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APPLICATIONS OF BOTT CONNECTION TO FINSLER GEOMETRY

TADASHI AIKOU

1. Introduction

In the present paper, we shall investigate connections theory in complex Finsler geometry. The basic tool in this paper is the so-called *Bott connection* which is a partial connection defined by a splitting on the fundamental sequence of vector bundles (see the definition below).

Let $\pi : E \to M$ be a holomorphic vector bundle over a complex manifold. We denote by T_E and T_M the holomorphic tangent bundles of E and M respectively. Moreover we denote by $T_{E/M}$ the relative tangent bundle of the holomorphic projection π. Then we get the fundamental sequence of vector bundles:

$$\mathbb{O} \to T_{E/M} \xrightarrow{i} T_E \xrightarrow{d\pi} \pi^* T_M \to \mathbb{O}. \tag{1.1}$$

We also denote by $\Omega^1_\bullet$ the corresponding holomorphic cotangent bundle.

We take an open covering $\mathcal{U} = \{U, V, \cdots\}$ of M with a local frame field $s_U = (s_1, \cdots, s_r)$ of E on each U. The covering $\{(U, s_U)\}_{U\in\mathcal{U}}$ induces a complex coordinate system (z_U, ξ_U) on each $\pi^{-1}(U)$, where $z_U = (z^1_U, \cdots, z^n_U)$ is a coordinate on U and $\xi_U = (\xi^1_U, \cdots, \xi^r_U)$ is the fibre coordinate on $\pi^{-1}(z) = E_z$ $(z \in U)$. If we denote by

$$g_{UV} = \left({g_{UV}}^i_j\right) : U \cap V \to GL(r, \mathbb{C})$$

the transition functions relative to the covering $\{(U, s_U)\}_{U\in\mathcal{U}}$, the coordinate transformation law is given by the form

$$\begin{cases} z^\alpha_U = z^\alpha_U(z_V) \\ \xi^i_U = \sum {g_{UV}}^i_j(z_V)\xi^j_V. \end{cases}$$

We define a local section $\{\sigma_{UV}\}$ of $\pi^*\Omega^1_M \otimes T_{E/M}$ over $\pi^{-1}(U \cap V)$ by

$$\sigma_{UV} = \sum_{i,\alpha} \left(\sum_j \frac{\partial {g_{UV}}^i_j}{\partial z^\alpha_V} \xi^j_V \right) dz^\alpha_V \otimes \frac{\partial}{\partial \xi^i_U}.$$

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Then we can easily verify that the family $\{\sigma_{UV}\}$ satisfies $\sigma_{UV} + \sigma_{VW} + \sigma_{WU} = 0$ on $U \cap V \cap W \neq \phi$. A splitting h of the sequence (1.1) is defined by a local sections $\{N_U\}$ of $\pi^*\Omega^1_M \otimes T_{E/M}$ over $\pi^{-1}(U)$ satisfying

$$N_V - N_U = \sigma_{UV}. \tag{1.2}$$

If a splitting $h : \pi^*T_M \to T_E$ is given in this sequence, we have a natural connection $\nabla : \Gamma(T_{E/M}) \to \Gamma(T_{E/M} \otimes \Omega^1_E)$ on the bundle $\varpi : T_{E/M} \to E$ from the given splitting h. In Finsler geometry, such a connection ∇ plays an important role.

2. Bott connections

2.1. **Ehresmann connections.** Let $\pi : E \to M$ be a holomorphic vector bundle over a complex manifold M. An *Ehresmann connection* H for π is a smooth distribution $H \subset T_E$ for which the morphism $d\pi$ in the sequence (1.1) induces an isomorphism $H \cong \pi^*T_M$. If an Ehresmann connection H is given for π, we get a smooth splitting

$$T_E = H \oplus T_{E/M}. \tag{2.1}$$

For a fixed Ehresmann connection H, the smooth splitting (2.1) induces the dual splitting $\Omega^1_{\mathcal{X}} = H^* \oplus \pi^*\Omega^1_M$, and thus the differential operators $d : \mathcal{O}_E \to \Omega^1_E$ is decomposed as $d = d^v + d^h$ by the differential $d^h : \mathcal{O}_{\mathcal{X}} \to \pi^*\Omega^1_M$ along H and the differential $d^v : \mathcal{O}_E \to \Omega^1_{E/M}$ along vertical direction. We also decompose the operators ∂ and $\bar\partial$ as $\partial = \partial^v + \partial^h$ and $\bar\partial = \bar\partial^v + \bar\partial^h$ respectively.

