HILBERTIAN FRACTAL STRUCTURES AND APPROXIMATION

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ABSTRACT. Hilbertian Kernels are basic tools in functional Analysis (Quantum Mechanics, Probability, Partial Differential Equations) and especially in Approximation theory. In this lecture, I indicate a way to construct Hilbertian *fractal* Kernels relating to (classical) fractal processes. So, I can deduce the existence of a wide class of fractal splines -which are solutions of fractal variationnal problems. Those splines can be considered from the viewpoints of approximation or design.They have fractionnary Hausdorff dimensions. Below, I give only elementary examples of real or complex fractal splines. But, many other interesting developments can be deduced from our construction of Hilbertian fractal structures.

First Part. Hilbertian Fractal Structures.

1.Hilbertian Kernels (spaces).

Let $\mathcal{E}$ be a locally convex space , $\mathcal{E}'$ its topological dual and $< .,. >$ the duality bracket between $\mathcal{E}$ and $\mathcal{E}'$.
Let $(\mathcal{H}, < . \mid . >)$ be a Hilbert space and $\|.\| = (< . \mid . >)^{\frac{1}{2}}$.
Then $(\mathcal{H}, < . \mid . >) \in Hilb(\mathcal{E})$ is equivalent to

(i) $\mathcal{H}$ *is a subspace of* $\mathcal{E}$ (*we denote* j *the injection from* $\mathcal{H}$ *into* $\mathcal{E}$)

(ii) $\forall e' \in \mathcal{E}' , \exists M(e') \in \mathbb{R}_+^* \ s.t. : \forall h \in \mathcal{H} , |< jh, e' >| \leq M(e').\|h\|$

So there exists only one $H : \mathcal{E}' \to \mathcal{E}$ s.t. :

$$\forall h \in \mathcal{H} , \forall e' \in \mathcal{E}' , < jh, e' > = < h \mid \tilde{H} e' > \ \ with \ \ H = j\tilde{H}.$$

If Λ is the duality map from $\mathcal{H}'$ *onto* $\mathcal{H}$ we have the following scheme :

$$\mathcal{E}' \xrightarrow{j^t} \mathcal{H}' \xrightarrow{\Lambda} \mathcal{H} \xrightarrow{j} \mathcal{E}$$

and $H = j\Lambda j^t$ is called the Hilbertian Kernel of $(\mathcal{H}, < . \mid . >) \in Hilb(\mathcal{E})$.
Moreover :

$$(i) \forall e', f' \in \mathcal{E}' , < Hf', e' > = \overline{< He', f' >}$$

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and

$$(ii) < He', e' > \geq 0$$

If H is a linear w-continuous map from $\mathcal{E}'$ into $\mathcal{E}$ which verifies the previous properties (i) and(ii) then H is the Hilbertian Kernel of a Hilbert space $(\mathcal{H}, < . \mid . >) \in Hilb(\mathcal{E})$.

A special case :
Ω is an arbitrary set and $\mathcal{E} = \mathbb{R}^{\Omega}$. So:

$$\mathcal{E}' = \{\sum_{j \in J} \lambda_j . \delta_{t_j} \; ; \lambda_j \in \mathbb{R} \; , t_j \in \Omega \; , J finite\}.$$

We set :

$$\forall s, t \in \Omega \; , H(s,t) = < H\delta_t, \delta_s >$$

So:

$$\forall h \in \mathcal{H} \; , \forall t \in \Omega \; , h(t) = < h \mid H(.,t) >$$

H is called the Aronszjan (or reproducing) Kernel of $(\mathcal{H}, < . \mid . >) \in Hilb(\mathbb{R}^{\Omega})$

Example 1 : $a, b \in \mathbb{R} \; , a < b, \; \mathcal{E} = C^0[a,b]$

$$\mathcal{H} = \{f \in H^1(a,b) \; ; f(a) = 0\} \; ,$$

$$\forall f, g \in \mathcal{H} \; , < f \mid g > = \int_a^b f'.g' \; dt.$$

Then :

$$\forall s, t \in (a,b) \; , H(s,t) = t - a - (t-s)_+$$

Example 2 : $\mathcal{E} = \mathcal{D}'(\mathbb{R}) \; , \mathcal{E}' = \mathcal{D}(\mathbb{R})$

