Proceedings Ist International Meeting
on Geometry and Topology
Braga (Portugal)

Public. Centro de Matemática
da Universidade do Minho
p. 111–117, 1998

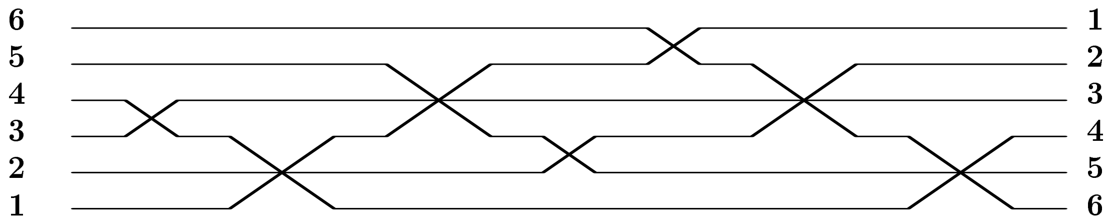
An interpretation of the monodromy group of a wiring diagram

António Guedes de Oliveira[1]

1 Introduction

Let $E = \{1, 2, \ldots, n\}$ and Σ be a sequence of strings which starts with $12 \cdots n$, the identity permutation of E, and, by reversing in each step one or more non-overlapping *increasing* substrings, ends with the reverse permutation $n(n-1) \cdots 1$. We may depict such an *allowable sequence* in a *wiring diagram* [2] as in the following example:

$\Sigma =$	Π_0,	Π_1,	Π_2,	Π_3,	Π_4,	Π_5,	Π_6,	Π_7
v	$12\underline{34}56$	$\underline{124}356$	$42\underline{135}6$	$4\underline{25}316$	$4523\underline{16}$	$45\underline{236}1$	$\underline{456}321$	654321



	Π_0^{-1}	Π_1^{-1}	Π_2^{-1}	Π_3^{-1}	Π_4^{-1}	Π_5^{-1}	Π_6^{-1}	Π_7^{-1}
$6 \mapsto$	6	6	6	6	6	5	3	1
$5 \mapsto$	5	5	5	3	2	2	2	2
$4 \mapsto$	4	3	1	1	1	1	1	3
$3 \mapsto$	3	4	4	4	4	4	4	4
$2 \mapsto$	2	2	2	2	3	3	5	5
$1 \mapsto$	1	1	3	5	5	6	6	6

Figure 1

[1]Partly supported by FCT and Project AGC/PRAXIS XXI/2/2.1/MAT/63/94

Now, this diagram can be thought of as *an arrangement of pseudolines* and hence, by Folkman-Lawrence theorem, as a rank 3 oriented matroid. In fact, every simple rank 3 voriented matroid can be represented this way [2](Theorem 6.3.3). When the lines are straight lines, of equation, say, $\alpha_j\, x + \beta_j\, y = \gamma_j$, and we consider the complement of the complexification of these lines,

$$\mathcal{M} \;=\; \{(z_1, z_2) \in \mathbb{C}^2 \,|\, \alpha_j\, z_1 + \beta_j\, z_2 \neq \gamma_j,\ j = 1, 2, \ldots, n\},$$

we know from work of Salvetti [7] that its topology lies in this oriented matroid. In particular, the fundamental group of $\mathcal{M}$ is a quotient of $\mathbb{F}_n$, the free group in n generators, where n is the number of lines, by a well-known theorem of Van Kampen-Zariski. Actual presentations were given by Randell [R] and Salvetti [7], and the latter one was considered under a different light, by using *monodromies*, by R. Cordovil and J. L. Fachada in [4]. My aim in here is to present a visualization of their construction:

Consider the manifold $\mathcal{M}(\mathcal{B}_n) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \neq x_j,$ for every $i \neq j\}$. Topologically, the manifold is the union of $n!$ connected (contractible) components or *chambers*, each one containing exactly one point of form $(i_1, i_2, \ldots, i_n) \in E^n \cap \mathcal{M}(\mathcal{B}_n)$ — a permutation of E. Now, since, for every allowable sequence Σ, two consecutive permutations ρ and σ differ only in the order of the elements of a substring of length, say, m, as elements of E^n the inverse permutations ρ^{-1} and σ^{-1} differ either in two coordinates, if $m = 2$, and the two coordinates are consecutive and so their chambers are separated by an hyperplane, or $m > 2$ and they are separated by the intersection of hyperplanes. Note that an allowable sequence has length $\binom{n}{2} + 1$ if and only if the latter never occurs in the sequence, in which case we call it *simple*.

