On composite stability of time-varying discrete interval systems

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Abstract

We consider a discrete dynamic interconnected system such that each of its isolated system is a time-varying discrete interval system. First, sufficient and also necessary conditions for the stability of isolated time-varying discrete interval systems are studied and the corresponding stability margins as well as bounds on robust stability are obtained. Our analysis heavily employs the Perron–Frobenius theorem for nonnegative matrices and its extensions to a class of interval matrices. Second, we find bounds on elements of (time-varying) interconnection matrices such that the composite system is stable with a given margin. Two illustrative numerical examples are presented.

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1 Introduction and Summary

An interval matrix is a real matrix in which all the elements are known only to the extent that each belongs to specified closed interval. In particular, an $n \times n$ interval matrix A_I is in fact a set of real matrices

$$A_I = \{A = [a_{ij}] : a_{ij} \in [b_{ij}, c_{ij}], \quad i, j = 1, \ldots, n\}$$

where $b_{ij} \le c_{ij}$ are given real numbers. Let $B = [b_{ij}]$, $C = [c_{ij}]$, and hence $A_I = [B \quad C]$. Observe that every $A \in A_I$ can be also written as $A = A_0 + \delta A$ where $A_0 = \frac{1}{2}(B + C)$ is the so called nominal matrix and $|\delta A| \le \Delta A = \frac{1}{2}(C - B)$ (the symbol $\le$ in a matrix relation is considered elementwise). Considering some $A \in A_I$, the symbol $|A|$ is reserved for the corresponding modulus matrix, i.e. $|A| = [|a_{ij}|]$, where $|a_{ij}|$ is the absolute value of a_{ij}.

The time-varying discrete interval system is given for $k = 0, 1, \ldots,$ by

$$x(k+1) = A(k)\, x(k), \quad x(0) = x_0 \tag{1}$$

where $x(k) = [x_i(k)]$ is an n-dimensional state vector and $A(k) \in A_I$. Obviously, using the nominal matrix A_0 instead of (1) we can also write

$$x(k+1) = (A_0 + \delta A(k))\, x(k), \quad x(0) = x_0. \tag{1a}$$

The system (1) (or equivalently (1a)) is (globally) asymptotically stable if for every $x(0) = x_0$ and every matrix sequence $\{A(k), k = 0, 1, \ldots : A(k) \in A_I\}$ $\lim_{k\to\infty} x(k) = 0$. The time-varying system (1) is said to be stable with stability margin h, where $0 \le h < 1$ (or to have the degree of stability h), if there exists a number $c \ge 0$ (depending on x_0) such that for an arbitrary matrix sequence $A(k) \in A_I$ it holds that $|x_i(k)| \le c\,(1-h)^k$ for $i = 1, \ldots, n$. Observe that from our definition of the stability margin we immediately conclude that the time-invariant discrete system (1) (i.e. if in (1) $A(k) \equiv A \in A_I$, $\forall k = 0, 1, \ldots$) is stable with stability margin h if and only if the modulus of every eigenvalue of any $A \in A_I$ is less than $1 - h$.

A matrix $A \in A_I$ is called extreme (vertex) if for every $i, j = 1, 2, \ldots, n$ either $a_{ij} = b_{ij}$ or $a_{ij} = c_{ij}$, and is (asymptotically) stable iff the system $x(k+1) = A\, x(k)$, $\forall k$ is (asymptotically) stable. In the sequel, we denote by $\bar{A}_I$ the (finite) set of all extreme matrices. Obviously, $\bar{A}_I \subset A_I$ and $\bar{A}_I$ possesses not more than 2^{n^2} different matrices. An extreme matrix $\hat{A} \in \bar{A}_I$ is called *dominating* if it is the extreme matrix with maximum modulus elements, i.e. $\hat{A} \in \bar{A}_I$ such that $|\hat{A}| \ge |A|$ for every $A \in \bar{A}_I$ (obviously then also $|\hat{A}| \ge |A|$ for every $A \in A_I$).

In recent years, stability of dynamic interval systems has been studied by many authors and some sufficients conditions for the stability have been obtained mostly for time-invariant interval systems. In particular, De Blasi and Schinas [5] applied general results on continuous time dynamical systems to study multi-valued discrete dynamical systems and Baranov [2] used the Lyapunov indicator approach for investigating stability of discrete systems under uncertainty. Jiang [9] has attempted to show that the time-invariant discrete system (1) is asymptotically stable, if and only if all extreme matrices are asymptotically stable. This result was shown to be incorrect (cf. e.g. Correspondence in *International*

Journal of Control [8]). However, it was shown in Juang *et al.* [11] and in Mori and Kokame [12] that the time-invariant system (1) is asymptotically stable if some norm of the extreme matrices is restricted to be less than unity.

The aim of the present contribution is the following.

