Bounds of the expected approximation error in optimal inventory policies

Carlos Bouza and Sira Allende
Departamento de Matematica Aplicada
Facultad de Matematica y Computacion
Universidad de la Habana

Abstract

Bounds for the approximation error of optimal inventory policies are derived. The unknown distribution functions are approximated by normal distributions. The expectation of the errors are of order $0[\log(N)^{\frac{1}{2}}]$.

1 Introduction

The present trend towards the introduction of new techniques in the production increases the need of setting optimal inventory levels. Generally the involved distribution functions are unknown and a normal approximation is used. This paper studies the effect of using, as approximation, a certain normal model in the solution of the corresponding optimization program.
The serial multiechelon inventory model due to Clark and Scarf [1962] is studied. A bound of the approximate error [AE] is obtained in Section 2. A Central Limit Theorem is obtained . It characterizes this bound.
The problem of the discrete dynamic lot size is analyzed. Two theorems permit to establish the AE for the proposed normal approximation and a bound for the AE when the empirical measure is used instead of the normal distribution.

2 Assembly systems

In this section we will follow essentially the notation used by Rosling [1989]. Take
μ =Expected demands per period for the end element
F(.)=Distribution function for the demands per period
F^n (.)=n-fold convolution of the distribution function per period
f(.)=Density function for the demands per period
f^n (.)=n-fold convolution of the density function per period

$\Phi(.)$ = Normal distribution function $N[0, V]$
$\varphi(.)$ =Normal density function of $\Phi[.]$
X_{it} =Echelon inventory position of item i in period t before ordering decisions are made
Y_{it} = *Echelon* inventory position of item i in period t after production decisions are made.
$X_{it}^{m(i)}$=Echelon inventory position before ordering in period t of assembled units of item i
m(i) =number of periods [lead time] for assembly [delivery] of item i.
ξ_t =i.i.d. positive random demand at period t with distribution F.
Note that X_{it} is the inventory on hand with all units assembled into other units, plus units in assembly or on order , minus backlog of the $X_{it} < Y_{it}$. The amount $Y_{it} - X_{it}$ is ordered at period t and the production is constrained by theavailable components to $X_{i+1t}^{m[i]} > Y_{it}$. the existence of an optimal policy Y^* with reorder level S_i was proved by Clark and Scarf [1962]. Therefore the optimal reorder levels can be computed recursively.
The decision maker [DM] is interested in minimizing the expected cost of running the system.The parameters related with the costs are:
c_i =unit echelon holding cost per period of item i
c_i^* =unit installation holding cost per period of item i
p =period discount backlogging cost per period of end item
α =period discount vector, $\alpha \in]0, 1[$
For example, for the first stage

$$c_1[Y - \Gamma[m(i+1)]] + [p + c_1^*] \int_Y^\infty [\xi - Y] f^{m(i+1)}[\xi] \mathrm{d}\xi$$

is the cost per period at the final stage when its inventory position is raised to Y in every period.
The total cost over the infinite horizon, see Rosling [1989] is given by the equation

$$\sum_{t=1}^{\infty} \alpha^{t-1} [\sum_{i=1}^{N} \alpha^{m(i)} [c_i(Y_{it} - \Gamma(m(i+1))] + \alpha^{m(1)} [p + c_1^* \int_{Y_{1t}}^{\infty} (\xi - Y_{it}) f^{m(i+1)}(\xi) d\xi] = W_1 + W_2 = W$$

In practice we can use the deviations of the original variable, with respect to its expectation, for characterizing Y_{it}. We will assume that c_i Y_{it} has a distribution function G with zero mean, finite variance σ^2 and that E$|$ Y_{it} $|^r$ *is* bounded for at least one $r > 2$.
The DM aims to fix the optimal reorder levels looking for the minimal expected costs. Then we have the optimization problem:

$$P1: \qquad Min_Y E[W|F, G] = M_1$$

subject to:

$$X_{it} < Y_{it}^* < X_{ht}, \quad \text{for any } \ h \in \Pi[i] \tag{2.1}$$

$$X_{it} = Y_{it-1} - \xi_{t-1} \tag{2.2}$$

where $\Pi(i)$ is the set of the immediate predecessors of item i and E denotes the expected value. [2.1] establishes logical relations between the echelon inventory position before ordering decisions are made, and the corresonding reorder level, see Rosling [1993].
Usually, F and G are unknown. Then a solution of P1 is derived by using distributions which are supposed to be sufficiently close to them. The corresponding effect in the solution is not considered. We seek for a Central Limit Theorem for establishing conditions that enable to accept that the solution of the corresponding model is close to M_1.
Let us analyze

$$\sum_{i=1}^{N} \alpha^{m(i)} c_i Y_{it} = \sum_{i=1}^{N} Y_{it}^{*} = Y_{tN}$$

It is a partial sum process. Therefore, it is possible to derive criteria that allow to establish the convergence of its distribution to a normal distribution function.
Note that $E[Y_{it}^{*}] = 0$ and $Var[Y_{it}^{*}] = [\alpha^{m(i)} c_i \sigma_0]^2 = \sigma_0^2$.
We will denote by λ the Lebesgue measure, by $\mid . \mid$ a norm on $\Re^d$ and the class $\Im$ given by the following definition:

Definition 2.1.: Take $\Im = \{]0, \tau[^d \mid \tau \in [0, 1]\}$ as a class with the properties:

(i) $D\{\Im\} = Sup\{n \in N \mid \#[A \cap \Im] = 2^n$ for some set A with #A=n}

(ii) $Sup_{s \in \Im} \lambda[\partial \Im]^{\epsilon} \leq K \epsilon^{\delta}$ for some $\delta \in]0, 1[$ and for any $\epsilon \in]0, 1[$.

(iii) $\int_0^1 [\epsilon^{-1} \log[N_{\#}(\epsilon, \Im)]^{\frac{1}{2}} d\epsilon < \infty$

(iv) For any $\tau \in]0, 1[$, $\forall s \in \Im$ we have that $\tau s \in \Im$

where #B represents the cardinal of B, $N_{\#}[\epsilon, \Im]$ denotes the minimal cardinality of a collection $\Im(\epsilon)$, $A \cap \Im = \{A \cap s \mid \forall s \in \Im\}$ and $[\partial \Im]^{\epsilon} = \{y \in \Re^d$ such that $\mid y - z \mid < \epsilon$ for some z$\in \partial \Im\}$.□

This family of Borelian subsets has nice properties for studying the convergence of partial sum processes. In our particular case d=1 and condition i of Definition 2.1 establishes that it is a Vapnik-Chernovenenkis class.
The following theorem is a direct result of the Central Limit Theorem of Alexander [1987] for independent arrays indexed by Vapnik-Chernovenenkis classes and the rates of convergence derived by del Rìo [1990].

Theorem 2.2.: Take $\Im$ as a family of Borelian subsets of]0,1[that satisfies Definition 2.1 for some $\delta \in]0, 1[$. Assume that G is a distribution function with zero mean and that, for some $r > 2$, $E_G[| Y^*_{it} |] < \infty$. Then there exists an array $\{Z_{it}\}_{i\geq 1}$ of i.i.d. random variables such that:

1. Z_{it} is normally distributed $E[Z_{it}] = 0$ and $Var[Z_{it}] = \sigma^2$, [Z_{it} follows as a distribution $N(0, \sigma^2)$].

2. $\theta_t = Sup_{s\in\Im} | Y^*_{tN} - Z_{tN} |= 0\{N^{\frac{1-\delta}{2}}(\log N)^{\frac{1}{2}}\}$ where $Z_{tN} = Z_{t1} + ... + Z_{tN}.$

A method for constructing such arrays was proposed by del Rìo [1990]. Using Wasserstein-type distances it is possible to derive bounds for the expectations of θ_t because G and N[0,σ^2] have finite means and variances.

