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On k-minimal submanifolds

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This is an account of joint work with U.Dursun. Details can be found in [1, 2, 3].

Let M^m be a smooth, connected, m-dimensional manifold without boundary, and let $f : M^m \to \mathbf{R}^p$ be a smooth immersion. I will consider immersions for which the focal set has certain symmetries. For example, in the case $m = 2$, in each normal plane the focal set is symmetrical about its base point if and only if the immersion is minimal. If $m > 2$, minimality can be interpreted as saying that, for all $x \in M$, on each normal line through $f(x)$ the focal points with base x balance out (strictly speaking it is the principal curvatures that balance out). Letting $N_x(f)$ and $T_x(f)$ denote the normal plane and tangent plane to f at x, the technical condition is that $\forall x \in M$, $\forall n \in N_x(f)$, $\operatorname{trace} A_n = 0$, where $A_n : T_x(f) \to T_x(f)$ is the shape operator of f in direction n. It is this interpretation of minimality that I will use.

Now let $\Lambda(f)$ be a line subbundle of $N(f)$, the normal bundle of f. I will give a condition which will imply that, for each $x \in M$, the focal points in $N_x(f)$ balance out along lines in $N_x(f)$ perpendicular to $\Lambda_x(f)$, the fibre of $\Lambda(f)$. So, for $m = 2$, the focal set in $N_x(f)$ has reflexive symmetry about $\Lambda_x(f)$. This certainly needs $\operatorname{trace} A_n = 0$ for all n perpendicular to $\Lambda_x(f)$. Further, assuming that $\Lambda(f)$ is parallel and u is a local parallel section, then $f_u : M \to \mathbf{R}^{m+d}$ defined by $f_u(x) = f(x) + u(x)$ is such that, for each $x \in M$, the normal planes and focal sets of f and f_u at x coincide. Then the required condition is that the trace of the shape operator of f_u in direction n is zero for all normals n perpendicular to $\Lambda_x(f)$. This can be expressed in terms of the shape operator of f, to give $\operatorname{trace}(1 - A_u)^{-1} A_n = 0$. If this condition holds for all u for which f_u is an immersion (so $(1 - A_u)$ is non-singular) I will say that f is 1-minimal. This can be generalized for symmetry about planes.

Definition 1 Let $\Lambda(f)$ be a parallel, flat k-plane subbundle of $N(f)$. The immersion f is *k-minimal with respect to* $\Lambda(f)$ if $\forall x \in M$, $\forall n \in N_x(f)$ with $n \perp \Lambda_x(f)$, $\forall u \in \Lambda_x(f)$ with $1 - A_u$ non-singular, $\operatorname{trace}(1 - A_u)^{-1} A_n = 0$.

Here is an equivalent definition.

Definition 2 The immersion f is k-minimal with respect to $\Lambda(f)$ if $\forall x \in M$, $\forall n \in N_x(f)$ with $n \perp \Lambda_x(f)$, $\forall u \in \Lambda_x(f)$, $\forall \ell = 0, 1, \ldots, (m-1), \ldots$ $\mathrm{trace}(A_u)^l A_n = 0$.

Yet another equivalent statement can be obtained by considering $\Omega = \{(x, u) \in \Lambda(f) : 1 - A_u$ is non-singular$\}$ and the immersion $F : \Omega \to \mathbf{R}^{m+d}$ defined by $F(x, u) = f(x) + u$.

Theorem 1([2]) The immersion f is k-minimal with respect to $\Lambda(f)$ if and only if F is minimal.

The image of F is "locally ruled" in the sense that it is locally foliated by k-planes. The conditions on $\Lambda(f)$ ensure a close relationship between the focal sets of f and F. It would be interesting to consider k-plane subbundles $\Lambda(f)$ of $N(f)$ which, locally, are minimal (in the Euclidean space) without any other conditions on $\Lambda(f)$, but then the focal sets of f and F may not be so closely related. For example, a curve in $\mathbf{R}^3$ with nowhere zero curvature satisfies the condition with $\Lambda_x(f)$ spanned by the principal normal at $f(x)$. But the surface obtained by taking lines through the principal normals will not, in general, be minimal. If the curve is a helix then the surface obtained is a helicoid which is minimal.

Examples More details are given in [3].

(1) If $f : M \to \mathbf{R}^p$ is pseudo-umbilical and $H/ \parallel H \parallel$ is parallel (where H is the mean curvature vector) then f is 1-minimal with respect to the line bundle determined by H.

(2) Let $f : S^1 \times S^1 \times S^1 \to \mathbf{R}^6$ be defined by

$$f(\alpha, \beta, \gamma) = (a \cos \alpha, a \sin \alpha, b \cos \beta, b \sin \beta, c \cos \gamma, c \sin \gamma)$$

where $a > 0$, $b > 0$, $c > 0$. Then f is 1-minimal if $a = b = c$, 2-minimal if two of a, b, c are equal and otherwise is 3-minimal.