First, we focus our attention on the stability of time-varying interval systems as well as on the corresponding stability margins and bounds on robust stability. This topic was recently studied in Bauer *et al.* [1], in Han and Lee [7] and in Zeng [14]. In [1] a necessary and sufficient condition for the stability of time-varying discrete interval matrices was presented. The condition is based on checking and evaluating the products of extreme matrices using standard matrix norms. Even the authors presented an algorithmic procedure for generating and checking these products, there are severe computational limits of an actual implementation of the algorithm since the computational complexity increases exponentially with the size of the considered interval matrix. On the other hand in [7] a simple sufficient condition for the stability of time-varying discrete interval systems was presented. Furthermore, in [14] maximal bound on robust stability was presented for the case of nonnegative nominal system with time-varying perturbations.

In Section 3 of the present paper we show that the sufficient condition on stability of time-varying discrete interval systems presented in [7] can be easily obtained from the Perron–Frobenius theorem on maximal eigenvalue of a nonnegative matrix. Moreover, our approach immediately yields also stability margins on the considered system as well bound on robust stability with a given stability margin. The simple test on stability suggested in [7] can be further amplified by combining this test with the approach presented in [1] and by evaluating the resulting matrices by weighted maximum norms. The obtained results are also useful for improving existing stability results on time-invariant systems.

Second, we investigate robust stability of composite systems. In particular, in Section 4 we consider an interconnected discrete dynamic system $\mathcal{S}$ such that each of its isolated subsystems $\mathcal{S}_i$ $(i = 1, \ldots, r)$ is a time-varying discrete interval system, i.e. $x_i(k+1) = A_{ii}(k)\, x_i(k)\, (\forall k = 0, 1, \ldots)$, where is $x_i(k)$ is an n_i-dimensional state vector and $A_{ii}(k) \in A^I_{ii}$ (the set of interval matrices of dimension $n_i \times n_i$). Under some assumptions on stability of every isolated subsystem $\mathcal{S}_i$, we focus our attention on finding bounds on elements of interconnection matrices such that the composite system is stable with a given margin. In particular, we may assume that the bounds are available on the absolute values of the maximum variations on the elements of the time-varying interconnection matrices $A_{ij}(k) \in I\!R^{n_i \times n_j}$, i.e. for every $i, j = 1, \ldots, r$ $|A_{ij}(k)| \leq \varepsilon_{ij}\, A_{ij}$ where $A_{ij} \in I\!R^{n_i \times n_j}$ are known nonnegative matrices and real numbers $\varepsilon_{ij} > 0$

are uncertain parameters. Our aim is to find the bounds of ε_{ij} under which the composite system $\mathcal{S}$ is stable with a given margin for all $|A_{ij}(k)| \le \varepsilon_{ij}\, A_{ij}$ with $i, j = 1, \ldots, r$.
Two illustrative examples are presented in Section 5.

2 Preliminaries

Recall that, according to the well-known Perron–Frobenius theorem (cf. e. g. [3] or [6]), the spectral radius, denoted $\rho(\cdot)$, of a nonnegative matrix is equal to its largest positive eigenvalue (called the Perron eigenvalue) and the corresponding left, right eigenvector (denoted $v(\cdot)$, $u(\cdot)$ and called the left, right Perron eigenvector respectively) can be selected nonnegative. In case that the matrix is irreducible, the Perron eigenvalue is simple, the Perron eigenvector is unique up to a multiplicative constant and can be selected positive.

In particular, for the dominating matrix $\hat{A}$ it holds (remember that symbols $\ge$, $>$ in a matrix relation are considered componentwise)

$$\left.\begin{aligned} \rho(|\hat{A}|)\, u(|\hat{A}|) &= |\hat{A}|\, u(|\hat{A}|) \ge |A|\, u(|\hat{A}|) \\ \rho(|\hat{A}|)\, v(|\hat{A}|) &= v(|\hat{A}|)\, |\hat{A}| \ge v(|\hat{A}|)\, |A| \end{aligned}\right\} \qquad (2)$$

where the left, resp. right, eigenvector $u(|\hat{A}|) \ge 0$, resp. $v(|\hat{A}|) \ge 0$. Observe that on premultiplying the first relation of (2) by $v(|A|)$, we immediately conclude that $\rho(|\hat{A}|) \ge \rho(|A|)$ for every $|\hat{A}| \ne |A|$, and in general $|A| \ge |B| \Longrightarrow \rho(|A|) \ge \rho(|B|)$. Moreover, if $|A|$ is irreducible $|A| > |B| \Longrightarrow \rho(|A|) > \rho(|B|)$ and $u(|A|) > 0$ is the unique positive eigenvector of $|A|$ (up to multiplicative constant). For details see [3] or [6].