Let : $\mu_0 \; , \mu_1 \in \mathbb{R}_+^*$ and $\forall u \in \mathcal{D}'(\mathbb{R}) \; \; , Hu = -\mu_1 u'' + \mu_0 u$
Then :

$$\forall \phi \in \mathcal{D}(\mathbb{R}) \; , < H\phi, \phi > = \int_{\mathbb{R}} (\mu_1 \phi'^2 + \mu_0 \phi^2) ds \geq 0$$

H is the Hilbertian Kernel of $\mathcal{H}^{-1}(\mathbb{R})$

Operations on Hilbert Kernels (Spaces) .

One can define many operations on Hilbert Kernels(spaces) such as the sum or the product by a scalar,the tensorial product.
The most useful is the image by a w-continuous map.

Let $\mathcal{E}$ $(resp. \; \mathcal{F})$ be a locally convex space and $\mathcal{E}'$ $(resp. \; \mathcal{F}')$ its topological dual.
Let $(\mathcal{H}, < .|. >_{\mathcal{H}}) \in Hilb(\mathcal{E})$ and H its Hilbertian Kernel.

If : $u^t : \mathcal{F}' \rightarrow \mathcal{E}'$ is the transpose of u ,we have the following scheme:

$$\mathcal{F}' \xrightarrow{u^t} \mathcal{E}' \xrightarrow{H} \mathcal{E} \xrightarrow{u} \mathcal{F}$$

$K = uHu^t$ is called the image of the Hilbert Kernel H by u .

Let $(\mathcal{K}, <.|.>_{\mathcal{K}})$ be the Hilbert space relating to K.

If $x \in \mathcal{H}$, $x = x^{\perp} + \tilde{x}$,with $x^{\perp} \in (Keru)^{\perp}$, $\tilde{x} \in Keru$,then

$$\forall x, y \in \mathcal{H}, < ux|uy >_{\mathcal{K}} = < x^{\perp}|y^{\perp} >_{\mathcal{H}}$$

2. Spline functions.

Let $(\mathcal{H}, <.|.>) \in Hilb(\mathcal{E})$, H its Hilbertian Kernel and $\alpha_1, ...\alpha_n \in \mathbb{C}$, $e'_1, ..., e'_n \in \mathcal{E}$'

$$\mathcal{P}_0 \ : \ Inf\Big\{||x||;\ x \in \mathcal{H}, < x, e'_k > = < x|He'_k > = \alpha_k, 1 \leq k \leq n\Big\}$$

$\mathcal{P}_0$ has a unique solution σ s.t:

$$\sigma = \sum_{j=1}^{n} \lambda_j.He'_j \ , \lambda_1, ..., \lambda_n \in \mathbb{C},$$

$$< \sigma, e'_k > = \alpha_k \ , 1 \leq k \leq n$$

A special case:

Ω is an arbitrary set, $\mathcal{E} = \mathbb{C}^{\Omega}$, $e'_j = \delta_{t_j}$, $t_j \in \Omega$, $1 \leq j \leq n$
Then:

$$\forall t \in \Omega,\ \sigma(t) = \sum_{j=1}^{n} \lambda_j.H(t, t_j) \ ,$$

$$\lambda_1, ..., \lambda_n \in \mathbb{C},\ \sigma(t_k) = \alpha_k \ , 1 \leq k \leq n$$

It can be proved that all classical spline functions(B-splines,Box-splines, etc..) could be calculated by the mean of previous formulas in which H is explicitely known(cf. [1])

3.Fractals .

Fractal curves or surfaces are essentially characterized by generic irregularities, fractionaly dimensions, fragmentations at all scales.

