Stability radii of positive discrete-time systems

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Abstract

In this note we study stability radii of positive linear discrete-time systems under real or complex parameter perturbations. A simple formula for the real stability radius is established and it is shown that the real and the complex stability radii of positive systems coincide. These results are derived for arbitrary perturbation norms induced by monotonic vector norms (e.g. p-norms, $1 \leq p \leq \infty$). Nonlinear and time-varying perturbations are also considered.

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1 Introduction

In recent years, considerable attention has been devoted to the study of positive systems, i.e. dynamical systems whose input, state and output variables are restricted to be nonnegative. In the linear discrete-time case such systems are described by equations of the form

$$x(t+1) = Ax(t) + Bu(t), \qquad t \in \mathbb{N} = \{0, 1, 2, ...\} \tag{1}$$

$$y(t) = Cx(t), \tag{2}$$

where A, B, C are matrices with nonnegative entries. Dynamical models of this kind play a role in various areas of application, e.g. in economics, population dynamics and biology, see [9], [1].

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Controllability and stabilizability properties of positive linear systems have been studied e.g. in [10], [12]. While questions of controllability and stabilizability are complicated by the positivity constraint, the analysis of stability properties is simplified. This will be illustrated by the quantitative results on robust stability presented in this note.
To our knowledge only a small number of papers have dealt with robustness issues for positive systems, we mention [13] and [2]. These papers follow a Kharitonov type approach and study the stability of intervals of nonnegative matrices. Here we apply a stability radius approach [4], [6]. More precisely, we determine the maximal $\rho > 0$ for which the family of systems

$$x(t+1) = (A + D\Delta E)x(t), \quad \|\Delta\| < \rho \tag{3}$$

is asymptotically stable. D and E are given nonnegative matrices specifying the structure of the pertubation, Δ is an unknown disturbance matrix and $\|\cdot\|$ a given operator norm. If complex perturbations are allowed the maximal number ρ is called the *complex stability radius* and denoted by $r_{\mathbb{C}} = r_{\mathbb{C}}(A; D, E)$. If only real perturbations are considered the *real stability radius* $r_{\mathbb{R}} = r_{\mathbb{R}}(A; D, E)$ is obtained. These two stability radii are in general distinct.
By definition the determination of $r_{\mathbb{C}}$ and $r_{\mathbb{R}}$ amounts to a *norm minimization problem* on the closed set of destabilizing matrices in $\mathbb{C}^{l\times q}$ and $\mathbb{R}^{l\times q}$, respectively. The problem is more easily solved in the complex case. A detailed analysis of the complex stability radius of discrete-time systems can be found in [7], an efficient algorithm for the computation of $r_{\mathbb{C}}(A; D, E)$ is described in [5] and the difference between the two stability radii is discussed in [4]. Only recently, a general formula for the real stability radius has been found by Qiu et al. [11]. The computation of $r_{\mathbb{R}}(A; D, E)$ requires the solution of a complicated global optimization problem.
Fortunately, the situation is much simpler for positive systems. The main result of this note is that the real and complex stability radii coincide for positive systems and that they can be determined via an easily computable formula (Theorem 3.4).
We proceed as follows. In the next section we recall some theorems on nonnegative matrices and derive preliminary results for later use. The main result will be presented in Section 3. In the final Section 4 it will be applied to investigate the robustness of stability of a positive discrete-time system under nonlinear and time-varying perturbations.

2 Preliminary results

In this section we present some properties of nonnegative matrices which will be used in the sequel.
We first introduce some notations. Let n, l, q be positive integers. A matrix $P = [p_{ij}] \in \mathbb{R}^{l\times q}$ is said to be *nonnegative* ($P \geq 0$) if all its entries p_{ij} are

nonnegative. For $P, Q \in \mathbb{R}^{l\times q}$, $P \geq Q$ means that $P - Q \geq 0$. The set of all nonnegative $l \times q$-matrices is denoted by $\mathbb{R}_+^{l\times q}$. A similar notation will be used for nonnegative vectors. Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, then for any $x \in \mathbb{K}^n$ and $P \in \mathbb{K}^{l\times q}$ we define $|x| \in \mathbb{R}_+^n$ and $|P| \in \mathbb{R}_+^{l\times q}$ by

$$|x| = (|x_i|), \quad |P| = (|p_{ij}|). \tag{4}$$

For any matrix $A \in \mathbb{K}^{n\times n}$ the spectral radius of A is defined by

$$\rho(A) = \max\{|\lambda|; \lambda \in \sigma(A)\}, \tag{5}$$

where $\sigma(A) \subset \mathbb{C}$ is the spectrum of A.
A norm $\|\cdot\|$ on $\mathbb{K}^n$ is said to be *monotonic* if it satisfies