Recall that the one norm (denoted $\|\cdot\|_1$) and the maximum (or infinity) norm (denoted $\|\cdot\|_\infty$) of the matrix $A = [a_{ij}]$ is given by $\max\limits_j \sum_{i=1}^n |a_{ij}|$ and by $\max\limits_i \sum_{j=1}^n |a_{ij}|$ respectively. Similarly, for every $u \in I\!R_+^n$ (i. e. $u > 0$, where $u = [u_1, \ldots, u_n]$) the u-weighted maximum norm of the matrix A is given by $\|A\|_u = \max\limits_{i=1,\ldots,n} \sum\limits_{j=1}^n |a_{ij}| \frac{u_j}{u_i}$. Supposing that $A = [a_{ij}] \in A_I$ is irreducible, let $u > 0$ be selected such that $\rho(|A|)\, u = |A|\, u$. Then $\|A\|_u = \rho(|A|)$ (observe that $\sum_{j=1}^n |a_{ij}| \frac{u_j}{u_i} = \rho(|A|) \quad \forall i = 1, \ldots, n$). In case of reducible A it may happen that no $u > 0$ exists (for details see [3] or [6]). In this specific case, recalling the continuous dependence of the matrix eigenvalues on its parameters, on replacing the null elements of $|A|$ by sufficiently small $\tilde{\varepsilon} > 0$, for an arbitrary small $\varepsilon > 0$ we can find $u^\varepsilon > 0$ such that $\rho(|A|) + \varepsilon \ge \|A\|_{u^\varepsilon} \ge \rho(|A|)$.

Observe that if $A \in A_I$ is asymptotically stable, there exists $u > 0$ such that $\|A\|_u < 1$, however it may happen that $\|A\| > 1$ both for the one and the maximum norm.

3 Stability of Discrete–Time Interval Systems

Stability criteria for time-invariant discrete systems based on examination of extreme matrices are summarized in the following propositions (unless otherwise stated, $\|\cdot\|$ denotes an arbitrary matrix norm):

Proposition 1 (cf. Juang *et al.* [11], Mori and Kokame [12])
The time-invariant discrete system (1) is globally asymptotically stable if there exists a matrix norm such that the norm of every extreme matrix in A_I is less than one.

Proof. Let $\bar{A}_I = \{A^{(i)} : i = 1, \ldots, p \equiv 2^{n^2}\}$ be the set of all extreme points of the interval matrix A_I. Obviously, if $A \in A_I$ then there exist α_i's such that $\alpha_i \geq 0$, $\sum_{i=1}^{p} \alpha_i = 1$ and $A = \sum_{i=1}^{p} \alpha_i A^{(i)}$. Recalling elementary properties of matrix norms we immediately have $\|A\| \leq \sum_{i=1}^{p} \alpha_i \|A^{(i)}\|$ and hence if $\|A^{(i)}\| < 1$ for every $A^{(i)} \in \bar{A}_I$, then also $\|A\| < 1$. Since the system (1) is supposed to be time-invariant, iterating (1) we get $x(k) = A^k x_0$ and taking the norms we immediately get $\|x(k)\| \leq \|A^k\| \|x_0\| \leq \|A\|^k \|x_0\|$. If $\|A\| < 1$, then the time-invariant system (1) is asymptotically stable. □

Now we present a stability criterion similar to that in Proposition 1 that is based on the dominating matrix $\hat{A}$ instead of matrix norms of extreme matrices. Since this criterion is a special case of the more general result for time-varying interval systems presented later (cf. Proposition 3 and Corollary 1) its proof is omitted.

Proposition 2 *The time-invariant discrete system (1) is asymptotically stable if the modulus matrix of the dominating matrix $\hat{A}$ is asymptotically stable, i.e. if $\rho(|\hat{A}|) < 1$. Then $h = 1 - \rho(|\hat{A}|)$ is the stability margin of (1) and there exists a matrix norm such that the norm of every matrix $A \in A_I$ (and hence also every extreme matrix in A_I) is less than one, i.e. the stability conditions due to Proposition 1 are fulfilled.*

Now we focus our attention on time-varying interval systems. The following proposition slightly extends some of the results reported in Han and Lee [7], in particular, the presented proof seems to be simpler and the part concerning the stability margins is believed to be new. On the contrary to the original proof in

[7], we show that that Proposition 3 immediately follows by iterating (2) where time-varying matrix $A(k)$ replaces the matrix A. Corollary 1 and Corollary 2 (perhaps the main results in Chen [4] and in Shafai et al. [13]) follow then trivially from Proposition 3, and, since in these references only time-invariant systems are considered, also from Proposition 2.

Proposition 3 (cf. Han and Lee [7] Theorem 1 and Lemma 1).
The time-varying discrete system (1) is globally asymptotically stable if $\rho(|\hat{A}|) < 1$. *Then* $h = 1 - \rho(|\hat{A}|)$ *is the stability margin of (1).*

Proof. Since $|A(k)| \le |\hat{A}|$ also $|\prod_{k=0}^{m-1} A(k)| \le \prod_{k=0}^{m-1} |A(k)| \le |\hat{A}|^m$. Supposing that $u(|\hat{A}|) > 0$, if $|x(0)| \le u(|\hat{A}|)$ we get from (1), (2)

$$|x(m)| = |\prod_{k=1}^{m} A(m-k)x(0)| \le |\prod_{k=1}^{m} A(m-k)|\, |x(0)| \le$$

$$\le |\prod_{k=1}^{m} A(m-k)|\, u(|\hat{A}|) \le |\hat{A}|^m\, u(|\hat{A}|) = \rho(|\hat{A}|)^m\, u(|\hat{A}|)$$