A crucial step in Cordovil-Fachada's construction is the following graph G: Let the chambers of $\mathcal{M}(\mathcal{B}_n)$ form the set of vertices of G, adjacent in G whenever separated in $\mathbb{R}^n$ by exactly one hyperplane. Then there is a one-to-one correspondence between *simple* allowable sequences and paths in G of minimal length between the chambers pertaining to $12\cdots n$ and to $n(n-1)\cdots 1$ [4], (Theorem 2). By viewing G as part of the 1-skeleton of the Salvetti complex [7] corresponding to the (new) manifold

$$\mathcal{M}_{\mathbb{C}}(\mathcal{B}_n) \;=\; \{(z_1, z_2 \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j,\ \text{for every } i \neq j\},$$

to such a minimal path there corresponds a decomposition of Δ as a positive word in the braid group, where Δ is the generator of the (infinite cyclic) center of this group [4]. Roughly, the positive word is obtained from the wiring diagram by replacing "crossings" with "whirling" (see figure 2, below), so that, if we label the strands $1, 2, \ldots, n$, the highest labeled strand goes over the lowest labeled one.

We make this more precise by looking at $\mathcal{M}(\mathcal{B}_n)$ as part of $\mathcal{M}_{\mathbb{C}}(\mathcal{B}_n)$ so as to realize in the latter space the graph G: For the (simple) sequence Σ

of permutations of E, $\Pi_0, \Pi_1, \ldots \Pi_N$ ($N = \binom{n}{2}$), the vertices are, as before, $\Pi_0^{-1}, \Pi_1^{-1}, \ldots \Pi_N^{-1}$ (see Figure 1), viewed as points in $E^n \cap \mathcal{M}_{\mathbb{C}}(\mathcal{B}_n)$; we have already mentioned that, if the vertex $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ follows the vertex $\rho = (\rho_1, \rho_2, \ldots, \rho_n)$ in a path from $(1, 2, \ldots, n)$ to $(n, n-1, \ldots, 1)$, then there exist j, k with $j > k$ for which $\rho(j) = \sigma(k) = \rho(j) + 1$, $\rho(k) = \sigma(j) (= \sigma(k) - 1)$ and $\rho(l) = \sigma(l)$ for every $l \neq j, k$. That is, $\sigma = \rho + (0, \ldots, 0, 1, 0, \ldots, 0, -1, 0, \ldots, 0)$. We define the path from ρ to σ to be the polygonal line $[\rho\pi\sigma]$ where $\pi = \rho + (0, \ldots, 0, 1, 0, \ldots, 0, -1, 0, \ldots, 0)\left(\frac{1}{2} + \frac{1}{2}i\right)$.

The graph G is part of $\tilde{G}$, the 1-skeleton of the Salvetti complex [7], which a deformation retract of $\mathcal{M}_{\mathbb{C}}(\mathcal{B}_n)$. This new graph is obtained by "doubling" the edges of G. Note that it can be embedded in this manifold by considering also the paths between vertices through the points (represented underneath the horizontal plane in figure 2) of form $\alpha + (0, \ldots, 0, 1, 0, \ldots, 0, -1, 0, \ldots, 0)\left(\frac{1}{2} - \frac{1}{2}i\right)$ for every permutation α of E, [5], [7].

