ORTHOGONAL POLYNOMIALS AND LINEAR FUNCTIONALS OF SECOND DEGREE

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Abstract

In this paper we deal with functionals verifying a second degree equation.We give characterizations of such a kind of functionals in terms of the Stieltjes function, in terms of the equation that satisfies the functional and in terms of the corresponding sequence of orthogonal polynomials.

Any functional of second degree is a semi-classical one , but the converse is not true in general.We find the conditions which should be fulfilled for a semi-classical functional to be a second degree functional.On the other hand we show that the set of second degree functionals is stable for the most usual operations;association, finite perturbations, inversion,.... Finally, some examples related to Tchebychev polynomials are given.
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1.Introduction

As much the linear functional corresponding to the Tchebychev polynomials of the first kind T_n as the linear functional corresponding to the Tchebychev polynomials of the second kind U_n, satisfy a second degree equation, and for this reason they are called second degree functionals. Other functionals, for example those related to the polynomials $P_n^{(-\frac{1}{2},\frac{1}{2})}$ and $P_n^{(\frac{1}{2},-\frac{1}{2})}$ have this property as well, (see [6]).

In this work, we define and characterize second degree functionals. They have been studied in [12] and [13] in the framework of the orthogonality on several intervals. Later on, in [10] and [11] an algebraic approach to such second degree functionals as an extension of the Tchebychev functionals is given.

We prove that for most usual operations (left multiplication by a polynomial, associated functional, perturbed functional, inverse, etc...) the set of second degree functionals is stable.

On the other hand, we shall see that every second degree functional is semi-classical. Furthermore, we will study under which conditions a semi-classical functional is of the second degree.

We analyze with special attention the classical polynomials. The impossibility that they can be of second degree in the cases of Hermite, Laguerre and Bessel polynomials is shown. We will give a necessary condition (in terms of α and β), so that the polynomials $P_n^{(\alpha,\beta)}$ can correspond to second degree functionals. The cases $\alpha=\beta=-\frac{1}{2}$, $\alpha=\beta=\frac{1}{2}$,$\alpha=-\beta=\frac{1}{2}$ and $\alpha=-\beta=-\frac{1}{2}$ are considered with more detail.

A second degree functional belongs to the Laguerre-Hahn class. We also analyze in this paper when a Laguerre-Hahn linear functional is of second degree and so is semi-classical. The first order associated polynomials to the classical polynomials belong to the Laguerre-Hahn class, i.e. the corresponding Stieltjes function fulfills the Riccati equation

$$\Phi(z)S'(u)(z)=B(z)S^2(u)(z)+C(z)S(u)(z)+D(z), \tag{1}$$

(where $\Phi(z)$, $B(z)$, $C(z)$ and $D(z)$ are polynomials with complex coefficients). For Bessel polynomials $B_n^{(\alpha)}$ if $\alpha=\frac{1}{2}$ the polynomial coefficient $B(z)$ vanishes, fulfilling the Stieltjes function an affine (linear) differential equation and so they are semi-classical.

The same reasoning is valid for the Tchebychev polynomials of first kind, (see [9], page 41).

But there can also appear associated of the first kind polynomials which are semi-classical, even if the polynomial $B(z)$ does not vanish, (for instance $(U_n)^{(1)}=U_n$). The justification of these cases is that they are polynomials corresponding to second degree functionals.

The equations (13) of the present work give the necessary and sufficient conditions for a Laguerre-Hahn functional to be (at the same time) a second degree functional and ,thus, to be semi-classical. Some examples for perturbations of Tchebychev polynomials complete this work.

2.-Definitions and characterizations.

Let u be a linear functional on the linear space P of polynomials with complex coefficients $u_n =< u, z^n >$ is called the n-th moment associated with u.

Definition 2.1. *A linear functional u is said to be quasi-definite if*

$$H_n = det \begin{vmatrix} u_0 & u_1 & \dots & u_{n-1} \\ \dots & \dots & \dots & \dots \\ u_{n-1} & u_n & \dots & u_{2n-2} \end{vmatrix}$$

for every $n \geq 1$.

Proposition 2.1. (See [3], page 7) *Let u be a quasi-definite linear functional. Then, there exists a sequence of monic polynomials $\{P_n\}_{n\geq 0}$ such that*

$$< u, P_n P_m >= 0 \quad n \neq m$$

$$< u, P_n^2 > \neq 0 \quad n \geq 0$$

Theorem 2.1. *If u is a quasi-definite linear functional, then for $n \geq 1$*

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x),$$

$$P_0(x) = 1, \quad P_1(x) = x - \beta_0.$$

where $\gamma_n \neq 0$.

Note: The converse is true and is known in the literature as Favard theorem, (see[3], page 21).

Algebraic structure in P^*.

