Prenucleolus and Nucleolus of a Cooperative Game: Characterizations by Tight Coalitions

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Abstract

A coalition in a balanced cooperative game with transferable utility is called tight if its payoff equals the coalition worth at every core allocation. In a nonbalanced game, the grand coalition is called tight. Non-tight coalitions are called slack. Structure and properties of the collection of tight coalitions are studied. For the prenucleolus, an independence property involving slack coalitions is introduced. Together with the continuity property and the requirement that to every additive game the unique core allocation is assigned, this independence property characterizes the prenucleolus. For the nucleolus, a characterization is obtained in a similar way by adding a property of independence involving least-core tight coalitions.

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1 Introduction

Game theory is concerned with studying formal, mathematical models of conflict and cooperation between several parties. The usual distinction between cooperative and noncooperative theory is mainly methodological: whereas noncooperative game theory is strategy-oriented, in cooperative theory the emphasis is on the payoffs that can be achieved. Game theory has important applications to economics (cf. the title of the seminal work by von Neumann and Morgenstern, 1944) as well as other social sciences.

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This paper belongs to a particular branch of cooperative game theory: cooperative games with transferable utility. A cooperative game with transferable utility or TU-game is described by a finite set of players and a function, assigning a real number to every coalition (i.e., a subset of the player set), the so-called worth of the coalition. If a coalition forms, then it may distribute its worth among its members in any way, and the central question in the theory of TU-games is: how will—or should—it do this? In studying this question, it is usually assumed that the grand coalition of all players forms. This is without loss of generality to the extent that also if smaller coalitions form, the question of how to distribute their worths needs to be answered. On the other hand, the assumption avoids the equally important question of coalition formation and the interaction between coalition formation and payoff distribution.

The question of payoff distribution can be handled by proposing a solution, i.e., a map or correspondence assigning a payoff distribution or set of payoff distributions to every TU-game. Two related issues play a major role in the study of a solution. First, a solution should be game-theoretically justified, which goal can be attained by axiomatic characterization(s) or noncooperative implementation(s). Second, the computation of a solution may be a difficult task, in particular because the number of coalitions grows exponentially with the number of players[1].

Among the best known solution concepts for TU-games are the core (Gillies, 1953, Bondareva, 1962, Shapley, 1967), the Shapley value (Shapley, 1953), and the nucleolus (Schmeidler, 1969). The core is a correspondence which has a "strategic" justification, as it assigns all payoff distributions that cannot be improved upon by any coalition. The Shapley value assigns a unique payoff distribution to every game. Roughly, this distribution gives each player his average marginal contribution in the game. However, also if the core is non-empty, the Shapley value distribution may fail to be in the core. The nucleolus is a one-point solution that is defined for all TU-games with non-empty imputation set—imputations are individually rational payoff distributions. If a game has a non-empty core, then the nucleolus is in the core. The latter is also true for the prenucleolus, which in that case coincides with the nucleolus. The prenucleolus, however, is defined for all games, because it is not individually rationally constrained.

The focus of this paper is on the prenucleolus and the nucleolus. In contrast with the Shapley value, both solutions are not defined by an explicit formula but as the solution of a minimization problem. Also, both solutions are difficult to axiomatize and to compute. Axiomatizations by means of so-called reduced games were given by Sobolev (1975) for the prenucleolus and by Potters (1990) and Snijders (1995) for the nucleolus. Kohlberg (1971) has characterized the

[1]For practical applications, general results with respect to axiomatization or computation may be of mainly theoretical interest, because the specific detailed structure of a particular class of games at hand may lead to a more satisfactory specific axiomatization and computation. See, for instance, Littlechild (1974) for an early reference.

nucleolus by a balancedness condition. Maschler, Peleg and Shapley (1979) have investigated further geometrical properties of the nucleolus, resulting in an algorithm to compute the nucleolus. This was generalized by Maschler, Potters and Tijs (1992). For still other results on the nucleolus, see Potters and Tijs (1992, 1994). Recently, a lot of effort has been invested in computational methods for the nucleolus. See Sankaran (1991), Solymosi and Raghavan (1994), Potters, Reijnierse and Ansing (1995), Kuipers, Solymosi, and Aarts (1995), and Hallefjord, Helming, and Jornsten (1995).

In this paper, we present alternative axiomatic characterizations of the prenucleolus and the nucleolus. In both characterizations, the main axiom is closely linked to the concept of tight coalitions (cf. for instance Potters and Tijs, 1994). Both characterizations are related to the Kohlberg criterion (Kohlberg, 1971) as well as to the algorithm proposed in Maschler, Peleg, and Shapley (1979). Basically, we present a way of converting the algorithm into an axiomatic characterization. Usually, this algorithm is performed by solving a system of linear programs. We suggest an alternative, based on minimal balanced collections, which might be convenient in case of repeated application for a small set of players.

The organization of the rest of the paper is as follows. Preliminaries are collected in Section 2. Section 3 deals with tight coalitions and the structure of the collection of tight coalitions. In Sections 4 and 5, we characterize the prenucleolus and the nucleolus, respectively.

2 Preliminaries

An *n-person transferable utility game* or, briefly, *game*, is a pair (N, v) where $N = \{1, \ldots, n\}$ is the set of players and $v : \mathcal{N} \to \mathbb{R}$ is the *characteristic function*. Here, $\mathcal{N}$ is the collection of all non-empty subsets of N. An element of $\mathcal{N}$ is called a *coalition*. For a coalition S, the number $v(S)$ is the *worth* of S. Instead of $v(\{i\})$ we often write $v(i)$. We denote by $\mathcal{G}^n$ the set of all n-person games, and write $v \in \mathcal{G}^n$ if (N, v) is an n-person game.[2]

A game is called *additive* if $v(S \cup T) = v(S) + v(T)$ for all $S, T \in \mathcal{N}$ with $S \cap T = \emptyset$. An additive game is determined by the vector $(v(1), \ldots, v(n))$. In the sequel we identify an additive game with this vector.

For a vector $x \in \mathbb{R}^n$, let $x(S) = \sum_{i \in S} x_i$. The set of *preimputations* is defined by

$$I^*(v) = \{x \in \mathbb{R}^n : x(N) = v(N)\},$$

and the set of *imputations* is the (possibly empty) set

$$I(v) = \{x \in \mathbb{R}^n : x(N) = v(N) \text{ and } x_i \geq v(i) \text{ for all } i \in N\}.$$

[2]We deviate from the usual model where, mainly for mathematical convenience, the empty set is included as a coalition, having worth zero by definition. In the present paper, including the empty coalition would be mathematically—in particular notationally—very inconvenient.

By $\mathcal{IG}^n$ we denote the set of games with a non-empty imputation set.

The *core* of a game is the set

$$C(v) = \{x \in I^*(v) : x(S) \geq v(S) \quad \text{for all } S \in \mathcal{N}\}.$$

Let $\mathcal{B} \subseteq \mathcal{N}$ be a collection of coalitions.[3] The game $\mathbf{1}_\mathcal{B}$ is defined by $\mathbf{1}_\mathcal{B}(S) = 1$ if $S \in \mathcal{B}$ and $\mathbf{1}_\mathcal{B}(S) = 0$ otherwise.

A collection $\mathcal{B} \subseteq \mathcal{N}$ is called *balanced* if there are positive weights λ_S for all $S \in \mathcal{B}$ such that $\sum_{S\in\mathcal{B}} \lambda_S e_S = e_N$, where $e_S \in \mathbb{R}^n$ is the indicator vector of S. The union of balanced collections is again balanced. A balanced collection is called *minimal* if does not contain a proper subcollection which is balanced. Every balanced collection is the union of minimal balanced collections. See Shapley (1967) for further details.

A game is called *balanced* if for every balanced collection $\mathcal{B}$

$$\sum_{S\in\mathcal{B}} \lambda_S v(S) \leq v(N),$$

where the weights λ_S satisfy the balancedness condition for $\mathcal{B}$.[4] Equivalently, we can replace "balanced" by "minimal balanced" in this definition. By a well-known result of Bondareva (1962) and Shapley (1967) a game has a non-empty core if and only if it is balanced. This result follows from the Duality Theorem of Linear Programming (or Farkas' Lemma). The balancing weights λ_S appear as dual variables in a linear programming problem. The following, nontechnical interpretation can also be given. Suppose each player has one unit of time available, which he can spend in various coalitions. A collection is balanced if it is feasible in the sense that, if all coalitions in that collection are formed (consecutively, where the balancing weights give the time that the coalition is active), each player spends exactly his one unit of time. Let $v(S)$ be interpreted as the productivity per unit of time of coalition S. Balancedness of a game then means that it is always more productive to form the grand coalition than to split up time among the various coalitions of a balanced collection. The Bondareva-Shapley characterization amounts to the, according to this interpretation intuitive, result that it is always at least as productive to form the grand coalition if, and only if, the core is non-empty.

By $\mathcal{BA}^n$ we denote the set of balanced games, i.e., the set of games with a non-empty core.

