Convexity, Lipschitz property and differentiability in two–stage stochastic nonlinear programming problems

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Abstract

In the paper, two–stage stochastic nonlinear programming problems are investigated. In particular, the optimization problems with two types of decision are investigated. They are the solution of the basic (outer) problem depending on the underlying random element only through the corresponding probability law (more precisely through the corresponding mathematical expectation) and the solution of the inner (recourse) problem that can depend on this realization. In addition, a new random element (connected to the inner problem) is considered. The solution of the inner problem can depend on this new random element only through the corresponding conditional mathematical expectation. The objective function of the outer problem includes the optimal value of the inner problem.

The aim of this paper is to investigate the behaviour of the objective function of the outer problem. A special attention is paid to the convexity, Lipschitz property and the existence of the partial derivatives. An explicit form of the gradient vector is introduced in the special convex case. The additional random element connected to the inner problem enables to generalize the achieved results to multistage stochastic programs. And, consequently, to employ (in future) the results on estimates and stability, achieved for one–stage case, to the general multistage programs.

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1 Introduction

Two–stage (generally multistage) stochastic programming problems correspond very well to many practical situations in engineering, biological, medical as well as economic and social practice in which a random element exists. Especially, two–stage models are employed everywhere when the basic solution should be determined without knowing the underlying random element realization and when the obtained effect can be corrected by a new (say inner) optimization problem depending already on this random element realization. If a new random element is connected with the inner problem, then its solution can depend on the new random element only through the conditional mathematical expectation.
Let $\xi^1 = \xi^1(\omega)$, $\xi^2 = \xi^2(\omega)$ be s_1 and s_2–dimensional random vectors defined on a probability space (Ω, S, P), $\bar{\xi}^2 = \bar{\xi}^2(\omega) = [\xi^1(\omega), \xi^2(\omega)]$,
$F^{\xi^1}(z^1)$, $F^{\xi^2}(z^2)$, $F^{\bar{\xi}^2}(\bar{z}^2)$, $z^1 \in E_{s_1}$, $z^2 \in E_{s_2}$, $\bar{z}^2 = [z^1, z^2] \in E_{s_1+s_2}$ denote the common distribution functions of $\xi^1(\omega)$, $\xi^2(\omega)$ and $\bar{\xi}^2(\omega)$,
$F^{\xi^2|\xi^1}(z^2|z^1)$, $z^2 \in E_{s_1}$, $z^1 \in E_{s_2}$ denote the conditional distribution function ($\xi^2(\omega)$ conditioned by $\xi^1(\omega)$),
$Z^1 \subset E_{s_1}$, $Z^2 \subset E_{s_2}$ denote the probability measure supports corresponding to the distribution functions $F^{\xi^1}(z^1)$, $F^{\xi^2}(z^2)$.

Let, moreover, $h_i(x^1, z^1)$, $i = 0, \ldots, l$ be real–valued, continuous functions defined on $E_{n_1} \times E_{s_1}$,
$f_0^1(\bar{x}^2, \bar{z}^2)$ and $f_i(x^2)$, $i = 1, 2, \ldots, l$ be real–valued, continuous functions defined on $E_{n_1+n_2} \times E_{s_1+s_2}$ and E_{n_2}, $\bar{x}^2 = [x^1, x^2]$,
$X^1 \subset E_{n_1}$, $X^2 \subset E_{n_2}$ be nonempty sets, $\bar{X}^2 = X^1 \times X^2$, $\bar{Z}^2 = Z^1 \times Z^2$.
(E_n, $n \geq 1$ denotes an n–dimensional Euclidean space.)

A rather general two–stage stochastic nonlinear (however partly saparable) programming problem can be introduced in the following form:
Find

$$\inf\{E_{F^{\xi^1}} Q(x^1, \xi^1(\omega)) | \, x^1 \in X^1\}, \tag{1}$$

where

$$Q(x^1, z^1) = h_0(x^1, z^1) + \inf\{E_{F^{\xi^2|\xi^1=z^1}} f_0^1(\bar{x}^2, \bar{\xi}^2(\omega)) | \, x^2 \in \mathcal{K}(x^1, z^1)\}, \tag{2}$$

$$\mathcal{K}(x^1, z^1) = \{x^2 \in X^2 : f_i(x^2) \leq h_i(x^1, z^1), \, i = 1, 2, \ldots, l\}, \; x^1 \in E_{n_1}, \, z^1 \in E_{s_1}. \tag{3}$$

$E_{F^{\xi^1}}$, $E_{F^{\xi^2|\xi^1}}$ denote the operators of the mathematical expectations corresponding to the $F^{\xi^1}(z^1)$, $F^{\xi^2|\xi^1}(z^2|z^1)$.

The problem defined by (1), (2) and (3) can be assigned, for example, to an underlying optimization problem with a random element in the form:
Find

$$\inf\{h_0(x^1, \xi^1(\omega)) | \, x^1 \in X^1 : g_i(x^1, \xi^1(\omega)) \le 0, \, i = 1, \ldots, l\},$$

where $g_i(x^1, z^1)$, $i = 1, \ldots, l$ are real–valued, continuous functions defined on $E_{n_1} \times E_{s_1}$. Evidently, then $h_i(x^1, z^1) = -g_i(x^1, z^1)$, $i = 1, 2, \ldots, l$.

The inner problem: Find

$$\inf\{E_{F^{\xi^2 | \xi^1 = z^1}} f_0^1(\bar{x}^2, \bar{\xi}^2(\omega)) | \, x^2 \in X^2 : f_i(x^2) \le h_i(x^1, z^1), \, i = 1, 2, \ldots, l\} \tag{4}$$

expresses (in the case $\xi^1(\omega) = z^1$) either an additional profit corresponding to the unexhausted constraints or it can correspond to a penalty.