Denote

$$\theta_a p(z) = [p(z) - p(a)]/(z - a).$$

We introduce some useful operations in the algebraic linear dual space P^* of P.

i)

$$< p(z)u, q(z) >=< u, p(z)q(z) >$$

for a fixed $p(z)$ and for every $q(z) \in P$.

ii)

$$< p^{-1}(z)u, q(z) >=< u, [q(z) - L_p q(z)]/p(z) >$$

where L_p means the Hermite interpolatory polynomial associated to the roots of p.In particular

$$< x^{-1}u, p(z) >=< u, (\theta_0 p)(z) >.$$

iii)

$$< uv, q(z) >=< v, uq(z) >,$$

where

$$(uq)(z) = \sum_{m=0}^{n} \sum_{j=m}^{n} a_j u_{j-m})z^m \text{ and } q(z) = \sum_{j=0}^{n} a_j z^j.$$

Definition 2.2. *The formal series* $S(u)(z) = -\sum_{n\geq 0} u_n/z^{n+1}$ *is called the Stieltjes function associated with* u.

Definition 2.3. *Let* $\{P_n\}_{n\geq 0}$ *be a sequence of monic orthogonal polynomials with respect to a quasi-definite functional* u. *The sequence* $\{P_n^{(1)}\}_{n\geq 0}$ *defined by*

$$P_n^{(1)}(x) = \tfrac{1}{u_0} < u_\epsilon, \tfrac{P_{n+1}(x)-P_{n+1}(\epsilon)}{x-\epsilon} >, \quad n \geq 0$$

is called the associated sequence of first order for the sequence $\{P_n\}_{n\geq 0}$, (see [3],page 88).

We shall denote by $u^{(1)}$ the normalized functional , $[u_0^{(1)} = 1]$, such that the sequence $\{P_n^{(1)}\}_{n\geq 0}$ is the corresponding sequence of monic orthogonal polynomials.

We give now the following results, (see [10] for a more comprehensive approach).

Definition 2.4. *A linear functional* u *on the linear space* P *is said to be of second degree if the Stieltjes function satisfies the quadratic equation*

$$b(z)S^2(u)(z) + c(z)S(u)(z) + d(z) = 0 \tag{2}$$

where b(z), c(z) and d(z) are polynomials with complex coefficients , with $b(z) \neq 0$, $c^2(z) - 4b(z)d(z) \neq 0$ *and* $d(z) = (u\theta_0 c)(z) - (u^2\theta_0^2 b)(z)$.

In terms of the functional we have

Proposition 2.2. (See [10]) *For a linear functional u defined on the linear space P the following assertions are equivalent:*

(i) u is of the second degree;

(ii)

$$b(z)u^2 - zc(z)u = 0; \tag{3}$$

and $< u^2, \theta_0 b(z) >=< u, c(z) >$

(iii)

$$b(z)u + z^2 d(z)u^{-1} = 0$$

$$c(z) = (u\theta_0 b)(z) + (u^{-1}zd)(z) \tag{4}$$

where $b(z)$, $c(z)$ and $d(z)$ are the polynomials defined in (2).

It is possible to characterize the second degree linear functional in terms of the corresponding sequence of orthogonal polynomials.

Proposition 2.3. *Let $\{P_n\}_{n\geq 0}$ be an orthogonal sequence. $\{P_n\}_{n\geq 0}$ is a second degree sequence of orthogonal polynomials if and only if there exist a polynomial B and a positive integer s such that*

$$B(z)P_n^{(1)}(z) = \sum_{j=n-s}^{n+r} \lambda_{n,j} P_j(z), \quad \lambda_{n,n-s} \neq 0, \quad n \geq s$$

where $B(z)=b(z)$, $r=deg\ B(z)$ and $s=deg\ d(z)$.

Notice that $U_n^{(1)}(z) = U_n(z)$ and $(1-z^2)T_n^{(1)}(z) = -T_{n+2}(z) + (1/4)T_n(z)$ and, then, the first and second kind Tchebychev polynomials are sequences of orthogonal polynomials of second degree.

Now , we will be concerned with the problem of simplification of equation (2) or (3), (see [10])

Proposition 2.4. *The equation (2) is irreducible if and only if the polynomials $b(z)$ and $c(z)$ have not a common root or whether for any common root a, the following condition holds*

$$< u, \theta_a c(z) > - < u^2, \theta_0\theta_a b(z) > \neq 0.$$

When the pair $[b(z), c(z)]$ verifies the above condition, it will be called a primitive pair associated to u. The primitive pair is unique.

3.- Reduction of a second degree functional to a semi-classical one

Proposition 3.1. *A linear functional of second degree u is a linear semi-classical functional satisfying the equation $D(\Phi(z)u)+\Psi(z)u=0$, where*

$$\Phi(z)=A(z),\quad \Psi(z)=-A'(z)-C(z)$$

$$A(z)=b(z)[c^2(z)-4b(z)d(z)] \tag{5}$$

$$C(z)=2b(z)[b'(z)d(z)-b(z)d'(z)]+c(z)[b(z)c'(z)-b'(z)c(z)]$$

and D denotes the derivative.