Next, we introduce the concepts of ϵ-core and least core as they are known from the literature. These can be defined either with respect to the preimputation set or with respect to the imputation set. In the former case, we use the prefix "pre-", as well as the asterisk * in the corresponding notations.

[3] By $\subseteq$ we denote inclusion, and by $\subset$ strict inclusion.

[4] In the remainder of the paper it will always be clear to which collection such balancing weights correspond.

For every $\epsilon \in I\!R$ the *pre-ϵ-core* of a game v is defined by

$$C^*_\epsilon(v) = \{x \in I^*(v) : x(S) \geq v(S) - \epsilon \text{ for all } S \in \mathcal{N},\ S \neq N\}.$$

This set is non-empty for ϵ large enough. The smallest non-empty pre-ϵ-core is the *least precore*

$$LC^*(v) = \bigcap \{C^*_\epsilon(v) : C^*_\epsilon(v) \neq \emptyset\}.$$

It is easy to see that $LC^*(v) = C^*_{\epsilon^*(v)}(v)$ where

$$\epsilon^*(v) := \min_{x \in I^*(v)} \max_{S \in \mathcal{N}, S \neq N} (v(S) - x(S)).$$

Alternatively, we can consider the games

$$v^\epsilon = v - \epsilon \mathbf{1}_{\mathcal{N} \setminus \{N\}}. \tag{1}$$

Then $C^*_\epsilon(v) = C(v^\epsilon)$ and $LC^*(v) = C(v^{\epsilon^*(v)})$.

For $v \in \mathcal{IG}^n$ the *ϵ-core* is defined by

$$C_\epsilon(v) = \{x \in I(v) : x(S) \geq v(S) - \epsilon \text{ for all } S \in \mathcal{N},\ S \neq N\}$$

and the *least core* by

$$LC(v) = \bigcap \{C_\epsilon(v) : C_\epsilon(v) \neq \emptyset\}.$$

Then $LC(v) = C_{\epsilon(v)}(v)$, where

$$\epsilon(v) = \min_{x \in I(v)} \max_{S \in \mathcal{N}, S \neq N} (v(S) - x(S)).$$

Alternatively, consider the games

$$v_1^\epsilon := \begin{cases} v^\epsilon & \text{if } \epsilon \leq 0 \\ v - \epsilon \mathbf{1}_{\mathcal{N} \setminus \{S : |S| \in \{1,n\}\}} & \text{if } \epsilon > 0. \end{cases}$$

Then $C_\epsilon(v) = C(v_1^\epsilon)$ and $LC(v) = C(v_1^{\epsilon(v)})$.

In order to introduce the prenucleolus and the nucleolus of a game, we define the *excess* of a coalition $S \subseteq N$ and a preimputation x to be the number $e^v(S, x) := v(S) - x(S)$. Let $\theta(e^v(x))$ be the vector of all excesses arranged in non-increasing order and let $\preceq_{\text{lex}}$ be the lexicographic order. The *prenucleolus* of a game is the unique preimputation $\nu^*(v)$ satisfying

$$\theta(e^v(\nu^*(v))) \preceq_{\text{lex}} \theta(e^v(x)) \quad \text{for all } x \in I^*(v).$$

Similarly, for a game with non-empty imputation set the *nucleolus* is the unique imputation $\nu(v)$ satisfying

$$\theta(e^v(\nu(v))) \preceq_{\text{lex}} \theta(e^v(x)) \quad \text{for all } x \in I(v).$$

Existence and uniqueness of the prenucleolus are proved in Sobolev (1975), and of the nucleolus in Schmeidler (1969).

The following properties are well known in literature.

(a) The prenucleolus and nucleolus are continuous functions $\nu^* : \mathcal{G}^n \longrightarrow \mathbb{R}^n$ and $\nu : \mathcal{IG}^n \longrightarrow \mathbb{R}^n$, respectively.

(b) The prenucleolus and nucleolus coincide and lie in the core whenever the core is not empty. Moreover, the prenucleolus is in every non-empty pre-ϵ-core and in the least precore. The nucleolus is in every non-empty ϵ-core and in the least core.

(c) The prenucleolus and nucleolus are covariant, i.e., $\nu^*(\lambda v + a) = \lambda\nu^*(v) + a$ and $\nu(\lambda v + a) = \lambda\nu(v) + a$ whenever $0 < \lambda \in \mathbb{R}$ and a is an additive game. In particular, this implies that $\nu^*(v) = v = \nu(v)$ for every additive game v. (This fact also follows from (b).)

(d) The prenucleolus satisfies

$$\nu^*(v) = \nu^*(v^\epsilon) \text{ for all } \epsilon \in \mathbb{R}, \tag{2}$$

as is straightforward from the definition of ν^*.

(e) The balancedness criterion for the prenucleolus (see Sobolev, 1975):

For $x \in I^*(v)$ let

$$\begin{aligned}
\mathcal{B}_1^v(x) &= \{S \in \mathcal{N} : e^v(S,x) = \max_{T \in \mathcal{N}} e^v(T,x)\} \\
\mathcal{B}_2^v(x) &= \{S \in \mathcal{N} \backslash \mathcal{B}_1^v(x) : e^v(S,x) = \max_{T \in \mathcal{N} \backslash \mathcal{B}_1^v(x)} e^v(T,x)\} \\
\mathcal{B}_3^v(x) &= \{S \in \mathcal{N} \backslash (\mathcal{B}_1^v(x) \cup \mathcal{B}_2^v(x)) : e^v(S,x) = \max_{T \in \mathcal{N} \backslash (\mathcal{B}_1^v(x) \cup \mathcal{B}_2^v(x))} e^v(T,x)\}
\end{aligned}$$

etc.

Then $x \in I^*(v)$ is the prenucleolus if, and only if, $\mathcal{B}_1^v(x)$, $\mathcal{B}_1^v(x) \cup \mathcal{B}_2^v(x)$, $\mathcal{B}_1^v(x) \cup \mathcal{B}_2^v(x) \cup \mathcal{B}_3^v(x)$, etc. are balanced collections.

(f) The Kohlberg criterion for the nucleolus (see Kohlberg, 1971):

Let, additionally,

$$\mathcal{B}_0^v(x) = \{\{i\} : x_i = v(i)\}.$$

Then $x \in I(v)$ is the nucleolus if, and only if, $\mathcal{B}_1^v(x)$, $\mathcal{B}_1^v(x) \cup \mathcal{B}_2^v(x)$, $\mathcal{B}_1^v(x) \cup \mathcal{B}_2^v(x) \cup \mathcal{B}_3^v(x)$, etc. can be extended to balanced collections by adding coalitions from $\mathcal{B}_0^v(x)$.

3 Tight coalitions

We start with the definition of tightness.

Definition 3.1 *Let v be a game with a non-empty core. A coalition $S \in \mathcal{N}$ is called tight if*

$$x(S) = v(S) \quad \textit{for all } x \in C(v).$$

If v has empty core, then a coalition S is called tight if, and only if, $S = N$. The collection of all tight coalitions is denoted by $\mathcal{T}(v)$.

Note that $\mathcal{T}(v)$ always contains N.

The collection of tight coalitions is closely connected to the dimension of the core. If $v \in \mathcal{BA}^n$ and $\mathcal{T}(v) = \{N\}$, then the core has full dimension. Also, $\mathcal{T}(v) = \mathcal{N}$ if, and only if, v is additive. In that case, the core consists of a single point. More generally, the core consists of a single point if, and only if, the dimension of $\mathcal{T}(v)$, i.e. the row rank of the matrix with column vectors e_S for $S \in \mathcal{T}(v)$, is equal to n.

Coalitions in a game that are not tight, will be called *slack*.

We are interested in the structure and the properties of $\mathcal{T}(v)$.

Lemma 3.1 *Let $v \in \mathcal{BA}^n$. Then there is an $x \in C(v)$ with $x(S) > v(S)$ for all $S \notin \mathcal{T}(v)$.*

Proof Let $\mathcal{T}(v) \neq \mathcal{N}$ (otherwise there is nothing to prove). For every $T \notin \mathcal{T}(v)$ there is an $x_T \in C(v)$ with $x_T(T) > v(T)$. Let

$$x := \frac{1}{\left|\overline{\mathcal{T}(v)}\right|} \sum_{T \notin \mathcal{T}(v)} x_T.$$

It is straightforward to check that x is as desired. □

Recall that a game is balanced if $\sum_{S \in \mathcal{B}} \lambda_S v(S) \leq v(N)$ for every balanced collection $\mathcal{B}$. The following lemma shows when this inequality is strict and when not.