Since, it follows from the optimization theory that in the problems of the type:
Find

$$\inf\{E_{F^{\xi^1}} Q(x^1, \xi^1(\omega)) | \, x^1 \in X^1\},$$

the (uniform) continuity, convexity, Lipschitz property and differentiability of the functions $Q(x^1, z^1)$, $E_{F^{\xi^1}} Q(x^1, \xi^1(\omega))$ are very suitable properties by the investigation of the stability, the statistical estimates and sufficient optimality conditions [1], [3], [9], [15] it is surely reasonable to investigate the problem introduced by the relations (1), (2) and (3) from the above mentioned point of view.

The aim of the paper is just this investigation. In the literature, these properties were discussed mostly in the linear case, see for example [4]. However, the papers [5], [6], [8] deal also with special types of the nonlinear case.

Remark 1 *In general, it may happen that some symbols in (1) and (2) are not well defined. However, this situation cannot appear under the assumptions considered in this paper.*

Remark 2 *Mostly, the random element $\xi^2(\omega)$ doesn't appear in the definition of the two–stage stochastic programming problem [4], [11] and [17]. Consequently, then also the conditional mathematical expectation is omitted in the inner problem. We consider the generalized definition to have the possibility to generalize the results of this paper to the multistage case.*

2 Problem Analysis

It is easy to see from the relations (1), (2) and (3) that the behaviour of the functions $Q(x^1, z^1)$, $E_{F^{\xi^1}} Q(x^1, \xi^1(\omega))$ depends on the behaviour of

1. the function $h_0(x^1, z^1)$,
2. the optimal value of the parametric problem (4),
3. the distribution function $F^{\bar{\xi}^2}(\bar{z}^2)$.

It follows from this analysis that to investigate $Q(x^1, z^1)$, $E_{F^{\xi^1}} Q(x^1, \xi^1(\omega))$ it is necessary to investigate the multifunction $\mathcal{K}(x^1, z^1)$. To this end we substitute

$$y_i = h_i(x^1, z^1), \; i = 1, 2, \ldots, l, \; x^1 \in E_{n_1}, \; z^1 \in E_{s_1} \tag{5}$$

and define a new multifunction

$$\bar{\mathcal{K}}(y) = \{x^2 \in X^2 : f_i(x^2) \le y_i, \; i = 1, 2, \ldots, l\}, \quad y = (y_1, \ldots, y_l) \in E_l. \tag{6}$$

Namely, if we denote

$$\begin{aligned} Y = \{y \in E_l : \quad & y = (y_1, y_2, \ldots, y_l), \; y_i = h_i(x^1, z^1), \; i = 1, 2, \ldots, l \\ & \text{for some } x^1 \in X^1, \; z^1 \in Z^1\}, \end{aligned} \tag{7}$$

then

$$\bar{\mathcal{K}}(y) = \mathcal{K}(x^1, z^1) \tag{8}$$

for $y = (y_1, \ldots, y_l) \in Y$ fulfilling the relation (5). Moreover, combining the behaviour of the multifunction $\bar{\mathcal{K}}(y)$ and the appropriate properties of the functions $h_i(x^1, z^1)$, $i = 1, 2, \ldots, l$ we obtain the behaviour of the multifunction $\mathcal{K}(x^1, z^1)$.

3 Some Auxiliary Assertions and Definitions

In this section, first, we shall recall some well–known results suitable for the next investigation.

Lemma 1 *Let*

1. *$X \subset E_n$, $Y \subset E_l$, $n, l \ge 1$ be nonempty, convex sets,*
2. *$\bar{\mathcal{K}}(y)$ be a multifunction mapping Y into the space of nonempty, closed subsets of X such that*

   $$\lambda \bar{\mathcal{K}}(y(1)) + (1-\lambda)\bar{\mathcal{K}}(y(2)) \subset \bar{\mathcal{K}}(\lambda y(1) + (1-\lambda) y(2))$$

   for every $y(1), y(2) \in Y$,
3. *$\bar{h}(x, y)$ be a convex function on $X \times Y$.*

If

$$\bar{\varphi}(y) = \inf\{\bar{h}(x, y) | \, x \in \bar{\mathcal{K}}(y)\} > -\infty \quad \text{for every} \quad y \in Y,$$

then $\bar{\varphi}(y)$ *is a real–valued, convex function on* Y.

The next auxiliary assertions deal with the Lipschitz property.

Lemma 2 *Let* $X \subset E_n$, $Y \subset E_l$, $n, l \geq 1$ *be nonempty sets. Let moreover*

1. $\bar{\mathcal{K}}(y)$ *be a multifunction mapping* Y *into the space of nonempty, closed subsets of* X *such that for every* $y(1), y(2) \in Y$ *and a* $C \geq 0$

$$\Delta[\bar{\mathcal{K}}(y(1)), \bar{\mathcal{K}}(y(2))] \leq C\|y(1) - y(2)\|,$$

2. $\bar{h}(x, y)$ *be a Lipschitz function on* $X \times Y$ *with the Lipschitz constant* L.

If

$$\bar{\varphi}(y) = \inf\{\bar{h}(x, y) | \, x \in \bar{\mathcal{K}}(y)\} > -\infty \quad \text{for every} \quad y \in Y,$$

then $\bar{\varphi}(y)$ *is a Lipschitz function on* Y *with the Lipschitz constant* $L(C+1)$.