Proof Let u be a second degree functional i.e. it satisfies the equation (3). We have:

$$2b(z)S(u)(z)+c(z)=\pm\sqrt{[c^2(z)-4b(z)d(z)]}$$

and

$$S^2(u)(z)=[1/b(z)][-c(z)S(u)(z)-d(z)] \tag{6}$$

If we take derivatives in the above expression

$$2b(z)S(u)(z)S'(u)(z)+b'(z)S^2(u)(z)+c'(z)S(u)(z)+c(z)S'(u)(z)+d'(z)=0$$

$$b(z)[2b(z)S(u)(z)+c(z)]S'(u)(z)+[b(z)c'(z)-b'(z)c(z)]S(u)(z)$$

$$+[b(z)d'(z)-b'(z)d(z)]=0.$$

Multiplying by $[2b(z)S(u)(z)+c(z)]$ we obtain

$$b(z)[c^2(z)-4b(z)d(z)]S'(u)(z)$$

$$+\{-c(z)[b(z)c'(z)-b'(z)c(z)]+2b(z)[b(z)d'(z)-b'(z)d(z)]\}S(u)(z) \tag{7}$$

$$-2d(z)[b(z)c'(z)-b'(z)c(z)]+c(z)[b(z)d'(z)-b'(z)d(z)]=0.$$

This shows that u is semi-classical. Its Stieltjes function satisfies the linear differential equation

$$A(z)S'(u)(z)=C(z)S(u)(z)+D(z),$$

where

$$A(z)=b(z)[c^2(z)-4b(z)d(z)]$$

$$C(z) = --c(z)[b(z)c'(z) - b'(z)c(z)] + 2b(z)[b(z)d'(z) - b'(z)d(z)]$$

and

$$D(z) = 2d(z)[b(z)c'(z) - b'(z)c(z)] - c(z)[b(z)d'(z) - b'(z)d(z)].$$

From here , the functional u verifies the equation $D[\Phi(z)u] + \Psi(z)u = 0$, where $\Psi(z) = A(z)$ and $\Psi(z) = -A'(z) - C(z)$.

The equations (5) are equivalent to

$$\begin{aligned} \Phi(z) &= b(z)[c^2(z) = 4b(z)d(z)] = b(z)M(z) \\ \Psi(z) &= -\tfrac{3}{2}[c^2(z) - 4b(z)d(z)]'b(z) = -\tfrac{3}{2}b(z)M'(z) \end{aligned} \tag{8}$$

where $m(z) = c^2(z) - 4b(z)d(z)$. $\square$.

Let us study the converse problem. Given an admissible semi-classical pair (see [9], page 38), under which conditions is the functional u of second degree?.

Proposition 3.2. *Let u be a semi-classical linear functional satisfying the equation $D[\Phi(z)u] + \Psi(z)u = 0$. Then , u is a second degree functional if and only if there exist polynomials p(z) , b(z), c(z) y d(z) such that*

$$\begin{aligned} &p(z)\Phi(z) = b(z)[c^2(z) - 4b(z)d(z)] = b(z)M(z) \\ &p(z)\Psi(z) = p'(z)\Phi(z) = -\tfrac{3}{2}b(z)M'(z) \\ &p(z)D(z) = 2d(z)[b(z)c'(z) - b'(z)c(z) = -c(z)[b(z)d'(z) - b'(z)d(z)] \\ &= \tfrac{1}{4}[M'(z)c(z) - 2(c'(z)M(z)] \end{aligned} \tag{9}$$

where $d(z) = -(u\theta_0\Phi)'(z) = (u\theta_0\Psi)(z)$, $b(z) \neq 0$, $c^2(z) - 4b(z)d(z) \neq 0$ and $d(z) = (u\theta_0 c)(z) - (u^2\theta_0^2 b)(z)$.

Proof Let u be a semi-classical functional satisfying the equation $D[\Phi(z)u] + \psi(z) = 0$. The corresponding Stieltjes function $S(u)(z)$ satisfies the equation $\Phi(z)S'(u)(z) = C(z)S(u)(z) + D(z)$, where $C(z) = -\Phi'(z) - \Psi(z)$ and $D(z)$ is defined in (9). Multiplying the last equation by $p(z)$, $p \in P$, and comparing with (7) and (8) the result follows. $\square$

Note The appearance of the polynomial $p(z)$ is justified as follows. The equation (7) may be reducible, that is to say, the polynomials $A(z)$, $C(z)$ y $D(z)$ may not be coprime. Calling its g.c.d. $p(z)$, $S(u)(z)$ fulfills the equation $(A/p)S' = (C/p)S + (D/p)$ from where the class of the functional is obtained (see [9], page 38).

If we begin with a semi-classical functional whose Stieltjes function fulfills the irreducible equation $AS' = CS + D$ and we want to prove its condition of second degree, we shall have to use the equations (9).

Examples: **Hermite Polynomials** $\mathbf{H}_n(z)$.