$$|x| \leq |y| \;\Rightarrow\; \|x\| \leq \|y\| \tag{6}$$

for all $x, y \in \mathbb{K}^n$. It can be shown that a vector norm $\|\cdot\|$ on $\mathbb{K}^n$ is monotonic if and only if $\|x\| = \|\,|x|\,\|$ for all $x \in \mathbb{K}^n$ (see e.g. [8]). It follows that every p-norm on $\mathbb{K}^n, 1 \leq p \leq \infty$, is monotonic. Recall that the p-norms on $\mathbb{K}^n$ are defined by

$$\|x\|_p = \begin{cases} (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}, & \text{if} \quad 1 \leq p < \infty \\ \max_{1\leq i\leq n} |x_i|, & \text{if} \quad p = \infty. \end{cases}$$

Throughout the paper $\|P\|$ denotes the operator norm of $P \in \mathbb{K}^{l\times q}$ induced by a given pair of norms on $\mathbb{K}^q$ and $\mathbb{K}^l$:

$$\|P\| = \max_{\|y\|=1} \|Py\|.$$

The operator norm of P with respect to the p-norm on $\mathbb{K}^q$ and $\mathbb{K}^l$ is denoted by $\|P\|_p$.
The following simple property of nonnegative matrices will be useful in the proof of our main result.

Lemma 2.1 *Suppose that $P \in \mathbb{R}_+^{l\times q}$ and $\|\cdot\|$ denotes the operator norm with respect to monotonic norms on $\mathbb{R}^l$, $\mathbb{R}^q$. Then there exists $e \in \mathbb{R}_+^q$ of norm $\|e\| = 1$ such that*

$$\|Pe\| = \|P\|. \tag{7}$$

Proof: By definition, there exists $y \in \mathbb{R}^q, \|y\| = 1$ such that $\|Py\| = \|P\|$. Set $e = |y| \in \mathbb{R}_+^q$. Then $\|e\| = 1$ and $\|Pe\| \geq \|Py\|$ by (6), hence

$$\|P\| \geq \|Pe\| \geq \|Py\| = \|P\|$$

which implies (7). □

Note that an operator norm induced by monotonic vector norms is not necessarily monotonic, i.e. for $P, Q \in \mathbb{K}^{l\times q}$, the implication

$$|P| \leq |Q| \quad \Rightarrow \quad \|P\| \leq \|Q\| \tag{8}$$

may *not* hold for such an operator norm. This implication holds, however, for *nonnegative* matrices. In fact, we have

Lemma 2.2 *Suppose that $\mathbb{K}^l, \mathbb{K}^q$ are provided with monotonic norms and $\|\cdot\|$ denotes the corresponding operator norm. If $P, Q \in \mathbb{K}^{l\times q}$ and $|P| \leq Q$, then*

$$\|P\| \leq \|\,|P|\,\| \leq \|Q\|. \tag{9}$$

Proof: We have, for any $y \in \mathbb{K}^q$, $\|Py\| = \|\,|Py|\,\| \leq \|\,|P|\,|y|\,\| \leq \|Q\,|y|\,\| \leq \|Q\|\,\|y\|$, which implies (9). □

In particular, we obtain for arbitrary matrices $P, Q \in \mathbb{R}^{l\times q}$,

$$0 \leq P \leq Q \;\Rightarrow\; \|P\| \leq \|Q\| \tag{10}$$

provided that $\mathbb{R}^l, \mathbb{R}^q$ are endowed with monotonic norms. Moreover, if the vector norms on $\mathbb{K}^l, \mathbb{K}^q$ are p-norms with $p = 1$ or ∞ then the left inequality in (9) is an equality. This holds true for arbitrary monotonic norms if the matrix under consideration is of rank one.

Lemma 2.3 *Let $\mathbb{K}^q, \mathbb{K}^l$ be provided with monotonic norms and $P \in \mathbb{K}^{l\times q}$ be of rank one. Then*

$$\|P\| = \|\,|P|\,\|.$$

Proof: There exist $y \in \mathbb{K}^q$ and $v \in \mathbb{K}^l$, $\|v\| = 1$ such that

$$Pz = \hat{y}(z)v, \quad z \in \mathbb{K}^q$$

where $\hat{y} : \mathbb{K}^q \;\to\; \mathbb{K}, \hat{y}(z) = \sum_{j=1}^q z_j y_j$, is the linear form on $\mathbb{K}^q$ associated with y. Let $w = |y|$ so that $\hat{w}(z) = \sum_{j=1}^q z_j\,|y_j|$, $z \in \mathbb{K}^q$. Then