(the symbols $\|\cdot\| = \|\cdot\|_n$ *and* $\Delta[\cdot, \cdot] = \Delta_n[\cdot, \cdot]$ *denote the Euclidean norm in* E_n *and the Hausdorff distance in the space of nonempty, closed subsets of* E_n*, for definition see e.g. [5] or [7].)*

To apply the assertion of Lemma 2 to the inner problem (4) it is necessary to verify the assumption 1. It means, first, to find the types of the functions $f_i(x^2)$, $i = 1, 2, \ldots, l$ for which the multifunction defined by the relation (6) fulfilled (for $n =: n_2$) the assumption 1 of Lemma 2. To this end we denote by Y_0 the convex hull of Y and by the symbol $Y_0(\varepsilon)$ the ε–neighbourhood of the set Y_0 (in detail, for $\varepsilon > 0$, $Y_0(\varepsilon) = Y + \mathcal{B}(\varepsilon) = \{y \in E_l, \, y = y(1) + y(2) : y(1) \in Y_0, \, y(2) \in \mathcal{B}(\varepsilon)\}$, where $\bar{B}(\varepsilon)$ denotes ε–neighbourhood of $0 \in E_l$). We shall introduce the following systems of the assumptions.

i.1
- a. $f_i(x^2)$, $i = 1, 2, \ldots, l$ are linear functions, $X^2 = E_{n_2}$, without loss of generality, we can consider in this case the constraints in (3) to be in the form of equalities,
- b. for every $y \in Y$, $\bar{\mathcal{K}}(y)$ is a nonempty, compact set,
- c. the matrix A of the type $(l \times n_2)$, $l \leq n_2$ fulfils the relation

$$\bar{\mathcal{K}}(y) = \{x^2 \in X^2 : Ax^2 = y\}, \; y \in Y$$

and, moreover, all its submatrices of the types $(l \times l)$, $A(1), \ldots, A(m)$ are nonsingular,

d. $C = l \max_{i\,r\,s} |a_{i\,r}(s)|$, where $a_{i\,r}(s)$ for $s \in \{1, 2, \ldots, m\}$ denote the elements of the inverse matrix to $A(s)$,

i.2 a. X^2 is a convex, compact set, $M_1 = \sup_{x^2(1),\, x^2(2) \in X^2} \|x^2(1) - x^2(2)\|$,

b. $f_i^2(x^2)$, $i = 1, 2, \ldots, l$ are convex functions on X^2,

c. $\bar{\mathcal{K}}(y)$ is a nonempty set for every $y \in Y_0(\varepsilon)$ and some $\varepsilon > 0$,

d. $C = \frac{M_1}{\varepsilon_0}$ for an $\varepsilon_0 \in (0, \varepsilon)$,

i.3 there exist real–valued constants $d_1, \gamma_2 > 0$ such that

a. if $x^2 \in X^2$, $y \in Y_0(d_1\gamma_2)$ fulfil the relations $f_j(x^2) \leq y_j$, $j = 1, \ldots, l$ and simultanously $f_i(x^2) = y_i$ for at least one $i \in \{1, \ldots, l\}$, then there exists a vector $x^2(0) = x^2(0, x^2) \in E_{n_2}$ such that

$$\|x^2(0)\| = 1, \quad x^2 + dx^2(0) \in X^2, \quad f_i(x^2) - f_i(x^2 + dx^2(0)) \geq \gamma_2 d$$

for every $d \in (0, d_1)$, $i = 1, 2, \ldots, l$,

b. for every $y \in Y_0(\gamma_2 d_1)$, $\bar{\mathcal{K}}(y)$ is a nonempty set,

c. $C = \frac{1}{\gamma_2}$,

Lemma 3 *If at least one of the assumptions (i.1), (i.2), (i.3) is fulfilled, then*

$$\Delta[\bar{\mathcal{K}}(y(1)),\, \bar{\mathcal{K}}(y(2))] \leq C\|y(1) - y(2)\|$$

for every $y(1), y(2) \in Y$.

Proof. If the systems of the assumptions (i.1) or (i.2) are fulfilled, then the assertion of Lemma 3 corresponds to well–known results on linear programming or convex analysis. The assertion of Lemma 3 under the assumption (i.3) is proven in [10]. □

It follows from the convex analysis and from [6] that the essential for the assertion of Lemma 3 (under the assumptions (i.2)) is the same as in the case of the Lipschitz property of the convex functions [16]. Obviously, the systems of the assumptions (i.1), (i.2) cover a great part of the multifunctions (3). However, surely, there exist many cases corresponding to applications that it is not possible to cover by the above mentiond cases. The system (i.3) try to include at least some of them. The idea of this system is (very simple) to include the cases in which for points $x^2 \in X^2$ there exist a neighbourhood (expressed by a constant d_1) and a direction $x^2(0)$ in which the functions $f_i(x^2)$, $i = 1, \ldots, l$ are dicreasing (enough quickly – the constant γ_2). Of course, this assumption can be further generalized. The direction $x(0)$ can be, for example, replaced by some curve along that the functions $f_i^2(\cdot)$, $i = 1, 2, \ldots, l$ are dicreasing.

To justify the idea of the system (i.3), let $l := 2$; $n_2 = 2$; $f_1(x^2) := x_1^2 x_2^2$; $f_2(x^2) := \log(x_1^2 + x_2^2)$; $X^2 := \langle 1, a_1\rangle \times \langle 1, a_2\rangle$, $e^3 \leq a_1, a_2$; $Y \subset \{y = (y_1, y_2) : y_1, y_2 \geq 2\}$. Evidently, the system of the assumptions (i.3) is (in this case) fulfilled, while the systems (i.1) and (i.2) are not satisfied.

4 Main Results

Combining the introduced auxiliary assertions and the appropriate behaviour of the function $f_0^1(\bar{x}^2, \bar{z}^2)$, $h_i(x^1, z^1)$, $i = 0, 1 \ldots, l$ we can obtain the behaviour of the optimal value

$$\inf\{E_{F^{\xi^2|\xi^1=z^1}} f_0^1(\bar{x}^2, \bar{z}^2)|\, x^2 \in \mathcal{K}(x^1, z^1)\}$$

and consequently the behaviour of the functions $Q(x^1, z^1)$, $E_{F^{\xi^1}} Q(z^1, \xi^1(\omega))$. We shall present the results on the convexity, the Lipschitz property and the partial vector derivatives existence, separately.

4.1 Convexity

First, for the completeness we present the following Theorem.

