Stability of the iterative process $x_{n+1} = Tx_n$ for Kannan maps in metric spaces

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Abstract. We prove the stability of an iterative process for a Kannan type map

1 Introduction

Let (X, d) be a metric space, $T : X \to X$ a map, $x_0 \in X$ and $x_{n+1} = Tx_n$, $n \in \mathbf{Z}_+$. Assume that T has one and only one fixed point p and that the sequence (x_n) converges to p. Let (y_n) be another sequence in X and $E_n = d(y_{n+1}, Ty_n)$, $\quad n \in \mathbf{Z}_+$. We say that the iterative process $x_{n+1} = Tx_n$ is *stable* if $\lim\limits_{n\to\infty} y_n = p$ whenever $\lim\limits_{n\to\infty} E_n = 0$.

The stability of the iterative process $x_{n+1} = Tx_n$, $n \in \mathbf{Z}_+$, when T is a contraction, was obtained by V. A. Ostrovski in 1968 ([2]). It also was obtained by A. Harder and T. Hicks in 1988 when the application T satisfies the Zamfirescu's condition or similar ones ([1]).

Here, our purpose is to prove the stability of this iterative process when the application T satisfies certain condition of Kannan type.

Let (X, d) be a metric space. A map $T : X \to X$ is said to be of Kannan type if for all $x, y \in X$

$$d(Tx, Ty) \leq b(x, y)d(x, Tx) + b(y, x)d(y, Ty)$$

where $b : X \times X \to \mathbf{R}_+$ satisfies the following conditions

i) $b(x, y) + b(y, x) < 1, \quad x, y \in X$

The existence of fixed points for Kannan type maps has been studied by R. Kannan, when $b(x,y) = b(y,x) = k < \frac{1}{2}$ ([3]), S. Reich, when $b(x,y) = b(y,x) = \frac{1}{2}$ ([5]), S. Massa ([4]) and others.

The above definition of Kannan type maps was introduced by D. Roux and E. Maluta in 1978 (cf [6]).

In this paper, however, we need a stronger condition. So, in the sequel, we say that T in of Kannan type if

ii) $\sup_{x,y\in X} [b(x,y) + b(y,x)] = \lambda < 1$

A Kannan type map can not have more than one fixed point. From theorem 2 of [7], we can easily obtain that a Kannan type map on a complete metric space (X,d) has a fixed point p and that any sequence (x_n), where $x_{n+1} = Tx_n$, $n \in \mathbf{Z}_+$ and $x_0 \in X$, converges to p

Harder and Hicks ([1], theorem 2) proved the stability of the iterative process $x_{n+1} = Tx_n$, $n \in \mathbf{Z}_+$ for a Kannan type map T such what $b(x,y) \leq k < \frac{1}{2}$ uniformly in $x, y \in X$.

The objetive of this papers is to show the stability of this iterative process when T is a Kannan type map.

2 Stability

2.1 Lemma ([1] lemma 1) *Let c be a real number, $|c| < 1$, and (b_n) a real sequence such that $\lim_{k\to\infty} b_k = 0$, then*

$$\lim_{n\to\infty} \left(\sum_{k=0}^{n} c^{n-k} b_k \right) = 0.$$

2.2 Remark *Let X be a set, $b : X \times X \to \mathbf{R}$ a map that satisfies ii) and $c \in [\lambda, 1)$. Then, for all $x, y \in X$*

$$\frac{b(y,x)}{1 - b(x,y)} \leq c.$$

2.3 Theorem *Let (X,d) be a complete metric space and $T : X \longrightarrow X$ a Kannan type map. Fix $x_0 \in X$ and set $x_{n+1} = Tx_n$, $n \in \mathbf{Z}_+$ and $p = \lim_{n\to\infty} x_n$. If (y_n) is any sequence in X, $E_n = d(y_{n+1}, Ty_n)$ and $\lambda < c < 1$. Then,*

A)

$$d(y_{n+1}, p) \le E_n + b(x_n, y_n)d(x_n, Tx_n) + d(x_{n+1}, p)$$
$$+\tfrac{E_{n-1}}{1-b(y_n,y_{n-1})} + \sum_{k=0}^{n-2} c^{n-k-1}\tfrac{E_k}{1-b(y_{k+1},y_k)} + c^n d(y_0, Ty_0)$$

B) If $\lim_{n\to\infty} E_n = 0$, then $\lim_{n\to\infty} y_n = p$.

