LOCAL INTERPOLATION OF NON SMOOTH FUNCTIONS

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Abstract.- By using a specific set of linear forms defined on a triangle, we introduce a local interpolant for non smooth functions in Sobolev spaces. Optimal finite element error estimates are derived for this interpolant by imposing some regularity conditions on a sequence of triangulation's.

Subject Classification: **AMS (MOS): 65 N 30, 65 N 50.**

Optimal finite element interpolants for none smooth functions were introduced by Clment 1972 via local L^2-projections . This result was later generalized to curved domains by Bernardi 1988. More recently, Scott & Zhang 1990 &1992, developed a general approach for these type of projectors which was later reviewed and restated in a constructive way by Brenner & Scott, 1994.

In this note we aim to build a particular local polynomial interpolant for non-smooth functions *which allows us to control the smoothness of the projected function.* As opposed to the above mentioned works, we introduce here a regularity condition which explicitly shows how the projector controls the smoothness of the interpolant. More precisely, we first define a set of*averaged* L^2 functionals defined on balls ω contained in a triangle $K \in \Delta$, where Δ is a sequence of triangulations of an open bounded domain in $\mathbf{R}^2$. Next, we use these functionals to define a specific projector from the Sobolev space $H^m(K), \quad m = 0, 1$, onto the space of first degree polynomials in K. Optimal finite element error estimates are derived for this operator, by imposing both the standard regularity condition on the triangles $K \in \Delta$ and *an extra condition which regulates the relative size between the diameters of* ω *and* K. The smoothness of the local interpolant is controlled by the size of this extra regularity condition, thus letting the local projector to act as a smoothing filter. The importance of this result is that it can be used to build global approximating schemes, capable of controlling in a local way the smoothness of the global approximating surface, (cf. Atteia et. al. 1993).

We remark that the proof of the result presented here differs from those in the above mentioned works, in that our proof is based on the construction of an infinite set of reference finite elements.

Through this note we shall use the notation of Ciarlet, 1978. Let $H^m(\Omega)$ the Sobolev space, where $\Omega \subset \mathbf{R}^2$ is an open bounded set. We shall denote by $\|\cdot\|_{H^m(\Omega)}$ and $|\cdot|_{H^m(\Omega)}$ the corresponding standard norm and semi-norm respectively. Further, let $\{\omega_i\}_{i=1}^n$ be a non empty set of disjoint balls contained in Ω having diameter d_i and each with center $t_i \in \mathbf{R}^2$. Assume that we can choose a set of points $\{s_j\}_{j=1}^l$ in $\partial\Omega$, such that a regular triangulation Δ, with regularity constant σ, (Ciarlet, 1978, p.132) relative to $\{t_i\}_{i=1}^n \cup \{s_j\}_{j=1}^l$ can be constructed. Let $\Delta_0 \subset \Delta$ be the set of triangles that do not have a vertex at $\partial\Omega$, ie. whose vertices belong to $\{t_i\}_{i=1}^n$. Then for any triangle $k \in \Delta_0$ with vertices $\{t_i^k\}_{i=1}^3$ and corresponding balls $\{\omega_i^k\}_{i=1}^3$, with diameters d_i^k, we define the finite element (K_k, P_k, Σ_k) by,

- $K_k = \{(x,y) \mid (x,y)\ is\ in\ the\ triangle\ with\ minimal\ area\ that\ contains\ the\ three\ balls, \omega_1^k, \omega_2^k, \omega_3^k\}$,
- P_k is the space of polynomials of first degree on K_k
- $\Sigma_k = \{\rho_i^k \mid \ \forall\ \ v \in H^1(\Omega), \ \ \rho_i^k(v) = \frac{1}{meas(\omega_i^k)} \int_{\omega_i^k} v(t)dt, \quad i = 1,2,3,\}$.

Let $\Pi^k : H^1(K_k) \to P_k$ be the unique operator defined by,

$$\rho_i^k(\Pi^k u) = \rho_i^k(u), \quad i = 1,2,3,$$

clearly $\Pi^k p = p \ \ \forall\ p \in P_k$. Let us denote by h_k the diameter of K_k and let us further assume that the triangulation Δ_0 satisfies,

$$\frac{h_k}{d_i^k} \le \mu \quad i = 1,2,3, \tag{1.1}$$

for all $k \in \Delta_0$. Another condition that we shall impose on Δ is that any vertex belonging to any triangle $k \in \Delta$ has at most n_l related sides, where $n_l < \infty$, is a positive integer. Let us now define the following affine transformation,