Lemma 3.2 *Let $v \in \mathcal{BA}^n$ and $\mathcal{B}$ be a balanced collection. Then*

$$\sum_{S \in \mathcal{B}} \lambda_S v(S) = v(N) \textit{ if, and only if, } \mathcal{B} \subseteq \mathcal{T}(v).$$

Proof For any preimputation x we have $\sum_{S \in \mathcal{B}} \lambda_S x(S) = \sum_{S \in \mathcal{B}} \lambda_S e_S \cdot x = e_N \cdot x = x(N) = v(N)$. Now let x be as in Lemma 3.1. Then $\sum_{S \in \mathcal{B}} \lambda_S v(S) = \sum_{S \in \mathcal{B}} \lambda_S x(S) = v(N)$ if $\mathcal{B} \subseteq \mathcal{T}(v)$ and $\sum_{S \in \mathcal{B}} \lambda_S v(S) < \sum_{S \in \mathcal{B}} \lambda_S x(S) = v(N)$ if $\mathcal{B} \not\subseteq \mathcal{T}(v)$. □

Theorem 3.3 *For $v \in \mathcal{BA}^n$ the collection $\mathcal{T}(v)$ of tight coalitions is balanced.*

Proof Suppose $\mathcal{T}(v)$ is not balanced. As the union of balanced collections is balanced there is a $T \in \mathcal{T}(v)$ with $T \notin \mathcal{B}$ for all balanced subcollections $\mathcal{B}$ of $\mathcal{T}(v)$. Also, $T \neq N$ because $\{N\}$ is a balanced subcollection of $\mathcal{T}(v)$. Now for every balanced collection $\mathcal{B}$ containing T (such a collection exists) we have $\sum_{S\in\mathcal{B}} \lambda_S v(S) < v(N)$ by Lemma 3.2. Increase $v(T)$ by a small amount, say $v'(T) = v(T) + \epsilon$ where

$$0 < \epsilon < \min_{\mathcal{B}\not\subseteq\mathcal{T}(v),\mathcal{B}\ni T} \frac{1}{\lambda_T}\left(v(N) - \sum_{S\in\mathcal{B}} \lambda_S v(S)\right).$$

Let $v'(S) = v(S)$ if $S \neq T$. Then still $\sum_{S\in\mathcal{B}} \lambda_S v'(S) \leq v'(N)$ for all balanced collections $\mathcal{B}$. Thus v' is balanced. Now $v'(N) = v(N)$ and $v' \geq v$ so $C(v') \subseteq C(v)$. For an arbitrary $x \in C(v')$ we have $x(T) \geq v'(T) > v(T)$ which contradicts $T \in \mathcal{T}(v)$. □

Corollary 3.4 *For $v \in \mathcal{BA}^n$, $\mathcal{T}(v)$ is the maximal balanced collection satisfying $\sum_{S\in\mathcal{B}} \lambda_S v(S) = v(N)$, i.e.,*

$$\mathcal{T}(v) = \bigcup\{\mathcal{B} : \mathcal{B} \textit{ is balanced and } \sum_{S\in\mathcal{B}} \lambda_S v(S) = v(N)\}.$$

Proof Lemma 3.2 and Theorem 3.3. □

This corollary provides a method for computing $\mathcal{T}(v)$ without knowing the core, by checking the number $\sum_{S\in\mathcal{B}} \lambda_S v(S)$ for every minimal balanced collection. Then $\mathcal{T}(v)$ is the union of all minimal balanced collections satisfying $\sum_{S\in\mathcal{B}} \lambda_S v(S) = v(N)$.

The following theorem shows that the collection of tight coalitions is characterized by balancedness.

Theorem 3.5 *Let $\mathcal{T}$ be a balanced collection containing N. Then there is a game $v \in \mathcal{BA}^n$ with $\mathcal{T}(v) = \mathcal{T}$.*

Proof Define v by $v(S) = |S|$ if $S \in \mathcal{T}$ and $v(S) = 0$ otherwise. Let $x = (1, \ldots, 1)$. Then $x(S) \geq v(S)$ for all $S \subseteq N$ and $x(N) = v(N)$, which shows that $x \in C(v)$. Further, $\sum_{S\in\mathcal{T}} \lambda_S v(S) = \sum_{S\in\mathcal{T}} \lambda_S |S| = \sum_{S\in\mathcal{T}} \lambda_S x(S) = v(N)$. By Lemma 3.2 we have $\mathcal{T} \subseteq \mathcal{T}(v)$.

Suppose $S \in \mathcal{T}(v)\backslash\mathcal{T}$ for some coalition $S \in \mathcal{N}$. Then $v(S) = 0$ and $x(S) = |S| = 0$ because $S \in \mathcal{T}(v)$. Hence $S = \emptyset$, a contradiction because $S \in \mathcal{N}$. □

We investigate what happens to the collection of tight coalitions when the core is expanded by lowering some coalition worths.

Theorem 3.6 *Let $v \in \mathcal{BA}^n$. Let $w \in \mathcal{G}^n$ satisfying $w \le v$ and $w(N) = v(N)$. Then $C(v) \subseteq C(w)$, and $\mathcal{T}(w)$ is the maximal balanced subcollection of $\mathcal{T}(v) \cap \mathcal{S}$ where $\mathcal{S} = \{S \subseteq N : w(S) = v(S)\}$, i.e.,*

$$\mathcal{T}(w) = \bigcup \{\mathcal{B} : \mathcal{B} \text{ is balanced and } \mathcal{B} \subseteq \mathcal{T}(v) \cap \mathcal{S}\}.$$

Proof Clearly, $C(v) \subseteq C(w)$, so $\mathcal{T}(w) \subseteq \mathcal{T}(v)$. Let $S \in \mathcal{T}(w)$ and $x \in C(v)$ arbitrary then $x(S) = w(S)$ and $x(S) = v(S)$, which shows that $w(S) = v(S)$. Thus, $\mathcal{T}(w) \subseteq \mathcal{T}(v) \cap \mathcal{S}$. By Theorem 3.3, $\mathcal{T}(w)$ is balanced, so we have one inclusion of the equality.

For the converse, let $\mathcal{B}$ be balanced with $\mathcal{B} \subseteq \mathcal{T}(v)$ and $v(S) = w(S)$ for all $S \in \mathcal{B}$. Then $\sum_{S \in \mathcal{B}} \lambda_S w(S) = \sum_{S \in \mathcal{B}} \lambda_S v(S) = v(N) = w(N)$, where the second equality follows by Lemma 3.2. So, again by Lemma 3.2, $\mathcal{B} \subseteq \mathcal{T}(w)$. This is the other inclusion of the equality. □

Corollary 3.7 *Let $v \in \mathcal{BA}^n$ and $w \in \mathcal{G}^n$ satisfying $w \le v$ and $w(S) = v(S)$ for all $S \in \mathcal{T}(v)$. Then $C(v) \subseteq C(w)$ and $\mathcal{T}(w) = \mathcal{T}(v)$.*

Proof Theorems 3.3 and 3.6. □

This corollary shows that the tight coalitions are essential for the core, in the sense that lowering some worths of coalitions outside of $\mathcal{T}(v)$ does not change the dimension of the core.

4 Characterization of the prenucleolus

Corollary 3.7 shows that worths of slack coalitions can be lowered without changing the collection of tight coalitions. Below, it will be shown that lowering the worths of *all* slack coalitions by the same amount does not change the prenucleolus.

In what follows we will frequently use the notation

$$v_\lambda := v - \lambda \mathbf{1}_{\overline{\mathcal{T}(v)}}.$$

Definition 4.1 *A solution σ is called relatively independent of slack coalitions (RIS) if for every game v we have*

$$\sigma(v) = \sigma(v_\lambda) \quad \text{for all } \lambda > 0. \tag{3}$$

For nonbalanced games, this condition simply means that we may "tax" all coalitions (apart from the grand coalition) by the same lump sum amount without changing the solution. Note that this lump sum tax is independent of the size of the coalition. This might seem unreasonable, but it is also true for the way in which coalitional excesses are taken into account in the definition of the prenucleolus and nucleolus. For balanced games, a similar interpretation holds.

There, however, only the slack coalitions are taxed. This may be justified if being in the core is regarded as compelling whenever the core is non-empty. In that case, there is no choice as far as the solution payoff to tight coalitions is concerned. Consequently, it is not clear how the solution should behave if worths of tight coalitions were changed, turning these coalitions into slack coalitions.

Theorem 4.1 *The prenucleolus is relatively independent of slack coalitions.*

Proof Let $v \in \mathcal{G}^n$. If v is not balanced, then the claim of the theorem follows immediately from (2). Now assume $v \in \mathcal{BA}^n$ and let $\lambda > 0$. Then by Corollary 3.7, $C(v) \subseteq C(v_\lambda)$ and $\mathcal{T}(v_\lambda) = \mathcal{T}(v)$. Recall that the prenucleolus is always an element of the core, so we have $\nu^*(v) \in C(v_\lambda)$. We show that in the game v_λ the ordered excess vectors of points in $C(v)$ are lexicographically smaller than those of other points in $C(v_\lambda)$, that is: For every $x \in C(v_\lambda)$:

$$x \in C(v) \quad \Leftrightarrow \quad \theta(e^{v_\lambda}(x)) \preceq_{\text{lex}} (0, \ldots, 0, -\lambda, \ldots, -\lambda)$$

where the number of zeros equals $|\mathcal{T}(v)|$.

Let $x \in C(v_\lambda)$.