We denote by $\mathcal{S}$ the sheaf of germs of linear functionals along the fibres of π. A splitting h is defined by the action of ∂^h on $\mathcal{S}$. If we put $\partial^h\xi^i = -\sum N^i_\alpha(z,\xi)dz^\alpha$ on each $\pi^{-1}(U)$, the functions $\{N^i_\alpha\}$ satisfy the homogeneity

$$N^i_\alpha(z, \lambda\xi) = \lambda N^i_\alpha(z,\xi) \tag{2.2}$$

for all $\lambda \in \mathbb{C}$. For such functions $\{N^i_\alpha\}$, we define a local section N_U of $\pi^*\Omega^1_M \otimes T_{E/M}$ by $N_U = \sum N^i_\alpha dz^\alpha \otimes (\partial/\partial\xi^i)$, then we can easily verify that $\{N_U\}$ satisfies (1.2). The splitting $h : \pi^*T_M \to T_{E/M}$ is defined by the *lift* X_α of natural frame fields $\{\partial/\partial z^\alpha\}$:

$$X_\alpha = \frac{\partial}{\partial z^\alpha} - \sum_i N^i_\alpha(z,\xi)\frac{\partial}{\partial \xi^i}. \tag{2.3}$$

A splitting h is said to be *linear* if $\partial^h\mathcal{S} \subset \mathcal{S} \otimes H^*$. In this case, since the functions $N^i_\alpha(z,\xi)$ is linear in (ξ^i) along the fibre E_z, there exist functions $\gamma^i_{j\alpha}(z)$ satisfying $N^i_\alpha = \sum \gamma^i_{j\alpha}(z)\xi^j$. Then we see that the $(1,0)$-form $\theta^i_j(z) = \sum \gamma^i_{j\alpha}(z)dz^\alpha$ defines a connection $\underline{\nabla} : \Gamma(E) \to \Gamma(E \otimes T^*_M)$ of E. In the present paper, we shall consider the case where h is a *non-linear* case.

An Ehresmann connection H of π is said to be *integrable* if it the distribution H is closed under the Lie bracket operator: $[\Gamma(H), \Gamma(H)] \subset \Gamma(H)$. The obstruction for H to be integrable is given by the torsion form $T = (T^i)$ defined by

$$T^i = -\frac{1}{2}\sum_{\alpha,\beta} R^i_{\alpha\beta} dz^\alpha \wedge dz^\beta - \sum_{\alpha,\beta} R^i_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta,$$

where the *inetgrability tensors* $R^i_{\alpha\beta}$ and $R^i_{\alpha\bar{\beta}}$ are defined by

$$R^i_{\alpha\beta} = X_\beta N^i_\alpha - X_\alpha N^i_\beta, \;\; R^i_{\alpha\bar{\beta}} = X_{\bar{\beta}} N^i_\alpha$$

respectively with the conjugate $X_{\bar{\alpha}}$ of X_α. Moreover, we define $\Phi = (\Phi^i_{\bar{j}})$ by

$$\Phi^i_{\bar{j}} = \sum_\alpha \frac{\partial N^i_\alpha}{\partial \bar{\xi}^j} dz^\alpha. \tag{2.4}$$

It is easily seen that, if $T = 0$ and $\Phi = 0$, then H is integrable and holomorphic.

Remark 2.1. If the bundle $\pi : E \to M$ admits a holomorphically integrable Ehresmann connection H, we have the holomorphic splitting $T_E = \pi^* T_M \oplus T_{E/M}$. Furthermore the fundamental group $\pi_1(M)$ acts on the canonical fibre X by complex automorphism, and so $\pi : E \to M$ is the fibre bundle associated with the universal covering $\tilde{M}$, that is, $E = (\tilde{M} \times X)/\pi_1(M)$.

2.2. **Bott connections.** A morphism $D : \Gamma(T_{E/M}) \to \Gamma(T_{E/M} \otimes \Omega^1_E)$ is called a *partial connection* on the relative tangent bundle $\varpi : T_{E/M} \to E$ if it satisfies the Leibniz condition $D(fs) = d^h f \otimes s + f Ds$ for $^\forall s \in \Gamma(T_{E/M})$ and $^\forall f \in C^\infty(E)$. An Ehresmann connection H for π induces a partial connection D on $T_{E/M}$ as follows.

Definition 2.1. ([Ai5]) A partial connection D of $(1,0)$-type on $T_{E/M}$ defined by

$$D_X Y = P([X, Y])$$

for $^\forall X \in \Gamma(H)$ and $^\forall Y \in \Gamma(T_{E/M})$ is called a *complex Bott connection*, where $P : T_E \to T_{E/M}$ is the natural projection.

The connection form $\omega = (\omega^i_j)$ of D is given by the $(1,0)$-form $\omega^i_j = \sum \Gamma^i_{j\alpha} dz^\alpha$ with

$$\Gamma^i_{j\alpha}(z,\xi) = \frac{\partial N^i_\alpha}{\partial \xi^j}. \tag{2.5}$$

By the homogeneity (2.2) of N^i_α, we have

$$\sum \Gamma^i_{j\alpha}(z,\xi)\xi^j = N^i_\alpha. \tag{2.6}$$

The curvature form $\Omega^D = D^2$ of D is given by $\Omega^D = d^h\omega + \omega \wedge \omega$. Then we have