Theorem 1 *Let $X^1 \subset E_{n_1}$, $X^2 \subset E_{n_2}$, $n_1, n_2 \geq 1$ be nonempty, convex sets. If*

1. *for every $z^1 \in Z^1$*

   a. *$h_0(x^1, z^1)$ is a convex function on X^1,*

   b. *$h_i(x^1, z^1)$, $i = 1, 2, \ldots, l$ are concave functions on X^1,*

2. *$f_i(x^2)$, $i = 1, 2, \ldots, l$ are convex functions on X^2,*

3. *for every $\bar{z}^2 \in \bar{Z}^2$, $f_0^1(\bar{x}^2, \bar{z}^2)$ is a convex function on $\bar{X}^2$,*

4. *for every $x^1 \in X^1$, $z^1 \in Z^1$, $\mathcal{K}(x^1, z^1)$ is a nonempty set,*

5. *for every* $\bar{x}^2 \in \bar{X}^2$, $z^1 \in Z^1$ *a finite* $E_{F^{\xi^2|\xi^1=z^1}} f_0^1(\bar{x}^2, \bar{\xi}^2(\omega))$ *exists and moreover for every* $x^1 \in X^1$, $z^1 \in Z^1$,

$$\inf\{E_{F^{\xi^2|\xi^1=z^1}} f_0^1(\bar{x}^2, \bar{\xi}^2(\omega)) | x^2 \in \mathcal{K}(x^1, z^1)\} > -\infty,$$

then

1. *for every* $z^1 \in Z^1$, $Q(x^1, z^1)$ *is a real–valued, convex function on* X^1,

2. *if moreover for every* $x^1 \in X^1$ *a finite* $E_{F^{\xi^1}} Q(x^1, \xi^1(\omega))$ *exists, then also* $E_{F^{\xi^1}} Q(x^1, \xi^1(\omega))$ *is a real–valued convex function on* X^1.

Proof. First, it follows from the definition of the mathematical expectation, from the assumptions of Theorem 1 and from the convex analysis that for every $\bar{x}^2(1), \bar{x}^2(2) \in \bar{X}^2$, $\bar{x}^2(1) = (x^1(1), x^2(1)), \bar{x}^2(2) = (x^1(2), x^2(2))$, $z^1 \in Z^1$ and every $\lambda \in \langle 0, 1 \rangle$

$$E_{F^{\xi^2|\xi^1=z^1}} f_0^1(\lambda \bar{x}^2(1) + (1-\lambda)\bar{x}^2(2), \bar{\xi}^2(\omega)) \leq$$
$$\lambda E_{F^{\xi^2|\xi^1=z^1}} f_0^1(\bar{x}^2(1), \bar{\xi}^2(\omega)) + (1-\lambda) E_{F^{\xi^2|\xi^1=z^1}} f_0^1(\bar{x}^2(2), \bar{\xi}^2(\omega))$$

and simultanously

$$\lambda \mathcal{K}(x^1(1), z^1) + (1-\lambda)\mathcal{K}(x^1(2), z^1) \subset \mathcal{K}(\lambda x^1(1) + (1-\lambda)x^1(2), z^1).$$

However, setting $x =: x^2$, $y =: x^1$, $\bar{\mathcal{K}}(y) =: \mathcal{K}(x^1, z^1)$, $\bar{h}(x, y) =: E_{F^{\xi^2|\xi^1=z^1}} f_0^1(\bar{x}^2, \bar{\xi}^2)$ in Lemma 1 we can see that for every $z^1 \in Z^1$ the assumptions of Lemma 1 are fulfilled. Consequently, for every $z^1 \in Z^1$, $Q(x^1, z^1)$ is a real–valued, convex function on X^1. The second part of the assertion follows from the just proven assertion and from the properties of the integral. □

It can be rather complicated to verify the existence of a finite mathematical expectations assumed in Theorem 1. Some assertions concerning this problem are presented in [10].

If we interchange the position x^1 and z^1 in Theorem 1 and if we assume that $\xi^1(\omega)$, $\xi^2(\omega)$ are stochastically independent, then we obtain the assumption om convexity of $Q(x^1, z^1)$ with respect to z^1.

4.2 Lipschitz property

In the previous part, we have dealt with the convexity of the functions $Q(x^1,\, z^1)$ and $E_{F^{\xi^1}}\, Q(x^1,\, \xi^1(\omega))$. Furthermore, we shall deal with the Lipschitz property.

Theorem 2 *If*

1. *at least one of the assumptions (i.1), (i.2), (i.3) is fulfilled,*
2. *for every* $z^1 \in Z^1$, $h_i(x^1,\, z^1),\, i = 0,\, 1,\, \ldots,\, l$ *are Lipschitz functions on* X^1 *with the Lipschitz constants* $\bar{L}_i(z^1)$,
3. *for every* $\bar{z}^2 \in \bar{Z}^2$, $f_0^1(\bar{x}^2,\, \bar{z}^2)$ *is a Lipschitz function on* $\bar{X}^2$ *with the Lipschitz constant* $\bar{L}_{f_0^1}(\bar{z}^2)$ *such that for every* $z^1 \in Z^1$ *there exists a finite* $E_{F^{\xi^2|\xi^1=z^1}} \bar{L}_{f_0^1}(\bar{\xi}^2)$,
4. *for every* $\bar{x}^2 \in \bar{X}^2,\, z^1 \in Z^1$ *a finite* $E_{F^{\xi^2|\xi^1=z^1}} f_0^1(\bar{x}^2,\, \bar{\xi}^2(\omega))$ *exists and moreover for every* $z^1 \in Z^1,\, x^1 \in X^1$,
$$\inf\{E_{F^{\xi^2|\xi^1=z^1}} f_0^1(\bar{x}^2,\, \bar{\xi}^2(\omega)) | x^2 \in \mathcal{K}(x^1,\, z^1)\} > -\infty,$$
5. *a finite* $E_{F^{\xi^1}}\, \{\bar{L}_0(\xi^1(\omega)) + E_{F^{\xi^2|\xi^1}}\, \bar{L}_{f_0^1}(\bar{\xi}^2(\omega))\, (C \sum\limits_{i=1}^{l} \bar{L}_i(\xi^1(\omega)) + 1)\}$ *exists,*

then $E_{F^{\xi^1}} Q(x^1,\, \xi^1(\omega))$ *is a Lipschitz function on* X^1 *with the Lipschitz constant*

$$E_{F^{\xi^1}}\, \{\bar{L}_0(\xi^1(\omega)) + E_{F^{\xi^2|\xi^1}}\, \bar{L}_{f_0^1}(\bar{\xi}^2(\omega))\, (C \sum_{i=1}^{l} \bar{L}_i(\xi^1(\omega)) + 1)\}.$$