Note that if Theorem 2.2 holds and $\delta = 1$, then

$$E[\sum_{t=1}^{\infty}\alpha^{t-1}(Y^*_{tN} - Z_{tN})] \leq \sum_{t=1}^{\infty}\alpha^{t-1}E \mid Y^*_{tN} - Z_{tN} \mid\leq 0[\log(N)^{\frac{1}{2}}]\alpha^* = \nu^*_1$$

where

$$\alpha^* = \sum_{t=1}^{\infty}\alpha^{t-1}$$

This result is obtained easily because , as Theorem 2.2 holds, then $\theta_t \geq| Y^*_{tN} - Z_{tN} |$ for any t.

Commonly G has a finite generating function. In such cases other convergence rates can be used. For example Komlos, Major and Tusnady [1975] proved that $\theta_t = 0[\log[N]]$, WP1.

Petrov [1975] obtained that when n=m(1)+1 is sufficiently large, μ_3 is defined *as* the the third moment of F and φ as the normal density function N[0, (m(1)+1)σ_0^2] $\int | f^{m(1)+1}(\xi) - \varphi(\xi) | d\xi = [\mu_3(1 + 4e^{-\frac{3}{2}}(18\pi(m(1) + 1)^{-\frac{1}{2}}]0[m(1) + 1]^{-\frac{1}{2}} = \gamma.$

Therefore a bound for the expectation AE of the second term of W when φ is usedas approximation, is :

$$E \mid W_2[F^{m(i)+1}, G] - W_2[\varphi] \mid\leq \sum_{t=1}^{\infty}\alpha^{t-1}\zeta_t\sum_{i=1}^{N}\alpha^*[m(i)](p + c_1)\gamma = \nu^*_2$$

where $\zeta_t \geq Max[\xi - Y_{1t}]$, is a value that can be fixed by the DM, and $\alpha^*[m(i)] = \alpha^{m(i)+1}\alpha^{m(1)}$.

These results support the following lemma:

Lemma 2.3.: Let the hypothesis of Theorem 2.2 hold, $\delta = 1$ and use the normal approximations $\Phi_G = N(0, \sigma^2)$ and $\Phi_F = N[0, (m(1)+1)\sigma^2_{0]}$ to G and F, respectively. The expected AE due to solving the optimization problem

$$P^*1 : \qquad Min_Y E[W \mid \Phi_1, \Phi_2] = M^*_1$$

subject to (2.1)-(2.3)

instead of P1 is E| $M_1 - M_1^*$ |< $\nu_1^* + \nu_2^*$ if m(1) is sufficiently large.
This result has a direct application in the study of the different particular multiechelon inventories models because the work with the convolution of normals is rather simple. In many practical situations it may be less costly to solve P^*1 than P1 because of the expected disminution in the computation time.

3 Discrete Dynamic Lot Size

Using the notation of Laserre, Bes and Roubellat [1985] the problem of determining optimal inventories levels is given by the solution of:

$$Q1: \qquad MinE[\sum_{t=1}^{N} c_t[X_t] + g_t[S_t]] = M_t \tag{3.1}$$

subject to:

$$X_t = X_{t-1} + S_t - \xi_t \tag{3.2}$$

$$S_t \in]0, S'_t[\tag{3.3}$$

$$\text{Prob}\{X_t \leq x'_t\} > \alpha_t \tag{3.4}$$

$$\text{Prob}\{X_t \geq x"_t\} > \beta_t \tag{3.5}$$

where S $'_t$, x $'_t$ and x $"_t$ are determined by the DM for maintaining the production and inventory level within a certain acceptable range, and
X_t =inventory level
S_t =production level
h_t =cost function, h=c, g
ξ_t =positive random demand
π_t =a probability level fixed by the DM, $\pi = \alpha, \beta$.
Laserre et. al. [1985] derived a deterministic equivalent problem of Q1. The following theorem permits to solve the deterministic equivalent problem when the distribution of the demands is unknown.