Example 1: Weierstrass functions

$$\omega, \delta \in \mathbb{R} \ , \omega > 1 \ , 0 < \delta < 1$$

Let:

$$\forall t \in \mathbb{R} \ ,\ \phi(t) = \sum_{n \in \mathbb{N}} \omega^{-n\delta}.cos(\omega^n t)$$

ϕ is called the Weierstrass function.

Let $\epsilon \in \mathbb{R}^*_+$ *and* $N_\epsilon(\phi)$ be the smallest number of sets (in $\mathbb{R}^2$) of diameters at most ϵ which can cover $graph(\phi)$. We set:

$$dim_B(graph(\phi)) = \lim_{\epsilon \to 0} \frac{Log\ N_\epsilon(\phi)}{-Log\epsilon}.$$

If ω is large enough,

$$dim_B(graph(\phi)) = 2 - \delta \ , 1 < 2 - \delta < 2.$$

$graph(\phi)$ is a curve of fractal type.

Let $\mathcal{B}(\mathbb{R})$ be the space of uniformly bounded functions defined on $\mathbb{R}$ and:

$$\forall x \in \mathcal{B}(\mathbb{R}),\ \forall t \in \mathbb{R}\ ,\ (T_{\omega,\delta})(t) = \omega^{-\delta}.x(\omega t)$$

$$\forall p \in \mathbb{N},\ u_p = \sum_{n=0}^{p} (T_{\omega,\delta})^n$$

Then:

$$\lim_{p \to \infty} u_p = (1 - T_{\omega,\delta})^{-1}$$

We set $u_F = (1 - T_{\omega,\delta})^{-1}$ and we say that u_F is a fractal linear map from $\mathcal{B}(\mathbb{R})$ *into* $\mathcal{B}(\mathbb{R})$.

Generally, we shall say that a (linear) map u is *fractal* if there exists a smooth function ϕ such that $u\phi$ is *fractal* function.

Example 2. Dubuc's fractal.

Let : $\forall p \in \mathbb{N}\ , D_p = \{m.2^{-p}; m \in \mathbb{Z}\}$ and $D = \bigcup_{p \in \mathbb{N}} D_p$

Then: $\mathbb{Z} = D_0 \subset D_1 \subset ... \subset D_p \subset ... \subset D$. D is dense in $\mathbb{R}$.

Let: $h_p = 2^{-p-1},\ \theta_p^1 = 3.h_p, \theta_p^2 = h_p,\ \theta_p^3 = -h_p,\ \theta_p^4 = -3.h_p$ and $a_1, a_2, a_3, a_4 \in \mathbb{C}$
We set :

$$\forall y \in \mathbb{C}^{D_p},\ \forall s \in D_{p+1}, \quad (v_{p+1}y)(s) = \begin{cases} y(s) & \text{if } s \in D_{p+1}, \\ \sum_{j=1}^4 a_j y(s - \theta_p^j) & \text{if } s \in D_{p+1} \setminus D_p \end{cases}$$

Let $\Phi_0 \in \mathbb{R}^{\mathbb{Z}}$ s.t.

$$\Phi_0(0) = 1\ ; \forall m \in \mathbb{Z}^*, \Phi_0(m) = 0$$

and

$$\Phi_1 = v_1\Phi_0,\ \Phi_2 = v_2\Phi_1,\ \Phi_{p+1} = v_{p+1}\Phi_p,$$

We set :

$$\forall f \in C^0(\mathbb{C})\ ,\ < f, \Phi_p >= \sum_{s \in D_p} f(s).\overline{\Phi_p}(s)$$

So , Φ_p is identified to a bounded positive measure on $\mathbb{C}$.
It can be proved that, for suitable choices of (a_1, a_2, a_3, a_4), Φ_p converges to a *fractal* bounded positive measure on $\mathbb{C}$ when p tends to infinity.

Let : $\forall p \in \mathbb{N},\ u_p = v_p ... v_1$ and $u_F = \lim_p u_p$.

Then u_F is a fractal linear mapping .

4.Fractal Hilbert spaces(Kernels).