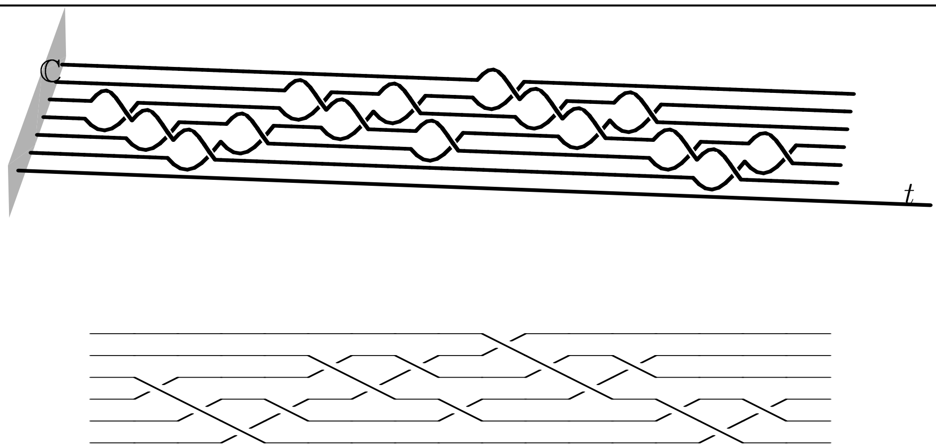


Figure 2

Finally, by using this setting, the path corresponding to a simple allowable sequence, i.e. the continuous function $c : [0, 1] \to \mathcal{M}_{\mathbb{C}}(\mathcal{B}_n)$ such that $c(t) = (c_1(t), c_2(t) \ldots, c_n(t))$, is a *geometric braid* when viewed as the union of the curves $(t, c_j(t)) \in [0, \binom{n}{2}] \times \mathbb{C}$, [1]. In case of a wiring diagram and its representation in G, this is the braid that corresponds to figure 2, as described above. We call it *the 3-dimensional picture* of the wiring diagram. Note that in figure 2 we have depicted (a smooth version of) the braid thus associated with a simple form of the initial sequence. This is done by untangling separately all pairs of strands whenever more than two cross at a given point in its wiring diagram. Note that, as braids, it is not relevant in which order the pairs are untangled[2][4], (p. 410).

[2]in the picture, we have used *colexicographical order*, i.e., the order where the first pair to

2 Geometric interpretation of the monodromy group of a wiring diagram

The *braid monodromy group* of Σ is defined in [4] as the subgroup of the (pure) braid group generated by the (ordered) set of given m elements of this group, $\Gamma_1, \Gamma_2, \ldots \Gamma_m$, where m is the number of cocircuits of the oriented matroid (or of "crossings" in the wiring diagram). These braids are walks (loops, in fact) in the 1-skeleton of the Salvetti complex of the complement of the complexification of the arrangement of real hyperplanes, i.e. in the graph $\tilde{G}$. Following our reasoning with the embedding of the latter graph in $\mathcal{M}_{\mathbb{C}}(\mathcal{B}_n)$ and its reading as a geometric braid, it is easy to see they correspond to what follows: Take the 3-dimensional picture of a braid together with its mirror image (see figure 3, where different colors were used for different strands).

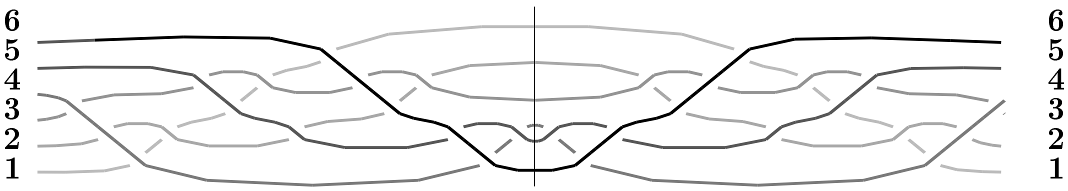


Figure 3

Naturally, we have obtained the *trivial braid*: by "pulling the extremities of the ropes", no two of them remain tangled.

Now, replace for every cocircuit of the oriented matroid C the mirror image of the whirling of the elements of C by the direct image of the same whirling. This is done so below, in figure 4, for $C = \{1, 3, 5\}$, the third cocircuit vof the wiring diagram. As a braid (denoted by Γ_3 in here), this comes to twisting 360^0 these three strands, to leaving strand 6 free and the remaining strands 2 and 4 *above* the crossing points of the former.

From these braids, say, $\Gamma_1, \Gamma_2, \ldots \Gamma_m$, a presentation of the fundamental group of $\mathcal{M} = \{(z_1, z_2) \in \mathbb{C}^2 \mid \alpha_j\, z_1 + \beta_j\, z_2 \neq \gamma_j,\ j = 1, 2, \ldots, n\}$, can be obtained as follows: let β_k $(k = 1, \ldots, n-1)$ be the (usual) k.th generator of the braid group. As a geometric braid, β_k twists 180^0 two consecutive strands labeled $k+1$ and k, the former above the latter, and leaves the others unchanged. Associate with it the unique automorphism $\overline{\beta_k} : \mathbb{F}_n \to \mathbb{F}_n$ such that, if $e_1, \ldots, e_n$ are the generators of $\mathbb{F}_n$:

$$\overline{\beta_k}(e_k) = {e_k}^{-1} \cdot e_{k+1} \cdot e_k, \overline{\beta_k}(e_{k+1}) = e_k,$$
$$\text{and } \overline{\beta_k}(e_j) = e_j, \text{ whenever } 1 \leq j \leq n, \quad j \neq k, k+1.$$

be untangled is the pair of lines labeled x and y (with $x < y$) where, firstly, y is as great as possible, and, finally, x is as great as possible.