Solving the system (9) we have

$$p = b(c^2 - 4bd) = bM$$

$$-p' + 2zp = -(3/2)b(c^2 - 4bd)' = -(3/2)bM'$$

$$-2p = 2d(bc' - b'c) - c(bd' - b'd)$$

$$d = (u\theta_0 c) - (u^2\theta_0 2b)$$

$$M = (c^2 - 4bd) = p/b$$

$$p'b - 3bp' + 4zbp = 0$$

Looking for solutions of the form

$$p(z) = \sum_{j=0}^{n} p_j z^j, \quad b(z) = \sum_{j=0}^{m} b_j z^j, \quad c(z) = \sum_{j=0}^{q} c_j z^j \tag{10}$$

we obtain that the leading coefficient $4b_m p_n$ vanishes; then $b_m = 0$ or $p_n = 0$,for every $n, m \geq 0$. By identifying coefficients , it follows that $b(z) = 0$ or $p(z) = 0$.

Conclusion The functional u , corresponding to the Hermite polynomials is not a second degree functional.

Laguerre Polynomials $\mathbf{L}_n^{(\alpha)}$.

In the similar way to the Hermite polynomials , solving the system (9), we obtain the equation $(-b'p + 3b'p - 2bp)z + (2\alpha - 1)bp = 0$; looking for polynomial solutions of the form (10), it follows that $b_m p_n = 0$, then $b_m = 0$ or $p_n = 0$, for every $n, m \geq 0$. By identifying coefficients , it follows that $b(z) = 0$ or $p(z) = 0$.

Conclusion The functional u , corresponding to the Laguerre polynomials is not a second degree functional.

Jacobi Polynomials $\mathbf{P}_n^{(\alpha,\beta)}$.

Solving (9)

$$\begin{array}{ll} -p'(z^2-1) - 2pz - (Az+B)p = -(3/2)bM' & p(z^2-1) = bM \\ (A+1)p = 2d(bc' - b'c) - c(bd' - b'd) & \\ \alpha + \beta A & \beta - \alpha = B \end{array}$$

from where we obtain

$$\begin{aligned} &2b(Az+B)p+(z^2-1)(3pb'-p'b)-2zpb=0 \\ &4b^2(A+1)p+(z^2-1)(2pbc'-p'bc+b'pc)-2zpbc=0 \end{aligned} \tag{11}$$

Looking for polynomial solutions and following the notation (10), we conclude that $n=3m+2(A-1)$, which is equivalent to

$$\alpha+\beta=A\in\{\pm\tfrac{2n+1}{2}\}\bigcup\{-1\}\bigcup\{0\}\bigcup I\!\!N,\ n=0,1,2,...$$

This is a necessary condition. Let us study the following cases:

i) For $\alpha=\beta=-1/2$, we have $(z^2-1)(3b'p-p'b)-4zpb=0$

Looking for polynomial solutions, as in the previous example $n=3m-4$. A solution is $m=n=2$, resulting in $b_0=-1$, $b_1=0$, $b_2=1$, $p_0=-4$, $p_1=0$, $p_2=4$. From where $p(z)=4(z^2-1)$, $b(z)=(z^2-1)$, $c(z)=0$, $d(z)=-1$.

$S(u)(z)$ fulfills the second degree equation $(z^2-1)S^2(u)(z)-1=0$ and then the functional u corresponding to $\mathbf{T}_n$ is of second degree fulfilling $(z^2-1)u2=0$, $u_1=0$.

ii) For $\alpha=\beta=1/2$, (11) gives $(z^2-1)(3pb'-p'b)=0$ and $8b^2p+(z^2-1)(2pbc'-p'bc+b'pc)-2zpbc=0$

Looking for polynomial solutions and studying the leading coefficient in the first of the last two equations, we have $3m-n=0$. One solution is $m=n=0$, that is $b(z)=1$, $p(z)=16$, $c(z)=4z$ and $d(z)=4$.

The functional u corresponding to $\mathbf{U}_n$ is of second degree fulfilling the equation $u^2=4z^2u$, $u_1=0$ and the Stieltjes function $S^2+4zS+4=0$.

iii)For $\alpha=1/2$, $\beta=-1/2$ it results (in a similar way to the previous one) $(z-1)u^2=2z(1-z)u$, $u_1=1/2$, $(z-1)S^2+2(1-z)S-2=0$

iv) For $\alpha=-1/2$, $\beta=1/2$ we get $(z+1)u^2=2z(z+1)u$, $u_1=-1/2$, $(z+1)S^2+2(z+1)S+2=0$.

Bessel Polynomials $\mathbf{B}_n^{(\alpha)}$.

Solving (9) we have
$-(pz^2)'-[2(\alpha-1)z+2]p=-(3/2)bM' \quad pz^2=bM$
$p(2\alpha-1)=(1/4)(M'c-2c'M)$
from where it results that $(p'b-3b'p)z^2+2bp[1-2(\alpha-1)]z-4bp=0$. Looking for solutions of the form (10) and evaluating the term of zero degree, we obtain either $b_0=0$ or $p_0=0$.

If $b_0 = 0$, dividing by z and evaluating at zero, it results that $b_1 = 0$. Iterating the procedure, we conclude that $b(z) = 0$. In the same way, if we suppose $p_0 = 0$, we deduce that $p(z) = 0$. The functional u corresponding to the Bessel polynomials is not of second degree.