Proposition 2.1. *The curvature Ω^D of the Bott connection D is given by*

$$\Omega^D = \Pi - \Phi \wedge \overline{\Phi}, \tag{2.7}$$

where $\Pi = (\Pi^i_j)$ is defined by

$$\Pi^i_j = \frac{\partial T^i}{\partial \xi^j} = -\frac{1}{2}\sum_{\alpha,\beta} \frac{\partial R^i_{\alpha\beta}}{\partial \xi^j} dz^\alpha \wedge dz^\beta - \sum_{\alpha,\beta} \frac{\partial R^i_{\alpha\bar{\beta}}}{\partial \xi^j} dz^\alpha \wedge d\bar{z}^\beta.$$

Proof. Since $X_\beta \Gamma^i_{j\alpha} = \partial(X_\beta N^i_\alpha)/\partial\xi^j + \sum_l \Gamma^i_{l\alpha}\Gamma^l_{j\beta}$, we have

$$\partial^h \omega^i_j + \sum_l \omega^i_l \wedge \omega^l_j = -\frac{1}{2}\sum_{\alpha,\beta} \frac{\partial R^i_{\alpha\beta}}{\partial z^j} dz^\alpha \wedge dz^\beta. \tag{2.8}$$

Similarly we have

$$\bar{\partial}^h \omega^i_j = -\sum_{\alpha,\beta} \frac{\partial R^i_{\alpha\bar{\beta}}}{\partial \xi^j} dz^\alpha \wedge d\bar{z}^\beta - \sum_l \Phi^i_l \wedge \overline{\Phi^l_{\bar{j}}}. \tag{2.9}$$

Hence we have $\Omega = \Pi - \Phi \wedge \bar{\Phi}$.

Q.E.D.

If the curvature Ω^D of D vanishes identically, then D is said to be *flat*.

3. Finsler geometry

3.1. **Finsler metrics.** Let $\pi : E \to M$ be a complex vector bundle with a complex structure $J_E \in \mathrm{End}(E)$ over a smooth manifold M.

Definition 3.1. A *Finsler metric* on E is a smooth assignment of a norm $\|\cdot\|_x$ to each fibre $E_x = \pi^{-1}(x)$ $(x \in M)$. A *complex Finsler metric* of a complex vector bundle (E, J_E) is a Finsler metric on E satisfying

$$\|(aI_E(x) + bJ_E(x))\,\xi\|_x = \sqrt{a^2+b^2}\,\|\xi\|_x \tag{3.1}$$

for $\forall \xi \in E_x$ and $\forall a, b \in \mathbb{R}$, where I_E is the identity morphism of E. The triplet $(E, J_E, \|\cdot\|)$ is called a *complex Finsler vector bundle*.

Let $E \otimes \mathbb{C}$ be the complexfication of E, and let $E \otimes \mathbb{C} = E^{1,0} \oplus \overline{E^{1,0}}$ be the canonical decomposition. The condition (3.1) is equivalent to

$$\left\|(a+\sqrt{-1}b)\xi\right\|_x = \sqrt{a^2+b^2}\,\|\xi\|_x\,, \tag{3.2}$$

where $a, b \in \mathbb{R}$ and $\xi \in E^{1,0}_x \cong \mathbb{C}^r$.

In the sequel, we assume that E is a holomorphic vector bundle of $\mathrm{rank}(E) = r$ (≥ 2) over a complex manifold M. For explicit expressions of Finsler metrics, we use the natural coordinate system on E induced from an open covering $\{(U, s_U)\}_{U \in \mathcal{U}}$.

Let $F : E \to \mathbb{R}$ be the fundamental function relative to the covering $\{(U, s_U)\}_{U\in\mathcal{U}}$ of $(E, \|\cdot\|)$ defined by $\|v\|_z^2 = F \circ \varphi_U(v)$, where $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{C}^r$ the local trivialization adapted to $\{(U, s_U)\}_{U\in\mathcal{U}}$. The function F satisfies the following conditions:

1. $F(z, \xi) \geq 0$ and $F(z, \xi) = 0$ if and only if $\varphi_U^{-1}(z, \xi) = 0$,
2. $F(z, \lambda\xi) = |\lambda|^2 F(z, \xi)$ for $^\forall \lambda \in \mathbb{C}$,
3. F is smooth on $E^\times = E - \{0\}$.

Then $\|\cdot\|$ is said to be *convex* if the Hermitian matrix $(F_{i\bar{j}})$ defined by

$$F_{i\bar{j}} = \frac{\partial^2 F}{\partial \xi^i \bar{\xi}^j} \tag{3.3}$$

is positive-definite. It is easily shown that the definition of convexity is independent on the choice of the open cover $\{\mathcal{U}, (s_U)\}$ of E. If $\|\cdot\|$ is convex, then $(F_{i\bar{j}})$ defines a Hermitian metric $\langle\cdot,\cdot\rangle$ on $\varpi : T_{E/M} \to E$ by

$$\left\langle \frac{\partial}{\partial \xi^i}, \frac{\partial}{\partial \xi^j} \right\rangle := F_{i\bar{j}}. \tag{3.4}$$

Remark 3.1. The complex structure J_E of E is naturally lifted to a complex structure $J_{\tilde{E}}$ on the bundle $T_{E/M} \cong \pi^* E = \tilde{E}$ by $J_{\tilde{E}} := \pi^* J_E$. If we denote by $g(X, Y)$ the real part of $\langle X, Y\rangle$, that is, $\langle X, Y\rangle = g(X, Y) + \sqrt{-1} g(J_{\tilde{E}} X, Y)$, the real metric g satisfies $g(J_{\tilde{E}} X, J_{\tilde{E}} Y) = g(X, Y)$, and thus the g is a so-called *generalized Finsler metric* on E by a theorem due to Ichijyō[Ic3] and Fukui[Fu].