Proof. First, it follows from the assumptions 3, 4 of Theorem 2 and from the properties of the integral that for every $z^1 \in Z^1$, $E_{F^{\xi^2|\xi^1=z^1}} f_0^1(\bar{x}^2,\, \bar{\xi}^2(\omega))$ is a Lipschitz function on $\bar{X}^2$ with the Lipshitz constant $E_{F^{\xi^2|\xi^1=z^1}}\, \bar{L}_{f_0^1}(\bar{\xi}^2(\omega))$.

Since, futhermore, it follows from the assumptions 1 and 2 of Theorem 2, from the assertion of Lemma 3 and from the elementary properties of the metric that

$$\Delta[\mathcal{K}(x^1(1),\, z^1),\, \mathcal{K}(x^1(2),\, z^1)] \le C \sum_{i=1}^{l} \bar{L}_i(z^1) \| x^1(1) - x^1(2) \| \tag{9}$$

for every $x^1(1),\, x^1(2) \in X^1,\, z^1 \in Z^1$, we can (according to Lemma 2 and the assumptions of Theorem 2) see that for every $z^1 \in Z^1$

$$\inf\{E_{F^{\xi^2|\xi^1=z^1}}\, f_0^1(\bar{x}^2,\, \bar{\xi}^2(\omega)) |\, x^2 \in \mathcal{K}(x^1,\, z^1)\} \tag{10}$$

is a real–valued, Lipschitz function on X^1 with the Lipschitz constant

$$E_{F^{\xi^2|\xi^1=z^1}} \bar{L}_{f_0^1}(\bar{\xi}^2(\omega))(C \sum_{i=1}^{l} \bar{L}_i(z^1(\omega)) + 1). \tag{11}$$

Combining this fact with the assumptions 2, 5 of Theorem 2 and the properties of integral we obtain the assertion of Theorem 2. □

Obviously, the Lipschitz property of the function $Q(x^1, z^1)$ with respect to $x^1 \in X^1$ can be suitable for a numerical treatment. However, mainly, this property vouches (under rather general assumptions) the exponential rate of convergence of empirical estimates [9].

To investigate the stability, generally, with respect to the distribution functions spase (equipped with the Kolmogorov metric) the Lipschitz property of the function $Q(x^1, z^1)$ with respect to $z^1 \in Z^1$ is appropriate [8]. However, the situation is in this case more comlicated. Consequently, to investigate the Lipschitz property of the function $Q(x^1, z^1)$ with respect to $z^1 \in Z^1$ it is suitable to consider the case when $\xi^1(\omega), \xi^2(\omega)$ are stochastically independent and the general case, separately. Of course, first we shall deal with the independent case.

Theorem 3 *If*

1. *at least one of the assumptions (i.1), (i.2), (i.3) is fulfilled,*
2. *for every* $x^1 \in X^1$, $h_i(x^1, z^1), i = 0, 1, \ldots, l$ *are Lipschitz functions on* Z^1 *with the Lipschitz constants* $L_i(x^1)$,
3. *for every* $x^1 \in X^1, z^2 \in Z^2$, $f_0^1(\bar{x}^2, \bar{z}^2)$ *is a Lipschitz function on* $X^2 \times Z^1$ *with the Lipschitz constant* $L_{f_0^1}(x^1, z^2)$ *such that for every* $x^1 \in X^1$ *a finite* $E_{F^{\xi^2}} L_{f_0^1}(x^1, \xi^2(\omega))$ *exists,*
4. *for every* $\bar{x}^2 \in \bar{X}^2, z^1 \in Z^1$ *a finite* $E_{F^{\xi^2}} f_0^1(\bar{x}^2, \bar{\xi}^2(\omega))_{|\xi^1=z^1}$ *exists and moreover for every* $z^1 \in Z^1, x^1 \in X^1$,
$$\inf\{E_{F^{\xi^2}} f_0^1(\bar{x}^2, \bar{\xi}^2(\omega))_{|\xi^1=z^1} | x^2 \in \mathcal{K}(x^1, z^1)\} > -\infty,$$
5. $\xi^1(\omega), \xi^2(\omega)$ *are stochastically independent,*

then for every $x^1 \in X^1$, $Q(x^1, z^1)$ *is a Lipschitz function on* Z^1 *with the Lipschitz constant*

$$L_0(x^1) + E_{F^{\xi^2}} L_{f_0^1}(x^1, \xi^2(\omega))\,(C \sum_{i=1}^{l} L_i(x^1) + 1).$$

Proof. First, it follows from the assumptions, from the assertions of Lemma 3 and from the elementary properties of the Euclidean norm that

$$\Delta[\mathcal{K}(x^1,\, z^1(1)),\, \mathcal{K}(x^1,\, z^1(2))] \leq C \sum_{i=1}^{l} L_i(x^1)\|z^1(1) - z^1(2)\|$$

for every $x^1 \in X^1,\ z^1(1),\, z^1(2) \in Z^1$.