Assume $x \in C(v)$. Then $e^v(S,x) = 0$ if $S \in \mathcal{T}(v)$ and $e^v(S,x) \leq 0$ otherwise. As $v(S)$ is lowered by λ for $S \notin \mathcal{T}(v)$ and $\mathcal{T}(v_\lambda) = \mathcal{T}(v)$ it holds that $e^{v_\lambda}(S,x) = 0$ if $S \in \mathcal{T}(v_\lambda)$ and $e^{v_\lambda}(S,x) \leq -\lambda$ otherwise. So $\theta(e^{v_\lambda}(x)) \preceq_{\text{lex}} (0, \ldots, 0, -\lambda, \ldots, -\lambda)$.

For the converse, assume $\theta(e^{v_\lambda}(x)) \preceq_{\text{lex}} (0, \ldots, 0, -\lambda, \ldots, -\lambda)$. Because $x \in C(v_\lambda)$ we have $e^{v_\lambda}(S,x) = 0$ if $S \in \mathcal{T}(v_\lambda)$. As $\mathcal{T}(v_\lambda) = \mathcal{T}(v)$, this implies $e^{v_\lambda}(S,x) \leq -\lambda$ if $S \notin \mathcal{T}(v_\lambda)$. So $e^v(S,x) = 0$ if $S \in \mathcal{T}(v)$ and $e^v(S,x) \leq 0$ if $S \notin \mathcal{T}(v)$ which shows that $x \in C(v)$.

Because the ordered excess vector of $\nu^*(v)$ is lexicographically minimal in $C(v)$, it also minimizes the ordered excess vectors in $C(v_\lambda)$, and the proof is complete. □

Corollary 3.7 states that lowering the worths of coalitions outside of $\mathcal{T}(v)$ as in (3) does not change the collection of tight coalitions. On the other hand, by increasing these worths, new coalitions may become tight at a certain point:

Lemma 4.2 *Let $v \in \mathcal{G}^n$ with $\mathcal{T}(v) \neq \mathcal{N}$. Let*

$$\lambda_0(v) := \begin{cases} \min_{x \in C(v)} \max_{S \notin \mathcal{T}(v)} (v(S) - x(S)) & \text{if } v \in \mathcal{BA}^n \\ \epsilon^*(v) & \text{if } v \notin \mathcal{BA}^n. \end{cases}$$

Then

(i) $C(v_\lambda) \neq \emptyset$ if, and only if, $\lambda \geq \lambda_0(v)$

(ii) $\mathcal{T}(v) = \mathcal{T}(v_\lambda)$ for all $\lambda > \lambda_0(v)$ and $\mathcal{T}(v) \subset \mathcal{T}(v_{\lambda_0(v)})$.

Proof If v is not balanced, then (i) is obvious from the definition of $\epsilon^*(v)$ in Section 2. Claim (ii) follows from the observation that $\mathcal{T}(v_\lambda) = \{N\}$ if, and only if, $\lambda \neq \epsilon^*(v)$.

Assume that v is balanced. Observe that always $v_\lambda(N) = v(N)$, and that $\lambda_1 \le \lambda_2$ implies $C(v_{\lambda_1}) \subseteq C(v_{\lambda_2})$. Also, $\lambda_0(v) < 0$, for there is an $x \in C(v)$ with $v(S) - x(S) < 0$ for all $S \notin \mathcal{T}(v)$.

To prove claim (i), observe that there is an $x \in C(v)$ with $\max_{S \notin \mathcal{T}(v)}(v(S) - x(S)) = \lambda_0(v)$. So $x(S) \ge v(S) - \lambda_0(v)$ if $S \notin \mathcal{T}(v)$ and $x(S) = v(S)$ otherwise. Thus, $x \in C(v_{\lambda_0(v)})$ which shows that $C(v_{\lambda_0(v)}) \neq \emptyset$. Also $C(v_\lambda) \neq \emptyset$ for every $\lambda \ge \lambda_0(v)$.
Suppose $C(v_\lambda) \neq \emptyset$ for some $\lambda < \lambda_0(v)$. Then there is an $x \in C(v_\lambda)$ with $x(S) \ge v(S) - \lambda > v(S) - \lambda_0(v)$ for all $S \notin \mathcal{T}(v)$. Now $x \in C(v)$ because $\lambda_0(v) < 0$, and $\max_{S \notin \mathcal{T}(v)}(v(S) - x(S)) < \lambda_0(v)$. This is a contradiction to the definition of $\lambda_0(v)$.

To prove claim (ii), observe that $C(v_{\lambda_0(v)}) \subseteq C(v)$, and therefore $\mathcal{T}(v) \subseteq \mathcal{T}(v_{\lambda_0(v)})$.
Suppose $\mathcal{T}(v) = \mathcal{T}(v_{\lambda_0(v)})$. There is an $x \in C(v_{\lambda_0(v)})$ with $x(S) > v_{\lambda_0(v)}(S)$ for all $S \notin \mathcal{T}(v_{\lambda_0(v)})$. Because $x(S) > v(S) - \lambda_0(v)$ for all $S \notin \mathcal{T}(v)$, we have $\max_{S \notin \mathcal{T}(v)}(v(S) - x(S)) < \lambda_0(v)$. Also $x \in C(v)$. This contradicts the definition of $\lambda_0(v)$ and shows that $\mathcal{T}(v) \subset \mathcal{T}(v_{\lambda_0(v)})$.

Finally, let $\lambda > \lambda_0(v)$. We want to apply Theorem 3.6. We have $v_\lambda \le v_{\lambda_0(v)}$, $v_\lambda(N) = v_{\lambda_0(v)}(N)$, and $\mathcal{S} = \mathcal{T}(v)$. It follows that $\mathcal{T}(v_\lambda)$ is the maximal balanced subcollection of $\mathcal{T}(v_{\lambda_0(v)}) \cap \mathcal{T}(v)$. Now $\mathcal{T}(v)$ is a balanced subcollection of $\mathcal{T}(v_{\lambda_0(v)})$, so we have $\mathcal{T}(v_\lambda) = \mathcal{T}(v)$. □

Note that $\lambda_0(v) < 0$ if v is balanced and $\mathcal{T}(v) \neq \mathcal{N}$, and $\lambda_0(v) > 0$ if v is not balanced.

RIS combined with Lemma 4.2 suggests the following algorithm to compute the prenucleolus. If the game is not balanced, decrease the worths of all slack coalitions (all coalitions except the grand coalition) by the amount $\lambda_0(v) = \epsilon^*(v)$. The resulting game has a core, which is the least precore of the original game. Next, decrease the worths of all slack coalitions in the resulting game by $\lambda_0(v)$ (this actually means an increase). The core becomes smaller and new coalitions are getting tight while the prenucleolus does not change. By repeating this step for each new game, the collection of tight coalitions keeps growing. Finally, this collection equals $\mathcal{N}$, which means that the game is additive and the core consists of one single point: the prenucleolus.

Algorithm 4.1

$v_0 := v,\ k := 0$
while $\mathcal{T}(v_k) \neq \mathcal{N}$ *do*
  $v_{k+1} := v_k - \lambda_0(v_k)\mathbf{1}_{\overline{\mathcal{T}(v_k)}}$
  increase k
$\nu^*(v) = v_k$

Remark Observe that a different termination criterion for this algorithm might be: Stop if the core consists of a single point. A check can be based on the dimension of $\mathcal{T}(v_k)$. By this criterion, redundant passes of the while loop are avoided. For the purpose of this paper the criterion in Algorithm 4.1 is more suited.

Termination and correctness of the algorithm are proved implicitly by the following characterization.

Theorem 4.3 *The prenucleolus is the only solution $\sigma : \mathcal{G}^n \longrightarrow \mathbb{R}^n$ satisfying*

(i) $\sigma(v) = v$ for every additive game v

(ii) σ is relatively independent of slack coalitions

(iii) σ is continuous.

Proof Necessity of the axioms follows by Theorem 4.1 and the properties stated in the preliminaries. Now let $\sigma : \mathcal{G}^n \longrightarrow \mathbb{R}^n$ be a solution satisfying (i)–(iii). Let $v \in \mathcal{G}^n$. We prove $\sigma(v) = \nu^*(v)$ by induction over $|\overline{\mathcal{T}(v)}|$.

If $|\overline{\mathcal{T}(v)}| = 0$ then $\mathcal{T}(v) = \mathcal{N}$, and v is additive. By (i) it follows that $\sigma(v) = v = \nu^*(v)$.

