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Some Generalizations of Poisson and Jacobi Structures

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1 Introduction

More than twenty years ago, Nambu [35] proposed a generalization of Hamiltonian mechanics by considering brackets involving three or more functions. The main motivation of Nambu to propose such a generalization was to study other forms of dynamics enforcing the Liouville theorem as in the case of statistical mechanics. The first example of a dynamical system which admits a Nambu formulation was given by Nambu himself and it is the Euler equations for the angular momentum of a rigid body. Another examples can be seen in [7, 18, 20, 41, 42]. Immediatly after the publication of Nambu's paper, many authors have investigated its relation with the hamiltonian theory of constraints (see for instance [6, C, 11, 26, 33, 17]).

The subject was almost dormant for many years till the appearence of Takhtajan's paper [42]. Indeed, Takhtajan gave a geometric approach to Nambu brackets by establishing the so-called Fundamental identity, which is a generalization of the Jacobi identity. Therefore, the notion of Nambu-Poisson manifold was introduced. It is remarkable that the fundamental identity was previously considered by Filippov [15] (see [31]) in a pure algebraic context.

On the other hand, Azcárraga *et al* [3, 4] have introduced a different generalization of the Jacobi identity, the so-called generalized Jacobi identity, which leads to the notion of generalized Poisson manifold.

The brackets considered above are derivations on each argument. However, one can weak this assumption in the same way that Poisson brackets can be

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extended to Jacobi brackets. Proceeding in this direction, one obtains the so-called Nambu-Jacobi and generalized Jacobi manifolds (see [30, B, 24]).

The aim of the present paper is to present these extensions of Poisson and Jacobi brackets, and show the main examples. The dynamics and the geometry of multibrackets have been studied in several recent papers. A huge list of references is included at the end.

2 Nambu-Poisson and generalized Poisson manifolds

Let M be a differentiable manifold of dimension m. We will use the following notations:

- $C^\infty(M, \mathbb{R})$ is the algebra of C^∞ real-valued functions on M.
- $\mathfrak{X}(M)$ is the $C^\infty(M, \mathbb{R})$-module of vector fields on M.
- $\mathcal{V}^k(M)$ is the space of k-vectors on M.
- $\Omega^k(M)$ is the space of k-forms on M.

An *almost Poisson bracket of order* n [21] on M is a n-linear mapping

$$\{\,,\dots,\,\} : C^\infty(M,\mathbb{R}) \times \dots^{(n} \dots \times C^\infty(M,\mathbb{R}) \longrightarrow C^\infty(M,\mathbb{R})$$

satisfying the following properties:

(1) (*Skew-symmetry*)

$$\{f_1, \dots, f_n\} = \varepsilon_\sigma \{f_{\sigma(1)}, \dots, f_{\sigma(n)}\} ,$$

for all $f_1, \dots, f_n \in C^\infty(M,\mathbb{R})$ and $\sigma \in \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the group of the permutations of order n and ε_σ is the signature of the permutation σ.

(2) (*Leibniz rule*)

$$\{f_1 g_1, \dots, f_n\} = f_1\{g_1, \dots, f_n\} + g_1\{f_1, \dots, f_n\} ,$$

for all $f_1, \dots, f_n, g_1 \in C^\infty(M,\mathbb{R})$.

An alternative way to define an n-bracket of functions is to consider the skew-symmetric tensor Λ of type $(n, 0)$ defined by

$$\Lambda(df_1, \dots, df_n) = \{f_1, \dots, f_n\} ,$$

for all $f_1, \dots, f_n \in C^\infty(M,\mathbb{R})$. We call Λ an *almost or pseudo Poisson n-tensor*, and (M, Λ) a *generalized almost Poisson manifold.* An almost Poisson n-tensor Λ induces a linear mapping

$$\# : \Omega^{n-1}(M) \longrightarrow \mathfrak{X}(M)$$

by defining

$$\langle \#(\alpha_1 \wedge \ldots \wedge \alpha_{n-1}), \beta \rangle = \Lambda(\alpha_1, \ldots, \alpha_{n-1}, \beta) \ ,$$

for all $\alpha_1, \ldots, \alpha_{n-1}, \beta \in \Omega^1(M)$. Therefore, given $n-1$ functions $f_1, \ldots, f_{n-1}$, we have a vector field

$$X_{f_1 \cdots f_{n-1}} = \#(df_1 \wedge \cdots \wedge df_{n-1})$$

which is called the *Hamiltonian vector field* associated with the Hamiltonian functions $f_1, \ldots, f_{n-1}$.