(these are the inverses of the automorphisms defined in [4].) Now, for every braid written as a product of the generators, consider the product on the left of the corresponding automorphisms: $\overline{\beta_k\,\beta_l}(e_i) = \overline{\beta_l}(\overline{\beta_k}(e_i))$. According to [4], $\pi(\mathcal{M})$ admits a presentation with generators $e_1, \ldots, e_n$ and defining relations $\overline{\Gamma_j}(e_k) = e_k$, for $k = 1, \ldots, n$, $j = 1, \ldots, m$.

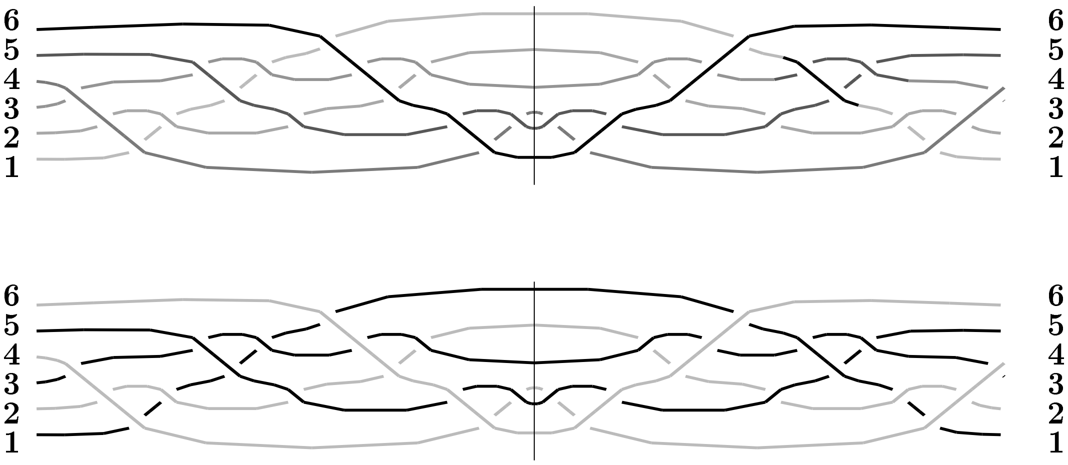


Figure 4

An easy way to construct these relations by using our drawings is as follows: Realize the elements of $\mathbb{F}_n$ as the labels of the strands of the 3-dimensional drawing of a wiring diagram, starting with the generators of $\mathbb{F}_n$. Now, for every "half-twist" or "crossing" of consecutive ropes labeled α and β with $\alpha > \beta$, relabel α by $\beta^{-1}\,\alpha\,\beta$ and keep the label of β if α crosses β *over it* and keep the label of α and replace β by $\alpha\,\beta\,\alpha^{-1}$ if it crosses it underneath.

Here is an example; it is easy to see the ends of the other strands remain as in the beginning. In general, if the monodromy braid is written in terms of the generators as $\beta_{i_1}, \beta_{i_k}, \ldots, \beta_{i_k}$, this construction relabels at the end the strand number i by $\overline{\beta_{i_1}, \beta_{i_k}, \ldots, \beta_{i_k}}(e_i)$. Hence, the non-trivial relations $\Gamma_3(e_i) = e_i$ are $e_1^{-1}\,e_3^{-1}\,e_5\,e_3\,e_1 = e_5, e_1^{-1}\,e_3^{-1}\,e_5^{-1}\,e_3\,e_5\,e_3\,e_1 = e_3$ and $e_1^{-1}\,e_3^{-1}\,e_5^{-1}\,e_1\,e_5\,e_3\,e_1 = e_1$, or, equivalently, $e_5\,e_3\,e_1 = e_3\,e_1\,e_5 = e_1\,e_5\,e_3$.

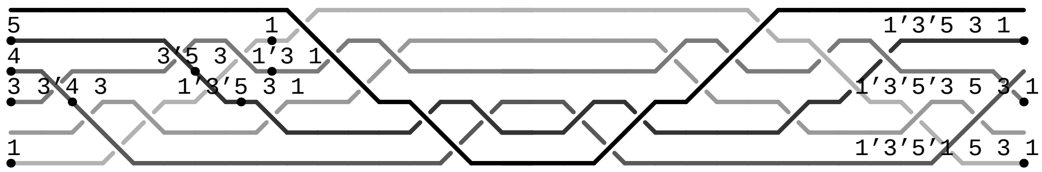


Figure 5

It is not difficult to see that, in general, with both constructions, the relations $\Gamma_j(e_i) = e_i$ read as those of Salvetti's presentation of $\pi(\mathcal{M})$.