Let $|\overline{\mathcal{T}(v)}| \geq 1$. By Lemma 4.2, $\mathcal{T}(v) \subset \mathcal{T}(v_{\lambda_0(v)})$. This implies $|\overline{\mathcal{T}(v_{\lambda_0(v)})}| < |\overline{\mathcal{T}(v)}|$, so by induction, $\sigma(v_{\lambda_0(v)}) = \nu^*(v_{\lambda_0(v)})$.
Now we prove that $\sigma(v) = \sigma(v_{\lambda_0(v)})$. If $\lambda_0(v) > 0$ then $\sigma(v) = \sigma(v_{\lambda_0(v)})$ by RIS. If $\lambda_0(v) < 0$ then we consider the games v_λ with $\lambda_0(v) < \lambda < 0$. By Lemma 4.2, $\mathcal{T}(v_\lambda) = \mathcal{T}(v)$. So we can write $v = v_\lambda + \lambda \mathbf{1}_{\overline{\mathcal{T}(v_\lambda)}}$. By RIS, $\sigma(v) = \sigma(v_\lambda)$. Now $v_\lambda \rightarrow v_{\lambda_0(v)}$ for $\lambda \rightarrow \lambda_0(v)$. By continuity $\sigma(v) = \sigma(v_{\lambda_0(v)})$.

As the prenucleolus also satisfies (i)–(iii), we have $\nu^*(v) = \nu^*(v_{\lambda_0(v)})$. So $\sigma(v) = \sigma(v_{\lambda_0(v)}) = \nu^*(v_{\lambda_0(v)}) = \nu^*(v)$ and the proof is complete. □

Remark In Theorem 4.3, continuity can be omitted if the following stronger independence property of the prenucleolus is imposed: For every balanced collection $\mathcal{T}$ with $\{N\} \subseteq \mathcal{T} \subseteq \mathcal{T}(v)$, $\sigma(v) = \sigma(v - \lambda \mathbf{1}_{\overline{\mathcal{T}}})$ for all $\lambda > 0$.

For a game $v - \lambda \mathbf{1}_{\overline{\mathcal{T}}}$ it can be seen that $\mathcal{T}(v - \lambda \mathbf{1}_{\overline{\mathcal{T}}}) = \mathcal{T}$ by Theorem 3.6. Now in the proof of Theorem 4.1 replace $\mathcal{T}(v)$ by $\mathcal{T}$ everywhere and the stronger independence property follows.

The proof of Theorem 4.3 runs along the same lines as before. Observe that in the case $\lambda_0(v) < 0$ we can write $v = v_{\lambda_0(v)} + \lambda_0(v) \mathbf{1}_{\overline{\mathcal{T}(v)}}$, where $\mathcal{T}(v)$ is a balanced subcollection of $\mathcal{T}(v_{\lambda_0(v)})$.

The following examples show the independence of the axioms in Theorem 4.3 for $n \geq 3$.

Example 4.1 $\sigma \equiv 0$ is continuous and satisfies RIS, but it does not satisfy (i).

Example 4.2 The Shapley value (see Shapley, 1953) is continuous and satisfies (i). It does not conincide with the prenucleolus unless $n \leq 2$. So it cannot satisfy RIS.

Example 4.3 Let

$$\sigma(v) = \begin{cases} v^{\epsilon^*(v)} & \text{if } v^{\epsilon^*(v)} \text{ is additive} \\ 0 & \text{otherwise} \end{cases}$$

where v^ϵ is defined as in (1). For every additive game we have $v^{\epsilon^*(v)} = v$, so (i) is fulfilled.

To prove RIS, let $\lambda > 0$. If $\mathcal{T}(v) = \{N\}$, then $(v_\lambda)^{\epsilon^*(v_\lambda)} = v^{\epsilon^*(v)}$, hence $\sigma(v_\lambda) = \sigma(v)$. If $\mathcal{T}(v) = \mathcal{N}$ then $v_\lambda = v$, hence again $\sigma(v_\lambda) = \sigma(v)$. Now let $\mathcal{T}(v) \neq \{N\}, \mathcal{N}$. Then $v \in \mathcal{BA}^n$ and by Corollary 3.7 $\mathcal{T}(v_\lambda) = \mathcal{T}(v)$. It is easy to see that $\epsilon^*(v_\lambda) = \epsilon^*(v) = 0$ and therefore $(v_\lambda)^{\epsilon^*(v_\lambda)} = v_\lambda$ and $v^{\epsilon^*(v)} = v$. Because $\mathcal{T}(v_\lambda) = \mathcal{T}(v) \neq \mathcal{N}$, v_λ and v are not additive. It follows that $\sigma(v_\lambda) = \sigma(v) = 0$. Hence, σ satisfies RIS.

Obviously, σ is not continuous if $n \geq 3$.

In fact, Algorithm 4.1 is a version of the algorithm given in Maschler, Potters, and Tijs (1992) for the general nucleolus. The value $\lambda_0(v)$ can be obtained by solving a linear program at every step. The Bondareva-Shapley characterization of games with non-empty core allows a direct computation of $\lambda_0(v)$:

Lemma 4.4 *Let $v \in \mathcal{G}^n$ with $\mathcal{T}(v) \neq \mathcal{N}$. Then*

$$\lambda_0(v) = \max_{\substack{\mathcal{B} \text{ balanced} \\ \mathcal{B} \not\subseteq \mathcal{T}(v)}} \frac{\sum_{S \in \mathcal{B}} \lambda_S v(S) - v(N)}{\sum_{S \in \mathcal{B} \setminus \mathcal{T}(v)} \lambda_S}. \tag{4}$$

Proof $\lambda_0(v)$ is the minimal value satisfying $C(v_\lambda) \neq \emptyset$. Let λ_0 denote the right-hand side of (4). For every balanced collection $\mathcal{B} \not\subseteq \mathcal{T}(v)$ we have

$$\begin{aligned} \sum_{S \in \mathcal{B}} \lambda_S v_{\lambda_0}(S) &= \sum_{S \in \mathcal{B} \setminus \mathcal{T}(v)} \lambda_S (v(S) - \lambda_0) + \sum_{S \in \mathcal{B} \cap \mathcal{T}(v)} \lambda_S v(S) \\ &= \sum_{S \in \mathcal{B}} \lambda_S v(S) - \sum_{S \in \mathcal{B} \setminus \mathcal{T}(v)} \lambda_S \cdot \lambda_0 \\ &\leq \sum_{S \in \mathcal{B}} \lambda_S v(S) - (\sum_{S \in \mathcal{B}} \lambda_S v(S) - v(N)) = v(N). \end{aligned}$$

If $\mathcal{B}$ is a balanced collection with $\mathcal{B} \subseteq \mathcal{T}(v)$ and $v \in \mathcal{BA}^n$ then

$$\sum_{S \in \mathcal{B}} \lambda_S v_{\lambda_0}(S) = \sum_{S \in \mathcal{B}} \lambda_S v(S) \leq v(N)$$

by balancedness.
If $\mathcal{B}$ is a balanced collection with $\mathcal{B} \subseteq \mathcal{T}(v)$ and $v \notin \mathcal{BA}^n$ then

$$\sum_{S \in \mathcal{B}} \lambda_S v_{\lambda_0}(S) = \sum_{S \in \mathcal{B}} \lambda_S v(S) = v(N)$$

because $\mathcal{T}(v) = \{N\}$. Hence, $C(v_{\lambda_0}) \neq \emptyset$.

Now let $\lambda < \lambda_0$. There is a $\mathcal{B}_0$ with $\lambda_0 = \frac{\sum_{S\in\mathcal{B}_0} \lambda_S v(S) - v(N)}{\sum_{S\in\mathcal{B}_0\setminus\mathcal{T}(v)} \lambda_S}$. Then

$$\begin{aligned}
\sum_{S\in\mathcal{B}_0} \lambda_S v_\lambda(S) &> \sum_{S\in\mathcal{B}_0} \lambda_S v_{\lambda_0}(S) \\
&= \sum_{S\in\mathcal{B}_0\setminus\mathcal{T}(v)} \lambda_S(v(S) - \lambda_0) + \sum_{S\in\mathcal{B}_0\cap\mathcal{T}(v)} \lambda_S v(S) \\
&= \sum_{S\in\mathcal{B}_0} \lambda_S v(S) - \sum_{S\in\mathcal{B}_0\setminus\mathcal{T}(v)} \lambda_S \cdot \lambda_0 \\
&= \sum_{S\in\mathcal{B}_0} \lambda_S v(S) - (\sum_{S\in\mathcal{B}_0} \lambda_S v(S) - v(N)) \\
&= v(N).
\end{aligned}$$

Hence $C(v_\lambda) = \emptyset$, and the proof is complete. □

Equivalently, "balanced" can be replaced by "minimal balanced" in this lemma, because the same is true for the Bondareva-Shapley theorem. The collection $\mathcal{T}(v)$ of tight coalitions can be obtained by Corollary 3.4. For an implementation of the algorithm the minimal balanced collections have to be listed for a given number of players. This seems to be feasible if the number of players is not too large. In Peleg (1965) an inductive method for computing these collections is described.

Remark There is a simple relation between tight coalitions and the balancedness criterion for the prenucleolus, namely

$$\mathcal{T}(v) = \mathcal{B}_1^v(\nu^*(v)) \quad \text{if } v \in \mathcal{BA}^n.$$

All proofs in this section could have been based on the balancedness criterion. The present treatment, however, is self-contained.