Furthermore, it follows from the integral theory that for every $x^1 \in X^1,\, z^1 \in Z^1,\ E_{F^{\xi^2|\xi^1=z^1}} f_0^1(\bar{x}^2,\, \bar{\xi}^2(\omega)) = E_{F^{\xi^2}} f_0^1(\bar{x}^2,\, \bar{\xi}^2(\omega))_{|\xi^1(\omega)=z^1}$ is a Lipschitz function on $X^2 \times Z^1$ with the Lipschitz constant $E_{F^{\xi^2}} L_{f_0^1}(x^1,\, \xi^2(\omega))$. Consequently employing the assertion of Lemma 2 and the assumptions of Theorem 3 we can see that the assertion of Theorem 3 is valid. □

In the life situations many problems develop in time and consequently also the corresponding random elements are stochastically connected each other. Consequently, it is necessary also to consider $\xi^1(\omega),\, \xi^2(\omega)$ stochastically dependent. To this end, let $\hat{Z}^2 \subset E_{s_2}$ be arbitrary such that

$$P\{\omega : \xi^2(\omega) \in \hat{Z}^2\} = 1, \quad .$$

Theorem 4 *Let $\hat{Z}^2$ be a compact set. If*

1. *at least one of the assumptions (i.1), (i.2), (i.3) is fulfilled,*
2. *for every $x^1 \in X^1,\ h_i(x^1,\, z^1),\, i = 0,\, 1,\, \ldots,\, l$ are Lipschitz functions on Z^1 with the Lipschitz constants $L_i(x^1)$,*
3. *for every $x^1 \in X^1,\, z^2 \in \hat{Z}^2,\ f_0^1(\bar{x}^2,\, \bar{z}^2)$ is a Lipschitz function on $X^2 \times Z^1$ with the Lipschitz constant $L_{f_0^1}(x^1)$ not depending on $z^2 \in Z^2$,*
4. *for every $z^1 \in Z^1,\, x^1 \in X^1$,*
$$\inf\{E_{F^{\xi^2|\xi^1=z^1}} f_0^1(\bar{x}^2,\, \bar{\xi}^2(\omega)) | x^2 \in \mathcal{K}(x^1,\, z^1)\} > -\infty,$$
5. *for every $z^1 \in Z^1$ the probability measure corresponding to $F^{\xi^2|\xi^1}(z^2|z^1)$ is absolutely continuous with respect to the Lebesgue measure in E_{s_2} and moreover*
   a. *for every $z^2 \in \hat{Z}^2,\ \bar{f}(z^2|z^1)$ is a Lipschitz function on Z^1 with the Lipschitz constant $L_{\bar{f}}$ not depending on $z^2 \in \hat{Z}^2$,*
   b. *$|\bar{f}(z^2|z^1)| \leq M_{\bar{f}}$ for every $z^1 \in Z^1,\, z^2 \in Z^2$,*
6. *$|f_0^1(\bar{x}^2,\, \bar{z}^2)| \leq M_{f_0^1}(x^1)$ for every $\bar{x}^2 \in \bar{X}^2,\, \bar{z}^2 \in \bar{Z}^2,\, \bar{x}^2 = (x^1,\, x^2)$,*

then for every $x^1 \in X^1$, $Q(x^1, z^1)$ *is a Lipschitz function on* Z^1 *with the Lipschitz constant*

$$L_0(x^1) + (L_{f_0^1}(x^1)\, M_{\bar{f}} + L_{\bar{f}}\, M_{f_0^1}(x^1))\ \mu(Z^2)\, (C \sum_{i=1}^{l} L_i(x^1) + 1).$$

($\bar{f}(z^2|z^1)$ *denotes the probability density corresponding to* $F^{\xi^2|\xi^1}(z^2|z^1)$*, the symbol* $\mu(\cdot) = \mu_{s_2}(\cdot)$ *is reserved for the Lebesgue measure in* E_{s_2}*.)*

Proof. It follows from the assumptions of Theorem 4, from the assertion of Lemma 3 and from the elementary properties of the Euclidean norm that

$$\Delta[\mathcal{K}(x^1,\, z^1(1)),\, \mathcal{K}(x^1,\, z^1(2))] \le C \sum_{i=1}^{l} L_i(x^1) \|z^1(1) - z^1(2)\| \tag{12}$$

for every $x^1 \in X^1$, $z^1(1), z^1(2) \in Z^1$.

Since $\hat{Z}^2$ is a compact set, we can see that for every $\bar{x}^2 \in \bar{X}^2$, $z^1 \in Z^1$ a finite $E_{F^{\xi^2|\xi^1 = z^1}} f_0^1(\bar{x}^2,\, \bar{\xi}^2(\omega))$ exists. Futhermore, according to the assumptions 2 and 3 of Theorem 4 we obtain for $x^1 \in X^1$, $x^2(1), x^2(2) \in X^2$, $z^1(1), z^1(2) \in Z^1$ arbitrary, $\bar{x}^2(1) = (x^1,\, x^2(1))$, $\bar{x}^2(2) = (x^1,\, x^2(2))$ successively the following system of inequlities

$$|\, E_{F^{\xi^2|\xi^1 = z^1(1)}}\, f_0^1(\bar{x}^2(1),\, \bar{\xi}^2(\omega)) - E_{F^{\xi^2|\xi^1 = z^1(2)}} f_0^1(\bar{x}^2(2),\, \bar{\xi}^2(\omega))\, | =$$

$$\int_{E_{s_2}} f_0^1(\bar{x}^2(1),\, (z^1(1),\, z^2)) dF^{\xi^2|\xi^1 = z^1(1)}(z^2|z^1(1)) -$$

$$\int_{E_{s_2}} f_0^1(\bar{x}^2(2),\, (z^1(2),\, z^2))\, dF^{\xi^2|\xi^1 = z^1(2)}(z^2|z^1(2))| =$$