In case that $u(|\hat{A}|) \ge 0$ replace all null elements of $|\hat{A}|$ by sufficiently small $\tilde{\varepsilon} > 0$ such that for any given $\varepsilon > 0$ for the resulting matrix $|\hat{A}^{\varepsilon}|$ it holds $\rho(|\hat{A}^{\varepsilon}|) < \rho(|\hat{A}|) + \varepsilon$ (obviously, $\rho(|\hat{A}^{\varepsilon}|) > \rho(|\hat{A}|)$, $u(|\hat{A}^{\varepsilon}|) > 0$) and repeat the above proof for $|\hat{A}| = |\hat{A}^{\varepsilon}|$. $\square$

Recalling that since $|\hat{A}|$ is a nonnegative matrix, we can find a u-weighted matrix norm such that for every $\varepsilon > 0$ $\|\hat{A}\|_u - \varepsilon \le \rho(|\hat{A}|) \le \|\hat{A}\|_u$. In particular, Proposition 3 yields immediately the following corollary.

Corollary 1 *The time-varying discrete system (1) is globally asymptotically stable if there exists* $\hat{u} \in \mathbb{R}_+^n$ *such that* $|\hat{A}|\hat{u} < \hat{u}$. *Then* $|A|\hat{u} < \hat{u}$ *and* $\|A\|_{\hat{u}} \le \|\hat{A}\|_{\hat{u}} < 1$ *for every* $A \in \hat{A}_I$.

Proposition 3 also has another important corollary reported in [7].

Corollary 2 (cf. Han and Lee [7] Corollary 1 and Corollary 2).
(i) Let $C \ge |B|$. *Then (1) is asymptotically stable if and only if* $\rho(C) < 1$.
(ii) Let $|B| \ge |C|$. *Then (1) is asymptotically stable if* $\rho(|B|) < 1$.

Remark 1 *Observe that if the nominal matrix* A_0 *of the system (1b) is nonnegative (resp. nonpositive) then* $C \ge |B|$ *(resp.* $|B| \ge |C|$*).*

¿From Corollary 1 we immediately obtain necessary and sufficient conditions for stability of nonnegative time-invariant discrete interval systems studied in [4] and in [13].

Corollary 3 (cf. Shafai *et al.* [13] and Chen [4] Theorem 1).
Let $B \geq 0$. Then (1) is globally asymptotically stable if and only if $\rho(C) < 1$. In particular, also the time-invariant system given by (1) is asymptotically stable.

Of course, the obtained results are also closely connected with the necessary and sufficient conditions for the stability of time-varying interval matrices reported in [1]. These results can be summarized as follows.

Proposition 4 (cf. Bauer *et al.* [1] Theorem and Lemma).
The time-varying discrete system (1) is globally asymptotically stable if and only if there exists a finite m_0 such that $\|\prod_{k=0}^{m_0-1} A(k)\| < 1 \quad \forall A(k) \in A_I,\ k = 0, 1, \ldots, m_0 - 1$, where $\|\cdot\|$ is reserved for the one norm or for maximum norm. This condition is fulfilled if and only if

$$\|\prod_{k=0}^{m_0-1} A(k)\| < 1 \quad \textit{for every} \quad A(k) \in \bar{A}_I,\ k = 0, 1, \ldots, m_0 - 1. \tag{3}$$

The following theorem is an extension of Propositions 3 and 4. Observe that for $m_0 = 1$ the sufficiency part of Theorem 1 reduces to Corollary 1; on setting $\hat{u}$ equal to the unit vector Theorem 1 reduces to Proposition 4 where $\|\cdot\|$ is the maximum norm.

Theorem 1 *The time-varying discrete system (1) is globally asymptotically stable if and only if there exists a finite m_0, $\hat{\rho} \in (0, 1)$ and $\hat{u} \in I\!R^n_+$ such that*

$$|\prod_{k=0}^{m_0-1} A(k)|\hat{u} < \hat{\rho}\,\hat{u} \quad \textit{for every} \quad A(k) \in A_I,\ k = 0, 1, \ldots, m_0 - 1. \tag{4}$$

Then $h = 1 - (\hat{\rho})^{1/m_0}$ is the stability margin of (1). Condition (4) is fulfilled if and only if

$$|\prod_{k=0}^{m_0-1} A(k)|\hat{u} < \hat{\rho}\,\hat{u} \quad \textit{for every} \quad A(k) \in \bar{A}_I,\ k = 0, 1, \ldots, m_0 - 1. \tag{5}$$

Since $|\prod_{k=0}^{m_0-1} A(k)|\hat{u} < \hat{\rho}\,\hat{u} \iff \|\prod_{k=0}^{m_0-1} A(k)\|_{\hat{u}} < \hat{\rho}$ we immediately get the following corollary.