Example 1 (as in paragraph 3 example 1)

Let : $\omega,\ \delta \in \mathbb{R},\ \omega > 1,\ 0 < \delta < 1$, and $(\mathcal{H}, < . \mid . >_{\mathcal{H}}) \in Hilb(\mathbb{R}^{\mathbb{R}})$ with H as its Hilbertian Kernel.

For each $p \in \mathbb{N}$, denote by $\mathcal{B}(\mathbb{R}^p)$ the linear space of functions on $\mathbb{R}^p$,uniformly bounded on $\mathbb{R}^p$.

For any $x \in \mathcal{B}(\mathbb{R})$, we set :

$$\forall t \in \mathbb{R} \quad ,(T_{\omega,\delta}\ x)(t) = \omega^{-\delta}.x(\omega t)$$

and

$$u_F x = \sum_{n \in \mathbb{N}} (T_{\omega,\delta})^n x = (1 - T_{\omega,\delta})^{-1} x.$$

Then :

$$\forall x \in \mathcal{B}(\mathbb{R}) \ ,\forall t \in \mathbb{R} \ ,\ (u_F x)(t) =< u_F x, \delta_t >=< x, u_F^t \delta_t >$$
$$= \sum_{n \in \mathbb{N}} \omega^{-n\delta}.x(\omega^n t) =< x, \sum_{n \in \mathbb{N}} \omega^{-n\delta} \delta_{\omega^n t} > .$$

and

$$u_F^t \delta_t = \sum_{n \in \mathbb{N}} \omega^{-n\delta} \delta_{\omega^n t}$$

Assume that $H \in \mathcal{B}(\mathbb{R}^2)$. Let $K = u_F.H.u_F^t$.

Then:

$$\forall s,t \in \mathbb{R},\ K(s,t) = \sum_{m,n \in \mathbb{N}} \omega^{-(m+n)\delta} H(\omega^n s, \omega^n t)$$

But:

$$\forall s \in \mathbb{R} \ ,\ K(s,0) = (1 - \omega^{-\delta})^{-1} [u_F.H(.,0)](s).$$

If $\forall s,t \in \mathbb{R},\ H(s,t) = cos(s-t)$,then :

$$K(s,0) = (1 - \omega^{-\delta})^{-1} [u_F.cos(.)](s) = (1 - \omega^{-\delta})^{-1} \phi(s)$$

with the notations of paragraph 3, example 1.

So, we say that K is a fractal Hilbertian Kernel.

If $(\mathcal{K}, < . \mid . >_{\mathcal{K}})$ is the Hilbertian subspace relating to K , we have:

$$\forall z_1, z_2 \in \mathcal{K} \ , < z_1 \mid z_2 >_{\mathcal{K}} =< (u_F)^{-1} z_1 \mid (u_F)^{-1} z_2 >_{\mathcal{H}} =< z_1 - T_{\omega,\delta} z_1 \mid z_2 - T_{\omega,\delta} z_2 >_{\mathcal{H}}$$

.

Définition : Let $\mathcal{K}$ be a Hilbert space and K its Hilbertian Kernel.We shall say that $(\mathcal{K}\ , < . \mid . >_{\mathcal{K}})$ or K is *fractal* if K is obtained by a fractal process.

In the following we shall consider only linear fractal processes.

Example 2

Let : $\quad \forall t \in \mathbb{R}\ , \phi(t) = \sum_{n \in \mathbb{N}} n^{-2}.sin(n^2 t)$

It is known that ϕ is a function of fractal type.