$$|P|z = |v|\hat{w}(z), \qquad z \in \mathbb{K}^q$$

and

$$\|P\| = \|\hat{y}\|, \qquad \|\,|P|\,\| = \|\hat{w}\|$$

where $\|\hat{y}\|, \|\hat{w}\|$ are the operator norms of the linear forms $\hat{y}, \hat{w}$, respectively. Thus it only remains to prove $\|\hat{y}\| = \|\hat{w}\|$. Since

$$|\hat{y}(z)| \leq \sum_{j=1}^{q} |z_j|\,|y_j| = \hat{w}(|z|) \leq \|\hat{w}\|\,\|z\|, \quad z \in \mathbb{K}^q$$

we have $\|\hat{y}\| \leq \|\hat{w}\|$. Conversely, by Lemma 2.1, there exists $z \in I\!R^q_+$ such that $\|z\| = 1$ and $\hat{w}(z) = \|\hat{w}\|$. Let $y_j = e^{i\varphi_j}|y_j|$ and define $\tilde{z}_j = z_j e^{-i\varphi_j}$. Then $\|\tilde{z}\| = \|\,|\tilde{z}|\,\| = 1$ and

$$\hat{y}(\tilde{z}) = \sum_{j=1}^{q} z_j y_j e^{-i\varphi_j} = \hat{w}(z) = \|\hat{w}\|,$$

hence $\|\hat{y}\| = \|\hat{w}\|$. □

The next two theorems are standard results from the literature on nonnegative matrices, see e.g. [9].

Theorem 2.4 *Let $A, B \in I\!R^{n\times n}_+$, $t \in I\!R$. Then*

(a) $A \leq B$ implies $\rho(A) \leq \rho(B)$;

(b) Given $\alpha > 0$, there exists a nonzero vector $x \geq 0$ such that $Ax \geq \alpha x$ if and only if $\rho(A) \geq \alpha$;

(c) $(tI - A)^{-1}$ exists and is nonnegative if and only if $t > \rho(A)$.

As a corollary we obtain the following inequalities analogous to (9) for the spectral radius of matrices.

Corollary 2.5 *Let $A, B \in I\!K^{n\times n}$. If $|A| \leq B$ then*

$$\rho(A) \leq \rho(|A|) \leq \rho(B). \tag{11}$$

Proof: Only the left inequality remains to be proved. Suppose $\lambda = \rho(A)e^{i\varphi}$, $\varphi \in [0, 2\pi]$, is an eigenvalue of A and $Ax = \lambda x$ with $x \in I\!K^n, x \neq 0$. Then

$$|A||x| \geq |Ax| = \rho(A)|x|$$

and $\rho(|A|) \geq \rho(A)$ follows from Theorem 2.4(b). □

Finally we recall the Perron-Frobenius Theorem.

Theorem 2.6 *Suppose that $A \in I\!R^{n\times n}_+$. Then*

(a) $\rho(A)$ is an eigenvalue of A and there exists a nonnegative eigenvector $x \geq 0$, $x \neq 0$ such that $Ax = \rho(A)x$.

(b) If $\lambda \in \sigma(A)$ and $|\lambda| = \rho(A)$ then the algebraic multiplicity of λ is not greater than the algebraic multiplicity of the eigenvalue $\rho(A)$.

3 Real and complex stability radii

Consider a dynamical system described by the following linear difference equation

$$x(t+1) = Ax(t), \quad t \in \mathbb{N} \tag{12}$$

where $A \in \mathbb{R}_+^{n\times n}$. We assume that the system (12) is *Schur stable*, i.e. the spectrum $\sigma(A)$ of A lies in the open unit disk $\mathbb{C}_1 = \{s \in \mathbb{C}; |s| < 1\}$ or, equivalently, $\rho(A) < 1$.

Since a dynamical model is never an exact portrait of the real process, it is important to investigate the robustness of the stable nominal system (12) under various classes of perturbations. In this section we will determine to which extent the stability of (12) is preserved under parameter perturbations of *linear output feedback type*

$$A \rightsquigarrow A + D\Delta E. \tag{13}$$

Here $D \in \mathbb{R}_+^{n\times l}$ and $E \in \mathbb{R}_+^{q\times n}$ are given nonnegative matrices defining the structure of the perturbations and $\Delta \in \mathbb{K}^{l\times q}$ is an unknown (real or complex) disturbance matrix. The assumption that the perturbations under consideration are of positive structure (D, E) is quite natural for positive systems and is not too restrictive since we do not restrict the disturbance matrix Δ to be nonnegative. The perturbations (12) include unstructured perturbations (for which $D = E = I_n$) as well as all perturbations which affect individual elements, rows and columns of A as particular cases.