Corollary 4 *Condition (5) is fulfilled if and only if there exists a $\hat{u}$-weighted maximum norm such that*

$$\| \prod_{k=0}^{m_0-1} A(k) \|_{\hat{u}} < \hat{\rho} \quad \textit{for every} \ \ A(k) \in \bar{A}_I, \ k = 0, 1, \dots, m_0 - 1. \tag{6}$$

Proof of Theorem 1. The first part of the theorem follows similarly to the proof of Proposition 3. Iterating (1) we get for fixed m_0, $\ell = 0, 1, \dots, m_0 - 1$ and $p = 1, 2, \dots$

$$x(p\, m_0 + \ell) = \prod_{j=0}^{p-1} \prod_{k=1}^{m_0} A((p-j)m_0 - k + \ell)\, x(\ell) \tag{7}$$

Since we can select $\hat{u} > 0$ such that $x(\ell) \le \hat{u}$ for $\ell = 0, 1, \dots, m_0 - 1$, if (4) is fulfilled we immediately get

$$|x(p\, m_0 + \ell)| \le \prod_{j=0}^{p-1} | \prod_{k=1}^{m_0} A((p-j)m_0 - k + \ell))|\, \hat{u} \le \hat{\rho}^p\, \hat{u}. \tag{8}$$

implying that if (4) is fulfilled the system (1) is stable with margin h.

If condition (4) is not satisfied for the maximum norm we get $\| \prod_{k=0}^{m_0-1} A(k) \| \ge 1$ and the necessity part of Theorem 1 follows from Proposition 4.

To finish the proof if suffices to verify that for finding extreme values in (7), (8) we can restrict our attention only on $A(\cdot) \in \bar{A}_I$. To this order assume by contradiction that for some $k_0 \le m_0$

$$| \prod_{k=1}^{m_0} A(k) | = | \prod_{k=1}^{k_0-1} A(k) \left(\sum_j \alpha^{(j)}(k_0) A^{(j)} \right) \prod_{k=k_0+1}^{m_0} A(k) |$$

where $A^{(j)}$ are vertex matrices and $\alpha^{(j)}(k_0)$ are nonnegative numbers such that $\sum_j \alpha^{(j)}(k_0) = 1$ and $\alpha^{(j)}(k_0) \ne 0, 1$ for at least one j. Then the RHS in the above equation depends linearly on each $\alpha^{(j)}(k_0)$, and hence the maximum must be attained within the class $\bar{A}_I$ (the same can be applied in (7), (8)). □

Up to now we have concentrated on discrete systems (1), (1a) where $A(k) = A_0 + \delta A(k)$, $\delta A(k) \le \Delta A$ with A_0, ΔA given, and obtained some sufficient stability conditions along with the corresponding stability margins. Supposing that the nominal matrix A_0 is fixed and $|A_0|$ is stable, the following natural question arises:

Find the largest values of elements of ΔA such that (1a) remains stable with a given margin h. In particular, supposing that ΔA is given find $\varepsilon_{\max}(h) > 0$ such that

$$x(k+1) = (|A_0| + \varepsilon(k)\delta A(k))\, x(k), \qquad x(0) = x_0 \tag{9}$$

is stable with given margin h for every $\varepsilon(k) \in [0, \varepsilon_{\max}(h)]$. The following theorem yields the value $\varepsilon_{\max}(h)$; for $h = 0$ the problem was studied in Zeng [14].

Theorem 2 *Let the matrix $|A_0|$ be stable with margin h_0. Then the time-varying discrete system (9) is globally asymptotically stable with a given margin $h < h_0$ if*

$$\varepsilon(k) < \varepsilon_{\max}(h) = \frac{1}{[\rho((1-h)I - |A_0|)^{-1}\Delta A]} < 1. \tag{10}$$

Proof. Obviously for $\varepsilon \in [0, \varepsilon_{\max}(h)]$

$$\rho(|A_0| + \varepsilon\Delta A) \leq \rho(|A_0| + \varepsilon_{\max}(h)\Delta A)$$

and

$$\rho(|A_0| + \varepsilon_{\max}(h)\Delta A) < 1 - h$$

if and only if there exists $u \in \mathbb{R}^n_+$ such that

$$(|A_0| + \varepsilon_{\max}(h)\Delta A)\, u < (1-h)u.$$

Since $|A_0|$ is assumed stable with margin h_0, if $h \leq h_0$ there exists $((1-h)I - |A_0|)^{-1} = (1-h)^{-1}\sum_{m=0}^{\infty} |(1-h)^{-1}A_0|^m > 0$ (hereafter I is reserved for an identity matrix of an appropriate dimension). After some algebra we conclude that

$$\varepsilon_{\max}(h)\, \Delta A\, u \leq ((1-h)I - |A_0|)\, u \Longleftrightarrow \varepsilon_{\max}(h)\, ((1-h)I - |A_0|)^{-1}\Delta A\, u < u$$

Recalling that $((1-h)I - |A_0|)^{-1}\Delta A > 0$, the last inequality holds if and only if

$$\rho[((1-h)I - |A_0|)^{-1}\Delta A] < \frac{1}{\varepsilon_{\max}(h)}$$

that finishes our proof. □

Remark 2 *Observe that if $A_0 = |A_0|$ (i.e. if the nominal matrix is nonnegative) the stability conditions of Theorem 2 are necessary and sufficient.*