Proposition 3.3. *The equations (9) are equivalent to*

$$\left(\frac{c(z)}{b(z)}\right)^2 - 4\left(\frac{d(z)}{b(z)}\right) = \frac{1}{f(z)} \qquad \frac{c(z)}{b(z)} = g(z)$$

$$d(z) = (u\theta_0 c)(z) - (u^2\theta_0^2 b)(z) \qquad f(z) = A\left[e^{\int \frac{\psi(z)+\varphi'(z)}{\varphi(z)} dz}\right]$$

$$e^{\int \frac{\psi(z)+\varphi'(z)}{\varphi(z)} dz}\left[B + \int e^{-\int \frac{\psi(z)+\varphi'(z)}{\varphi(z)} dz} \frac{2D(z)}{\varphi(z)} dz\right].$$

(A and B constants).

Proof The system (9) can be integrated as follows. From (9) we have

$$p(z)\varphi(z) = b(z)M(z)$$

$$p(z)\psi(z) + \varphi'(z)] = b'(z)M(z) - (1/2)b(z)M'(z)$$

$$p(z)D(z) = (1/4)[M'(z)c(z) - 2c'(z)M(z)]$$

Eliminating $p(z)$ we obtain

$$-\frac{1}{2}\frac{M'(z)}{M(z)} + \frac{b'(z)}{b(z)} = \frac{\Psi(z)+\varphi'(z)}{\varphi(z)}$$

$$(\frac{c(z)}{b(z)})' - \left[\frac{\Psi(z)+\varphi'(z)}{\varphi(z)}\right](\frac{c(z)}{b(z)} = \frac{2D(z)}{\varphi(z)}.$$

Integrating both equations

$$\begin{aligned} \frac{b^2(z)}{M(z)} &= A\left[e^{\int \frac{\Psi(z)+\varphi'(z)}{\varphi(z)} dz}\right]^2 \\ \frac{c(z)}{b(z)} &= e^{\int \frac{\Psi(z)+\varphi'(z)}{\varphi(z)} dz}\left[B + \int e^{-\int \frac{\Psi(z)+\varphi'(z)}{\varphi(z)} dz} \frac{2D(z)}{\varphi(z)} dz\right] \end{aligned} \tag{12}$$

Notice that from $D[\Phi(z)u] + \Psi(z)u = 0$if $w(z)$ is a weight function associated to u, then we have

$$-\int_a^b \Phi(z)p'(z)w(z)dz + \int_a^b \Psi(z)p(z)w(z)dz = 0, \quad \text{for each } p(z) \in P,$$

being $w(z)$ the corresponding weight function. Integrating by parts

$$(\Phi(z)w(z))' + \Psi(z)w(z) = 0 \quad \text{and} \quad \frac{w'(z)}{w(z)} = -\frac{\Psi(z)+\Phi'(z)}{\Phi(z)}$$

Substituting in (12) we obtain $kw^2(z) = \frac{M(z)}{b^2(z)}$. From here we deduce that $w^2(z)$ must to be a rational function. For instance, if we search which Jacobi weight functions $[w(z) = (1-z)^\alpha(1+z)^\beta]$ give second degree functionals, a necessary condition is that $2\alpha \in \mathbb{Z}$ and $2\beta \in \mathbb{Z}$.

Furthermore, a sufficient condition would follow from the fact that

$$\frac{c(z)}{b(z)} = e^{Lnw(z)}\left[B + \int e^{Lnw(z)}\frac{2D(z)}{\varphi(z)}dz\right] = \frac{1}{w(z)}\left[B + \int w(z)\frac{2D(z)}{\varphi(z)}dz\right]$$

$$= \frac{B}{w(z)} + \int \frac{2D(s)}{\varphi(s)}\frac{w(s)}{w(z)}ds = \frac{B}{w(z)} + \int \frac{2D(s)}{\varphi(s)}\sqrt{\frac{(1-s)^{2\alpha}(1+s)^{1\beta}}{(1-z)^{2\alpha}(1+z)^{2\beta}}}ds$$

must be a rational function. 15

4.Stability of the second degree functional

The set of second degree functionals is closed for the most usual operations.In this paragraph we study the stability of the associated functional as well as the functional undergoing a series of perturbations.

Proposition 4.1. *Let u be a second degree functional fulfilling the equation (3). The functional $u^{(k+1)}$, associated of order (k+1) to the functional u, is a second degree functional.*

Proof The relation between the Stieltjes functions $S(u)$ y $S_{k+1} = S(u^{(k+1)})$ is

$$\gamma_{k+1}S_{k+1}(z) = \frac{-P_{k+1}(z)S(u)(z) - P_k^{(1)}(z)}{P_k(z)S(u)(z) + P_{k-1}^{(1)}}$$

