# THE OVERALL PROPERTIES OF PIEZOCOMPOSITE MATERIALS WITH CYLINDRICAL FIBRES

J. B. Castillero, J. A. Otero, R. R. Ramos, R. G. Diaz
Institute of Cybernetics, Mathematics and Physics (ICIMAF)
Acoustic Group, Calle D, No. 353 e/t 15 y 17
Vedado, Havana, Cuba.

**Abstract**

The purpose of this paper is to apply the technique of asymptotic homogenization to determine the effective elastic, piezoelectric and dielectric moduli of reinforced piezoelectric composite materials with unidirectional cylindrical fibres with periodic structure. Each periodic cell of the medium is a binary piezoelectric composite wherein both constituents are homogeneous piezoelectric materials with hexagonal symmetry (6mm). The analytical solution made use of series expansions of the doubly periodic Weierstrass elliptic functions to obtain overall averaged coefficients of the piezocomposite material. This result is a generalization of the main formulae published by [11] (Chapter 6) for the purely elastic case.

## 1 Introduction

In recent years, new piezoelectric composite materials have been intensively developed to join the requirements of electro- and hydroacoustic for more effective transducer materials. By varying the proportion of several homogeneous constituents one should be able to obtain effective macroscopic materials of desired homogenized properties, e.g. a piezoelectric inactive polymer matrix - piezoelectric ceramic fibres, as in [6], [3], [12], [13] and more recently, piezoelectric ceramic - piezoelectric polymer composites in [14] .
Different techniques have been adopted in the literature to estimate the effective electroelastic properties of fibrous composites. For instance, [5] derived the effective moduli of a transversaly isotropic piezocomposite with cylindrical inclusions based on the hypothesis of equivalent homogeneity. Benveniste and Dvorak ([2]) derived the effective behaviour of a binary piezoelectric composite with unidirectional fibres based on the theory of the creation of uniform fields

in heterogeneous media by proper boundary conditions. Moreover, the reader can find an extensive list of references on others methods in [4] .
The asymptotic averaging method ([1]) have been used for the determination of the effective coefficients of fibrous composites with isotropic phases by [11] , [8], [9]. The aim of this contribution is to present our preliminary results concerning effective properties of piezocomposite. In the present paper by means of this method of homogenization the analytical expressions for the effective coefficients of binary homogeneous piezoelectric composite materials with unidirectional cylindrical fibres have been obtained using the theory of functions of complex variable.

## 2 Local Problems and Effective Coefficients

The main problem of the asymptotic homogenization, for piezoelectric composite materials, consists finding the local periodic functions $N_{pmn}$, $M_{mn}$, $\Phi_{pn}$ and $P_n$ from the following local problems (see [15], [16], Galka et al. (1992), [10] ):
$P^1_{loc}$

$$\begin{aligned} (C_{ijpq}(x)N_{pmn,q}(\xi) + e_{pij}(\xi)M_{mn,p}(\xi))_{,j} &= -C_{ijmn,j}(\xi) \\ (e_{ipq}(\xi)N_{pmn,q}(\xi) - \epsilon_{ip}(\xi)M_{mn,p}(\xi))_{,i} &= -e_{imn,i}(\xi) \end{aligned} \tag{1}$$

$P^2_{loc}$

$$\begin{aligned} (C_{ijpq}(\xi)\Phi_{pn,q}(\xi) + e_{pij}(\xi)P_{n,p}(\xi))_{,j} &= -e_{nij,j}(\xi) \\ (-e_{ipq}(\xi)\Phi_{pn,q}(\xi) + \epsilon_{ip}(\xi)P_{n,p}(\xi))_{,i} &= -\epsilon_{in,i}(\xi) \end{aligned} \tag{2}$$

The moduli: $C_{ijpq}(\xi)$ (elastic), $e_{pij}(\xi)$ (piezoelectric), $\epsilon_{ip}(\xi)$ (dielectric) satisfy the usual symmetry conditions

$$C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk}, \quad e_{kij} = e_{kji}, \quad \epsilon_{ij} = \epsilon_{ji}. \tag{3}$$