The Hermitian metric defined by (3.4) induces a Kähler metric on each fibre $E_z \cong \mathbb{C}^r$. Hence the $\partial\bar{\partial}$-exact real $(1,1)$-form $\omega_E = \sqrt{-1}\partial\bar{\partial} F$ defines a pseudo-Kähler metric on the total space E whose restriction ω_z to each fibre E_z is a Kähler metric on E_z.

Example 3.1. Let $h = (h_{i\bar{j}}(z))$ be an arbitrary Hermitian metric on E. The function $L : E \to \mathbb{R}$ defined by

$$F(z, \xi) = \sum h_{i\bar{j}}(z) \xi^i \bar{\xi}^j \tag{3.5}$$

defines a convex Finsler metric on E. □

Example 3.2. Let $\mathcal{D}$ be a strictly convex domain of $\mathbb{C}^{n+1}$ with smooth boundary. The *Kobayashi-Royden metric* $K_\mathcal{D}$ is defined by $K_\mathcal{D}(z, \xi) := \inf\{1/R\}$, where the infimum is taken all holomorphic maps $\varphi : \Delta(R) \to M$ satisfying $\varphi(0) = z$ and $d\varphi(d/dt)_0 = \xi$ for $^\forall(z, \xi) \in T_\mathcal{D}$. By the early work due to Lempert[Le], the function $F = K_\mathcal{D}^2$ defines a convex Finsler metric on the tangent bundle $T_\mathcal{D}$. □

3.2. **Bott connections of Finsler bundles.** Let $(E, \|\cdot\|)$ be a convex Finsler vector bundle. We define a splitting $h : \pi^* T_M \to T_E$ of the sequence (1.1) so that its Bott connection D is metrical with respect to the Hermitian metric $\langle\cdot,\cdot\rangle$:

$$d^h\langle X, Y\rangle = \langle DX, Y\rangle + \langle X, DY\rangle \tag{3.6}$$

for all $X, Y \in \Gamma(T_{E/M})$. Since this condition can be written as $\partial^h F_{i\bar{j}} = \sum F_{m\bar{j}}\omega_i^m$, we have

$$\omega_j^i = \sum_m F^{i\bar{m}}\partial^h F_{j\bar{m}}. \tag{3.7}$$

Then, from (2.6) we have

Proposition 3.1. *For the fundamental function F, the functions*

$$N_\alpha^i := \sum_m F^{\bar{m}i}\frac{\partial^2 F}{\partial z^\alpha \partial\bar{\xi}^m} = \sum_{j,m} F^{\bar{m}i}\frac{\partial F_{j\bar{m}}}{\partial z^\alpha}\xi^j \tag{3.8}$$

define an Ehresmann connection H for π, where $(F^{\bar{m}i})$ is the inverse of $(F_{i\bar{j}})$.

We define a splitting h of the sequence (1.1) by the coefficients N_α^i in (3.8), from which we get a canonical Bott connection D of $\{T_{E/M}, \langle\cdot,\cdot\rangle\}$ by the formula (2.5). If we define a section $\mathcal{E}$ of $T_{E/M}$ by

$$\mathcal{E} = \sum_j \xi^j\frac{\partial}{\partial\xi^j},$$

the relations (2.6) is equivalent to

$$D\mathcal{E} \equiv 0, \tag{3.9}$$

and, since $\langle\mathcal{E},\mathcal{E}\rangle = F(z,\xi)$ and D satisfies the metrical condition (3.6), we have

$$d^h F = \partial^h F + \bar{\partial}^h F \equiv 0. \tag{3.10}$$

For the Bott connection D of $(E, \|\cdot\|)$, we have proved the following proposition.

Proposition 3.2. ([Ai3]) *Let D be the complex Bott connection of $(E, \|\cdot\|)$ defined by the connection H of (4.11). Then we have*

1. $R^i_{\alpha\beta} \equiv 0$,
2. $\partial^h\omega + \omega\wedge\omega \equiv 0$.