Now, as a final remark, we note the following: the wiring diagram corresponding to figure 6, below, corresponds to the same (unoriented) matroid as the initial one, above. In particular, the third cocircuit of the initial configuration and the fifth of the second refer to the same cocircuit: $\{1, 3, 5\}$. As braids, their braid monodromies are similar, but not equal: in the first one, strands 2 and 4 lay in a superior plane to that of 1, 3 and 5; in the second one, it happens the opposite. Following Cordovil [3], let us see they are conjugate in one of the braid monodromy groups, $\alpha\,\Gamma_3\,\alpha^{-1} = \Gamma'_5$: In fact, if we take $\alpha = \Gamma_2^{-1}$, firstly, and then $\alpha = \Gamma_1^{-1}\,\Gamma_2^{-1}$, in $\alpha\,\Gamma_3\,\alpha^{-1}$ the strands 2 and 4 cross 1 underneath, firstly, and then under 3.

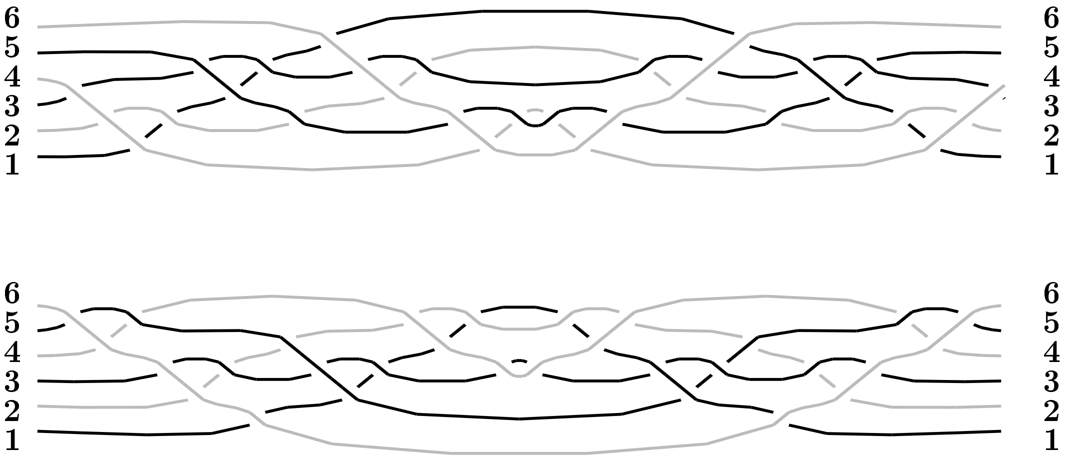


Figure 6

References

[1] Birman, J.: *Braids, Links and Mapping Class Groups*, Annals of Mathematical Studies **82**, Princeton Univ. Press (1975).

[2] Björner, A., Las Vergnas, M., Sturmfels, B., White, N. and Ziegler, G.M.: *Oriented Matroids*, Encycl. of Math. and its Appl. **46** (1993), Cambridge University Press.

[3] Cordovil, R.: "Braid Monodromy Groups of Arrangement of Hyperplanes", manuscript, 1995, to be published in *Adv. in Applied Math.*

[4] Cordovil, R. and Fachada, J. L.: "Braid Monodromy Groups of Wiring Diagrams", *Bull. U.M.I.* (7) **9-B** (1995), 399–416.

[5] Cordovil, R., Guedes de Oliveira, A. and Las Vergnas, M.: "A generalized Desargues configuration and the pure braid group", *Discrete Mathematics* **160** (1996) 105–113.

[6] Randel, R.: "The Fundamental Group of the Complement of a Union of Complex Hyperplanes", *Inv. Math.* **69** (1982), 103–108; correction, *Inv. Math.* **80** (1985), 467–468.

[7] Salvetti, M.: "Topology of the complement of real hyperplanes in $\mathbb{C}^N$", *Inv. Math.* **88** (1987), 603–618.

António Guedes de Oliveira
Centro de Matemática do Porto
Faculdade de Ciências da Univ. do Porto
PORTUGAL
Email: agoliv@fc.up.pt