$$T_k(x,y) = \begin{pmatrix} \frac{1}{h_k} & 0 \\ 0 & \frac{1}{h_k} \end{pmatrix} \begin{pmatrix} x - c_x^k \\ y - c_y^k \end{pmatrix} = (\hat{x}, \hat{y}), \tag{1.2}$$

where (c_x^k, c_y^k) is the centroid of the triangle K_k. Then we define a set of reference finite elements $(\hat{K}_k, \hat{P}_k, \hat{\Sigma}_k)$ by $\hat{K}_k = T_k(K_k)$ and $\hat{\Sigma}_k$ the corespondent linear forms. Note that $\hat{K}_k \subset [-1,1] \times [-1,1]$. We shall use the correspondence, $\hat{v} := T_k v$. With this notation we have,

Proposition 1.1. *For any reference finite element $\hat{K}_k$, $\ k \in \Delta_0$ there is a constant c, that depends only on σ and μ, such that for all $\hat{v} \in H^{m+1}(\hat{K}_k)$, $m \in \{0,1\}$,*

$$|\hat{v} - \hat{\Pi}^k \hat{v}|_{H^m(\hat{K}_k)} \le c|\hat{v}|_{H^{m+1}(\hat{K}_k)}. \tag{1.3}$$

Proof. Let $\hat{K}_k$ be defined as above. Since $\hat{\Pi}^k \hat{p} = \hat{p}, \quad \forall\ \hat{p} \in \hat{P}_k$, we have that for all $\hat{v} \in H^{m+1}(\hat{K}_k)$, (Ciarlet 1978, p.117),

$$|\hat{v} - \hat{\Pi}^k \hat{v}|_{H^m(\hat{K}_k)} \leq ||I - \hat{\Pi}^k||_{\mathcal{L}(H^2(\hat{K}_k), H^m(\hat{K}_k))} c(\hat{K}_k)|\hat{v}|_{H^{m+1}} \tag{1.4}$$

We shall prove first that,

$$\frac{||\hat{\Pi}^k \hat{v}||_{H^m(\hat{K}_k)}}{||\hat{v}||_{H^{m+1}(\hat{K}_k)}} \leq c. \tag{1.5}$$

Note that from the definition of $\hat{\Pi}^k$,

$$\hat{\Pi}^k \hat{v} = \sum_{j=1}^{3} \hat{\rho}_j^k(\hat{v}) \hat{p}_j^k,$$

where $\hat{\rho}_i^k(\hat{p}_j^k) = \delta_{ij}, \quad i, j = 1, 2, 3$ and $\hat{p}_j^k \in \hat{P}^k$. Now let,

$$\hat{p}_j^k(\hat{x}, \hat{y}) = \hat{a}_j^k \hat{x} + \hat{b}_j^k \hat{y} + \hat{c}_j^k.$$

Observe now that,

$$\hat{\rho}_i^k(\hat{x}) = \hat{x}_i^k \quad and \quad \hat{\rho}_i^k(\hat{y}) = \hat{y}_i^k \quad, i = 1, 2, 3,$$

hence letting,

$$A = \det \begin{pmatrix} \hat{x}_1^k & \hat{y}_1^k & 1 \\ \hat{x}_2^k & \hat{y}_2^k & 1 \\ \hat{x}_3^k & \hat{y}_3^k & 1 \end{pmatrix},$$

we have that,

$$|A| \geq 2\pi(\hat{r}^k)^2,$$

where $\hat{r}^k$ is the radius of the supremum ball that can be inscribed in $\hat{K}_k$. Hence if we denote by $\hat{h}_k$ the diameter of $\hat{K}_k$ and we use Cramer's rule on the system $\hat{\rho}_i^k(\hat{p}_j^k) = \delta_{ij}, \quad i, j = 1, 2, 3$ we obtain that,

$$|\hat{a}_j^k| \leq \frac{\hat{h}_k}{2\pi(\hat{r}^k)^2} \ ; \quad |\hat{b}_j^k| \leq \frac{\hat{h}_k}{2\pi(\hat{r}^k)^2} \ ; \quad |\hat{c}_j^k| \leq \frac{\hat{h}_k}{2\pi(\hat{r}^k)^2}. \tag{1.6}$$

Note now that since $\hat{K}_k \subset [-1, 1] \times [-1, 1]$, we have that,

$$\int_{\hat{K}_k} x^2 dxdy \leq \int_{\hat{K}_k} dxdy = meas(\hat{K}_k) \ \ ; \ \ \int_{\hat{K}_k} y^2 dxdy \leq \int_{\hat{K}_k} dxdy = meas(\hat{K}_k).$$