$\xi = (\xi_1, \xi_2, \xi_3)$ is the local variable on the periodic cell. The subscripts assume the values 1-3 and the comma denotes partial differentiation. The uniqueness of the solution of the above local problems is achieved by imposing the conditions:

$$\langle N_{pmn}(\xi)\rangle = 0\ ,\ \langle M_{mn}(\xi)\rangle = 0\ ,\ \langle \Phi_{pn}(\xi)\rangle = 0\ ,\ \langle P_n(\xi)\rangle = 0 \tag{4}$$

where $\langle f\rangle = \frac{1}{|Y|}\int_Y f dY$, Y is the periodic cell.
Once solved the local problems, it is possible to determine the homogenized (effective) moduli by using the following expressions:

$$\begin{aligned} \overline{C}_{ijmn} &= \langle C_{ijmn} + C_{ijpq}N_{pmn,q} + e_{pij}M_{mn,p}\rangle, & (5)\\ \overline{e}_{imn} &= \langle e_{imn} + e_{ipq}N_{pmn,q} - \epsilon_{ip}M_{mn,p}\rangle, & (6)\\ \overline{\epsilon}_{in} &= \langle \epsilon_{in} + \epsilon_{ip}P_{n,p} - e_{ipq}\Phi_{pn,q}\rangle. & (7) \end{aligned}$$

# 3 Formulation of the problem

The previously described homogenization technique is capable of treating the three-dimensional piezoelectric behaviour of reinforced piezocomposite materials with periodic structure. In order to demonstrate the applicability of this method, the unidirectional fibre-reinforced composite shown in Fig. 1 is considered.
In Fig. 1, the periodic cell of the fibre-reinforced piezocomposite material consists of a single circular fibre " F" embeded into a square matrix " M". In the regions " F" and " M" in virtue of the homogeneity of the components of the piezocomposite we can obtain, from the equations (1), (2), the following systems:
Equations $I_{pq}$:

$$\begin{aligned} (C_{ijkl}(\xi)N_{kpq,l}(\xi)+e_{kij}(\xi)M_{pq,k}(\xi))_{,j}=0 \\ (e_{ikl}(\xi)N_{kpq,l}(\xi)-\epsilon_{ik}(\xi)M_{pq,k}(\xi))_{,i}=0 \end{aligned} \tag{8}$$

Equations $I_n$:

$$\begin{aligned} (C_{ijkl}(\xi)\Phi_{kn,l}(\xi)+e_{kij}(\xi)P_{n,k}(\xi))_{,j}=0 \\ (e_{ikl}(\xi)\Phi_{kn,l}(\xi)-\epsilon_{ik}(\xi)P_{n,k}(\xi))_{,i}=0 \end{aligned} \tag{9}$$

The solutions of $I_{pq}$ and $I_n$ must be double periodic in $\xi_1$ and $\xi_2$ subjected to the following perfect bonding conditions at the interface $\Gamma$:
For equations $I_{pq}$:

$$N^1_{kpq}|_\Gamma = N^2_{kpq}|_\Gamma\ , \qquad M^1_{pq}|_\Gamma = M^2_{pq}|_\Gamma, \tag{10}$$

$$\begin{aligned} (\sigma^1_{ij(pq)}+C^1_{ijpq}n^1_j|_\Gamma = -(\sigma^2_{ij(pq)}+C^2_{ijpq})n^2_j|_\Gamma, \\ (D^1_{j(pq)}+e^1_{jpq})n^1_j|_\Gamma = -(D^2_{j(pq)}+e^2_{jpq})n^2_j|_\Gamma, \end{aligned} \tag{11}$$

For equations $I_n$:

$$\Phi^1_{pn}|_\Gamma = \Phi^2_{pn}|_\Gamma\ , \qquad P^1_n|_\Gamma = P^2_n|_\Gamma, \tag{12}$$

$$\begin{aligned} (\sigma^1_{ij(n)}+e^1_{ijn})n^1_j|_\Gamma &= -(\sigma^2_{ij(n)}+e^2_{ijn})n^2_j|_\Gamma, \\ (D^1_{j(n)}-\epsilon^1_{jn})n^1_j|_\Gamma &= -(D^2_{j(n)}-\epsilon^2_{jn})n^2_j|_\Gamma. \end{aligned} \tag{13}$$

where

$$\begin{aligned}
\sigma_{ij(pq)} &= C_{ijk\ell}N_{kpq,\ell} + e_{kij}M_{pq,k},\\
\sigma_{ij(n)} &= C_{ijk\ell}\Phi_{kn,\ell} + e_{kij}P_{n,k},\\
D_{j(pq)} &= e_{ik\ell}N_{kpq,\ell} - \epsilon_{ik}M_{pq,k},\\
D_{j(n)} &= e_{ik\ell}\Phi_{kn,\ell} - \epsilon_{ik}P_{n,k},
\end{aligned}$$