For $h = 0$ we obtain

Corollary 5 (cf. Zeng [14] Theorems 3 and 4).
The time-varying discrete system (9) with $A_0 > 0$ stable is globally asymptotically stable if and only if

$$\varepsilon(k) < \varepsilon_{\max}(h) = \frac{1}{[\rho(I - |A_0|)^{-1}\Delta A]} < 1$$

4 Composite Stability of Systems

Now we focus our attention on composite stability of interconnected systems. In particular, we shall consider for $k = 0, 1, \ldots$ and $i = 1, \ldots, r$ an interconnected time-varying discrete dynamic system $\mathcal{S}$

$$x_i(k+1) = A_{ii}(k)\, x_i(k) + \sum_{j=1, j\neq i}^{r} A_{ij}(k)\, x_j(k) \tag{11}$$

We assume that

(i) Every isolated subsystem $\mathcal{S}_i$ $(i = 1, \ldots, r)$ is a time-varying discrete interval system, i.e.

$$x_i(k+1) = A_{ii}(k)\, x_i(k), \quad \forall k = 0, 1, \ldots \tag{12}$$

where is $x_i(k)$ is an n_i-dimensional state vector and $A_{ii}(k) \in A_{ii}^I$ (the set of interval matrices of dimension $n_i \times n_i$). Let $\hat{A}_{ii} \in A_{ii}^I$ be the dominating matrix of the isolated subsystem $\mathcal{S}_i$.

(ii) For every isolated subsystem $\mathcal{S}_i$ there exists $\hat{u}_i \in \mathbb{R}_+^{n_i}$ along with a number $\hat{\rho}_i \in (0,1)$, $\hat{\rho}_i \geq \rho(|\hat{A}_{ii}|)$ such that

$$\hat{\rho}_i\, \hat{u}_i \geq |\hat{A}_{ii}|\, \hat{u}_i \geq |A_{ii}|\, \hat{u}_i \qquad \forall A_{ii} \in A_{ii}^I \tag{13}$$

Hence every subsystem $\mathcal{S}_i$ is globally asymptotically stable with the stability margin $h_i = 1 - \hat{\rho}_i$ (cf. Proposition 3).

(iii) $A_{ij}(k)$ of dimension $n_i \times n_j$ is a time-varying interconnection matrix, such that for every $i, j = 1, \ldots, r$ $|A_{ij}(k)| \leq A_{ij}$, where $A_{ij} \in \mathbb{R}^{n_i \times n_j}$ are known nonnegative matrices.

We are looking for real numbers $\varepsilon_{ij}(h) > 0$ $(i,\, j = 1, \ldots, r,\, j \neq i)$ such that the composite system (11) remains globally asymptotically stable with a given margin h if

$$|A_{ij}(k)| \leq \varepsilon_{ij}(h) A_{ij} \qquad \text{for all} \;\; i, j = 1, \ldots, r$$

To this end we shall follow an approach similar to that in the proof of Proposition 3 and Theorem 1, and introduce column vectors $x(k)$, $\hat{u}(h)$ and a square matrix $A(h)$ being of dimensions $\sum_{j=1}^{r} n_j$ such that

$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_r(k) \end{bmatrix} \qquad \hat{u}(h) = \begin{bmatrix} \hat{u}_1(h) \\ \hat{u}_2(h) \\ \vdots \\ \hat{u}_r(h) \end{bmatrix}$$

$$\hat{A}(h) = \begin{bmatrix} |\hat{A}_{11}| & \varepsilon_{12}(h)A_{12} & \dots & \varepsilon_{1r}(h)A_{1r} \\ \varepsilon_{21}(h)A_{21} & |\hat{A}_{22}| & \dots & \varepsilon_{2r}(h)A_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{r1}(h)A_{r1} & \varepsilon_{r2}A_{r2} & \dots & |\hat{A}_{rr}| \end{bmatrix}$$

where for $\hat{u}(h) > 0$

$$\hat{A}(h)\,\hat{u}(h) \;\leq\; \hat{\rho}\,\hat{u}(h). \tag{14}$$

Theorem 3 *Let condition (13) hold for every isolated subsystem (12). Then the composite system (11) is globally asymptotically stable with a given stability margin $h \leq \min_i(1-\hat{\rho}_i)$ if the following conditions are satisfied*
(i) $|A_{ij}(k)| \leq \varepsilon_{ij}(h)A_{ij}$ for every $i, j = 1, \dots, r$, $i \neq j$
(ii) Spectral radius $\rho(\hat{A}(h))$ of the matrix $\hat{A}(h)$ is less than $1-h$,
i.e. there exists $\hat{u}(h) > 0$ such that (14) holds for $\rho(\hat{A}(h)) \leq \hat{\rho} < 1-h$. Furthermore, $1-\hat{\rho}$ is the stability margin of composite system (11).

Condition (ii) can be awkward to check if large scale composite systems are considered. To this end we present a simple sufficient condition for the stability of composite system (11).