Consequently,

$$||\hat{x}||_{H^m(\hat{K}_k)} \leq (2meas(\hat{K}_k))^{1/2} \leq 2\hat{h}_k$$

and

$$||\hat{y}||_{H^1(\hat{K}_k)} \leq (2meas(\hat{K}_k))^{1/2} \leq 2\hat{h}_k, \quad m \in \{0, 1\}. \tag{1.7}$$

Now, the regularity condition that holds in Δ is satisfied by any $\hat{K}_k$, i.e. $1 \leq \frac{\hat{h}_k}{\hat{r}^k} \leq \sigma$. Then by using equations, (1.5) and (1.6), we obtain,

$$|\hat{a}_j^k| \; ||\hat{x}||_{H^m(\hat{K}_k)} \leq \frac{\sigma^2}{\pi}; \quad |\hat{b}_j^k| \; ||\hat{y}||_{H^m(\hat{K}_k)} \leq \frac{\sigma^2}{\pi}; \quad |\hat{c}_j^k| \; meas(\hat{K}_k)^{1/2} \leq \frac{\sigma^2}{\pi}.$$

¿From equations (1.1) and (1.2) we have that,

$$\frac{1}{meas(\hat{\omega}_i^k)} = \frac{4}{\pi(\hat{d}_i^k)^2} = \frac{4h_k^2}{\pi(d_i^k)^2} \leq \frac{4\mu^2}{\pi},$$

hence by using,

$$|\hat{\rho}_i^k \hat{v}| = \left| \frac{1}{meas(\hat{\omega}_i^k)} \int_{\hat{\omega}_i^k} \hat{v}(\hat{t}) d\hat{t} \right| \leq c(\mu) ||\hat{v}||_{H^2(\hat{K}_k)},$$

we obtain (1.4). Now the constant $c(\hat{K}_k)$ of equation (1.3) comes from the Bramble-Hilbert lemma, (see, Ciarlet 1978, p.115). Let us note that $\hat{K}_k$ is a star shaped domain with respect to a ball of diameter $\hat{R}_k \hat{h}_k = \hat{r}^k$, then an estimate of this constant is given by Dupont & Scott 1979, (see also Dupont & Scott 1980),

$$c = (1 + \left(\frac{2}{\hat{R}_k} \right) \mathcal{C}(m))^{1/2}.$$

Thus by noting that, $\frac{\hat{h}_k}{\hat{R}_k \hat{h}_k} \leq \sigma$, we obtain $c(\hat{K}_k) \leq c(\sigma) \; \forall \; k \in \Delta_0$. And hence from equation (1.3) one can obtain equation (1.2). ■

Theorem 1.1. *For any finite element $K_k, \; k \in \Delta_0$ and for all $u \in H^{m+1}(K_k)$, $m \in \{0, 1\}$, there is a constant c, that depends only on σ and μ, such that:*

$$|u - \Pi^k u|_{H^m(K_k)} \leq ch |u|_{H^{m+1}(K_k)}, \tag{1.8}$$

where

$$h = \max_{k \in \Delta} \{h_k\}.$$

Proof. It is a classical result of the finite element method (see Ciarlet 1978, p. 121), that if K_k and $\hat{K}_k$ are two affine equivalent sets then,

$$|v|_{H^m(K_k)} \leq c(m, 2) \frac{\hat{h}_k^m}{(r^k)^m} \cdot \frac{meas(K_k)^{1/2}}{meas(\hat{K}^k)^{1/2}} |\hat{v}|_{H^m(\hat{K}_k)}, \tag{1.9}$$

holds and

$$|\hat{v}|_{H^m(\hat{K}_k)} \leq c(m,2)\frac{h_k^m}{(\hat{r}^k)^m} \cdot \frac{meas(\hat{K}^k)^{1/2}}{meas(K_k)^{1/2}}|v|_{H^m(K_k)}. \tag{1.10}$$

Then letting m equal 0 or 1 and $v := v - \Pi^k v$ in equation (1.8), using Proposition 1.1, and equation (1.9) and also noting that,

$$\frac{\hat{h}_k}{(\hat{r}^k)^2} \cdot \frac{(h_k)^2}{r^k} \leq \frac{\sigma^3}{2} h_k,$$

we obtain equation (1.7). $\blacksquare$

ACKNOWLEDGMENTS

The authors wishes to express their gratitude to Prof. C. Bernardi for useful commentaries on this article. We would also like to acknowledge the referees for their precise remarks on this work which considerable improve its final version.

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