Proof

A) For any possitive integer n,

$$\begin{aligned} d(y_{n+1}, p) &\le d(y_{n+1}, Ty_n) + d(Ty_n, Tx_n) + d(Tx_{n+1}, p) \\ &= E_n + d(Ty_n, Tx_n) + d(x_{n+1}, p), \end{aligned}$$

Then, since T is a Kannan type map,

$$\begin{aligned} d(y_{n+1}, p) &\le E_n + b(y_n, x_n)d(y_n, Ty_n) + b(x_n, y_n)d(x_n, Tx_n) \\ &\quad + d(Tx_{n+1}, p). \end{aligned} \tag{1}$$

On the other hand,

$$\begin{aligned} d(y_n, Ty_n) &\le d(y_n, Ty_{n-1}) + d(Ty_{n-1}, Ty_n) \\ &= E_{n-1} + d(Ty_{n-1}, Ty_n) \\ &\le E_{n-1} + b(y_{n-1}, y_n)d(y_{n-1}, Ty_{n-1}) + b(y_n, y_{n-1})d(y_n, Ty_n). \end{aligned}$$

Thus,

$$\{1 - b(y_n, y_{n-1})\}d(y_n, Ty_n) \le E_{n-1} + b(y_{n-1}, y_n)d(y_{n-1}, Ty_{n-1})$$

and then

$$d(y_n, Ty_n) \le \frac{E_{n-1}}{1-b(y_n,y_{n-1})} + \frac{b(y_{n-1},y_n)}{1-b(y_n,y_{n-1})} d(y_{n-1}, Ty_{n-1}), \quad n > 1. \quad (2)$$

Now, using (2), we deduce

$$\begin{aligned} d(y_n, Ty_n) & \le \tfrac{E_{n-1}}{1-b(y_n,y_{n-1})} + \sum_{i=0}^{n-2} \left[\prod_{j=i+1}^{n-1} \tfrac{b(y_j,y_{j+1})}{1-b(y_{j+1},y_j)} \right] \tfrac{E_i}{1-b(y_{i+1},y_i)} \\ & + \prod_{j=0}^{n-1} \tfrac{b(y_j,y_{j+1})}{1-b(y_{j+1},y_j)} d(y_0, Ty_0). \end{aligned} \quad (3)$$

From the remark 2.2 we have that

$$\frac{b(y_j, y_{j+1})}{1-b(y_{j+1},y_j)} \le c \quad \text{for all} \quad j = 1, 2, \cdots, n.$$

and then it follows from (3) that

$$d(y_n, Ty_n) \le \frac{E_{n-1}}{1-b(y_n,y_{n-1})} + \sum_{i=0}^{n-2} c^{n-1-i} \frac{E_i}{1-b(y_{i+1},y_i)} + c^n d(y_0, Ty_0). \quad (4)$$

Taking into account that $b(y_n, x_n) < 1$, $n \in \mathbf{Z}_+$,we finally obtain from (1) and (4) that

$$\begin{aligned} d(y_{n+1}, p) & \le E_n + b(x_n, y_n) d(x_n, Tx_n) + d(x_{n+1}, p) \\ & + \tfrac{E_{n-1}}{1-b(y_n,y_{n-1})} + \sum_{k=0}^{n-2} c^{n-k-1} \tfrac{E_k}{1-b(y_{k+1},y_k)} + c^n d(y_0, Ty_0) \end{aligned}$$

B) Suppose that $\lim\limits_{n\to\infty} E_n = 0$. Take c such that $\lambda < c < 1$. Since $b(x,y) \le \lambda < 1$ for all $x, y \in X$, we obtain from A) that

$$\begin{aligned} 0 \le d(y_{n+1}, p) & \le E_n + \lambda d(x_n, Tx_n) \\ & + d(x_{n+1}, p) + \tfrac{E_{n-1}}{1-\lambda} \\ + \sum_{k=0}^{n-2} c^{n-k-1} \tfrac{E_k}{1-\lambda} & \\ + c^n d(y_0, Ty_0). & \end{aligned}$$

According to lemma 2.1, $\lim_{n\to\infty}\left(\sum_{k=0}^{n-2} c^{n-k-1}\frac{E_k}{1-\lambda}\right) = 0$. Since $x_n \to p$, we deduce that $d(x_{n+1}, p) \to 0$ and $d(x_n, Tx_n) \to 0$ if $n \to \infty$. Therefore $\lim_{n\to\infty} d(y_{n+1}, p) = 0$ ■

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