2.1 Nambu-Poisson manifolds

A generalized almost Poisson manifold (M, Λ) of order n is called *Nambu-Poisson* if the following *fundamental identity* is satisfied [21, 41, 42]:

$$\{f_1, \ldots, f_{n-1}, \{g_1, \ldots, g_n\}\} = \sum_{i=1}^{n} \{g_1, \ldots, \{f_1, \ldots, f_{n-1}, g_i\}, \ldots, g_n\} \ , \tag{2.1}$$

for all functions $f_1, \ldots, f_{n-1}, g_1, \ldots, g_n$ on M. This is equivalent to say that the Hamiltonian vector fields $X_{f_1 \cdots f_{n-1}}$ are infinitesimal automorphisms of Λ, that is, $\mathcal{L}_{X_{f_1 \cdots f_{n-1}}} \Lambda = 0$.

For $n = 2$ the fundamental identity is the well-known Jacobi identity, so (M, Λ) is a Poisson manifold. Symplectic manifolds are the main examples of Poisson manifolds.

Example: [17]

The constant Nambu-Poisson tensors of order n on $\mathbb{R}^{n+s}$ are just the decomposable ones.

Example: (Volume forms) [17, 21]

Let M be an oriented n-dimensional manifold and ν a volume form. Given n functions $f_1, \ldots, f_n$ on M, we define a Nambu-Poisson bracket by the formula

$$\{f_1, \ldots, f_n\} \, \nu = df_1 \wedge \cdots \wedge df_n \ .$$

If we denote by Λ_ν the associated Nambu-Poisson tensor, then $\Lambda_\nu \neq 0$ at every point of M. Moreover, each Nambu-Poisson tensor $\Lambda \neq 0$ of order n over an n-dimensional manifold M comes from a volume form.

If we take $M = \mathbb{R}^n$, and ν is the standard volume form $\nu = dx^1 \wedge \cdots \wedge dx^n$, we recover the example originally discussed by Nambu [35]:

$$\Lambda_\nu = \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^n}.$$

In some sense, the Nambu-Poisson structure associated to a volume form (for $n \geq 3$) can be considered as the analogous to the symplectic structure in Poisson geometry. Indeed, in this case the associated morphism $\# : \Omega^{n-1}(M) \longrightarrow \mathfrak{X}(M)$ is an isomorphism. Moreover, there is a generalized foliation whose leaves are points or volume manifolds.

Example:(Completely integrable systems) [19, 22]

Let (M, ω, H) be a completely integrable Hamiltonian system, that is, M is a $2n$-dimensional symplectic manifold with symplectic form ω such that there exist $f_1, \ldots, f_n$ independent functions pairwise in involution, that is, $\{f_i, f_j\} = 0$ for $i, j = 1, \ldots, n$. Then the associated Hamiltonian vector fields $X_i = X_{f_i}$ commute, that is, $[X_i, X_j] = 0$. Therefore, $\Lambda = X_1 \wedge \ldots \wedge X_n$ is a Nambu-Poisson tensor on M of order n.

More generally, let M be a differentiable manifold of dimension m and $X_1, \ldots, X_n$ independent vector fields that commute, that is, $[X_i, X_j] = 0$, for $i, j = 1, \ldots, n$. Then the n-tensor $\Lambda = X_1 \wedge \cdots \wedge X_n$ is a Nambu-Poisson structure of order n. Notice that the simultaneous flow-box theorem says that for each $p \in M$ there exists a local coordinate system $(x^1, \ldots, x^m)$ on a neighborhood U of p such that $X_1 = \frac{\partial}{\partial x^1}, \cdots, X_n = \frac{\partial}{\partial x^n}$; and therefore, $\Lambda = \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^n}$.