5 Characterization of the nucleolus

If the core of a game is not empty, the nucleolus coincides with the prenucleolus. So by Theorem 4.1, the nucleolus is relatively independent of slack coalitions, provided attention is restricted to balanced games. Note that if $v \in \mathcal{BA}^n$, then also $v_\lambda \in \mathcal{BA}^n$ for $\lambda > 0$, so there is no problem in restricting RIS to the class $\mathcal{BA}^n$. Furthermore, by Theorem 4.3 a characterization of the nucleolus on the class $\mathcal{BA}^n$ is already available. Consequently, Algorithm 4.1 yields the nucleolus for balanced games.

For non-balanced games, property (2) was essential for the characterization of the prenucleolus. This property, however, does not hold for the nucleolus.

Therefore, we propose an independence property involving coalitions that are tight in the least core. This leads to a characterization of the nucleolus also for non-balanced games.

Definition 5.1 *Let $v \in \mathcal{IG}^n$ with least core $LC(v) = C_{\epsilon(v)}(v)$. A coalition is called least-core tight if*

$$x(S) = v(S) - \epsilon(v) \quad \textit{for all } x \in LC(v).$$

The collection of least-core tight coalitions is denoted by $\mathcal{LT}(v)$.

For a game $v \in \mathcal{IG}^n$ and $\lambda \in \mathbb{R}$ we introduce the notation

$$\bar{v}_\lambda := v + \lambda \mathbf{1}_{\mathcal{LT}(v)}.$$

Definition 5.2 *A solution $\sigma : \mathcal{IG}^n \to \mathbb{R}^n$ is called relatively independent of least-core tight coalitions for non-balanced games (RILT) if for every $v \in \mathcal{IG}^n \backslash \mathcal{BA}^n$ it holds that*

$$\sigma(v) = \sigma(\bar{v}_\lambda) \qquad \textit{for all } \lambda > 0.$$

It is possible to derive similar results for least-core tight coalitions as was done in Section 3 for tight coalitions, and to use these results to prove the characterization of the nucleolus below in an elementary, self-contained manner. The least core and least-core tight coalitions, however, are much more difficult to handle than the core and the tight coalitions. Therefore, for simplicity and brevity, we will use the Kohlberg criterion for this (see Section 2).

There is a simple relation between least-core tight coalitions and the Kohlberg criterion.

Lemma 5.1 *Let $v \in \mathcal{IG}^n \backslash \mathcal{BA}^n$. Then*

$$\mathcal{LT}(v) = \mathcal{B}_1^v(\nu(v)).$$

Proof First note that $\epsilon(v) > 0$ because v is not balanced. So $\nu(v)$ has maximal excess $\epsilon(v)$ because N has excess 0. Analogously to Lemma 3.1, there is an $x \in LC(v)$ with $x(S) = v(S) - \epsilon(v)$ for all $S \in \mathcal{LT}(v)$, $x(S) > v(S) - \epsilon(v)$ for all $S \notin \mathcal{LT}(v)$, and $x(N) = v(N)$. So $e^v(S,x) = \epsilon(v)$ if $S \in \mathcal{LT}(v)$, and $e^v(S,x) < \epsilon(v)$ if $S \notin \mathcal{LT}(v)$. As the nucleolus lexicografically minimizes the ordered excess vector and $e^v(S,\nu(v)) = \epsilon(v)$ for $S \in \mathcal{LT}(v)$, it follows that $e^v(S,\nu(v)) < \epsilon(v)$ if $S \notin \mathcal{LT}(v)$. So $\mathcal{LT}(v) = \mathcal{B}_1^v(\nu(v))$. □

Theorem 5.2 *The nucleolus is relatively independent of least-core tight coalitions for non-balanced games.*

Proof Let $v \in \mathcal{IG}^n \backslash \mathcal{BA}^n$ and $\lambda > 0$. Because v is not balanced, $N, \{i\} \notin \mathcal{LT}(v)$ for every $i \in N$. So $I(\bar{v}_\lambda) = I(v)$ and $\nu(v) \in I(\bar{v}_\lambda)$. In the game $\bar{v}_\lambda$ all excesses of coalitions in $\mathcal{LT}(v)$ are increased by λ. The excesses of coalitions outside of $\mathcal{LT}(v)$ are equal to those in v. Consider the collections $\mathcal{B}_1^v(\nu(v))$, $\mathcal{B}_2^v(\nu(v))$, $\mathcal{B}_3^v(\nu(v))$, etc., and $\mathcal{B}_0^v(\nu(v))$. By Lemma 5.1, $\mathcal{LT}(v) = \mathcal{B}_1^v(\nu(v))$. $\mathcal{B}_1^v(\nu(v))$ is the collection of coalitions which have maximal excess. They still have maximal excess when this is increased by λ. Thus $\mathcal{B}_1^{\bar{v}_\lambda}(\nu(v)) = \mathcal{B}_1^v(\nu(v))$. The excesses of coalitions in $\mathcal{B}_2^v(\nu(v))$, $\mathcal{B}_3^v(\nu(v))$, etc., and $\mathcal{B}_0^v(\nu(v))$ have not changed. Thus, $\mathcal{B}_2^{\bar{v}_\lambda}(\nu(v)) = \mathcal{B}_2^v(\nu(v))$, $\mathcal{B}_3^{\bar{v}_\lambda}(\nu(v)) = \mathcal{B}_3^v(\nu(v))$, etc., and $\mathcal{B}_0^{\bar{v}_\lambda}(\nu(v)) = \mathcal{B}_0^v(\nu(v))$. Because $\nu(v)$ is the nucleolus of the game v, it is also the nucleolus of the game $\bar{v}_\lambda$ in view of the Kohlberg-criterion. □

Lemma 5.3 *Let $v \in \mathcal{IG}^n \backslash \mathcal{BA}^n$. Let*

$$\lambda_1(v) = \min_{x \in LC(v)} \max_{S \notin \mathcal{LT}(v)} (v(S) - x(S)) \; - \; \epsilon(v).$$

Then

$$\mathcal{LT}(v) = \mathcal{LT}(\bar{v}_\lambda) \textit{ for all } \lambda > \lambda_1(v) \textit{ and } \mathcal{LT}(v) \subset \mathcal{LT}(\bar{v}_{\lambda_1(v)}).$$

Proof As in the proof of Theorem 5.2, $I(\bar{v}_\lambda) = I(v)$ and $\nu(v) \in I(\bar{v}_\lambda)$. In the game $\bar{v}_\lambda$ the excesses of coalitions in $\mathcal{LT}(v)$ are increased by λ and the excesses of coalitions outside of $\mathcal{LT}(v)$ are equal to those in v. By Lemma 5.1 we have $\mathcal{LT}(v) = \mathcal{B}_1^v(\nu(v))$. Consider the collections $\mathcal{B}_1^{\bar{v}_\lambda}(\nu(v))$, $\mathcal{B}_2^{\bar{v}_\lambda}(\nu(v))$, $\mathcal{B}_3^{\bar{v}_\lambda}(\nu(v))$, etc., and $\mathcal{B}_0^{\bar{v}_\lambda}(\nu(v))$ when λ varies.

For coalitions in $\mathcal{B}_1^v(\nu(v))$, $\nu(v)$ has excess $\epsilon(v)$ because the nucleolus minimizes the maximal excess. For coalitions in $\mathcal{B}_2^v(\nu(v))$, $\nu(v)$ has excess $\mu(v) := \min_{x \in LC(v)} \max_{S \notin \mathcal{LT}(v)} (v(S) - x(S))$ because for points with maximal excess the nucleolos minimizes the second highest excess.

Let $\lambda > \lambda_1(v) = \mu(v) - \epsilon(v)$. In the game $\bar{v}_\lambda$, $\nu(v)$ has excess $\epsilon(v) + \lambda$ for coalitions in $\mathcal{B}_1^v(\nu(v))$. All other excesses remain unchanged. Because $\epsilon(v) + \lambda > \mu(v)$ the excess of $\nu(v)$ for coalitions in $\mathcal{B}_1^v(\nu(v))$ is still maximal. Thus, $\mathcal{B}_1^{\bar{v}_\lambda}(\nu(v)) = \mathcal{B}_1^v(\nu(v))$. Obviously, $\mathcal{B}_2^{\bar{v}_\lambda}(\nu(v)) = \mathcal{B}_2^v(\nu(v))$, $\mathcal{B}_3^{\bar{v}_\lambda}(\nu(v)) = \mathcal{B}_3^v(\nu(v))$, etc., and $\mathcal{B}_0^{\bar{v}_\lambda}(\nu(v)) = \mathcal{B}_0^v(\nu(v))$. As $\nu(v)$ is the nucleolus of v it is the nucleolus of $\bar{v}_\lambda$ by the Kohlberg-criterion. Note that $C(\bar{v}_\lambda) = \emptyset$ because $\epsilon(\bar{v}_\lambda) > 0$. By Lemma 5.1 it follows that $\mathcal{LT}(\bar{v}_\lambda) = \mathcal{LT}(v)$.