Let :

$$\forall t \in \mathbb{R}\ ,\forall x \in \mathcal{B}(\mathbb{R})\ ,\forall p \in \mathbb{N}^*\ ,(u_p x)(t) = p^{-2}.x(p^2 t)$$

and

$$u_F x = \sum_{n \in \mathbb{N}^*} u_n x$$

Let $(\mathcal{H},\ <.\mid.>_{\mathcal{H}})$ be a Hilbert space.
We assume that the Hilbertian Kernel $H \in \mathcal{B}(\mathbb{R}^2)$
Let : $K = u_F.H.u_F^t$. Then :

$$\forall s,t \in \mathbb{R}\ , K(s,t) = \sum_{p,q \in \mathbb{N}^*} p^{-2}.q^{-2}.H(p^2 s, q^2 t)$$

and

$$K(s,0) = (\sum_{q \in \mathbb{N}^*} q^{-2})(\sum_{p \in \mathbb{N}^*} H(p^2 s, 0))$$

Now , we suppose that :

$$\forall s,t \in \mathcal{R}\ \ , H(s,t) = exp(i(s-t))\ \ (i^2 = -1)$$

H is a complex Hilbertian Kernel and :

$$\mathcal{I}m(K(s,0)) = (\sum_{q \in \mathbb{N}^*} q^{-2})\phi(s).$$

Then, K is a fractal Hilbertian Kernel.

Example 3 .Dubuc's processes (Cf paragraph 3, example 2).
Let : $\forall p \in \mathbb{N}\ , D_p = \{m.2^{-p}; m \in \mathbb{Z}\}$

$\theta_p^1 = 3.h_p\ , \theta_p^2 = h_p\ , \theta_p^3 = -h_p\ , \theta_p^4 = -3.h_p\ , h_p = 2^{-p-1}$

and :

$\forall y \in \mathbb{C}^{D_{p+1}},\ \forall s \in D_{p+1},$

$$(v_{p+1}y)(s) = \begin{cases} y(s) & \text{if } s \in D_p, \\ \sum_{j=1}^4 a_j y(s-\theta_p^j) & \text{if } s \in D_{p+1} \setminus D_p \end{cases}$$

$$v_{p+1}^t \delta_s = \begin{cases} \delta_s & \text{if } s \in D_p, \\ \sum_{j=1}^4 a_j(\delta_s - \delta_{\theta_p^j}) & \text{if } s \in D_{p+1} \setminus D_p \end{cases}$$

We set : $\forall p \in \mathbb{N}^*$,

$$u_p = v_p ......v_1\ \ , u_p^t = v_1^t ......v_p^t,\ \ u_F = lim_p u_p\ , u_F^t = lim_p u_p^t$$

Let $\rho_0\ :\mathbb{C}^{\mathbb{R}} \to \mathbb{C}^{D_0}$ s.t.: $\forall y \in \mathbb{C}^{\mathbb{R}}$, $\rho_0 y = y\mid_{D_0}$ and :

$K_0 = \rho_0 H \rho_0^t$, $K_1 = v_1 K_0 v_1^t = u_1 K_0 u_1^t$,.........., $K_{p+1} = v_{p+1} K_p v_{p+1}^t = u_{p+1} K_0 u_{p+1}^t$,.....

It can be easily proved that :

(i) $\forall p \in \mathbb{N}$, K_p is a Hilbertian Kernel

(ii) For suitable choices of (a_1, a_2, a_3, a_4) the sequence $\{K_p ; p \in \mathbb{N}\}$ is convergent.

Moreover : $\forall s \in D_{p+1}$,

$$K_p(s,0) = \begin{cases} K_{p-1}(s,o) & \text{if } s \in D_p, \\ \sum_{j=1}^{4} a_j K_{p-1}(s-\theta_p^j, 0) & \text{if } s \in D_{p+1} \setminus D_p \end{cases}$$

Let : $\forall s,t \in \mathbb{R}$, $H(s,t) = exp(i\frac{4\pi}{5}(s-t))$ and $K = u_F K_0 u_F^t$ $(= \lim_p u_p K_0 u_{p^t})$

When :

$a_1 = a_4 = \frac{1}{2}$, $a_2 = a_3 = 0$
or $a_1 = a_4 = 0$, $a_2 = \frac{1}{2} + \frac{i}{2}$, $a_3 = \overline{a_2}$
or $a_1 = a_4 = 0$, $a_2 = \frac{1}{5} + \frac{i}{3}$, $a_3 = \overline{a_2}$

the curve Γ with coordinates $(x(s), y(s))$, $s \in \mathbb{R}^2$ s.t.