By this proposition, some quantities are simplified. By the identity $\partial^h\omega+\omega\wedge\omega \equiv 0$, the curvature Ω^D of D is given by

$$\Omega^D = \bar{\partial}^h\omega, \tag{3.11}$$

and its components R^i_j are given in the form $R^i_j = \sum R^i_{j\alpha\bar\beta} dz^\alpha \wedge d\bar z^\beta$, where

$$R^i_{j\alpha\bar\beta}(z,\xi) = -X_{\bar\beta}\Gamma^i_{j\alpha} \tag{3.12}$$

is the curvature tensor of Ω^D. ¿From (2.5) and the homogeneity (2.2) of N^i_α, we have

$$\sum_j \xi^j R^i_{j\alpha\bar\beta} \equiv R^i_{\alpha\bar\beta}. \tag{3.13}$$

We consider a Finsler bundle $(E, \|\cdot\|)$ whose Bott connection D is flat. By definitions and the identity (3.13), we have $R^i_{\alpha\bar\beta} = 0$ and thus $\Pi = 0$. Then, by (2.7), we have $\Phi = 0$. Hence we get

$$\frac{\partial N^i_\alpha}{\partial \bar\xi^j} = 0,$$

and

$$\frac{\partial N^i_\alpha}{\partial \bar z^\beta} = \frac{\partial N^i_\alpha}{\partial \bar z^\beta} - \sum_l \overline{N^l_\beta} \frac{\partial N^i_\alpha}{\partial \bar\xi^l} = R^i_{\alpha\bar\beta} = 0.$$

We see that, if $\Omega^D = 0$, then the coefficients N^i_α are holomorphic. Hence we have

Proposition 3.3. *If the Bott connection D of $(E, \|\cdot\|)$ is flat, then the sequence* (1.1) *splits holomorphically.*

If D is flat, since $R^i_{\alpha\beta} = R^i_{\alpha\bar\beta} = 0$, the PDE

$$\frac{\partial \Psi^i}{\partial z^\alpha} = -N^i_\alpha(z, \Psi(z)) \tag{3.14}$$

is completely integrable and has a holomorphic solution $\zeta^i(z) = \Psi^i(z,(z_0,\xi_0))$ for an arbitrary initial point $(z_0,\xi_0) \in E$. By the identity (2.6), this PDE is also written as

$$\frac{\partial \zeta^i}{\partial z^\alpha} + \sum_j \Gamma^i_{j\alpha}(z,\zeta(z))\zeta^j = 0. \tag{3.15}$$

For a solution $\zeta^i(z)$ of (3.14), we define a holomorphic section $\tilde\zeta$ of $T_{E/M}$ by

$$\tilde\zeta = \sum \zeta^i(z) \left(\frac{\partial}{\partial \xi^j}\right)_{(z,\zeta(z))}.$$

Then, by definition, we have $\|\zeta(z)\|^2 = \langle\tilde{\zeta},\tilde{\zeta}\rangle = F(z,\zeta(z))$ and

$$\begin{aligned}\frac{\partial}{\partial z^\alpha}\|\zeta(z)\|^2 &= \left(\frac{\partial F}{\partial z^\alpha}\right)_{(z,\zeta(z))} + \sum \frac{\partial \zeta^l}{\partial z^\alpha}\left(\frac{\partial F}{\partial \xi^l}\right)_{(z,\zeta(z))}\\ &= \left(\frac{\partial F}{\partial z^\alpha} - \sum_l N^l_\alpha \frac{\partial F}{\partial \xi^l}\right)_{(z,\zeta(z))}\\ &= (X_\alpha F)_{(z,\zeta(z))}\\ &= 0.\end{aligned}$$

Hence the norm of ζ is constant, and so ζ is a non-vanishing holomorphic section of E. On the other hand, since the condition (3.15) can be written as $D\tilde{\zeta} = 0$, we see that the flatness of D implies the existence of parallel section of $T_{E/M}$.

Theorem 3.1. *If the Bott connection D of $(E, \|\cdot\|)$ is flat, then there exists a flat Hermitian metric on E.*

Proof. For the solution $\zeta \in H^0(M, \mathcal{O}(E))$ of (3.14), we define a holomorphic map $f_\zeta : M \to E$ by $f_\zeta(z) = (z, \zeta(z))$. We introduce a Hermitian metric $g = (g_{i\bar{j}}(z))$ on E by

$$g_{i\bar{j}}(z) = F_{i\bar{j}}(z, \zeta(z)).$$

The Hermitian connection θ^i_j of (E, g) is given by

$$\theta^i_j = \sum_m g^{i\bar{m}}\partial g_{j\bar{m}} = \sum_m \left(F^{i\bar{m}}\partial^h F_{j\bar{m}}\right)_{(z,\zeta)} = f^*_\zeta \omega^i_j.$$

Hence the curvature $\Omega^g = f^*_\zeta \Omega^D$ of (E, g) vanishes identically.

Q.E.D.

3.3. **Complex Finsler connections and flat Finsler metrics.** The complex Bott connection D of $(E, \|\cdot\|)$ is extended to an ordinary connection ∇ on $T_{E/M}$. In fact, since $T_{E/M} \cong \pi^*E$, the relative tangent bundle $T_{E/M}$ admits a canonical relative flat connection $\nabla^0 : \Gamma(T_{E/M}) \to \Gamma(T_{E/M} \otimes \Omega^1_E)$ characterized by the property $\nabla^0(\pi^{-1}s) = 0$ for every $s \in \Gamma(E)$. The connection $\nabla : \Gamma(T_{E/M}) \to \Gamma(T_{E/M} \otimes \Omega^1_E)$ is given by

$$\nabla = D \oplus \nabla^0.$$

It is noted that ∇ is not compatible with the Hermitian metric $\langle\cdot,\cdot\rangle$ on $T_{E/M}$.