$n_j$ are components of the outward unit normal vector to the interface, the superscripts "1" and "2" denotes the values of the functions over the matrix and the fibre respectively.
In general, the exact number of problems of the type $I_{pq}$ and $I_n$ that must be solved in each particular case will depend on the symmetry properties of the matrix and fibre materials. If both are homogeneous piezoelectric materials with hexagonal symmetry (6mm), then, it is possible to obtain the overall averaged coefficients by solving five similar strain problems involving $I_{\beta\beta}(\beta = 1,2,3)$, $I_{12}$ and $I_3$; and two similar "antiplane" problems involving $I_{13}$ and $I_1$. The materials with hexagonal symmetry (6mm) are characterized by the following independent constants: five elastic constants $(C_{1111} = C_{2222}$, $C_{1122}$, $C_{1133} = C_{2233}$, $C_{3333}$, $C_{2323} = C_{1313}$, $C_{1212} = \frac{(C_{1111}-C_{1122})}{2})$, three piezoelectric constants $(e_{311} = e_{322}$, $e_{333}$, $e_{113} = e_{223})$, and two dielectric constants $(\epsilon_{11} = \epsilon_{22}$, $\epsilon_{33})$.

## 4 The antiplane problems

Let us consider the **problem** $I_{13}$. For this problem we can assume that $N^\alpha_{313} \equiv N^\alpha_3$ and $M^\alpha_{13} \equiv M^\alpha$ are independent of $\xi_3$, and $N^\alpha_{113} = N^\alpha_{213} = 0$. The corresponding problem can be expressed as:

$$C^\alpha_{1313}N^\alpha_{3,KK} = 0, \qquad e^\alpha_{113}M^\alpha_{,KK} = 0, \tag{14}$$

$$N^1_3|_\Gamma = N^2_3|_\Gamma\,, \qquad M^1|_\Gamma = M^2|_\Gamma, \tag{15}$$

$$\begin{aligned}
(\sigma^1_{3J(13)}n^1_J + C^1_{1313}n^1_1)|_\Gamma &= -(\sigma^2_{3J(13)}n^2_J + C^2_{1313}N^2_1)|_\Gamma,\\
(D^1_{J(13)}n^1_J + e^1_{113}n^1_1)|_\Gamma &= -(D^2_{J(13)}n^2_J + e^2_{113}n^2_1)|_\Gamma,
\end{aligned} \tag{16}$$

with $\alpha, J, K = 1,2$ and

$$\begin{aligned}
\sigma_{3J(13)} = C_{1313}N_{313,J} + e_{113}M_{13,J}\;,\; \sigma_{33(13)} = 0\;,\; \sigma_{IJ(13)} = 0,\\
D_{J(13)} = e_{113}N_{313,J} - \epsilon_{11}M_{13,J}\;,\; D_{3(13)} = 0,
\end{aligned}$$

$I, J = 1, 2$. The solution of (14)-(16) can be achieved by representing the harmonic functions $N_3^\alpha$ and $M^\alpha$ by the following series expansion:

$$N_3^1 = Re\left\{A\frac{z}{R} + \frac{1}{C_{1313}^1}f_1(z)\right\}, \tag{17}$$

$$N_3^2 = \frac{1}{C_{1313}^2}Re\{f_2(z)\}, \tag{18}$$

$$M^1 = Re\left\{B\frac{z}{R} + \frac{1}{e_{113}}g_1(z)\right\}, \tag{19}$$

$$M^2 = \frac{1}{e_{113}^2}Re\{g_2(z)\}, \tag{20}$$

where

$$f_1(z) = \sum_{k=1}^{\infty}{}^{*} a_k R^k \frac{\zeta^{(k-1)}(z)}{(k-1)!}\ ,\ f_2(z) = \sum_{k=1}^{\infty}{}^{*} c_k \left(\frac{z}{R}\right)^k,$$

$$g_1(z) = \sum_{k=1}^{\infty}{}^{*} b_k R^k \frac{\zeta^{(k-1)}(z)}{(k-1)!}\ ,\ g_2(z) = \sum_{k=1}^{\infty}{}^{*} d_k \left(\frac{z}{R}\right)^k. \tag{21}$$