$x(s) = \mathcal{R}e(K(s,0))$, $y(s) = \mathcal{I}m(K(s,0))$

is a curve of fractal type and K is a fractal Hilbertian Kernel.

Let $\hat{\Phi}_p$ be the Fourier's transform of Φ_p.

Then we know (Bochner's theorem) that :

$(\zeta, \eta) \rightarrow \hat{\Phi}_p(\zeta, \eta)$ is a complex Hilbertian Kernel.

When $a_1, a_2, a_3, a_4 \in \mathbb{R}_+$ and $a_1 = a_4, a_2 = a_3$, then $(\zeta, \eta) \rightarrow \hat{\Phi}_p(\zeta, \eta)$ is a real Hilbertian Kernel.

For suitable choices of (a_1, a_2, a_3, a_4) , $\hat{\Phi}_p$ converges to a fractal Hilbertian Kernel when p tends to infinity.

Remark : In the previous examples we have easily calculated fractal Hilbertian Kernels.
Generally, the determination of $(\mathcal{K}, < . \mid . >_{\mathcal{K}})$ from a fractal Hilbertian Kernel is often difficult but provides many useful informations when it is possible.

Second Part. Applications.

5.Design of fractal curves (surfaces) .

Above, we have proved that classical fractal functions can be regarded as restrictions of some fractal Hilbertian Kernels to straight lines.

But, from the previous study, we can deduce, more generally that : there exists a wide class of fractal functions, of spline type, denoted by σ , s.t. :

$$\sigma(.) = \sum_{j \in J} \lambda_j \ l_{j,*} K(.,*)$$

with: $K = \mathcal{F}(K_1, ..., K_p)$, $K_j = u_{F_j} H_j u_{F_j}^t$ is for $j \in \{1, ...p\}$ a fractal Hilbertian Kernel and l_k , $k \in J$ are linear functionals.

In that class ,we can adjust σ to satisfy some technical or aesthetic conditions,for example, by choosing convenient $K_1, ..., K_p$ and $l_j, j \in J$.

6. About interpolating fractal splines.

Let $t_0, ...t_n \in \mathbb{R}$, and $\sigma = \sum_{j=0}^{n} \lambda_j K(., t_j)$ a fractal spline defined on $\mathbb{R}$.

The two following properties are equivalent :

(i) $\forall \alpha = (\alpha_0, ..., \alpha_n) \in \mathbb{R}^{n+1}$, $\exists$ only one $\lambda = (\lambda_0, ..., \lambda_n) \in \mathbb{R}^{n+1}$ s.t. $\sigma(t_j) = \alpha_j$, $0 \leq j \leq n$ and:

(ii) the matrix $\tilde{K}_n = (K(t_i, t_j))_{0 \leq j \leq n}$ is invertible . .

To prove property (ii) is not generally easy and needs, in each case,specific considerations.

Example (Cf example 1 paragraph 3)

$\omega, \delta \in \mathbb{R}$, $\omega > 1$, $0 < \delta < 1$

$$\forall s, t \in \mathbb{R} \text{ , } K(s,t) = \sum_{p,q \in \mathbb{N}} \omega^{-(p+q)\delta} H(\omega^p s, \omega^q t)$$

We cannot assert, a priori, that $\tilde{K}_n$ is invertible or equivalently ,strictly positive definite.

But, let :$\forall s, t \in \mathbb{R}$, $L(s,t) = \sum_{p \in \mathbb{N}} \omega^{-2p\delta} H(\omega^p s, \omega^p t)$

Denote by $\tilde{H}_n(p, \omega)$, the matrix $(H(\omega^p t_j, \omega^p t_l)_{0 \leq j,l \leq n}$

Assume that $\tilde{H}_n(p, 0)$ is strictly positive definite.