Let us first recall the notion of stability radius [4]. Given a pair of norms on $\mathbb{K}^q$, $\mathbb{K}^l$, the *stability radius* of a matrix $A \in \mathbb{K}^{n\times n}$ with respect to perturbations of the structure $(D, E) \in \mathbb{K}^{n\times l} \times \mathbb{K}^{q\times n}$ is defined by

$$r_{\mathbb{K}} = r_{\mathbb{K}}(A; D, E) = \inf\{\|\Delta\|; \Delta \in \mathbb{K}^{l\times q}, \rho(A + D\Delta E) \geq 1\} \tag{14}$$

where $\|\Delta\|$ denotes the operator norm of $\Delta \in \mathbb{K}^{l\times q}$ with respect to the given norms on $\mathbb{K}^q$, $\mathbb{K}^l$ and, by definition, $\inf \emptyset = \infty$. By continuity of the spectrum $r_{\mathbb{K}}(A; D, E) > 0$ if and only if $\sigma(A) \subset \mathbb{C}_1$. If A, D, E are nonnegative matrices then the *stability radius of A with respect to nonnegative perturbations of structure* (D, E) is defined by

$$r_{\mathbb{R}_+} = r_{\mathbb{R}_+}(A; D, E) = \inf\{\|\Delta\|; \Delta \in \mathbb{R}_+^{l\times q}, \rho(A + D\Delta E) \geq 1\}. \tag{15}$$

Since the function $\Delta \to \rho(A + B\Delta C)$ is continuous, the set

$$\mathcal{U} = \{\Delta \in \mathbb{R}_+^{l\times q}; \rho(A + D\Delta E) \geq 1\}$$

is closed. Therefore, if $\mathcal{U} \neq \emptyset$, there always exists $\Delta_0 \in \mathcal{U}$ such that $\|\Delta_0\| = r_{\mathbb{R}_+}(A; D, E)$. Such a matrix is called a *minimum norm nonnegative destabilizing perturbation.*

The following example shows that the distance of the spectrum of A from the unit circle is a misleading indicator of stability robustness, even in the unstructured case $(D = E = I_n)$.

Example 3.1 Consider

$$A = \begin{bmatrix} \frac{1}{4} & 9 \\ 0 & \frac{1}{4} \end{bmatrix} \quad \text{and} \quad \Delta = \begin{bmatrix} 0 & 0 \\ \frac{1}{16} & 0 \end{bmatrix}.$$

Then A is Schur stable with $\rho(A) = \frac{1}{4}$, but Δ is destabilizing ($\sigma(A+\Delta) = \{1, -1/2\}$) with norm $\|\Delta\|_2 = \frac{1}{16} < \frac{3}{4} = 1 - \rho(A)$. □

Given nonnegative matrices $(A; D, E) \in \mathbb{R}_+^{n\times n} \times \mathbb{R}_+^{n\times l} \times \mathbb{R}_+^{q\times n}$ we obtain three different stability radii via the definitions (14) ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and (15). $r_{\mathbb{C}}(A; D, E)$, $r_{\mathbb{R}}(A; D, E)$ and $r_{\mathbb{R}_+}(A; D, E)$ measure, respectively, the robustness of stability of A with respect to complex, real and nonnegative perturbations of the form $A \rightsquigarrow A + D\Delta E$. Obviously

$$r_{\mathbb{C}}(A; D, E) \leq r_{\mathbb{R}}(A; D, E) \leq r_{\mathbb{R}_+}(A; D, E). \tag{16}$$

At first sight it seems unnatural to consider complex perturbations of a positive system (i.e. the case $\mathbb{K} = \mathbb{C}$ in (14)). However, it has been shown in [6] that the *complex stability radius* $r_{\mathbb{C}}(A; D, E)$ remains invariant when more general (e.g. nonlinear, time-varying and/or dynamic) perturbations are considered instead of the disturbances (13). In fact it has been proved that the complex stability radius $r_{\mathbb{C}}(A; D, E)$ is equal to the stability radius with respect to *real* dynamic perturbations (ibid., Theorem 4.4). Therefore the complex stability radius is of interest, when nonlinear, time-varying or dynamic perturbations of the nominal system (12) have to be dealt with, cf. Section 4.
In general the complex and the real stability radii are different. It can be shown that, for every $n \geq 2$, there exist nonnegative structure matrices D, E such that the ratio $r_{\mathbb{R}}(A; D, E)/r_{\mathbb{C}}(A; D, E)$ is unbounded on the set of Schur stable matrices $A \in \mathbb{R}^{n\times n}$. We will now see that the situation is much simpler for nonnegative triplets. In fact, we will prove that all the three stability radii introduced above are equal for positive systems.