In the game $\bar{v}_{\lambda_1(v)}$, $\nu(v)$ has excess $\epsilon(v) + \lambda_1(v)$ for coalitions in $\mathcal{B}_1^v(\nu(v))$. This implies that now the excess of $\nu(v)$ for coalitions in $\mathcal{B}_1^v(\nu(v))$ and $\mathcal{B}_2^v(\nu(v))$ is the same. So $\mathcal{B}_1^{\bar{v}_{\lambda_1(v)}}(\nu(v)) = \mathcal{B}_1^v(\nu(v)) \cup \mathcal{B}_2^v(\nu(v))$. Clearly $\mathcal{B}_2^{\bar{v}_{\lambda_1(v)}}(\nu(v)) = \mathcal{B}_3^v(\nu(v))$, $\mathcal{B}_3^{\bar{v}_{\lambda_1(v)}}(\nu(v)) = \mathcal{B}_4^v(\nu(v))$, etc., and $\mathcal{B}_0^{\bar{v}_{\lambda_1(v)}}(\nu(v)) = \mathcal{B}_0^v(\nu(v))$. As $\nu(v)$ is the nucleolus of v it is the nucleolus of $\bar{v}_{\lambda_1(v)}$ by the Kohlberg-criterion.

Observe that $\mu(v) \geq 0$ because $N \notin \mathcal{LT}(v)$. If $\mu(v) > 0$ then $C(\bar{v}_{\lambda_1(v)}) = \emptyset$ and by Lemma 5.1 $\mathcal{LT}(\bar{v}_{\lambda_1(v)}) = \mathcal{B}_1^{\bar{v}_{\lambda_1(v)}}(\nu(v))$. If $\mu(v) = 0$ then $\epsilon(\bar{v}_{\lambda_1(v)}) = 0$, so again $\mathcal{LT}(\bar{v}_{\lambda_1(v)}) = \mathcal{B}_1^{\bar{v}_{\lambda_1(v)}}(\nu(v))$. Thus in both cases it follows that $\mathcal{LT}(v) \subset \mathcal{LT}(\bar{v}_{\lambda_1(v)})$. □

We consider the following algorithm for computing the nucleolus. (Cf. also the Remark following Algorithm 4.1.) Termination and correctness are proved implicitly in Theorem 5.4.

Algorithm 5.1

$v_0 := v,\ k := 0$
while $C(v_k) = \emptyset$ *do*
  $v_{k+1} := v_k + \lambda_1(v_k)\mathbf{1}_{\mathcal{LT}(v_k)}$
  increase k
while $\mathcal{T}(v_k) \neq \mathcal{N}$ *do*
  $v_{k+1} := v_k - \lambda_0(v_k)\mathbf{1}_{\overline{\mathcal{T}(v_k)}}$
  increase k
$\nu(v) = v_k$

Theorem 5.4 *The nucleolus is the only solution* $\sigma : \mathcal{IG}^n \to \mathbb{R}^n$ *satisfying*

(i) $\sigma(v) = v$ *for every additive game* v

(ii) σ *is continuous*

(iii) σ *is relatively independent of slack coalitions for balanced games*

(iv) σ *is relatively independent of least-core tight coalitions for non-balanced games.*

Proof Necessity of the axioms follows by Theorem 4.1, Theorem 5.2, and the properties stated in the preliminaries.

For sufficiency, let $\sigma : \mathcal{IG}^n \to \mathbb{R}^n$ be a solution satisfying (i)–(iv). As the nucleolus conicides with the prenucleolus on $\mathcal{BA}^n$ we have $\sigma(v) = \nu(v)$ for every $v \in \mathcal{BA}^n$ by Theorem 4.3. Let $v \in \mathcal{IG}^n$. We show $\sigma(v) = \nu(v)$ by induction on $|\overline{\mathcal{LT}(v)}|$.

If $|\overline{\mathcal{LT}(v)}| = 0$ then $\mathcal{LT}(v) = \mathcal{N}$. v must be balanced for otherwise $\mathcal{LT}(v)$ would not contain N, and any one player coalition. It follows that $\sigma(v) = \nu(v)$.

Let $|\overline{\mathcal{LT}(v)}| \geq 1$. If v is balanced then $\sigma(v) = \nu(v)$. Otherwise, by Lemma 5.3, $\mathcal{LT}(v) \subset \mathcal{LT}(\bar{v}_{\lambda_1(v)})$. This implies $|\overline{\mathcal{LT}(\bar{v}_{\lambda_1(v)})}| < |\overline{\mathcal{LT}(v)}|$ so by induction, $\sigma(\bar{v}_{\lambda_1(v)}) = \nu(\bar{v}_{\lambda_1(v)})$.

Consider the games $\bar{v}_\lambda$ for $\lambda_1(v) < \lambda < 0$. By Lemma 5.3 it follows that $\mathcal{LT}(\bar{v}_\lambda) = \mathcal{LT}(v)$. So we can write $v = \bar{v}_\lambda - \lambda\mathbf{1}_{\mathcal{LT}(\bar{v}_\lambda)}$. Also, as in the proof of Lemma 5.3, $C(\bar{v}_\lambda) = \emptyset$. By RILT it follows that $\sigma(v) = \sigma(\bar{v}_\lambda)$.

Now $\bar{v}_\lambda \to \bar{v}_{\lambda_1(v)}$ for $\lambda \to \lambda_1(v)$. By continuity, $\sigma(v) = \sigma(\bar{v}_{\lambda_1(v)})$.

As the nucleolus also satisfies (i)–(iv), $\nu(v) = \nu(\bar{v}_{\lambda_1(v)})$. So we have $\sigma(v) = \sigma(\bar{v}_{\lambda_1(v)}) = \nu(\bar{v}_{\lambda_1(v)}) = \nu(v)$ and the proof is complete. □

Algorithm 5.1 is a version of the algorithm given in Maschler, Potters, and Tijs (1992) for the general nucleolus. Maschler, Peleg, and Shapley (1979) call the outcome of this algorithm the "lexicografic center" of a game and show its equivalence with the nucleolus.

Like the number $\lambda_0(v)$ in the case of the prenucleolus (see Lemma 4.4), the number $\lambda_1(v)$ of Lemma 5.3 can be calculated if we know all balanced collections.

Lemma 5.5 *Let $v \in \mathcal{IG}^n \backslash \mathcal{BA}^n$. Then*
(i)

$$\epsilon(v) = \max_{\substack{\mathcal{B} \text{ balanced} \\ \mathcal{B} \not\subseteq \{S:|S|\in\{1,n\}\}}} \frac{\sum_{S\in\mathcal{B}} \lambda_S v(S) - v(N)}{\sum_{S\in\mathcal{B},|S|\notin\{1,n\}} \lambda_S} \tag{5}$$

(ii)
$\lambda_1(v)$ is the maximum of $-\epsilon(v)$ and the number

$$\max_{\substack{\mathcal{B} \text{ balanced} \\ \mathcal{B} \not\subseteq \mathcal{LT}(v)\cup\{S:|S|\in\{1,n\}\}}} \frac{\sum_{S\in\mathcal{B}} \lambda_S v(S) - v(N) - \epsilon(v)\cdot\sum_{S\in\mathcal{B},|S|\notin\{1,n\}} \lambda_S}{\sum_{S\in\mathcal{B}\backslash\mathcal{LT}(v),|S|\notin\{1,n\}} \lambda_S}. \tag{6}$$

Proof
(i) $\epsilon(v)$ is the minimal number ϵ for which $C(v_1^\epsilon) \neq \emptyset$, where $v_1^\epsilon = v - \epsilon \mathbf{1}_{\mathcal{N}\backslash\{S:|S|\in\{1,n\}\}}$, see Section 2. Let κ denote the right hand side of equation (5). Then for every balanced collection $\mathcal{B}$ with $\mathcal{B} \not\subseteq \{S : |S| \in \{1, n\}\}$,

$$\begin{aligned}
\sum_{S\in\mathcal{B}} \lambda_S v_1^\kappa(S) &= \sum_{S\in\mathcal{B},|S|\notin\{1,n\}} \lambda_S(v(S)-\kappa) + \sum_{S\in\mathcal{B},|S|\in\{1,n\}} \lambda_S v(S) \\
&= \sum_{S\in\mathcal{B}} \lambda_S v(S) - \sum_{S\in\mathcal{B},|S|\notin\{1,n\}} \lambda_S \cdot \kappa \\
&\leq \sum_{S\in\mathcal{B}} \lambda_S v(S) - (\sum_{S\in\mathcal{B}} \lambda_S v(S) - v(N)) = v(N).
\end{aligned}$$

If $\mathcal{B}$ is balanced and $\mathcal{B} \subseteq \{S : |S| \in \{1, n\}\}$ then

$$\sum_{S\in\mathcal{B}} \lambda_S v_1^\kappa(S) = \sum_{S\in\mathcal{B}} \lambda_S v(S) = \alpha v(N) + (1-\alpha)\sum_{i\in N} v(i) \leq v(N)$$

for some $0 \leq \alpha \leq 1$, where the inequality follows because $v \in \mathcal{IG}^n$.
Hence, $C(v_1^\kappa) \neq \emptyset$.
Next, suppose $\epsilon < \kappa$. There is a $\mathcal{B}_0 \not\subseteq \{S : |S| \in \{1, n\}\}$ with

$$\kappa = \frac{\sum_{S\in\mathcal{B}_0} \lambda_S v(S) - v(N)}{\sum_{S\in\mathcal{B}_0,|S|\notin\{1,n\}} \lambda_S}.$$