Definition 3.2. The connection ∇ is called the *Finsler connection* of $(E, \|\cdot\|)$.

Since $\partial^h\omega + \omega \wedge \omega \equiv 0$, the curvature form $\Omega^\nabla = d\omega + \omega \wedge \omega$ of ∇ is given by

$$\Omega^\nabla = \Omega^D + d^v\omega. \tag{3.16}$$

In the case of $d^v\omega \equiv 0$, then we have $\Omega^\nabla \equiv \Omega^D$. In this case, such a $(E, \|\cdot\|)$ is said to be *modeled on a complex Minkowski space*, and it is proved that there exists a Hermitian metric h_F on E such that the Finsler connection ∇ is given by pull-back $\pi^*\nabla^{h_F}$ of the Hermitian connection ∇^{h_F} of h_F (cf. [Ai1]). If the curvature Ω^∇ of ∇ vanishes identically, then ∇ is said to be *flat*. In this case, $(E, \|\cdot\|)$ is modeled on a complex Minkowski space and its associated Hermitian metric h_F is flat.

A Hermitian bundle (E, h) is flat if and only if there exists an open cover $\{(U, s_U)\}_{U\in\mathcal{U}}$ of E with a parallel orthonormal holomorphic frame s_U. Then its norm function F_h with respect to $\{(U, s_U)\}_{U\in\mathcal{U}}$ is independent on the base point $z \in M$. In the case of Finsler metrics, the flatness of Finsler metrics is defined as follows:

Definition 3.3. A complex Finsler vector bundle $(E, \|\cdot\|)$ is said to be *flat* if there exists an open cover $\{(U, s_U)\}$ of E such that the pseudo-Kähler potential F of the Kähler morphism $E \to M$ relative to $\{(U, s_U)\}_{U\in\mathcal{U}}$ is independent on the base point $z \in M$.

Unlike the case of Hermitian metrics, by Theorem 4.2, the flatness of D does not implies the flatness of Finsler metrics. In [Ai3] and [Ai4], we have discussed the flatness of the connection ∇ with relation to the flatness of Finsler metrics. The following theorem is given in [Ai3]. The proof is essentially the same as the one in [Ai3], however, we shall reproduce here for the convenience.

Theorem 3.2. *A complex Finsler vector bundle $(E, \|\cdot\|)$ is flat if and only if the curvature R^∇ of ∇ vanishes identically.*

Proof. We denote by F the fundamental function of $\|\cdot\|$ relative to a fixed covering $\{\mathcal{U}, (s_U)\}$. We suppose that $(E, \|\cdot\|)$ is flat. The fundamental function F relative to the adapted covering $\{(U, s_U)\}_{U\in\mathcal{U}}$ is independent on $z \in M$. Then, by (3.8) the coefficients N^i_α vanish identically, and so by (2.5) the connection forms ω vanish on each $U \in \mathcal{U}$. Hence, from (3.11) and (3.16) the curvature R^∇ of ∇ vanishes identically.

Conversely we shall prove that the flatness of ∇ implies the flatness of $(E, \|\cdot\|)$. For this purpose, we shall fix an open covering $\{(U, s_U)\}_{U\in\mathcal{U}}$. Then it defines a local trivialization $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{C}^r$ by $\varphi_U(v) = (z^\alpha, \xi^i)$ for ${}^\forall v = \sum \xi^i s_i(z) \in \pi^{-1}(U)$. Since ∇ is flat, $(E, \|\cdot\|)$ is modeled on a complex Minkowski space and its associated Hermitian metric h_F is flat. Thus ∇ is the Hermitian connection of this flat Hermitian metric h_F. Hence we can introduce a holomorphic frame field $\tilde{s}_U = (\tilde{s}_1, \cdots, \tilde{s}_r)$ on each $U \in \mathcal{U}$ with respect to which the connection form $\tilde{\omega}$ of

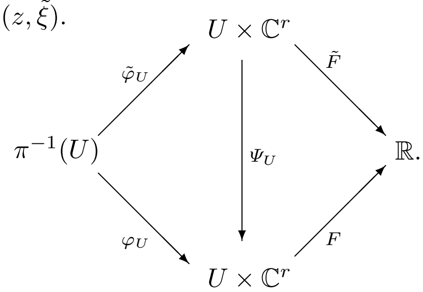
∇ vanishes identically on each $U \in \mathcal{U}$. Hence we have another local trivialization $\tilde{\varphi}_U : \pi^{-1}(U) \to U \times \mathbb{C}^r$ by $\tilde{\varphi}_U(v) = (z^\alpha, \tilde{\xi}^i)$ for $^\forall v = \sum \tilde{\xi}^i \tilde{s}_i(z) \in \pi^{-1}(U)$. We denote by $A_U = \left(A^i_j\right)$ the transition function between the frame fields s_U and $\tilde{s}_U$, that is, $\tilde{s}_j(z) = \sum s_i(z) A^i_j(z)$. The coordinate transition $\Psi_U = \varphi_U \circ \tilde{\varphi}_U^{-1}$ is given by

$$\Psi_U(z, \tilde{\xi}) = \left(z^\alpha, \sum_j A^i_j(z) \tilde{\xi}^j \right).$$

We note that, since $\tilde{s}_U$ is a parallel frame filed on U, the functions $A^i_j(z)$ satisfy

$$\frac{\partial A^i_j}{\partial z^\alpha} + \sum_l \Gamma^i_{l\alpha}(z) A^l_j(z) = 0 \tag{3.17}$$

on each $U \in \mathcal{U}$. The fundamental function $\tilde{F}$ of $\|\cdot\|$ relative to $\{(U, \tilde{s}_U)\}$ is given by $\tilde{F}(z, \tilde{\xi}) = (\Psi_U^* F)(z, \tilde{\xi})$.