2.2 Generalized Poisson manifolds

A generalized almost Poisson manifold (M, Λ) of order n (even) is called a *generalized Poisson manifold* if the *generalized Jacobi identity* is satisfied [3, 4, 21]:

$$\text{Alt}\left(\{f_1, \ldots, f_{n-1}, \{g_1, \ldots, g_n\}\}\right) = 0 , \tag{2.2}$$

for all functions $f_1, \ldots, f_{n-1}, g_1, \ldots, g_n$ on M. This is equivalent to the following property: $[\Lambda, \Lambda] = 0$, where $[\,,\,]$ is the Schouten-Nijenhuis bracket. Again, for $n = 2$ we recover Poisson manifolds.

Example:(Nambu-Poisson manifolds) [2, 21]

Every Nambu-Poisson manifold (of even order) is generalized Poisson.

Example:(Compatible Poisson structures) [22, 23]

Let M be a differentiable manifold and Λ_1, Λ_2 two compatible Poisson structures, that is, $[\Lambda_1, \Lambda_2] = 0$, then $\Lambda_1 \wedge \Lambda_2$ is a generalized Poisson structure of order 4.

Example:(Hyperkähler manifolds)[22, 23]

Let M be a hyperkähler manifold, that is, M is a differentiable manifold of dimension $4n$ with a Riemannian metric g and three complex structures J_1, J_2, J_3 compatible with g and such that:

i) the complex structures satisfy $J_3 = J_1 J_2 = -J_2 J_1$.

ii) the Kähler forms ω_i defined by $\omega_i(X, Y) = g(X, J_i Y)$, for $X, Y \in \mathfrak{X}(M)$ are closed.

If Λ_i is the Poisson structure associated with the Kähler form ω_i, we have that Λ_1, Λ_2 and Λ_3 are compatible, i.e., $[\Lambda_i, \Lambda_j] = 0$, for all i, j, and consequently, the 4-vector

$$\Lambda = \Lambda_1 \wedge \Lambda_1 + \Lambda_2 \wedge \Lambda_2 + \Lambda_3 \wedge \Lambda_3,$$

is a generalized Poisson structure of order 4 on M.

Example:(Generalized Lie-Poisson structures)[4, 5, 24]

Let $\mathfrak{g}$ be a real vector space of dimension m. A skew-symmetric Lie $2n$-bracket on $\mathfrak{g}$ is a $2n$-linear skew-symmetric mapping

$$[\ ,\ldots,\ ] : \mathfrak{g} \times \ldots^{(2n} \ldots \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

satisfying the following condition:

$$Alt[a_1, \ldots, a_{2n-1}, [b_1, \ldots, b_{2n}]] = 0, \tag{2.3}$$

for all $a_1, \ldots, a_{2n-1}, b_1, \ldots, b_{2n} \in \mathfrak{g}$. A vector space endowed with such a bracket is called a *generalized Lie algebra of order* $2n$.

Now, on the dual space $\mathfrak{g}^*$, we define the generalized almost Poisson bracket of order $2n$

$$\{,\ldots,\} : C^\infty(\mathfrak{g}^*, \mathbb{R}) \times \ldots \times C^\infty(\mathfrak{g}^*, \mathbb{R}) \longrightarrow C^\infty(\mathfrak{g}^*, \mathbb{R})$$

by putting

$$\{f_1, \ldots, f_{2n}\}_\alpha = \alpha([(df_1)_\alpha, \ldots, (df_{2n})_\alpha]) \tag{2.4}$$

for all $f_1, \ldots, f_{2n} \in C^\infty(\mathfrak{g}^*, \mathbb{R})$ and $\alpha \in \mathfrak{g}^*$. Note that $(df_i)_\alpha \in T^*_\alpha \mathfrak{g}^* \cong (\mathfrak{g}^*)^* \cong \mathfrak{g}$. Let $\bar{\Lambda}$ be the $2n$-vector associated with the bracket defined in (2.4). Then, $(\mathfrak{g}^*, \bar{\Lambda})$ is a generalized Poisson manifold of order $2n$.

3 Nambu-Jacobi and Generalized Jacobi manifolds

A *generalized almost or pseudo Jacobi bracket of order* n [24] on M is an n-linear mapping

$$\{,\ldots,\} : C^\infty(M, \mathbb{R}) \times \ldots^{(n} \ldots \times C^\infty(M, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R})$$

satisfying the following properties:

(i) *Skew-symmetry:*

$$\{f_1, \ldots, f_n\} = \varepsilon_\sigma \{f_{\sigma(1)}, \ldots, f_{\sigma(n)}\}$$

for all $f_1, \ldots, f_n \in C^\infty(M, \mathbb{R})$ and $\sigma \in \mathfrak{S}_n$.