(see [4] and [11]). Substituting in (2) we have that S_{k+1} fulfills $b(k+1,z)S_{k+1}^2 + c(k+1,z)S_{k+1} + d(k+1,z) = 0$, where

$$b(k+1,z) = (bN^2 - cNQ + dQ^2)(z)$$

$$c(k+1,z) = (2MNb - MQc - NPc + 2PQd)(z)$$

$$d(k+1,z) = (bM^2 - MPc + dP^2)(z) \qquad M(z) = P_k^{(1)}(z)$$

$$N(z) = \gamma_{k+1} P_{k-1}^{(1)}(z) \qquad P(z) = P_{k+1}(z)$$

$$Q(z) = P_{k+1}(z) \qquad b(z) = b(0,z)$$

$$c(z) = c(0,z) \qquad d(z) = d(0,z)\square.$$

An affine transformation in the variable leaves invariant the equation (2). Let us consider the linear functional: $u^* = (h_{1/\alpha} o \tau_{-\beta})u$, where $< \tau_\beta u, p(z) >=< u, p(z+\beta) >$ and $< h_\alpha u, p(z) >=< u, p(\alpha z) >$.

Proposition 4.2. *Let u be a linear functional and $u^{=}(h_{1/\alpha} o \tau_{-\beta})u$. If u is a second degree functional, so is u^*, fulfilling the corresponding Stieltjes function $S(u^*)(z)$,the equation*

$$b^*(z)S^2(u^*)(z) + c^*(z)S(u^*)(z) + d^*(z) = 0,$$

where

$$b^*(z) = \alpha^{-r} b(\alpha z + \beta)$$

$$c^*(z) = \alpha^{1-r} c(\alpha z + \beta), \quad d^*(z) = \alpha^{2-r} d(\alpha z + \beta)$$

and $b(z)$, $c(z)$, $d(z)$ the polynomials defined in (2) and r the degree of $b(z)$.

Proof The relation between the corresponding Stieltjes functions is

$$S(u^*)(z){=}\alpha S(u)(\alpha z + \beta),$$

(see [1], page 54). Substituting in (2) we obtain $b(\alpha z + \beta)S^2(u^*) + c(\alpha z + \beta)\alpha S(u^*) + d(\alpha z + \beta)\alpha^2 = 0$. Dividing by α^r we obtain the result.$\square$

Let u be a second degree functional and let $\{P_n\}_{n\geq 0}$ be an orthogonal sequence with respect to u. It verifies the recurrence relation:

$$P_0(z) = 1, \qquad P_1(z) = z = \beta_0,$$

$$P_{n+2}(z) = (z - \beta_{n+1})P_{n+1}(z) - \gamma_{n+1}P_n(z), \qquad n \geq 0.$$

On calls r-order perturbed sequence of $P_{n\,n\geq 0}$, the sequence $P^*_{n\,n\geq 0}$ defined by

$$\beta_0^* = \beta_0 + \mu_0, \quad \beta_n^* = \beta_n + \mu_n, \quad \gamma_n^* = \gamma_n \lambda_n \quad \lambda_0 \neq 0 \quad 1 \leq n \leq r, \text{if } r \geq 1.$$

$\beta_n^* = \beta_n \;, \quad \gamma_n^* = \gamma_n, \;, \; n \geq r+1.$

The sequence $\{P_n^*\}n \geq 0$ is orthogonal with respect to $u^* = u(\mu_0;^{\mu}_{\lambda};r)$, where $\mu = (\mu_1, ..., \mu_r)$ and $\lambda = (\lambda_1, ..., \lambda_r)$, (see [9] as well as a particular and interesting example in [14]).

Proposition 4.3. *(r-order perturbed functional) Let u be a second degree functional fulfilling (3). The functional $u^* = u(\mu_0;^{\mu}_{\lambda};r)$, r-order perturbed functional of u, is a second degree functional.*

Proof The relation between the corresponding Stieltjes functions is, (see [11])

$$S(u^*)(z) = -\tfrac{U_r(z)+V_r(z)S(u)(z)}{X_r(z)+Y_r(z)S(u)(z)}.$$

with

$$U_r(z) = \gamma_r\{P_{r-1}^{*(1)}(z)P_{r-2}^{(1)}(z) - \lambda_r P_{r-1}^{(1)}(z)P_{r-2}^{*(1)}(z)\} - \mu_r P_{r-1}^{(1)}(z)P_{r-1}^{*(1)}(z)$$

$$V_r(z) = \gamma_r\{P_{r-1}^{*(1)}(z)P_r^{(1)}(z) - \lambda_r P_r^{(1)}(z)P_{r-2}^{*(1)}(z)\} - \mu_r P_r^{(1)}(z)P_{r-1}^{*(1)}(z)$$

$$X_r(z) = \gamma_r\{P_r^*(z)P_{r-2}^{(1)}(z) - \lambda_r P_{r-1}^{(1)}(z)P_{r-1}^*(z)\} - \mu_r P_{r-1}^{(1)}(z)P_r^*(z)$$

$$Y_r(z) = \gamma_r\{P_r^*(z)P_{r-1}(z) - \lambda_r P_r(z)P_{r-1}*(z)\} - \mu_r P_r(z)P_r^*(z)$$

Substituting in (2) we obtain:

$$[bX_r^2 - cX_rY_r + dY_r^2]S^{*2}+$$

$$[2bU_rX_r - cU_rY_r - cX_rV_r + 2dY_rV_r]S^*+$$

$$[bU_r^2 - cU_rV_r + dV_r^2] = 0$$

Next we show that if u is a second degree funcional then $u^* = u + \mu\delta_c$ and $u^* = u + \mu\delta_c'$, where $\mu, c \in \mathbb{C}$, are also second degree functionals. Conditions about the quasi-definite character of u^* have been given in [7].We give the second degree equation satisfied by the corresponding Stieltjes functions.