Corollary 6 *Spectral radius $\rho(\hat{A}(h))$ of the matrix $\hat{A}(h)$ is less than $1-h$ if there exist $\hat{u}_j \in \mathbb{R}_+^{n_j}$ $(j = 1, \dots, r)$ such that*

$$\sum_{j=1,\dots,r,j\neq i}^{r} \varepsilon_{ij}(h)A_{ij}\,\hat{u}_j + |\hat{A}_{ii}|\hat{u}_i < (1-h)\hat{u}_i \quad \textit{for every} \quad i = 1, \dots, r \tag{15}$$

Proof. Without loosing generality we may suppose that $\hat{u}(h) \geq x(0)$. Similarly as in the proof of Proposition 3 we get

$$|x(m)| \leq |\hat{A}(h)|^m\,\hat{u}(h) = (\hat{\rho})^m\,\hat{u}(h)$$

establishing the assertion of Theorem 3.
Recalling that the spectral radius of a matrix must be nongreater than any its norm, employing the $\hat{u}$-weighted maximum norm, we immediately obtain the assertion of Corollary 6. □

For the case of two isolated subsystems, Corollary 7 presents another condition replacing condition (ii) in Theorem 3.

Corollary 7 *Let the assumptions of Theorem 3 be fulfilled and let there exist only two subsystems, i.e. $r = 2$. Then the composite system (11) is stable with a given margin $h \leq \min(1-\hat{\rho_1}, 1-\hat{\rho_2})$ if the spectral radius*

$$\rho(((1-h)I - |\hat{A}_{11}|)^{-1}A_{12}\,((1-h)I - |\hat{A}_{22}|)^{-1}A_{21}) \;<\; \frac{1}{\varepsilon_{12}(h)\,\varepsilon_{21}(h)} \tag{16}$$

Proof. From (15) of Corollary 6 the composite system $\mathcal{S}$ is stable with a given margin h if

$$\begin{aligned} |\hat{A}_{11}|\,\hat{u}_1 + \varepsilon_{12}(h)\,A_{12}\,\hat{u}_2 &< (1-h)\hat{u}_1 \\ \varepsilon_{21}(h)\,A_{21}\hat{u}_1 + |\hat{A}_{22}|\,\hat{u}_2 &< (1-h)\hat{u}_2 \end{aligned}$$

implying that

$$\begin{aligned} \varepsilon_{12}(h)\,((1-h)I - |\hat{A}_{11}|)^{-1}\,A_{12}\,\hat{u}_2 &< \hat{u}_1 \\ \varepsilon_{21}(h)\,((1-h)I - |\hat{A}_{22}|)^{-1}\,A_{21}\,\hat{u}_1 &< \hat{u}_2 \end{aligned}$$

The proof can be finished by inserting from the second to the first equation and recalling that the resulting matrix is nonnegative. □

5 Illustrative Examples

We shall test our results on two numerical examples of composite systems. The respective isolated subsystems are interval systems borrowed from [1] and [10].

Example 1. Consider a composite system (11) with two isolated subsystems (see Example 3 in [10])

$$A_{11}^I = A_{22}^I = \begin{bmatrix} [-0.20 \quad 0.16] & [-0.34 \quad 0.02] \\ [-0.24 \quad 0.12] & [-0.16 \quad 0.20] \end{bmatrix}.$$

and the interconnecting matrices $A_{12}(k)$, $A_{21}(k)$ bounded by

$$A_{12} = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & 0.4 \end{bmatrix} \qquad A_{21} = \begin{bmatrix} 0.4 & 0.8 \\ 0.2 & 1.0 \end{bmatrix}$$

Then for the isolated subsystems we get

$$|\hat{A}_{11}| = |\hat{A}_{22}| = \begin{bmatrix} 0.2 & 0.34 \\ 0.24 & 0.2 \end{bmatrix}$$

$\|\hat{A}_{11}\|_1 = \|\hat{A}_{22}\|_\infty = 0.54$ and $\rho(|\hat{A}_{11}|) = 0.485$, hence each isolated subsystem is stable with stability margin 0.515. We are looking for $\varepsilon_{12}(h)$, $\varepsilon_{21}(h)$ such that the composite system (11) with $A_{ij}(k) \le \varepsilon_{ij}(h)A_{ij}$, $(i \ne j)$ is stable with margin $h = 0.2$.

Recalling that for $h = 0.2$

$$((1-h)I - |\hat{A}_{11}|)^{-1} = ((1-h)I - |\hat{A}_{22}|)^{-1} = \begin{bmatrix} 2.155 & 1.221 \\ 0.8621 & 2.155 \end{bmatrix}$$

and

$$((1-h)I - |\hat{A}_{11}|)^{-1} A_{12} \, ((1-h)I - |\hat{A}_{11}|)^{-1} A_{21} = \begin{bmatrix} 4.228 & 12.981 \\ 3.683 & 11.016 \end{bmatrix}$$

Calculating the spectral radius of the above matrix we conclude (cf. (16)) that the composite system (11) is stable if $\varepsilon_{12}(h) \cdot \varepsilon_{21}(h) < 0.06525$.