Lemma 3.2 *Suppose* $(A, D, E) \in \mathbb{R}_+^{n\times n} \times \mathbb{R}_+^{n\times l} \times \mathbb{R}_+^{q\times n}$, $\rho(A) < 1$ *and let*

$$G(s) = E(sI - A)^{-1}D \in \mathbb{R}(s)^{q\times l}, \quad s \in \mathbb{C} \setminus \sigma(A).$$

If $\mathbb{R}^l, \mathbb{R}^q$ *are provided with monotonic norms and* $\rho(A) < t_1 < t_2$ *then*

$$G(t_1) \geq G(t_2) \geq 0 \quad \textit{and} \quad \|G(t_1)\| \geq \|G(t_2)\| \geq 0. \tag{17}$$

Proof: Using the resolvent equation and Theorem 2.4 (c) we have, for $t_2 > t_1 > \rho(A)$,

$$(t_1 I - A)^{-1} - (t_2 I - A)^{-1} = (t_2 - t_1)(t_1 I - A)^{-1}(t_2 I - A)^{-1} \geq 0.$$

Thus (17) follows by means of (10). □

Lemma 3.3 *Suppose* $(A, D, E) \in I\!R_+^{n\times n} \times I\!R_+^{n\times l} \times I\!R_+^{q\times n}$, $\rho(A) < 1$ *and let* $G(s) = E(sI - A)^{-1}D$. *Then a matrix* $\Delta \in I\!R_+^{l\times q}$ *is destabilizing if and only if there exist a nonzero vector* $y \in I\!R_+^q$ *and* $t \geq 1$ *such that*

$$y = G(t)\Delta y. \tag{18}$$

Proof: If $\Delta \in I\!R_+^{l\times q}$ is destabilizing then $t := \rho(A + D\Delta E) \geq 1$. By Theorem 2.6 t is an eigenvalue of $A+D\Delta E$ and there exists an eigenvector $x \in I\!R_+^n$, $x \neq 0$, such that

$$(A + D\Delta E)x = tx. \tag{19}$$

Since $\rho(A) < 1$, $(tI - A)$ is invertible and hence (19) is equivalent to

$$x = (tI - A)^{-1}D\Delta Ex.$$

Multiplying this equation by E from the left we obtain (18) for $y = Ex \geq 0$, $y \neq 0$.
Conversely, if (18) holds for some $t \geq 1$ and $y \in I\!R^q, y \neq 0$ then by setting $x = (tI - A)^{-1}D\Delta y$ we get $Ex = y$ and hence $(A + D\Delta E)x = tx$. Thus $\rho(A + D\Delta E) \geq 1$ and Δ is destabilizing. □

We are now in a position to prove the main result of this note.

Theorem 3.4 *Suppose that* $(A, D, E) \in I\!R_+^{n\times n} \times I\!R_+^{n\times l} \times I\!R_+^{q\times n}$, $\rho(A) < 1$ *and* $I\!K^l, I\!K^q$ *are provided with monotonic norms,* $I\!K = I\!R$ *or* $\mathbb{C}$. *Then*

$$r_{\mathbb{C}}(A; D, E) = r_{I\!R}(A; D, E) = r_{I\!R_+}(A; D, E) = \|G(1)\|^{-1}, \tag{20}$$

where $G(s) = E(sI - A)^{-1}D$, $\|G(1)\|$ *is the operator norm of* $G(1) : I\!K^l \to I\!K^q$ *and, by definition,* $0^{-1} = \infty$.

Proof: Since

$$r_{\mathbb{C}}(A; D, E) = \left[\max_{|s|=1} \|G(s)\|\right]^{-1}$$

(see [4]) it suffices by (16) to prove

$$r_{I\!R_+}(A; D, E) \leq r_{\mathbb{C}}(A; D, E) \quad \text{and} \quad \|G(1)\|^{-1} \leq r_{\mathbb{C}}(A; D, E).$$

If $r_{\mathbb{C}}(A; D, E) = \infty$ then (20) is trivial. Suppose $r_{\mathbb{C}}(A; D, E) < \infty$ and let $\Delta \in \mathbb{C}^{l\times q}$ be a minimal norm destabilizing matrix of rank 1 for the triplet (A, D, E) [4]. Then

$$1 = \rho(A + D\Delta E) \leq \rho(A + D|\Delta|E)$$

by Corollary 2.5, so that $|\Delta| \in I\!R_+^{l\times q}$ is also destabilizing. Since Δ has rank 1 it follows from Lemma 2.3 and definition (15) that

$$r_{\mathbb{C}}(A; D, E) = \|\Delta\| = \|\,|\Delta|\,\| \geq r_{I\!R_+}(A; D, E).$$

Moreover, by Lemma 3.3, there exists a nonzero vector $y \in \mathbb{R}_+^q$ such that $G(t)|\Delta|y = y$ for some $t \geq 1$. Applying Lemma 3.2 we obtain

$$\|G(1)\|^{-1} \leq \|G(t)\|^{-1} \leq \| \, |\Delta| \, \| = r_{\mathbb{C}}(A; D, E)$$

as required. □

The proof shows that, under the conditions of the theorem, the function $s \mapsto \|G(s)\|$ attains its maximum value on the unit circle at $s = 1$.
We note the following consequence of Theorem 3.4 which illustrates that a Schur stable nonnegative triplet $(A_+; D_+, E_+)$ is the closest to instability amongst all $(A; D, E)$ satisfying $|A| \leq A_+$, $|D| \leq D_+$ and $|E| \leq E_+$.