(ii) $\{,\ldots,\}$ is a first order linear differential operator on M with respect to each argument, i.e.,

$$\{f_1g_1, f_2, \ldots, f_n\} = f_1\{g_1, f_2, \ldots, f_n\} + g_1\{f_1, \ldots, f_n\} - f_1g_1\{1, f_2, \ldots, f_n\}$$

for all $f_1, g_1, f_2, \ldots, f_n \in C^\infty(M, \mathbb{R})$.

If $\{,\ldots,\}$ is a generalized almost Jacobi bracket of order n then we can define an n-vector Λ and an $(n-1)$-vector $\Box$ as follows

$$\begin{aligned} \Box(df_1, \ldots, df_{n-1}) &= \{1, f_1, \ldots, f_{n-1}\} \\ \Lambda(df_1, \ldots, df_n) &= \{f_1, \ldots, f_n\} + \sum_{i=1}^{n}(-1)^i f_i \Box(df_1, \ldots, \widehat{df_i}, \ldots, df_n) \end{aligned}$$

for all $f_1, \ldots, f_n \in C^\infty(M, \mathbb{R})$. Conversely, any pair $(\Lambda, \Box) \in \mathcal{V}^n(M) \oplus \mathcal{V}^{n-1}(M)$ defines a v*generalized almost Jacobi bracket of order n* given by

$$\{f_1, \ldots, f_n\} = \Lambda(df_1, \ldots, df_n) + \sum_{i=1}^{n}(-1)^{i-1} f_i \Box(df_1, \ldots, \widehat{df_i}, \ldots, df_n).$$

Thus, the pair $(\Lambda, \Box) \in \mathcal{V}^n(M) \oplus \mathcal{V}^{n-1}(M)$ is called a *generalized almost Jacobi structure of order n* and $(M, \Lambda, \Box)$ is a *generalized almost Jacobi manifold.* If $\Box = 0$, then we obtain a *generalized almost Poisson structure of order n.*

Given $n-1$ functions $f_1, \cdots, f_{n-1} \in C^\infty(M, \mathbb{R})$, the associated *Hamiltonian vector field $X_{f_1,\ldots,f_{n-1}}$* is defined by

$$X_{f_1\cdots f_{n-1}} = X^\Lambda_{f_1\cdots f_{n-1}} + \sum_{i=1}^{n-1}(-1)^{i-1} f_i X^\Box_{f_1\cdots\widehat{f_i}\cdots f_{n-1}} \tag{3.1}$$

where we denote by $X^\Lambda_{f_1\cdots f_{n-1}}$ (respectively, $X^\Box_{f_1\cdots\widehat{f_i}\cdots f_{n-1}}$) the Hamiltonian vector field on (M, Λ) (respectively, $(M, \Box)$) associated with the functions $f_1, \ldots, f_{n-1}$ (respectively, $f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_{n-1}$).

3.1 Nambu-Jacobi manifolds

A generalized almost Jacobi manifold $(M, \Lambda, \Box)$ of order n is called *Nambu-Jacobi* [24, 30] if the fundamental identity (2.1) is satisfied. It can be proved [30] that in this case, Λ (resp. $\Box$) is a Nambu-Poisson structure of order n (resp. $n-1$) and for $x \in M$ satisfying $\Lambda(x) \neq 0$, there exists $\theta \in T^*_xM$ such that $i_\theta\Lambda(x) = \Box(x)$.

For $n = 2$ we recover Jacobi manifolds, which main examples are provided by contact and locally conformal symplectic manifolds.

Example: [24, 30]

Let (M, Λ) be a Nambu-Poisson manifold of order n, $n > 2$. If $f \in C^\infty(M, \mathbb{R})$ then $i_{df}\Lambda$ is also a Nambu-Poisson structure on M. Furthermore, it can be proved that the pair $(\Lambda, i_{df}\Lambda)$ is a Nambu-Jacobi structure of order n. Since a closed 1-form θ is locally exact, the above result can be extended to $(\Lambda, i_\theta\Lambda)$.