Theorem 3.1.: Let t be the distribution function of $D_t = \sum_{h<t} \xi_h$ and take λ_{tP}=$F_{t\theta}^{-1}[P]$. The deterministic equivalent to Q1 is:

$$Q2: \qquad MinE[\sum_{t=1}^{N} c_t^*[Y_t] + g_t[S_t]] = M'_2$$

subject to:
$Y_t = Y_{t-1} + S_t - \lambda_{tP}$
$S_t \in]0, S'_t[$
$Y_t \in]y'_t, y"_t[$
where $y_0 = x_0$, y $"_t = x"_t$, y $'_t = x'_t + \lambda_{t(1-\alpha_t)} - \lambda_{t\beta_t}$ for t=,...N, $c_t^*[y_t] = E[c_t(y_t - \lambda_{t\beta_t} - \xi_t]]$ and $\lambda_{t0} = \lambda_{1\beta_t}$ if t=1 $\lambda_{t0} = \lambda_{t\beta_t} - \lambda_{(t-1)\beta_t}$ if $t > 1$. If:

(i) $\Im$ is a family of Borelian subset of [0,1] fixed by Definition 2.1.

(ii) $F_{t\theta}$ has a zero mean and E| ξ_t $|^r < \infty$ for some $r > 2$. Then:

(a) The involved quantiles can be approximated by those of a Normal distribution .

(b) For the difference between M '$_2$ and the optimal value calculated under the normal approximation M_2^* is granted that $\mid M_2' - M_2^* \mid = 0[\log(N)^{\frac{1}{2}}]$.

Proof:

a) Changing slightly the notation of Theorem 2.2: G=F, Y^*=D. Then

$$\theta_t = Sup \mid D_t - Z_t \mid = 0[\log(N)^{\frac{1}{2}}],$$

where Z_t is distributed normally with mean equal to zero, finite variance and p-th quantile z_{p_t} . Then, for an adequate choice of N, Theorem 2.2 ensures that $\lambda_{tp_t} \approx z_{p_t}$.

b] $\sum\limits_{t=1}^{N} c_t[y_t] = C_N$

is a partial sum indexed by a Vapnik-Chernovenenkis class. Constructing the convenient array

$C_N^* = \sum\limits_{t=1}^{N} c_t[y_t^*]$

we have that $\mid M_2' - M_2^* \mid <$ E| $C_N - C_N^* \mid Sup \mid C_N - C_N^* \mid = 0[\log(N)^{\frac{1}{2}}]\Box$

We are dealing with the unidimensional case. Hence t he Wasserstein-type distance between F and Φ is given by:

$$W_1[F, \Phi] = \int_0^1 [F^{-1}(u) - \Phi^{-1}(u)]du = \delta^*$$

Denote by μ_ς the expectation and by σ_ς the standard deviation of a random variable with distribution ς. Take $\hat{\mu}_\varsigma$as the estimator of the mean. Frèchet [1957] proved that $W_2[F, G] = [\mid \mu_F - \mu_G \mid^2 + (\sigma_F - \sigma_G)^2]^{\frac{1}{2}}$. If we deal with a class of distribution functions $\wp = \{\phi_\theta \mid W_2[\phi, \Phi] < \varepsilon, \theta \in \Theta\}$ the robustness of mean based statistics ensures that there exists a positive real constant K such that

$$K \mid \hat{\mu}_\phi - \hat{\mu}_\Phi \mid \leq \varepsilon.$$

These hypothesis permit the use of techniques of inserted intervals. Theorem 3.2 fixes a bound for the AE when a certain normal based test is used. The results of procedures based on such tests will be similar for any distribution F$\in \wp$.

Theorem 3.2. Suppose that

1. $\xi_1, \ldots, \xi_N$ is an i.i.d. sample selected from a distribution function which belongs to $\wp$.

2. The DM fixes a positive constant Δ such that $\Omega_\theta = \{M_\theta \mid \theta \in \Theta\} \subset]M_\theta, M_\theta + \Delta[=]w', w''[$

3. The hypotheses of Theorem 3.1 are satisfied.

Then there exists a positive constant ρ such that

$$E[\mid M_n - M_2^* \mid] \leq \rho \beta_N[\Delta] + 0[\log(N)^{\frac{1}{2}}],$$

where $\beta_N[\Delta_N] = \mid M_N - M_2^* \mid$ and M is the solution of Q_2 based on the empirical measure.