Now if f is k-minimal with respect to $\Lambda(f)$ then, from the equivalent definition, $\mathrm{trace}\, A_n = 0$, $\forall n \perp \Lambda_x(f)$. It follows that, $\forall x \in M$, the mean curvature vector $H(x) \in \Lambda_x(f)$. In the 1-minimal case if the mean curvature vector H is not zero at x, then $\Lambda_x(f)$ is the line determined by this vector, $H/ \parallel H \parallel$ is parallel and, $\forall n \perp H$, $\mathrm{trace}(A_H)^l A_n = 0$, $\ell = 1, \ldots, m-1$. Immersions for which $\mathrm{trace}\, A_H A_n = 0$ $\forall n \in N_x(f)$ with $\langle H, n \rangle = 0$ have been studied by a number of people.

Definition 3 An immersion $f : M^m \to \mathbf{R}^p$ is a *Chen immersion* (A-immersion) if $\mathrm{trace}\, A_H A_n = 0$, $\forall x \in M$ and $\forall n \in N_x(f)$ with $\langle H, n \rangle = 0$.

These immersions were introduced by B.Y.Chen in [4] and have been developed by, among others, L.Gheysens, P.Verheyen and L.Verstraelen [5], C.S.Houh [6], S.J.Li [7], and B.Rouxel [8]. It is immediate from the above comment that a 1-minimal immersion is Chen. For an immersion $f : M^2 \to \mathbf{R}^p$ with $H/\parallel H\parallel$ parallel then f is 1-minimal if and only if f is Chen. All known examples of Chen immersions of higher dimensional manifolds with $H/\parallel H\parallel$ parallel are 1-minimal but it seems very unlikely that these ideas are the same.

Finally I will describe a method for generating many examples of k-minimal immersions. Start with a pseudoumbilical immersion $f : M^m \to \mathbf{R}^p$ (or, more generally, let f be 1-minimal) with $u = H/\parallel H\parallel$ parallel. Take any immersion $g : W^w \to \mathbf{R}^{1+w}$, $g(y) = (g_0(y), \ldots, g_w(y))$, with unit normal field $\psi : W \to \mathbf{R}^{1+w}$ and include $\mathbf{R}^p$ in $\mathbf{R}^{p+w}$ so that f can be thought of as an immersion into $\mathbf{R}^p \times \mathbf{R}^w$. Then, at each $x \in M$, put the image of g into the $1 + w$-subplane of the normal plane to f at x spanned by $H(x)$ and $\mathbf{R}^w$. More formally, define $h : M \times W \to \mathbf{R}^{p+w}$ by

$$h(x, y) = (f(x) + g_0(y)u(x), g_1(y), \ldots, g_w(y)).$$

If M and W are compact, by replacing g by ϵg if necessary, for sufficiently small ϵ, h is an immersion. Then $\mu : M \times W \to \mathbf{R}^{p+w}$ defined by

$$\mu(x, y) = (\psi_0(y)u(x), \psi_1(y), \ldots, \psi_w(y))$$

is a parallel normal field for h. It can be shown that h is 1-minimal with respect to the line bundle determined by μ [1]. This construction can be repeated, replacing f by h and, possibly, using a different g. A similar construction can be done if g has codimension k, which gives h is k-minimal. Even more generally we can start with f being k-minimal. These results are summarized in the following theorem.

Theorem 2([2]) Let $f : M^m \to \mathbf{R}^p$ be an immersion with an orthonormal set of parallel normal fields $u_1, \ldots, u_k$ $(1 \le k \le p - m)$, let $q \ge k$ and let $g : W^w \to \mathbf{R}^q$, $g(y) = (g_1(y), \ldots, g_q(y))$, be an immersion with an orthonormal set of parallel normal fields $\psi_1, \ldots \psi_{q-w}$ such that $\forall (x, y) \in M \times W$, $1 - \sum_{i=1}^{k} g_i(y)A_{u_i}$ is non-singular, where $A_{u_i} : T_x(f) \to T_x(f)$ is the shape operator of f.
Let $h : M \times W \to \mathbf{R}^{p+q-k}$ be defined by

$$h(x, y) = (f(x) + \sum_{i=1}^{k} g_i(y)u_i(x), g_{k+1}(y), \ldots, g_q(y)).$$

Then h is an immersion and for $j = 1, \ldots, q - w$, $\mu_j : M \times W \to \mathbf{R}^{p+q-k}$ defined by

$$\mu_j(x, y) = (\sum_{i=1}^{k} (\psi_j)_i(y)u_i(x), (\psi_j)_{k+1}(y), \ldots, (\psi_j)_q(y))$$

is a parallel normal field for h and $\mu_1, \ldots, \mu_{q-w}$ is orthonormal. Let $\Lambda(f)$ be the k-plane subbundle of $N(f)$ with fibre

$$\Lambda_x(f) = \mathrm{span}\{u_1(x), \ldots, u_k(x)\}$$

and let $\tilde{\Lambda}(h)$ be the $(q-w)$-plane subbundle of $N(h)$ with fibre

$$\tilde{\Lambda}_{(x,y)}(h) = \mathrm{span}\{\mu_1(x,y), \ldots, \mu_{q-w}(x,y)\}.$$

Then, if f is k-minimal with respect to $\Lambda(f)$, h is $(q-w)$-minimal with respect to $\tilde{\Lambda}(h)$.

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