Corollary 3.5 *(i) Suppose* $(A_+; D_+, E_+) \in \mathbb{R}_+^{n\times n} \times \mathbb{R}_+^{n\times l} \times \mathbb{R}_+^{q\times n}$ *is Schur stable and* $\mathbb{C}^l, \mathbb{C}^q$ *are provided with monotonic norms. Then, for all* $(A, D, E) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times l} \times \mathbb{C}^{q\times n}$, $|A| \leq A_+$, $|D| \leq D_+$, $|E| \leq E_+$ *the matrix* A *is Schur stable and*

$$r_{\mathbb{C}}(A; D, E) \geq r_{\mathbb{R}_+}(A_+; D_+, E_+). \tag{21}$$

(ii) If $(A, D, E) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times l} \times \mathbb{C}^{q\times n}$ *is given with* $|A|$ *Schur stable then*

$$\| \, |E|(I - |A|)^{-1}|D| \, \|^{-1} \leq r_{\mathbb{C}}(A; D, E) \leq r_{\mathbb{R}}(A; D, E) \leq \|E(I - A)^{-1}D\|^{-1}. \tag{22}$$

Proof: Let $(A, D, E) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times l} \times \mathbb{C}^{q\times n}$ and $|A| \leq A_+$, $|D| \leq D_+$, $|E| \leq E_+$. Then $\rho(A) \leq \rho(A_+)$ by Corollary 2.5 so that A is Schur stable. If $r_{\mathbb{C}}(A; D, E) = \infty$ then $\|E(I - A)^{-1}D\| = 0$ and hence (21), (22) hold. If $r_{\mathbb{C}}(A; D, E) < \infty$ there exists a rank 1 destabilizing matrix $\Delta_0 \in \mathbb{C}^{l\times q}$ for (A, D, E) such that $\|\Delta_0\| = r_{\mathbb{C}}(A; D, E)$. But

$$|A + D\Delta_0 E| \leq |A| + |D| \, |\Delta_0| \, |E| \leq A_+ + D_+|\Delta_0|E_+,$$

hence by Corollary 2.5

$$1 \leq \rho(A + D\Delta_0 E) \leq \rho(A_+ + D_+|\Delta_0|E_+).$$

Therefore $|\Delta_0|$ is destabilizing for the nonnegative triplet $(A_+; D_+, E_+)$ and by Lemma 2.3

$$r_{\mathbb{C}}(A; D, E) = \|\Delta_0\| = \| \, |\Delta_0| \, \| \geq r_{\mathbb{R}_+}(A_+; D_+, E_+).$$

This proves (21). As a consequence we obtain by means of Theorem 3.4

$$r_{\mathbb{C}}(A; D, E) \geq r_{\mathbb{R}_+}(|A|; |D|, |E|) = \| \, |E|(I - |A|)^{-1}|D| \, \|^{-1}$$

whence the first inequality in (22). The latter two inequalities follow from $r_{\mathbb{R}}(A; D, E) \leq [\max\{\|G(-1)\|, \|G(1)\|\}]^{-1}$ where $G(s) = E(sI - A)^{-1}D$ is the associated transfer matrix, see [4]. □

4 Nonlinear and time-varying perturbations

In this section we combine the previous results with known results about the complex stability radius in order to analyze the robustness of stable positive systems under nonlinear and time-varying perturbations.
Consider a perturbed system of the form

$$x(t+1) = Ax(t) + DN(Ex(t), t), \qquad t \in \mathbb{N} \tag{23}$$

where $(A, D, E) \in \mathbb{R}_+^{n\times n} \times \mathbb{R}_+^{n\times l} \times \mathbb{R}_+^{q\times n}$ is given, $\rho(A) < 1$ and $N : \mathbb{K}^q \times \mathbb{N} \to \mathbb{K}^l$ is an unknown continuous time-varying nonlinearity satisfying

(a) $N(0, t) = 0$ for all $t \in \mathbb{N}$;

(b) N is of finite gain, i.e. there exists $\gamma \geq 0$ such that

$$\|N(y, t)\| \leq \gamma \|y\|, \quad y \in \mathbb{K}^q,\ t \in \mathbb{N}. \tag{24}$$