Proof:
Note that the functional is bounded and $\beta_N[\Delta]$ is a decreasing function of Δ. Using Proposition 2.2 of Bouza [1993] it is possible to define a test for the hypothesis

$$H_0 : M_\theta \leq m \quad \text{vs} \quad H_1 : M_\theta \leq m + \Delta.$$

θ identifies the element of $\wp$ that sustains the test procedure. If we use Φ, ad.hoc. procedures are available in the literature for constructing regions]z $'_k, z_k" + 2\Delta_k$[and]z $"_{k-2\Delta_k}, z"_k$[for each stage k$\geq$ 1. A confidence interval [CI]]z $'_k, z"_k$[is obtained and the test permits to accept one of the two hypotheses or to develop another experiment with $\Delta_{k+1} = [z"_k - z'_k]/9$. The test uses the technique of inserted CI. From the results of Birgè [1991] and Bouza [1993] is easily obtained that

$$E \mid M_N - M_2^* \mid \leq Q_N[\frac{2^{k_0}}{3^{k_0+1}}][\sum_{k \in]0,k_0[} [\frac{2}{3}]^k \frac{2^{k_0-1}}{3^{k_0}}]\beta_N(\Delta) = \kappa \beta_N[\Delta]$$

where:

$$Q_N[a] = Sup \inf_{h \in]01[} \{Sup_{F \in A(F)} \int h \partial F_\theta^N + Sup_{G \in A(G)} \int (1-h) \partial G_\theta^N\}.$$

A(F)={ $F_\theta \mid M_\theta \leq m$} and A(G)={$G_\theta \mid M_\theta \geq m + \Delta$}. The procedure is stopped at $k = k_0$. See Birgè [1991] and Bouza [1993] for further details on the computational procedure.
To conclude the proof it only remains to establish that $\mid M_N - M_2^* \mid \leq \mid M_N - M_2' \mid + \mid M_2' - M_2^* \mid$ and to calculate its expectation. From Theorem 3.1 we have that E$\mid M_2' - M_2^* \mid = 0[\log(N)^{\frac{1}{2}}]$. Hence the needed bound is obtained $\square$

Acknowledgements: This paper was elaborated during a visit to the Fachbereich Mathematik supported by a research project between Humboldt-Universitat and Universidad de La Habana. The authors gratefully acknowledge the careful reading of the manuscript by the referees and their suggestions, which significantly improved the final presentation of the paper.

References

Alexander K.S. [1987]: Central limit theorems for stochastic processes under random entropy conditions. Prob. Theory and Related Fields. 75, 35-78.

Birgè, L. [1991]: Sur l 'estimation des fonctionelles d 'une loi de probabilité. Prepublications Mathemàtique 91-55. Université Paris Sud. 143-152.

Bouza C. [1993]: Optimum lot size determination under unknown distribution of the demands. Accepted for publication in the Serie Approximation and Optimization.

Clark A.J. and Scarf, H. [1962]:Approximate solutions to a simple multiechelon inventory problem. In "Studies in Applied Probability and Management Sc. '(K.J.Arrow et. al. eds) Stanford University Press. California 88-110

Frèchet M. [1957]: Sur la distance de deux lois de probabilités. C.R. Acad. de Sc. Paris. 244, 689-692.

Komlos J., Major P. and Tusnady, G. [1975] An approximation of partial sums of independent random variables and the sample distribution function. I. Z. Warsh. ver. Geb. 32, 110-131.

Laserre, J.B., Bess C. and Roubellat, F. [1985] The stochastic discrete dynamic lot size problem: an open loop solution. Operation Res. 33, 684-89.

Petrov, V.V. [1975] Sums of independent random variables. Springer Verlag.N.York.

Rosling K. [1989]:Optimal inventory policies for essembly systems under random demands. Operation Res. 37, 565-579.

Rosling K. [1993]: Bounds of Clark and Scarf's multiechelon inventory model. Res. Paper. Linkoping Institute of Technollogy.