Then

$$\sum_{S\in\mathcal{B}_0} \lambda_S v_1^\epsilon(S) \quad > \quad \sum_{S\in\mathcal{B}_0} \lambda_S v_1^\kappa(S)$$

$$\begin{aligned}
&= \sum_{S\in\mathcal{B}_0,|S|\notin\{1,n\}} \lambda_S(v(S)-\kappa) + \sum_{S\in\mathcal{B}_0,|S|\in\{1,n\}} \lambda_S v(S) \\
&= \sum_{S\in\mathcal{B}_0} \lambda_S v(S) - \sum_{S\in\mathcal{B}_0,|S|\notin\{1,n\}} \lambda_S\cdot\kappa \\
&= \sum_{S\in\mathcal{B}_0} \lambda_S v(S) - (\sum_{S\in\mathcal{B}_0} \lambda_S v(S) - v(N)) \\
&= v(N).
\end{aligned}$$

Hence $C(v_1^\epsilon)=\emptyset$. It follows that $\kappa=\epsilon(v)$.

(ii) As in the proof of Lemma 5.3, $\lambda_1(v)=\mu(v)-\epsilon(v)$ where

$$\mu(v) := \min_{x\in LC(v)} \max_{S\notin\mathcal{LT}(v)} [v(S)-x(S)].$$

In the games $\bar{v}_\lambda$, $\nu(v)$ has excess $\epsilon(v)+\lambda$ for coalitions in $\mathcal{B}_1^v(\nu(v))$ and excess $\mu(v)$ for coalitions in $\mathcal{B}_2^v(\nu(v))$. $\nu(v)$ is the nucleolus of all games $\bar{v}_\lambda$ with $\lambda\geq\lambda_1(v)$. Also, $\mu(v)\geq 0$ because $N\notin\mathcal{LT}(v)$.

Let λ_1 denote the number in (6).

If $\mu(v)=0$ then $\lambda_1(v)=-\epsilon(v)$. So we have to show that $\lambda_1(v)=\lambda_1$. The game $\bar{v}_{-\epsilon(v)}$ is balanced because the excess of $\nu(v)$ in $\bar{v}_{-\epsilon(v)}$ is smaller than or equal to zero for every coalition. This implies that for every balanced collection $\mathcal{B}$ we have

$$\sum_{S\in\mathcal{B}} \lambda_S \bar{v}_{-\epsilon(v)}(S) \leq \bar{v}_{-\epsilon(v)}(N).$$

By expanding these inequalities it follows that every expression over which the maximum is taken in (6) is smaller than or equal to $-\epsilon(v)$. Hence, $\lambda_1\leq-\epsilon(v)$, as was to be proved.

If $\mu(v)>0$ then $\lambda_1(v)>-\epsilon(v)$. So we have to show that $\lambda_1(v)=\lambda$. We first claim that

$$C(\bar{v}_\lambda-(\epsilon(v)+\lambda)\mathbf{1}_{\mathcal{N}\setminus\{S:|S|\in\{1,n\}\}})\neq\emptyset \Leftrightarrow \lambda\geq\lambda_1(v). \tag{7}$$

If $\lambda\geq\lambda_1(v)$ then $\nu(v)$ is the nucleolus of the game $\bar{v}_\lambda$ with highest excess $\epsilon(v)+\lambda$. Hence $\epsilon(\bar{v}_\lambda)=\epsilon(v)+\lambda$ and $C(\bar{v}_\lambda-(\epsilon(v)+\lambda)\mathbf{1}_{\mathcal{N}\setminus\{S:|S|=1,n\}})\neq\emptyset$.

Suppose $\lambda<\lambda_1(v)$ and $C(\bar{v}_\lambda-(\epsilon(v)+\lambda)\mathbf{1}_{\mathcal{N}\setminus\{S:|S|\in\{1,n\}\}})\neq\emptyset$. Then there is an $x\in I(v)$ with $x(S)\geq v(S)-\epsilon(v)$ if $S\in\mathcal{LT}(v)$, $x(S)\geq v(S)-\epsilon(v)-\lambda$ if $S\notin\mathcal{LT}(v)$ and $|S|\neq 1,n$, and $x(S)\geq v(S)$ if $|S|=1,n$. Because $\lambda<0$ it follows that $x\in LC(v)$. x has an excess of less than $\mu(v)$ for all coalitions that are not least-core tight. Hence, $\max_{S\notin\mathcal{LT}(v)} e^v(S,x)<\mu(v)$, which contradicts the definition of $\mu(v)$. So we have proved claim (7).

Next, we show that (7) holds if we replace $\lambda_1(v)$ by λ_1. For every balanced collection $\mathcal{B}\not\subseteq\mathcal{LT}(v)\cup\{S:|S|=1,n\}$ we have

$$\sum_{S\in\mathcal{B}} \lambda_S(\bar{v}_{\lambda_1}-(\epsilon(v)+\lambda_1)\mathbf{1}_{\mathcal{N}\setminus\{S:|S|=1,n\}})(S) \quad = \quad \sum_{S\in\mathcal{B}} \lambda_S v(S) - \epsilon(v) \sum_{S\in\mathcal{B},|S|\neq 1,n} \lambda_S$$

$$
\begin{aligned}
& -\lambda_1 \sum_{S\in\mathcal{B}, S\notin\mathcal{LT}(v), |S|\neq 1,n} \lambda_S \\
\leq\ & v(N)
\end{aligned}
$$

where the inequality follows by definition of λ_1.

For $\mathcal{B} \subseteq \mathcal{LT}(v) \cup \{S : |S| = 1, n\}$ we have

$$
\begin{aligned}
\sum_{S\in\mathcal{B}} \lambda_S(\bar{v}_{\lambda_1} - (\epsilon(v) + \lambda_1)\mathbf{1}_{\mathcal{N}\setminus\{S:|S|=1,n\}})(S) &= \sum_{S\in\mathcal{B}} \lambda_S v(S) - \epsilon(v) \sum_{S\in\mathcal{B}, |S|\neq 1,n} \lambda_S \\
&= \sum_{S\in\mathcal{B}} \lambda_S v_1^{\epsilon(v)}(S) \\
&\leq v^{\epsilon(v)}(N) \\
&= v(N)
\end{aligned}
$$

because the game $v_1^{\epsilon(v)}$ is balanced. Hence,

$$C(\bar{v}_{\lambda_1} - (\epsilon(v) + \lambda_1)\mathbf{1}_{\mathcal{N}\setminus\{S:|S|\in\{1,n\}\}}) \neq \emptyset.$$

Let $\lambda < \lambda_1$. There is a balanced collection $\mathcal{B}_0 \not\subseteq \mathcal{LT}(v) \cup \{S : |S| = 1, n\}$ with

$$\lambda_1 = \frac{\sum_{S\in\mathcal{B}} \lambda_S v(S) - v(N) - \epsilon(v) \cdot \sum_{S\in\mathcal{B}, |S|\notin\{1,n\}} \lambda_S}{\sum_{S\in\mathcal{B}\setminus\mathcal{LT}(v), |S|\notin\{1,n\}} \lambda_S}. \tag{8}$$

Then

$$
\begin{aligned}
\sum_{S\in\mathcal{B}_0} \lambda_S(\bar{v}_{\lambda} - (\epsilon(v) + \lambda)\mathbf{1}_{\mathcal{N}\setminus\{S:|S|=1,n\}})(S) &> \sum_{S\in\mathcal{B}_0} \lambda_S v(S) - \epsilon(v) \sum_{S\in\mathcal{B}_0, |S|\neq 1,n} \lambda_S \\
&\quad -\lambda_1 \sum_{S\in\mathcal{B}_0, S\notin\mathcal{LT}(v), |S|\neq 1,n} \lambda_S \\
&= v(N)
\end{aligned}
$$

where the equality follows from (8). Hence,

$$C(\bar{v}_{\lambda} - (\epsilon(v) + \lambda)\mathbf{1}_{\mathcal{N}\setminus\{S:|S|\in\{1,n\}\}}) = \emptyset.$$

It follows that $\lambda_1 = \lambda_1(v)$. □

In the preceding lemma, "balanced" may again be replaced by "minimal balanced". $\mathcal{LT}(v)$ is the union of all minimal balanced collections which attain the maximum in (5) with the exception of all one-player coalitions and the grand coalition.

Like in the case of the prenucleolus, in order to base Algorithm 5.1 on this lemma, one could list all minimal balanced collections. For a small number of players for which the nucleolus has to be computed repeatedly (say, a new game every day), such an algorithm may be convenient, but otherwise, the number of minimal balanced collections soon becomes very large.

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