$$|\int_{E_{s_2}} f_0^1(\bar{x}^2(1),\, (z^1(1),\, z^2)) \bar{f}(z^2|z^1(1))\, dz^2 -$$
$$\int_{E_{s_2}} f_0^1(\bar{x}^2(2),\, (z^1(2),\, z^2)) \bar{f}(z^2|z^1(2))\, dz^2| \le$$
$$\int_{E_{s_2}} |f_0^1(\bar{x}^2(1),\, (z^1(1),\, z^2)) \bar{f}(z^2|z^1(1)) - f_0^1(\bar{x}^2(1),\, (z^1(1),\, z^2)) \bar{f}(z^2|z^1(2))|\, dz^2 +$$

$$\int_{E_{s_2}} |f_0^1(\bar{x}^2(1),\, (z^1(1),\, z^2)) \bar{f}(z^2|z^1(2)) - f_0^1(\bar{x}^2(2),\, (z^1(2),\, z^2)) \bar{f}(z^2|z^1(2))|\, dz^2 =$$

$$\int\limits_{E_{s_2}} |f_0^1(\bar{x}^2(1),\,(z^1(1),\,z^2))|\,|\bar{f}(z^2|z^1(1)) - \bar{f}(z^2|z^1(2))|\,dz^2 +$$

$$\int\limits_{E_{s_2}} |\bar{f}(z^2|z^1(2))|\,|f_0^1(\bar{x}^2(1),\,(z^1(1),\,z^2)) - f_0^1(\bar{x}^2(2),\,(z^1(2),\,z^2))|\,dz^2 \le$$

$$(M_{f_0^1}(x^1)L_{\bar{f}} + L_{f_0^1}(x^1)M_{\bar{f}}) \int\limits_{Z^2} \|(x^2(1),\,z^1(1)) - (x^2(2),\,z^1(2))\|\,dz^2.$$

Consequently we can see that for every $x^1 \in X^1$, $E_{F^{\xi^2|\xi^1=z^1}} f_0^1(\bar{x}^2(1),\,\bar{\xi}^2(\omega))$ is a Lipschitz function on $X^2 \times Z^1$ with the Lipschitz constant

$$(L_{f_0^1}(x^1)\,M_{\bar{f}} + L_{\bar{f}}\,M_{f_0^1}(x^1))\,\mu(Z^2). \tag{13}$$

Combining now the relations (12), (13), the assertion of Lemma 2 and the assumptions of Theorem 4 we can see that the assertion of Theorem 4 holds. □

To illustrate the application of Theorem 4, we consider the well–known simple recourse problem (for definition see e.g. [4]) with compact $\bar{X}^2$, $Z^1 \times \hat{Z}^2$, "positive" random elements $\xi_1^1(\omega),\,\xi_2^1(\omega),\,\ldots,\xi_{s_1}^1(\omega)$ on the left–hand side of the constraints (in the underlying problems), the additional random elements $q_1^+(\omega),\,\ldots,q_{s_1}^+(\omega)$ and deterministic positiv $q_1^-,\,q_2^-,\,\ldots,\,q_{s_1}^-$ coefficients. If $q_j^+(\omega) = \xi_j^1(\omega) + \eta_j(\omega)$ (where $\eta_j(\omega)$, $j = 1,\,2,\,\ldots,\,s_1$ are random values with exponentional distributions restricted to compact interval $\langle 0,\,A\rangle$, $A > 0$, $Z^1 + \langle 0,\,A\rangle \subset \hat{Z}^2$), then the assumptions of Theorem 4 are fulfilled. However, if ($\eta_j(\omega)$, $j = 1,\,2,\,\ldots,\,s_1$ are exponentially distributed, then the assumptions are not fulfilled.

The assertions of Theorems 2, 3, 4 introduced the assumptions under which the objective functions of the problem (1) is Lipschitz. The corresponding proofs are nearly straightforwwad, but just these assertions enable to apply many of the one–stage stochastic programming results to the problems introduced by (1), (2) and (3) (see e.g. [8], [9]) and, consequently also to the multistage case.

4.3 Existence of Partial Derivatives

It is well–known that the existence of the partial derivatives vector of the objective function of any optimization problem is very important from the numerical point of view as well as for the sensitivity analysis. In the literature this problem has been already discussed in the case of two–stage stochastic models. We recall here [4], [6], [7]. In this part of the paper we shall try to present some new results in this direction. To this end we restrict our consideration to the case when

$$f_0^1(\bar{x}^2,\,\bar{z}^2) =: \bar{f}_0^1(x^2,\,z^2), \tag{14}$$

where $\bar{f}_0^1(x^2, z^2)$ is a function defined on $E_{n_2} \times E_{s_2}$. We denote

$$\varphi(y) = \inf\{E_{F^{\xi^2}} \bar{f}_0^1(x^2, \xi^2(\omega)) | x^2 \in \bar{\mathcal{K}}(y)\}, \quad y \in Y, \tag{15}$$

$$h(x^1, z^1) = [h_1(x^1, z^1), h_2(x^1, z^1), \ldots, h_l(x^1, z^1)], x^1 \in X^1, z^1 \in Z^1. \tag{16}$$

Furthermore, we introduce the following system of assumptions.

D.1. for every $x^1 \in X^1$ the probability measure corresponding to the random vector $[h_1(x^1, \xi^1(\omega)), h_2(x^1, \xi^1(\omega)), \ldots, h_l(x^1, \xi^1(\omega))]$ is absolutely continuous with respect to the Lebesgue measure in E_l,

D.2. for every $z^1 \in Z^1$, $h_i(x^1, z^1), i = 0, 1, \ldots, l$ are differentiable functions on X^1,

D.3. $\xi^1(\omega), \xi^2(\omega)$ are stochastically independent.