Example 2. Consider a composite system (11) with two isolated subsystems (see Example 3 in [10] and Example 2 in [1])

$$A_{11}^I = \begin{bmatrix} [-0.20 & 0.16] & [-0.34 & 0.02] \\ [-0.24 & 0.12] & [-0.16 & 0.20] \end{bmatrix}, \qquad A_{22}^I = \begin{bmatrix} [-0.7 & 0.6] & 0.6 \\ 0.1 & & 0.5 \end{bmatrix}.$$

and the interconnecting matrices $A_{12}(k)$, $A_{21}(k)$ bounded by

$$A_{12} = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & 0.4 \end{bmatrix} \qquad A_{21} = \begin{bmatrix} 0.4 & 0.8 \\ 0.2 & 1.0 \end{bmatrix}$$

Then for the first, resp. second, isolated subsystem we get

$$|\hat{A}_{11}| = \begin{bmatrix} 0.2 & 0.34 \\ 0.24 & 0.2 \end{bmatrix}, \qquad |\hat{A}_{22}| = \begin{bmatrix} 0.7 & 0.6 \\ 0.1 & 0.5 \end{bmatrix}$$

$\|\hat{A}_{11}\|_1 = \|\hat{A}_{11}\|_\infty = 0.54$ and $\rho(|\hat{A}_{11}|) = 0.485$, hence the first isolated subsystem is stable with stability margin 0.515. Similarly, for the second isolated subsystem we get $\|\hat{A}_{22}\|_1 = 1.1$, $\|\hat{A}_{22}\|_\infty = 1.3$ and $\rho(|\hat{A}_{22}|) = 0.865$, hence the second isolated system is stable with stability margin 0.135. We are looking for $\varepsilon_{12}(h)$, $\varepsilon_{21}(h)$ on interconnecting matrices A_{12}, A_{21} such that the composite system (11) with $A_{ij}(k) \leq \varepsilon_{ij}(h) A_{ij}$, $(i \neq j)$ is stable with margin 0.05.
Recalling that for $h = 0.05$

$$((1-h)I - |\hat{A}_{11}|)^{-1} = \begin{bmatrix} 1.559 & 0.7070 \\ 0.4991 & 1.559 \end{bmatrix}$$

$$((1-h)I - |\hat{A}_{22}|)^{-1} = \begin{bmatrix} 8.5714 & 11.428 \\ 1.9047 & 4.762 \end{bmatrix}$$

and

$$((1-h)I - |\hat{A}_{11}|)^{-1} A_{12} ((1-h)I - |\hat{A}_{22}|)^{-1} A_{21} = \begin{bmatrix} 11.2028 & 37.0793 \\ 10.5944 & 34.7206 \end{bmatrix}$$

Calculating the spectral radius of the above matrix (cf. (16)) we conclude that the composite system (11) is stable if $\varepsilon_{12}(h) \cdot \varepsilon_{21}(h) < 0.0403$.

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References

[1] P. H. Bauer, K. Premaratne, J. Durán: *A necessary and sufficient condition for robust asymptotic stability of time-variant discrete systems.* IEEE Trans. Automat. Control AC-38, 1993, 1427–1430.

[2] M. E. Barabanov: *Lyapunov indicator of discrete inclusions. Parts I, II, III.* Autom. Remote Control 49, 1988, 152–157 (Part I), 283–287 (Part II), 558–565 (Part III).

[3] A. Berman, R. J. Plemmons: *Nonnegative Matrices in the Mathematical Sciences.* Academic Press, New York, 1979.

[4] J. Chen: *Comments on 'A necessary and sufficient conditions for the stability of nonnegative interval discrete systems.'* IEEE Trans. Automat. Control AC-38, 1993, 189.

[5] F. S. De Blasi, J. Schinas: *Stability of multivalued discrete dynamical systems.* J. Differential Equations 14, 1973, 245–262.

[6] F. R. Gantmakher: *Teoriya matric.* Second edition. Nauka, Moscow 1966. English translation: *The Theory of Matrices, Volume I, II.* Chelsea, New York 1959.

[7] H. S. Han, J. G. Lee: *Necessary and sufficient conditions for stability of time-varying discrete interval matrices.* Internat. J. Control 59, 1994, 1021–1029.

[8] *Internat. J. Control* 49, 1989, 1095–1108, Correspondence.

[9] C. L. Jiang: *Sufficient and necessary condition for the asymptotic stability of discrete linear interval systems.* Internat. J. Control 47, 1988, 1563–1565.

[10] Y.-T. Juang, C.-S. Shao: *Stability analysis of dynamic interval systems.* Internat. J. Control 49, 1989, 1401–1408.

[11] Y.-T. Juang, S.-H. Tung, T.-C. Ho: *Sufficient condition for asymptotic stability of discrete interval systems.* Internat. J. Control 49, 1989, 1799–1803.

[12] T. Mori, H. Kokame: *Convergence properties of interval matrices and interval polynomials.* Internat. J. Control 45, 1987, 243–248.

[13] B. Shafai, K. Perev, J. Cowley, Y. Chehab: *A necessary and sufficient conditions for the stability of nonnegative interval discrete systems.* IEEE Trans. Automat. Control AC-36, 1991, 742–746.

[14] X.-J. Zeng: *Robust stability for linear discrete-time systems with structured perturbations.* Internat. J. Control 61, 1995, 739–748.