Then, since $A^i_j(z)$ satisfy (3.17) we have

$$\Psi_{U*}\left(\frac{\partial}{\partial z^\alpha}\right) = \frac{\partial}{\partial z^\alpha} - \sum_{j,l} \Gamma^i_{j\alpha}(z) A^j_l(z) \tilde{\xi}^l \frac{\partial}{\partial \xi^i} = \frac{\partial}{\partial z^\alpha} - \sum_i N^i_\alpha(z, \xi) \frac{\partial}{\partial \xi^i} = X_\alpha$$

and $X_\alpha F = 0$ by (3.10), we have

$$\frac{\partial \tilde{F}}{\partial z^\alpha} = \left(\frac{\partial}{\partial z^\alpha}\right)(\Psi_U^* F) = \Psi_{U*}\left(\frac{\partial}{\partial z^\alpha}\right) F = X_\alpha F = 0,$$

which shows $\tilde{F} = \tilde{F}(\tilde{\xi})$. Hence $(E, \|\cdot\|)$ is flat.

Q.E.D.

By the proof of Theorem 4.3, we have

Corollary 3.1. *A complex Finsler bundle $(E, \|\cdot\|)$ is flat if and only if it is modeled on a complex Minkowski space and its associated Hermitian bundle (E, h_F) is flat.*

4. Some remarks

Let $\pi_{P_E} : P_E \to M$ be the projective bundle associated with E. We denote by $\pi_{L_E} : L_E \to P_E$ the tautological line bundle, and we also denote by $L_E^\times$ the open submanifold of L_E consisting of the non-zero elements. The holomorphic map $\tau : E^\times \to P_E \times E$ defined by $\tau(v) = ([v], v)$ maps $E^\times$ to $L_E^\times$ bi-holomorphically. Then it is shown that any Finsler metric on E is naturally identified as a Hermitian metric on L_E as follows(cf. [Ko1]).

For the projective bundle P_E associated with E, we introduce a standard open covering $\{U_j; U \in \mathcal{U}, 1 \leq j \leq r\}$ of P_E from an open cover $\{(U, ts_U)\}$ of E by putting $U_j = \pi^{-1}(U) \cap \{\xi^j \neq 0\} = \{(z, [\xi]) \in P_E; z \in U,\ \xi^i \neq 0\}$. On each open set U_j, we define a holomorphic section $t_j : U_j \to L_E$ by

$$t_j(z, [\xi]) = \left((z, [\xi]), \left(\frac{\xi^1}{\xi^j}, \cdots, \frac{\xi^r}{\xi^j} \right) \right).$$

Then $\{t_j\}$ defines a local trivialization $\varphi_j : U_j \times \mathbb{C} \to \pi_{L_E}^{-1}(U_j)$ by $\varphi_j((z, [\xi]), \lambda) = \lambda t_j(z, [\xi])$. Hence the bi-holomorphism τ can be written as

$$\tau(z, \xi) = ((z, [\xi]), \xi) = \varphi_j \left((z, [\xi]), \xi^j \right) \cong \left((z, [\xi]), \xi^j \right).$$

On the other hand, any Hermitian metric h_{L_E} on L_E is defined by the family of positive functions $\{h_{L_E,j}\}$ on each U_j satisfying

$$h_{L_E,i} = \left| \frac{\xi^j}{\xi^i} \right|^2 h_{L_E,j} \tag{4.1}$$

on $U_i \cap U_j \neq \phi$. Hence, by using the map τ and identification

$$L^2(z, \xi) = h_{L_E}(\tau(z, \xi)) = |\xi^j|^2 h_{L_E,j}(z, [\xi]), \tag{4.2}$$

any Finsler metric $\| \cdot \|$ on E is identified with a Hermitian metric h_{L_E} on the tautological line bundle L_E.

Since $\mathcal{E}$ is generated by the action μ_λ and $\mathcal{L}_{\mathcal{E}} \partial\bar{\partial} \log F = 0$, the real $(1,1)$-form

$$\omega_{P_E,F} = \sqrt{-1}\partial\bar{\partial} \log F \tag{4.3}$$

is invariant by the action μ_λ for $^\forall \lambda \in \mathbb{C}$, and so $\omega_{P_E,F}$ may be considered as a real $(1,1)$-form on P_E. We suppose that $(E, \| \cdot \|)$ is convex. Since

$$\sqrt{-1}\partial\bar{\partial} F_z = \sqrt{-1} F_z (\partial\bar{\partial} \log F_z + \partial \log F_z \wedge \bar{\partial} \log F_z), \tag{4.4}$$

the function $\log F_z$ is strictly subharmonic on each fibre $P(E_z)$. Thus we have obtained a Kähler morphism $\pi_{P_E} : P_E \to M$ with a pseudo-Kähler metric $\omega_{P_E,F}$ from an arbitrary $(E, \| \cdot \|)$.