Theorem 5 *If*

1. *the assumptions D.1, D.2 and D.3 are fulfilled,*
2. *at least one of the assumption (i.1), (i.2), (i.3) is fulfilled,*
3. *for every $z^1 \in Z^1$, $h_i(x^1, z^1)$, $i = 0, 1, \ldots, l$ are Lipschitz function on X^1 with the Lipschitz constant $L_i(z^1)$,*
4. *for every $x^2 \in X^2$ a finite $E_{F^{\xi^2}} \bar{f}_0^1(x^2, \xi^2(\omega))$ exists and moreover $\varphi(y) > -\infty$ for every $y \in Y$,*
5. *for every $z^2 \in Z^2$, $\bar{f}_0^1(x^2, z^2)$ is a Lipschitz function on X^2 with the Lipschitz constant $L_{\bar{f}_0^1}(z^2)$ such that a finite $E_{F^{\xi^2}} L_{\bar{f}_0^1}(\xi^2(\omega))$ exists,*
6. *for every $x^1 \in X^1$ finite $E_{F^{\xi^1}} \varphi(h(x^1, \xi^1(\omega)))$, $E_{F^{\xi^1}} h_0(x^1, \xi^1(\omega))$, $E_{F^{\xi^1}} L_i(\xi^1)$, $i = 0, 1, \ldots, l$ exist,*

then for every $x^1 \in X^1$ (inner point) there exist

$$\frac{\partial E_{F^{\xi^1}} Q(x^1, \xi^1(\omega))}{\partial x_j^1}, \quad j = 1, 2, \ldots, n_1.$$

If moreover

7. *X^2 is a convex set, $\bar{\mathcal{K}}(y)$ are nonempty sets for every $y \in Y_0(\varepsilon)$ and some $\varepsilon > 0$,*

8. *for every $z^2 \in Z^2$, $\bar{f}_0^1(x^2, z^2)$, $f_i(x^2)$, $i = 1, 2, \ldots, l$ are convex functions on X^2,*

then for every $j \in \{1, 2, \ldots, n_1\}$

$$\frac{\partial E_{F^{\xi^1}} Q(x^1, \xi^1(\omega))}{\partial x_j^1} = \frac{\partial E_{F^{\xi^1}} h_0(x^1, \xi^1(\omega))}{\partial x_j^1} + E_{F^{\xi^1}} \{ \sum_{i=1}^{l} v_j(h(x^1, \xi^1(\omega))) \frac{\partial h_i(x^1, \xi^1(\omega))}{\partial x_j^1} \}, \tag{17}$$

where $v = v(y) = [v_1(y), v_2(y), \ldots, v_l(y)]$ is the Kuhn–Tucker vector (for definition see [7]) of the parametric optimization problem:
Find

$$\inf\{E_{F^{\xi^2}} f_0^1(x^2, \xi^2(\omega)) | x^2 \in \bar{\mathcal{K}}(y)\}.$$

Proof. Let $x^1(0) \in X^1$ be an arbitrary inner point. It follows from the assumptions 1, 5 of this Theorem and from the elementary properties of the integral that for every $z^1 \in Z^1$, $E_{F^{\xi^2}} \bar{f}_0^1(x^2, \xi^2(\omega))$ is a Lipschitz function on X^2 with the Lipschitz constant $E_{F^{\xi^2}} L_{\bar{f}_0^1}(\xi^2(\omega))$ not depending on $x^1 \in X^1$, $z^1 \in Z^1$.
By the technique employed in the proof of Theorems 2, 3, and 4 we can see that $\varphi(y)$ is a Lipschitz function on Y with the Lipschitz constant $E_{F^{\xi^2}} L_{\bar{f}_0^1}(\xi^2(\omega))(C +1)$ and simultanously for every $z^1 \in Z^1$, $\varphi(h(x^1, z^1))$ is a Lipschitz function on X^1 with the Lipschitz constant $E_{F^{\xi^2}} L_{\bar{f}_0^1}(\xi^2(\omega))(C \sum_{i=1}^{l} L_i(z^1) + 1)$.
It follows from the theory of the Lipschitz functions [2] (Rademacher's theorem) that there exists a set $R \subset Y(\varepsilon)$ (for a $\varepsilon > 0$) of the corresponding Lebesgue measure 0 such that a vector of the partial derivatives of the function $\varphi(y)$ exists on $Y - R$. If we denote $Z^1(x^1(0)) = \{z^1 \in Z^1 : h(x^1(0), z^1) \in R\}$, then according to the assumption D.1 we can see that $P_{F^{\xi^1}}\{\omega : \xi^1(\omega) \in Z^1(x^1(0))\} = 0$. Consequently, according to D.2 there exist

$$\frac{\partial \varphi(h(x^1, \xi^1(\omega)))}{\partial x_j^1}|_{x^1 = x^1(0)}, \quad j = 1, 2, \ldots, l \quad \text{a.s.}, \quad x^1 = (x_1^1, \ldots, x_{n_1}^1).$$

Employing the assumption 3, the Lipschitz property of $\varphi(h(x^1, z^1))$ and the Lebesgue limit theorem we can see that the first assertion of Theorem 5 holds.

Since it follows from the assumptions of Theorem 5 and from the assertion of Lemma 1 that $E_{F^{\xi^2}} \bar{f}_0^1(x^2, \xi^2(\omega))$ is a convex function on X^2 the validity of the relation (17) follows from Theorem in [7].

($P_{F^{\xi^1}}(\cdot)$ denotes the probability measure corresponding $F^{\xi^1}(\cdot)$.)

4.4 Discussion

The aim of the paper was to introduce the assumptions under which the objective function (connected to the outer problem) in two–stage stochastic programming problem (defined by the relation (1), (2) and (3)) is convex, Lipschitz and differentiable. In particular, the presented results concern the case when the additional random element (connected to the inner problem) appears in the definition of the two–stage stochastic programs. A purpose of this generalization (in the definition of two–stage stochastic programming problems) is in the potential application of the achieved results to the multistage stochastic nonlinear programs that seems to be very useful. Namely, the decomposition of the multistage stochastic programming problem (with respect to the individual time points) lead to (parametric) problems of the type just introduced by the relation (1), (2) and (3) [10]. Consequently, it seems to be possible to utilize some of the results (on stability and estimates) connected to the one–stage to the multistage case. The classical two–stage case was (from this point of view) discussed in [8].

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