Proposition 4.4. *Let u be a second degree functional. Then $u^* = u + \mu\delta_c$ is also a second degree functional.*

Proof Let $S(z) = S(u)(z)$ and $S^*(z) = S(u^*)(z)$ be the corresponding Stieltjes functions.

From $< u^*, z^n > = < u + \mu\delta_c, z^n > = u_n + \mu c^n$, $n \geq 0$, we obtain $S(z) = S^*(z) + \mu/(z-c)$.

Substituting in (2) we have that $S^*(z)$ fulfills $b^*(z)S^{*2}(z) + c^*(z)S^*(z) + d^*(z) = 0$ where $b^*(z) = b(z-c)^2$, $c^*(z) = 2\mu b(z-c) + c(z-c)^2$ and $d^*(z) = \mu^2 b + c\mu(z-c) + d(z-c)^2$.$\square$

Proposition 4.5 *Let u be a second degree functional. Then $u^* = u + \mu\delta'_c$ is also a second degree functional.*

Proof Let $S(z) = S(u)(z)$ and $S^*(z) = S(u^*)(z)$ be the corresponding Stieltjes functions.

From $< u^*, z^n >=< u + \mu\delta'_c, z^n >= u_n - \mu n c^{n-1}$, $n \geq 0$, we obtain $S(z) = S^*(z) - \mu/(z-c)^2$.

Then, substituting in (2) $S^*(z)$ fulfills $b^*(z)S^{*2}(z) + c^*(z)S^*(z) + d^*(z) = 0$ where $b^*(z) = b(z-c)^4$, $c^*(z) = -2\mu b(z-c)^2 + c(c-z)^4$ and $d^*(z) = \mu^2 b - c\mu(z-c)^2 + d(z-c)^4$.□

Proposition 4.6. *(Left multiplication by a polynomial) Let u be a second degree functional and v=p(z)u where p(z) is a polynomial. Then v is a second degree functional fulfilling the corresponding Stieltjes function $S^*(z) = S(v)(z)$ the equation $b(z)S^{*2}(z) + (cp-2bq)(z)S^*(z) + (bq^2 - cpq + p^2 d)(z) = 0$, where b(z), c(z) and d(z) are the polynomials defined in (2) and $q(z) = (u\theta_0 p)(z)$.*

Proof Remember that the functional pu has been defined when we introduced some operations in P^*.

If $p(z) = \sum_{j=0}^{n} p_j z^j$, then $(pu)_m = \sum_{j=0}^{n} p_j u_{j+m}$ and $S^*(z) = S(v)(z) = S(u)(z)p(z) + q(z)$, whith $q(z) = \sum_{j=1}^{n} p_j \sum_{k=0}^{j-1} u_k z^{j-k-1} = (u\theta_0 p)(z)$.

Substituting in (2) the result follows.□

Proposition 4.7. *(The inverse functional) Let u be a second degree functional such that $u+0 \neq 0$. Then the inverse functional u^{-1} is also a second degree functional.*

Proof From $S(uv) = -zS(u)S(v)$, (see [9]), we have $S(uu^{-1}) = S(\delta_0) = -(1/z) == -zS(u)S(u^{-1})$ and $S(u) = 1/[z^2 S(u^{-1})]$. Substituting in (2), we obtain that $S(u^{-1})$ satisfies the equation $z^4 d(z)S^2(u^{-1}) + z^2 c(z)S(u^{-1}) + b(z) = 0$.□

Examples A perturbation of the second kind Tchebychev polynomials

Let u be the Tchebychev functional of second kind and let $S(u)(z)$ be the corresponding Stieltjes function. Then $D[(z^2-1)u] - 3zu = 0$, $S^2(z) + 4zS(z) + 4 = 0$ and $u^2 = 4z^2 u$, $u_1 = 0$.

i) Let us consider only one modification in the coefficient β_0 as $\beta_0^* = \beta_0 + \mu$, and let $u^* = u(\mu;_0^0; 0)$ be the new functional. $S^*(z) = S(u^*)(z)$ fulfills the equation

$$(1-4\mu z+4\mu^2)S^{*2}(z)+4(z-2\mu)S^*(z)+4=0$$

This kind of perturbation leads to co-recursive orthogonal polynomials and they have been studied in [4] and [15].

