moscow, (1976).
[3] V. Hlavaty: Theorie d'immersion d'une W_m dans W_n, Ann. Soc. Polon. Math. **21** (1949), 196-206.
[4] G. Zlatanov: Networks in the two-dimensional space of Weyl. Comptes Rendus de l'Academie Bulgare des Sciences. **29** (1976), 619-622, (in Russian).
[5] G. Zlatanov: Nets in the n-dimensional space of Weyl, C.R. Acad. Bulgare Sci., **41** No.10, (1988), 29-32.
[6] E.Ö. Canfes and A. Özdeger: Some applications of prolonged differentiation in Weyl spaces, Journal of Geometry, **60** (1997), 7-16.
[7] H. Pedersen and Y.S. Poon, A. Swann: Einstein-Weyl Deformations and Submanifolds. Preprint No.11, Marts 1995.
[8] G. Zlatanov: On the Conformal Curvature Geometry of Nets in an n-dimensional Weyl space, Izv. Vyssh. Uchebn. Zaved. Math., No. 8, (1991), 19-26.
[9] F. Özen and S.A. Uysal: On conharmonic transformations of Weyl spaces, Tensor (in print).
[10] F. Özen and S. A. Uysal: Conharmonically recurrent and birecurrent Weyl spaces, Tensor (to appear).
[11] T. Miyazawa and G. Chuman: On certain subspaces of Riemannian recurrent spaces, Tensor,N.S., **23** (1972).
[12] F. Özen: Conharmonic Transformations of Weyl Spaces, PhD Thesis, September, 1999.

Istanbul Technical University, Faculty of Sciences and Letters, Department of Mathematics, 80626 Maslak Istanbul, Turkey
E-mail address: fozen@sariyer.cc.itu.edu.tr