Here, $\zeta(z)$ is Weierstrass's zeta function, satisfying the quasiperiodicity conditions: $\zeta(z+1) - \zeta(z) = \pi$, $\zeta(z+\mathbf{i}) - \zeta(z) = -\mathbf{i}\pi$. The constants $A$, $B$, $a_k$, $c_k$, $b_k$, $d_k(k = 1, 3, 5, ..)$ are real, $\sum^*$ means the sum for the odd subscripts,

$$\zeta(z) = \frac{1}{z} + \sum_{m,n}{}' \left\{\frac{1}{z - \beta_{mn}} + \frac{1}{\beta_{mn}} + \frac{z}{\beta_{mn}^2}\right\}$$

where $\sum_{m,n}'$ means that $m, n$ are integers except m=n=0, $\beta_{mn} = m + \mathbf{i}n$, and $\mathbf{i}^2 = -1$.
Taking into account, the periodicity conditions for $N_3$ and $M$ the constants $A$ and $B$ are related to $a_1$ and $b_1$ respectively by the formulae

$$A = -\frac{\pi R^2 a_1}{C_{1313}^1}\ , \qquad B = -\frac{\pi R^2 b_1}{e_{113}^1}$$

The perfect bonding conditions at the interface $\Gamma$, (15) and (16), can be written in the following form:

$$A\frac{t+\overline{t}}{2R} + \frac{1}{C_{1313}^1}Re\{f_1(t)\} = \frac{1}{C_{1313}^1}Re\{f_2(t)\}, \tag{22}$$

$$B\frac{t+\overline{t}}{2R} + \frac{1}{e_{113}^1}Re\{g_1(t)\} = \frac{1}{e_{113}^2}Re\{g_2(t)\}, \tag{23}$$

$$Im\{f_1(t)+g_1(t)\}+\left(\frac{C^1_{1313}}{R}A-\frac{e^1_{113}}{R}B+C^1_{1313}-C^2_{1313}\right)\frac{t-\bar{t}}{2\mathbf{i}}= Im\{f_2(t)+g_2(t)\}, \quad (24)$$

$$Im\left\{\frac{e^1_{113}}{C^1_{1313}}f_1(t)-\frac{\epsilon^1_{11}}{e^1_{113}}g_1(t)\right\} +\left(\frac{e^1_{113}}{R}A-\frac{\epsilon^1_{11}}{R}B+e^1_{113}-e^2_{113}\right)\frac{t-\bar{t}}{2\mathbf{i}}= Im\{\frac{e^2_{113}}{C^2_{1313}}f_2(t)- \frac{\epsilon^2_{11}}{e^2_{113}}g_2(t)\}, \quad (25)$$

where $t=Re^{\mathbf{i}\theta}$, "Re" means the real part and "Im" means the imaginary part.
The Laurent series expansions of the functions $N^1_3$ and $M^1$ about the point $z=0$ are given by

$$N^1_3 = Re\left\{z\frac{A}{R}+\frac{1}{C^1_{1313}}\left[\sum_{k=1}^{\infty}{}^{*} a_k\left(\frac{R}{z}\right)^k+\sum_{p=1}^{\infty}{}^{*}\sum_{k=1}^{\infty}{}^{*} a_k\eta_{kp}\left(\frac{z}{R}\right)^p\right]\right\}, (26)$$

$$M^1 = Re\left\{z\frac{B}{R}+\frac{1}{e^1_{113}}\left[\sum_{k=1}^{\infty}{}^{*} b_k\left(\frac{R}{z}\right)^k+\sum_{p=1}^{\infty}{}^{*}\sum_{k=1}^{\infty}{}^{*} b_k\eta_{kp}\left(\frac{z}{R}\right)^p\right]\right\}, (27)$$

where,

$$\eta_{kp} \equiv \frac{(-1)^k(p+k-1)!}{p!(k-1)!}R^{p+k}S_{p+k},$$

$$S_{p+k} \equiv \sum_{m,n}\frac{1}{(\beta_{mn})^{p+k}}.$$

Substituting the expansions (18), (20), (26) and (27) into the conditions (22)-(25) on the interface $\Gamma$, the following system of algebraic equations for the constants $a_k$ and $b_k$ (k=1,3,5,...) can be obtained:

$$\frac{C^2_{1313}+C^1_{1313}}{C^1_{1313}}a_k + \frac{e^2_{113}+e^1_{113}}{e^1_{113}}b_k + \frac{C^2_{1313}-C^1_{1313}}{C^1_{1313}}(\sum_{n=1}^{\infty}{}^{*} a_n\eta_{nk} - \pi R^2 a_1\delta_{1k}) + \frac{e^2_{113}-e^1_{113}}{e^1_{113}}(\sum_{n=1}^{\infty}{}^{*} b_n\eta_{nk} - \pi R^2 b_1\delta_{1k}) = (C^1_{1313}-C^2_{1313})R\delta_{1k},$$

$$\frac{e^1_{113}+e^2_{113}}{C^1_{1313}}a_k - \frac{\epsilon^1_{11}+\epsilon^2_{11}}{e^1_{113}}b_k + \frac{e^2_{113}-e^1_{113}}{C^1_{1313}}(\sum_{n=1}^{\infty}{}^{*} a_n\eta_{nk} - \pi R^2 a_1\delta_{1k}) - \frac{\epsilon^2_{11}-\epsilon^1_{11}}{e^1_{113}}(\sum_{n=1}^{\infty}{}^{*} b_n\eta_{nk} - \pi R^2 b_1\delta_{1k}) = (e^1_{113}-e^2_{113})R\delta_{1k}. \quad (28)$$

Having solved the above system, one can determine the constants $c_k$ and $d_k$ (k=1, 3, 5, ...) by

$$\begin{aligned} c_k &= \frac{C^2_{1313}}{C^1_{1313}}(-\pi R^2 a_1\delta_{1k} + a_k + \sum_{k=1}^{\infty}{}^{*} a_n\eta_{nk}), \\ d_k &= \frac{e^2_{113}}{e^1_{113}}(-\pi R^2 b_1\delta_{1k} + b_k + \sum_{k=1}^{\infty}{}^{*} b_n\eta_{nk}). \end{aligned}$$

On the other hand, from formulae (5), (6) the following expressions for the effective coefficients $\overline{C}_{1313}$ and $\overline{e}_{113}$ can be obtained:

$$\overline{C}_{1313} = \langle C_{1313} + C_{1313}N_{3,1} + e_{113}M_{,1}\rangle, \quad (29)$$

$$\overline{e}_{113} = \langle e_{113} + e_{113}N_{3,1} - \epsilon_{11}M_{,1}\rangle. \quad (30)$$

Now, solving the above integrals (29), (30) and substituting the expansions (26) and (27) we obtain:

$$\overline{C}_{1313} = \langle C_{1313}\rangle + \frac{C^2_{1313}-C^1_{1313}}{C^1_{1313}}\pi R(a_o+\overline{A}) + \frac{e^2_{113}-e^1_{113}}{e^1_{113}}\pi R(b_o+\overline{B}),$$

$$\overline{e}_{113} = \langle e_{113}\rangle + \frac{e^2_{113}-e^1_{113}}{C^1_{1313}}\pi R(a_o+\overline{A}) - \frac{\epsilon^2_{11}-\epsilon^1_{11}}{e^1_{113}}\pi R(b_o+\overline{B}),$$

where $a_o \equiv -a_1 R^2\pi$, $b_o \equiv -b_1 R^2\pi$ and $\overline{A} \equiv a_1 + \sum_{n=1}^{\infty}{}^* a_n\eta_{n1}$, $\overline{B} \equiv b_1 + \sum_{n=1}^{\infty}{}^* b_n\eta_{n1}$.
Finally, by using the equations (28) for $k=1$, we can obtain

$$\begin{aligned}\overline{C}_{1313} &= C^1_{1313} - 2\pi R(a_1+b_1), &(31)\\ \overline{e}_{113} &= e^1_{113} - 2\pi R(\frac{e^1_{113}}{C^1_{1313}}a_1 - \frac{\epsilon^1_{11}}{e^1_{113}}b_1). &(32)\end{aligned}$$