ii) Let us consider only one modification in the coefficient γ_1 as $\gamma_1^*=\lambda\gamma_1$, $\lambda\neq 0$, and let $u^*=u(0;^0_\lambda;1)$ be the new functional. $S^*(z)=S(u^*)(z)$ fulfills the equation

$$[\lambda^2+4z^2(1-\lambda]S^{*2}(z)+4z(2-\lambda)S^*(z)+4=0.$$

iii) Next, if we introduce a modification in the coefficients β_1 and γ_1, $\beta_1^*=\beta_1+\mu$, $\gamma_1=\lambda\gamma_1$, let $u^*=u(0;^\mu_\lambda;1)$ be the new functional. $S^*(z)=S(u^*)(z)$ fulfills the equation

$$b^*(z)S^{*2}(z)+c^*(z)S^*(z)+d^*(z)=0$$

where

$$b^*(z)=-\mu z^3+(1/4)(4\mu^2-\lambda+1)z^2+(1/2)\mu\lambda z+(1/16)\lambda^2.$$

$$c^*(z)=-2\mu z^2+[2\mu^2+(1/4)(2-\lambda)]z+(1/2)\mu\lambda,$$

$$d^*(z)=-z+\mu+(1/4\lambda).$$

iv) A modification of the functional u as $u^*=u+\mu\delta_c$ leads to

$$(z-c)^2S^{*2}(z)+[2\mu(z-c)+4z(z-c)^2]S^*(z)+[\mu^2+4\mu z(z-c)+4(z-c)^2]=0.$$

v) A modification of the functional u as $u^*=u+\mu\delta_c'$ yields

$$(z-c)^4S^{*2}(z)+[-2\mu(z-c)^2+4z(z-c)^4]S^*(z)+[\mu^2+4\mu z(z-c)^2+4(z-c)^4]=0$$

5.The reduction of a Laguerre-Hahn functional to a semi-classical linear functional

Let u be a Laguerre-Hahn functional.The corresponding Stieltjes function fulfills the Riccati ecuation (1), (see [1],[2],[5],[8]). We shall study under which conditions the Stieltjes function satisfies a quadratic equation (2), being the functional u of second degree and, consequently, semi-classical.

Proposition 5.1. *Let u be a Laguerre-Hahn functional, i.e. the corresponding Stieltjes function satisfies the equation (1). If there exist b(z), c(z), d(z) and p(z) polynomials of complex coefficients such that:*
$c^2(z)-4b(z)d(z)=\Phi(z)p(z)$,

$-2b(z)[b'(z)d(z) - b(z)d'(z)] - c(z)[b(z)c'(z) - b'(z)c(z)] = [c(z)B(z) - b(z)C(z)]p(z)$,

$-2d(z)[b(z)c'(z) - b'(z)c(z)] + [c(z)[b(z)d'(z) - b'(z)d(z)] = [d(z)B(z) - b(z)D(z)]p(z)$,

$$d(z) = (u\theta_0 c)(z) - (u^2\theta_0^2 b)(z) \tag{13}$$

then u satisfies the equation (3), i.e. is a second degree functional and, as a consequence, it is a semi-classical functional.

Proof Substituting in (1) the term $S^2(u)(z)$ given by (6) we obtain

$$\begin{array}{l} b(z)\Phi(z)S'(u)(z) + [-b(z)C(z) + c(z)B(z)]S(u)(z) \\ +[-b(z)D(z) + d(z)B(z)] = 0 \end{array} \tag{14}$$

Comparing (14) with (7) the system follows.□

We consider a functional u such that $zu = v$ where v is the linear functional relative to Tchebychev polynomials of the second kind.

This shows a Laguerre-Hahn functional of class $s = 2$, (see[1], page 58 and [8]), which is reducible to a semi-classical one.

Let v be the Tchebychev functional of the second kind. Then $v(1) = v$ (see [7]) and, as a consequence v is a Laguerre-Hahn functional.

The Stieltjes function satisfies

$$(z^2 - 1)S'(v)(z) = (1/2)S^2(v)(z) + 3zS(z)(v) + 4$$

and $u = z^{-1}v + \delta$ is a Laguerre-Hahn functional of class $s = 2$. In fact the corresponding Stieltjes function $S^*(u)(z)$ fulfills the equation

$$z(z^2 - 1)S'^*(u)(z) = (1/2)z^2S^{*2}(u)(z) + (2z^2 + z + 1)S^*(u)(z) + 3z + (9/2).$$

Applying the formulas (13) we obtain

$$z(z^2 - 1)p = c^2 - 4bd$$

$$-2b[b'd - bd'] - c[bc' - b'c] = p[(1/2)z^2c - b(2z^2 + z + 1)]$$

$$-2d[bc' - b'c] + c[bd' - b'd] = p[(1/2)z^2d - b(3z + 9/2)]$$

Taking a polynomial $p(z)$ with $degp = 1$, the system has polynomial solutions
$p(z) = 16z, \quad b(z) = z^2, \quad c(z) = 2z(2z + 1), \quad d(z) = 4z + 5.$

The functional u is a second degree functional and , consequently, semi-classical.

The corresponding Stieltjes function fulfills the equation

$$z^2 S^{*2}(u)(z) + 2z(2z+1)S^*(u)(z) + 4z + 5 = 0.$$

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