By $x_N(\cdot; t_0, x_0)$ we denote the solution of (23) starting at $x(t_0) = x_0 \in \mathbb{C}^n$.
Throughout this section we assume that all vector spaces $\mathbb{K}^n, \mathbb{K}^q, \mathbb{K}^l$ are provided with their usual 2-norms. The norms of matrices are the induced operator norms (spectral norms). The size of a nonlinear perturbation $N(\cdot, \cdot)$ is measured by the norm

$$\|N\| = \inf\{\gamma \in \mathbb{R}_+; \forall y \in \mathbb{K}^q\ \forall t \in \mathbb{N} : \|N(y, t)\| \leq \gamma \|y\|\}.$$

Note that in the special case where $N(y, t) = \Delta y$ is time-invariant and linear we have $\|N\| = \|\Delta\|$where $\|\Delta\|$ is the operator norm of $\Delta \in \mathbb{K}^{l\times q}$. Hence there exists a destabilizing perturbation N of norm $\|N\| = r_{\mathbb{K}}(A; D, E)$. The question is whether the system is more easily destabilized by time-varying and/or nonlinear perturbations than by time-invariant linear ones. It has been shown in [6] that in general the answer is "yes" if $\mathbb{K} = \mathbb{R}$, but "no" if $\mathbb{K} = \mathbb{C}$. Since in the case of positive systems the real and the complex stability radius coincide, we do not expect that (23) can become unstable for nonlinearities N with $\|N\| < r_{\mathbb{R}}(A; D, E)$.
To be more precise we say that the nonlinear system (23) is *globally asymptotically stable* if (i) the equilibrium $\bar{x} = 0$ is stable in the sense of Lyapunov and (ii) $\bar{x} = 0$ is globally attractive, i.e. for every $x_0 \in \mathbb{K}^n$ the solution $x(t) = x_N(t; t_0, x_0)$ satisfies $\lim_{t\to\infty} \|x(t)\| = 0$.

Proposition 4.1 *Suppose that* $(A, D, E) \in \mathbb{R}_+^{n\times n} \times \mathbb{R}_+^{n\times l} \times \mathbb{R}_+^{q\times n}$, $\rho(A) < 1$ *and* $\mathbb{C}^l$,$\mathbb{C}^q$ *are provided with their 2-norms. If the continuous nonlinearity* $N : \mathbb{C}^q \times \mathbb{N} \to \mathbb{C}^l$ *satisfies* $\|N\| < \|E(I-A)^{-1}D\|^{-1}$ *then the system (23) is globally asymptotically stable. Moreover, if* $\rho < \|E(I-A)^{-1}D\|^{-1}$ *then there exists a joint quadratic Lyapunov function for all the systems (23) with* $\|N\| \leq \rho$ *and there exist constants* $\alpha > 0, \omega > 0$ *such that all solutions of these systems satisfy:*

$$\|x_N(t; t_0, x_0)\| \leq \alpha e^{-\omega(t-t_0)} \|x_0\|, \qquad t \geq t_0, \quad t_0 \in \mathbb{N},\ x_0 \in \mathbb{C}^n. \tag{25}$$

Proof: The first statement is a direct consequence of $r_{\mathbb{C}}(A; D, E) = \|E(I - A)^{-1}D\|^{-1}$ and Theorem 2.5 in [6]. Now suppose $\rho < \|E(I - A)^{-1}D\|^{-1}$. Then $\rho < [\max_{|s|=1} \|E(sI - A)^{-1}D\|]^{-1}$ by Theorem 3.4. For $\varepsilon > 0$ choose $E_\varepsilon \in \mathbb{R}^{n\times n}$ such that $E_\varepsilon^* E_\varepsilon = E^*E + (\varepsilon^2/\rho^2)I_n$ (Cholesky factorization). If $\varepsilon > 0$ is sufficiently small then $\rho < [\max_{|s|=1} \|E_\varepsilon(sI - A)^{-1}D\|]^{-1}$. Hence, applying Proposition 3.7 of [7], there exist $\varepsilon > 0$ and a stabilizing negative definite solution P of the discrete algebraic Riccati equation

$$P - A^*PA + A^*PD\,(I + D^*PD)^{-1}\,D^*PA + \rho^2 E^*E + \varepsilon^2 I = 0 \tag{26}$$

i.e. $P \prec 0$ satisfies

$$I + D^*PD \succ 0, \quad \sigma\left(A - D\,(I + D^*PD)^{-1}\,D^*PA\right) \subset \mathbb{C}_1 \tag{27}$$

and solves (26). Define $V : \mathbb{C}^n \mapsto \mathbb{R}_+$ by $V(x) = -\langle x, Px\rangle,\ x \in \mathbb{C}^n$. Then, for every solution $(x(t))_{t\in\mathbb{N}}$ of (23) with $\|N\| \le \rho$, we obtain