**The problem** $I_1$: For this problem we assume that $\phi^\alpha_{n(1)} = 0$ (n=1,2) and $\phi^\alpha_{3(1)}$, $P^\alpha_1$ are independent of $\xi_3$. The equations can be expressed as

$$e^\alpha_{113}\phi^\alpha_{3(1),kk} = 0; \qquad -\epsilon^\alpha_{11}P^\alpha_{(1),kk} = 0, \tag{33}$$

and the conditions at the interface $\Gamma$ are

$$\phi^1_{3(1)}|_\Gamma = \phi^2_{3(1)}|_\Gamma; \qquad P^1_{(1)}|_\Gamma = P^2_{(1)}|_\Gamma \tag{34}$$

$$\begin{aligned}(\sigma^1_{3J(1)}n^1_J + e^1_{113}n^1_1)|_\Gamma &= -(\sigma^2_{3J(1)}n^2_J + e^2_{113}n^2_1)|_\Gamma,\\ (D^1_{J(1)}n^1_J - \epsilon^1_{11}n^1_1)|_\Gamma &= -(D^2_{J(1)}n^2_J - \epsilon^2_{11}n^2_1)|_\Gamma, &(35)\end{aligned}$$

with

$$\begin{aligned}\sigma_{IJ(1)} = 0, &\qquad \sigma_{3J(1)} = C_{1313}\phi_{3(1),J} + e_{113}P_{(1),J},\\ D_{3(1)} = 0, &\qquad D_{J(1)} = e_{113}\phi_{3(1),J} - \epsilon_{11}P_{(1),J},\end{aligned}$$

$(I, J, K = 1, 2)$.
The solution of this problem is proposed as

$$\begin{aligned}\phi^1_{3(1)} &= Re\{A\frac{z}{R} + \frac{1}{e^1_{113}}f_1(z)\}, \phi^2_{3(1)} = \frac{1}{e^2_{113}}Re\{f_2(z)\},\\ P^1_{(1)} &= Re\{B\frac{z}{R} - \frac{1}{\epsilon^1_{11}}g_1(z)\}, P^2_{(1)} = -\frac{1}{\epsilon^2_{11}}Re\{g_2(z)\},\end{aligned}$$

where $f_1$, $f_2$, $g_1$ and $g_2$ have the same expressions as in (21). Following an analogous procedure shown in $I_{13}$ it is possible to prove that the effective coefficient $\overline{\epsilon}_{11}$ has the form

$$\overline{\epsilon}_{11} = \epsilon^1_{11} + 2\pi(a_1+b_1), \tag{36}$$

where $a_1$, $b_1$ are solutions of the following system:

$$\frac{e^1_{113}+e^2_{113}}{e^1_{113}}a_k + \frac{\epsilon^1_{11}+\epsilon^2_{11}}{\epsilon^1_{11}}b_k +$$

$$\frac{e^2_{113}-e^1_{113}}{e^1_{113}}(\sum_{n=1}^{\infty}{}^{*} a_n\eta_{nk}-\pi R^2 a_1\delta_{1k})+$$
$$\frac{\epsilon^2_{11}-\epsilon^1_{11}}{\epsilon^1_{11}}(\sum_{n=1}^{\infty}{}^{*} b_n\eta_{nk}-\pi R^2 b_1\delta_{1k})=$$
$$(\epsilon^1_{11}-\epsilon^2_{11})R\delta_{1k},$$
$$\frac{C^1_{1313}+C^2_{1313}}{e^1_{113}}a_k-\frac{e^1_{113}+e^2_{113}}{\epsilon^1_{11}}b_k+$$
$$\frac{C^2_{1313}-C^1_{1313}}{e^1_{113}}(\sum_{n=1}^{\infty}{}^{*} a_n\eta_{nk}-$$
$$\pi R^2 a_1\delta_{1k})-\frac{e^2_{113}-e^1_{113}}{\epsilon^1_{11}}(\sum_{n=1}^{\infty}{}^{*} b_n\eta_{nk}-\pi R^2 b_1\delta_{1k})=$$
$$(e^1_{113}-e^2_{113})R\delta_{1k},$$