Proposition 4.1. *Let $(E, \| \cdot \|)$ be a convex Finsler vector bundle. Then the projective bundle $\pi_{P_E} : P_E \to M$ is a Kähler morphism with the pseudo-Kähler metric $\omega_{P_E,F}$.*

Conversely, from an arbitrary pseudo-Kähler metric ω_{P_E} of the projective bundle P_E, it induces a convex Finsler metric $\|\cdot\|$ on E, that is, we have

Theorem 4.1. *A holomorphic vector bundle admits a convex Finsler metric if and only if the projective bundle $\pi_{P_E} : P_E \to M$ associated with E is a Kähler morphism.*

Proof. We express locally $\omega_{P_E} = \sqrt{-1}\partial\bar{\partial}G_j$ on U_j for a C^∞-function G_j on U_j. Since $G_j - G_i$ is pluri-harmonic, there exists a 1-cocycle $K_{ij} \in Z^1(\mathcal{U}_{P_E}, \mathcal{O}_{P_E})$ satisfying $G_j - G_i = K_{ij} + \overline{K_{ij}}$ on $U_i \cap U_j \neq \phi$. Restricting to each fibre $P(E_z)$, we have $\omega_z = \sqrt{-1}\partial\bar{\partial}G_{j,z}$ and $G_{j,z} - G_{i,z} = K_{ij,z} + \overline{K_{ij,z}}$. Then $\{K_{ij,z}\}$ is a 1-cocycle on $P(E_z) \cong \mathbb{P}^{r-1}$, and since $H^1(\mathbb{P}^{r-1}, \mathcal{O}) = 0$, we may put

$$K_{ij,z} = (K_{j,z} - \log \xi^j) - (K_{i,z} - \log \xi^i)$$

for a 0-cochain $\{K_{j,z}\}$ on $P(E_z)$. Hence we have

$$G_{j,z} - (K_{j,z} + \overline{K_{j,z}}) + \log |\xi^j|^2 = G_{i,z} - (K_{i,z} + \overline{K_{i,z}}) + \log |\xi^i|^2$$

If we put $F_{j,z}([\xi]) = \exp\{G_{j,z} - (K_{j,z} + \overline{K_{j,z}})\}$ on $U_{j,z}$, we have $|\xi^j|^2 F_{j,z}([\xi]) = |\xi^i|^2 F_{i,z}([\xi])$. Since each $|\xi^j|^2 F_{j,z}([\xi]) := F_z(\xi)$ depends on $z \in M$ smoothly, the function $F_z(\xi)$ defines a complex Finsler metric $F(z, \xi)$ on E. Consequently we have

$$F(z, \xi) = |\xi^j|^2 \exp\{G_j - (K_j + \overline{K_j})\} \tag{4.5}$$

for a family of local functions $\{K_j\}$ on U_j which are holomorphic in $[\xi]$ and smooth in z. Moreover, because of $\sqrt{-1}\partial\bar{\partial}\log F_z = \sqrt{-1}\partial\bar{\partial}\log F_{j,z} = \sqrt{-1}\partial\bar{\partial}G_{j,z} > 0$ and (4.4), F defines a convex Finsler metric on E.

Q.E.D.

The convex Finsler metric $\|\cdot\|$ determined from ω_{P_E} should be satisfy the relation $\omega_z = \sqrt{-1}\partial\bar{\partial}\log F_z(\xi)$ on each fibre P_{E_z}. Hence we have

Proposition 4.2. *A pseudo-Kähler metric ω_{P_E} on P_E determines a unique convex Finsler metric $\|\cdot\|$ on E up to a locally conformal equivalence.*

We say that the pseudo-Kähler metric $\mathit{\Pi}_{P_E}$ is flat if there exists an open covering $\{(U, s_U)\}_{U \in \mathcal{U}}$ of E such that the pseudo-Kähler potential $\{G_j\}$ of $\mathit{\Pi}_{P_E}$ is independent on the base point $z \in M$. We also say the Finsler metric induced from $\mathit{\Pi}_{P_E}$ is *projectively flat* if $\mathit{\Pi}_{P_E}$ is flat. The projective flatness of a Finsler metric $\|\cdot\|$ is characterized in terms of curvature of the Finsler connection ∇ of $(E, \|\cdot\|)$ (cf. [Ai5]), and it is easily shown that the projective flatness of $\|\cdot\|$ is equivalent to the locally-conformal flatness of $\|\cdot\|$ in the sense of [Ai3].

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Department of Mathematics and Computer Science, Faculty of Sciences, Kagoshima University

E-mail address: aikou@sci.kagoshima-u.ac.jp