$$\begin{aligned}
& V(x(t+1)) - V(x(t)) \\
= \;& -\langle Ax(t) + DN(Ex(t),t), P\,[Ax(t) + DN(Ex(t),t)]\rangle + \langle x(t), Px(t)\rangle \\
= \;& \langle x(t), Px(t)\rangle - \langle x(t), A^*PAx(t)\rangle - 2\,\mathrm{Re}\langle x(t), A^*PDN(Ex(t),t)\rangle \\
& \qquad\qquad\qquad\qquad - \langle N(Ex(t),t), D^*PDN(Ex(t),t)\rangle \\
= \;& -\varepsilon^2\|x(t)\|^2 - \rho^2\|Ex(t)\|^2 + \|N(Ex(t),t)\|^2 - \Big\langle\,(I + D^*PD)\,N(Ex(t),t) \\
& \quad + D^*PAx(t), (I + D^*PD)^{-1}\,[(I + D^*PD)\,N(Ex(t),t) + D^*PAx(t)]\,\Big\rangle \\
\le \;& -\varepsilon^2\|x(t)\| - \left[\rho^2\|Ex(t)\|^2 - \|N(Ex(t),t)\|^2\right] \le -\varepsilon^2\|x(t)\|.
\end{aligned} \tag{28}$$

Hence V is a joint Lyapunov function on $\mathbb{C}^n$ for all the systems (23) with $\|N\| \le \rho$. Moreover it follows from (28) and standard arguments of Lyapunov theory that there exist constants $\alpha > 0$, $\omega > 0$ such that (25) holds. □

For arbitrary (not necessarily positive) triplets we obtain the following result from Corollary 3.5.

Corollary 4.2 *Suppose* $(A, D, E) \in \mathbb{C}^{n\times n}\times\mathbb{C}^{n\times l}\times\mathbb{C}^{q\times n}$ *and* $|A|$ *is Schur stable. If* $N : \mathbb{K}^q \times \mathbb{N} \to \mathbb{K}^l$ *is a continuous time-varying nonlinearity satisfying* $N(0,t) = 0$ *for all* $t \in \mathbb{N}$ *and*

$$\|N\| < \left\|\, |E|(I - |A|)^{-1}|D|\, \right\|^{-1}.$$

Then the system (23) is globally asymptotically (and even exponentially) stable.

Proof: By assumption and (22) we have $\|N\| < r_{\mathbb{C}}(A; D, E)$ and so the same arguments as in the previous proof can be applied. □

Example 4.3 Consider

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad E = I_2$$

and endow $I\!R^2$ with the 2-norm. The lower bound in (22) is given by

$$\|\,|E|(I-|A|)^{-1}|D|\,\|_2^{-1} = \left\| \begin{bmatrix} 1 & -1 \\ -1/2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\|_2^{-1} = \left\| \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\|_2^{-1} = \frac{1}{2\sqrt{2}}. \tag{29}$$

The upper bound in (22) is given by

$$\|E(I-A)^{-1}D\|_2^{-1} = \left\| \begin{bmatrix} 1 & -1 \\ 1/2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\|_2^{-1} = \left\| \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix} \right\|_2^{-1} = \frac{3}{2\sqrt{2}}.$$

An elementary calculation shows that

$$r_{I\!R}(A,D,E) = \frac{1}{2}.$$

Hence there does *not* exist a real matrix $\Delta \in I\!R^{1\times 2}$ of norm smaller than $1/2$ such that the perturbed system

$$x(t+1) = (A + D\Delta E)x(t)$$

is unstable. Now consider the following *time-varying* linear perturbation of period 2

$$N(x,t) = \Delta(t)x, \qquad x \in I\!R^2,\ t \in I\!N \tag{30}$$

where

$$\Delta(2t) = \begin{bmatrix} -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}, \quad \Delta(2t+1) = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} \end{bmatrix}, \quad t \in I\!N.$$

Note that $\|N\| = (2\sqrt{2})^{-1} = \|\,|E|(I-|A|)^{-1}|D|\,\|_2^{-1} < 1/2$, see (29). Since

$$(A + D\Delta(1)E)(A + D\Delta(0)E) = \begin{bmatrix} 0 & 1 \\ -\frac{3}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -\frac{3}{4} & -\frac{1}{4} \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} & -\frac{1}{4} \\ -\frac{3}{16} & -\frac{13}{16} \end{bmatrix}$$

has the eigenvalues $-1, -9/16$, the time-varying perturbation (30) is destabilizing. It follows that in this case

$$r_{I\!\!C}(A,D,E) = \|\,|E|(I-|A|)^{-1}|D|\,\|_2^{-1}$$

and there is a *time-varying real* parameter pertubation of this norm which destabilizes the system. □

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