## 5 The plane problems

In **the problems** $I_{\beta\beta}$ and $I_{12}$ it can be assumed that $N^{\alpha}_{3\beta\beta}=M^{\alpha}_{\beta\beta}=N^{\alpha}_{312}=M^{\alpha}_{12}=0$, and consequently they represent plane strain problems of the theory of elasticity having the form

$$(C^{\alpha}_{1122}+C^{\alpha}_{1212})N^{\alpha}_{kmn,ki}+C^{\alpha}_{1212}N^{\alpha}_{imn,kk}=0, \tag{37}$$

where, $\alpha$,k,i=1,2; mn=11,22,33,12. Besides they satisfy the following perfect bonding conditions

$$N^1_{kmn}|_{\Gamma}=N^2_{kmn}|_{\Gamma}, \tag{38}$$

$$(\sigma^1_{ij(mn)}+C^1_{ijmn})n^1_j|_{\Gamma}=-(\sigma^2_{ij(mn)}+C^2_{ijmn})n^2_j|_{\Gamma}, \tag{39}$$

where

$$\sigma_{ij(mn)}=C_{1122}(\xi_1,\xi_2)\delta_{ij}N_{kmn}+C_{1212}(\xi_1,\xi_2)(N_{imn,j}+N_{jmn,i}), \tag{40}$$

i,j=1,2; $\beta=1,2,3$, $\sigma_{i3(mn)}=0$, $\sigma_{33(mn)}=\nu(\xi_1,\xi_2)(\sigma_{11(mn)}+\sigma_{22(mn)})$, $\nu(\xi_1,\xi_2)=\frac{C_{1133}}{C_{1111}+C_{1122}}$, $D_{i(mn)}=0$, $D_{3(mn)}=\frac{e_{311}(\xi_1,\xi_2)}{C_{1133}(\xi_1,\xi_2)}\sigma_{33(mn)}$, and $\delta_{ij}$ is the Kroneker's delta.

On the other hand, if we desire to obtain the formulation $I_3$ it is sufficient to change in equations (37)-(39) the functions $N^{\alpha}_{kmn}$ by $\phi^{\alpha}_{k3}$ and the components of the tensor $C^{\alpha}_{ijmn}$ in equation (39) by $e^{\alpha}_{3ij}$.
The analytic solutions of the plane problems for the case of isotropic fibre and matrix have been treated by several authors [see, for instance, [11] , [8] and [9]] by using the complex potencials of Kolosov-Muskhelishvili. Here, upon the basis of the results published in [[11] ], it is possible to obtain the analytic expressions for the rest of the effective coefficients.
The functions $N^{\alpha}_{kpq} \equiv N^{\alpha}_{k}$ for the problems $I_{\beta\beta}$, $I_3$ and $I_{12}$ are sought in the form

$$2C^{\alpha}_{1212}(N^{\alpha}_{1} + iN^{\alpha}_{2}) = \chi_{\alpha}\varphi_{\alpha}(z) - z\overline{\varphi'_{\alpha}} - \overline{\psi_{\alpha}(z)}, \tag{41}$$

where, $\chi_{\alpha} = 3 - 4\nu_{\alpha}$, the functions $\varphi_{\alpha}(z)$ and $\psi_{\alpha}(z)$ are differentiable functions of the complex variable $z = \xi_1 + i\xi_2$ and can be expressed as

$$\varphi_1(z) = \frac{a_0}{R}z + \sum_{k=1}^{\infty}{}^{*}\, a_k R^k \xi^{(k-1)}(z), \tag{42}$$

$$\psi_1(z) = \frac{b_0}{R}z + \sum_{k=1}^{\infty}{}^{*}\, b_k R^k \xi^{(k-1)}(z) + \sum_{k=1}^{\infty}{}^{*}\, a_k R^k \frac{Q^{(k-1)}(z)}{(k-1)!}, \tag{43}$$

$$\varphi_2(z) = \sum_{k=1}^{\infty}{}^{*}\, c_k \left(\frac{z}{R}\right)^k, \qquad \psi_2(z) = \sum_{k=1}^{\infty}{}^{*}\, d_k \left(\frac{z}{R}\right)^k. \tag{44}$$

Here

$$Q(z) = \sum_{m,n}{}' \left\{ \frac{\overline{\beta}_{mn}}{(z-\beta_{mn})^2} - 2z\frac{\overline{\beta}_{mn}}{\beta^3_{mn}} - \frac{\overline{\beta}_{mn}}{\beta^2_{mn}} \right\};$$