Next, let M be an oriented manifold of dimension n, and choose a volume form ν on M. Denote by Λ_ν the corresponding Nambu-Poisson tensor on M. If θ is a 1-form on M, then the generalized almost Jacobi manifold $(M, \Lambda_\nu, i_\theta\Lambda_\nu)$ is a Nambu-Jacobi manifold if and only if θ is closed.

Example:[24, 30]

Let (M, Λ) be a Nambu-Poisson manifold of order n, $n > 2$ (respectively, a symplectic manifold of dimension 2). It can be proved that the pair $(0, \Lambda)$ is a Nambu-Jacobi structure of order $n + 1$ (respectively, of order 3) on M.

3.2 Generalized Jacobi manifolds

A generalized almost Jacobi manifold $(M, \Lambda, \Box)$ of order n (even) is called *generalized Jacobi* [24, B] if the generalized Jacobi identity (2.2) is satisfied. This is equivalent to the properties:

$$[\Lambda, \Lambda] = 2(2s-1)\Lambda \wedge \Box \qquad \text{and} \qquad [\Lambda, \Box] = 0.$$

Example: [24]

Every Nambu-Jacobi manifold (of even order) is a generalized Jacobi manifold.

Example: [24]

Let $(M, \Lambda, \Box)$ be a generalized almost Jacobi manifold of even order n. For each $p \in \mathbb{R} - \{0\}$, we define on $M \times \mathbb{R}$ the n-vector

$$\widetilde{(\Lambda, \Box)}_p = e^{-pt}\Lambda + e^{-pt}\frac{\partial}{\partial t} \wedge \Box \tag{3.2}$$

where t is the usual coordinate on $\mathbb{R}$. Then, $(\Lambda, \dfrac{p}{n-1}\Box)$ is a generalized Jacobi structure on M if and only if $\widetilde{(\Lambda, \Box)}_p$ is a generalized Poisson structure on $M \times \mathbb{R}$.

Example:(Compatible Jacobi structures) [24]

Two Jacobi structures $(\Lambda_1, \Box_1)$ and $(\Lambda_2, \Box_2)$ on a manifold M are said to be *compatible* if $(\Lambda_1 + \Lambda_2, \Box_1 + \Box_2)$ is a Jacobi structure on M, that is, $[\Lambda_1, \Lambda_2] = \Box_1 \wedge \Lambda_2 + \Box_2 \wedge \Lambda_1$ and $\mathcal{L}_{\Box_1}\Lambda_2 + \mathcal{L}_{\Box_2}\Lambda_1 = 0$.

If we put

$$\Lambda = \Lambda_1 \wedge \Lambda_2, \qquad \Box = \Box_1 \wedge \Lambda_2 + \Box_2 \wedge \Lambda_1 \tag{3.3}$$

then $(M, \Lambda, \frac{2}{3}\square)$ is a generalized Jacobi manifold of order 4.

Example:(3-Sasakian manifolds)[24]

A 3-*Sasakian manifold* is a $(4n+3)$-dimensional Riemannian manifold (M, g) that admits three Sasakian structures $(\varphi_i, \square_i, \eta_i, g)_{i=1,2,3}$ whose vector fields $\square_1$, $\square_2$, $\square_3$ are mutually orthogonal and satisfy $[\square_{\sigma(1)}, \square_{\sigma(2)}] = 2\varepsilon_\sigma \square_{\sigma(3)}$, for all $\sigma \in \mathfrak{S}_3$ (see [27]).

Suppose that $(\Lambda_i, \square_i)$ is the associated Jacobi structure with the contact 1-form η_i, $i = 1, 2, 3$. Then, the structures $(\Lambda_1, \square_1)$, $(\Lambda_2, \square_2)$ and $(\Lambda_3, \square_3)$ are compatible and if Λ and $\square$ are the 4-vector and the 3-vector on M given by

$$\Lambda = \sum_{i=1}^{3} \Lambda_i \wedge \Lambda_i, \qquad \square = \sum_{i=1}^{3} \square_i \wedge \Lambda_i,$$

we have that $(\Lambda, \frac{2}{3}\square)$ is a generalized Jacobi structure on M of order 4.

Finally, if M is a 3-Sasakian manifold, then $M \times \mathbb{R}$ is an hyperkähler manifold. Moreover, if $\overline{\Lambda}$ is the generalized Poisson structure on $M \times \mathbb{R}$, then $\overline{\Lambda} = \widetilde{(\Lambda, \square)}_2$.

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