$a_0, b_0, a_k, b_k, c_k, d_k$ (k=1,3,5...) are certain constants. These constants are real numbers for the problems $I_{\beta\beta}$, $I_3$ and they are pure imaginary numbers for the problem $I_{12}$. From equations (41) we can write $a_0$ and $b_0$ through $a_1$ and $b_1$ respectively, taking into account the periodicity conditions for the functions $N^{\alpha}_{1}$ and $N^{\alpha}_{2}$. For the determination of $a_k, b_k, c_k, d_k$ it is necessary to use: the perfect bonding conditions, the relations among $\sigma^{\alpha}_{ij}$ (i,j=1,2) and $\varphi_{\alpha}(z)$, $\psi_{\alpha}(z)$ wich are given by:

$$\sigma^{\alpha}_{11} + \sigma^{\alpha}_{22} = 4Re\{\varphi'_{\alpha}(z)\}$$

$$\sigma^{\alpha}_{22} - \sigma^{\alpha}_{11} + 2i\sigma^{\alpha}_{12} = 2[\overline{z}\varphi''_{\alpha}(z) + \psi'_{\alpha}(z)]$$

and the Laurent series expansions of the functions (42)-(44). Taking into account the above ideas we can obtain a regular system with respect to $a_k$ ([7] ). The constants $b_k, c_k, d_k$ (k=1,3,5...) can be expressed through $a_k$.
Finally, the formulae for the effective coefficients were obtained from each problem as follows:
From $I_{\beta\beta}$:

$$
\begin{aligned}
\overline{C}_{11\beta\beta} + \overline{C}_{22\beta\beta} &= \langle C_{11\beta\beta} + C_{22\beta\beta} \rangle - 2\frac{\gamma_2(1)}{C^2_{1212}}\frac{\chi_2 - 1}{R}\gamma c_1, \\
\overline{C}_{22\beta\beta} + \overline{C}_{11\beta\beta} &= \langle C_{22\beta\beta} - C_{11\beta\beta} \rangle + 2\gamma\{(\chi_1 + 1)\frac{a_1}{R} + \gamma_1(\beta)\}, \qquad (45) \\
\overline{C}_{33\beta\beta} &= \langle C_{33\beta\beta} \rangle - \frac{\gamma_2(3)}{C^2_{1212}R}\gamma(\chi_2 - 1)c_1, \qquad (46) \\
\overline{e}_{3\beta\beta} &= \langle e_{3\beta\beta} \rangle + \frac{e^2_{311} - e^1_{311}}{C^2_{1212}R}\gamma(\chi_2 - 1)c_1, \qquad (47)
\end{aligned}
$$

From $I_3$:

$$\overline{\epsilon}_{33} = \langle \epsilon_{33} \rangle - \frac{e^2_{311} - e^1_{311}}{C^2_{1212}R}\gamma(\chi_2 - 1)c_1. \qquad (48)$$

From $I_{12}$:

$$\overline{C}_{1212} = \langle C_{1212} \rangle - \gamma\{(\chi_1 + 1)\frac{Im\{a_1\}}{R} + C^2_{1212} - C^1_{1212}\}, \qquad (49)$$

where

$$
\begin{aligned}
\gamma &= \pi R^2, \\
\gamma_1(\beta) &= \frac{(C^1_{22\beta\beta} - C^1_{11\beta\beta} + C^2_{11\beta\beta} - C^2_{22\beta\beta})}{2}, \\
\gamma_2(\beta) &= \frac{(C^1_{11\beta\beta} + C^1_{22\beta\beta} - C^2_{11\beta\beta} - C^2_{22\beta\beta})}{2}.
\end{aligned}
$$

The constants $a_1$ and $c_1$ that appear in the above formulae are obtained from the solution of infinite systems of algebraic equations corresponding to each problem.

# 6 Concluding remarks

In this work, the technique of asymptotic homogenization is applied to the problem of a unidirectional fibre-reinforced piezocomposite material. Analytical expressions for the effective coefficients of binary homogeneous piezoelectric composite materials with unidirectional cylindrical fibres have been obtained using the theory of complex variable. The formulae obtained for such effective coefficients are a generalization of the relations published by [11] (Chapter 6), for the purely elastic case. The global behaviour of the piezocomposite considered is characterized by: six elastic constants, three piezoelectric constants and two dielectric constants, equations (31), (32), (36), (45)-(49). Such material belongs to the crystalline group $4mm$ and it is called thetragonal system.

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