If $\tilde{L}_n = (L(t_i, t_j)_{0 \leq j,l \leq n}$, we have:

$$\forall x \in \mathbb{R}^{n+1} \text{ , } x \neq 0 \quad x^t \tilde{L}_n x = x^t \tilde{H}_n(p, 0) x + \sum_{p \in \mathbb{N}^*} \omega^{-2p\delta} (x^t \tilde{H}_n(p, \omega) x) > 0$$

If $\tau = \sum_{j=0}^{n} \lambda_j L(., t_j)$, $\lambda_0, ..., \lambda_n \in \mathbb{R}$,

then :$\forall \alpha = (\alpha_0, ..., \alpha_n) \in \mathbb{R}^{n+1}$, $\exists$ only one $\lambda = (\lambda_0, ..., \lambda_n) \in \mathbb{R}^{n+1}$ s.t.:

$\tau(t_j) = \alpha_j$, $0 \leq j \leq n$.

7. Approximation of fractal curves by fractal splines.

The classical approximation of functions by splines can be extended to fractals using schemes like the following :

Let $(\mathcal{H}, < . \mid . >_{\mathcal{H}}))$ be a Hilbert space and H its Hilbertian Kernel.

We set : $\|.\|_{\mathcal{H}} = (< . \mid . >_{\mathcal{H}})^{\frac{1}{2}}$

Let ϕ be a fractal function defined on an open set Ω of $\mathbb{R}$ s.t.:

$\phi = u_F f$ where $f \in \mathcal{H}$ and u_F is a linear fractal map.

Assume that :

$$\sigma = \sum_{j \in J} \lambda_j H(., t_j) \ , t_j \in \Omega \ , \lambda_j \in \mathbb{R} \ , j \in J \quad \text{is s.t. :}$$

$$\|\sigma - f\|_{\mathcal{H}} \leq cst.h^{\alpha} \|f\|_{\mathcal{H}} \ , \ \alpha \in \mathbb{R}_+^* \ , h \sim (Card J)^{-1}$$

We suppose, futhermore, that u_F is a (linear) continuous map from $(\mathcal{H}, \|.\|_{\mathcal{H}})$ into a vector normed space $(\mathcal{B}, \|.\|_{\mathcal{B}})$ .Then :

$$\|u_F \sigma - u_F f\|_{\mathcal{B}} \leq \|u_F\|_{\mathcal{L}(\mathcal{B},\mathcal{H})} . \|\sigma - f\|_{\mathcal{H}} \leq cst.h^{\alpha} \|u_F\|_{\mathcal{L}(\mathcal{B},\mathcal{H})} . \|f\|_{\mathcal{H}}$$

Example :

(i) We denote by $< . \mid . >_1$ $(resp. \|.\|_1)$ a scalar product (resp. a norm) on $\mathcal{H}^1(0,1)$ s.t.:

$(\mathcal{H}^1(0,1), < . \mid . >_1) \in Hilb(\mathbb{R}^{(0,1)})$.

Then: $\exists M \in \mathbb{R}_+^* \ , \forall t \in \mathbb{R} \ , \forall e \in \mathcal{H}^1(0,1), \ Sup\{\mid e(t) \mid ; t \in (0,1)\} \leq M.\|e\|_1$

Let $\mathcal{H} = \{f \in C^0(\mathbb{R}); \ f_{|(0,1)} = \tilde{f} \in \mathcal{H}^1(0,1) \ and \ \forall t \in \mathbb{R} \ , f(t+1) = f(t)\}$.

We set :

$\forall f, g \in \mathcal{H} \ , < f \mid g >_{\mathcal{H}} = < \tilde{f} \mid \tilde{g} >_1 \ , \ \|f\|_{\mathcal{H}} = \|\tilde{f}\|_1$.

Let : $\forall f \in \mathcal{B}(\mathbb{R}) \ , \|f\|_{\mathcal{B}} = \sup_{t \in \mathbb{R}} \mid f(t) \mid$

Then : $\mathcal{H} \subset \mathcal{B} \ and \ : \forall f \in \mathcal{H}$,

$\|f\|_{\mathcal{B}} = Sup\{\mid f(t) \mid ; t \in (0,1)\} \leq M.\|\tilde{f}\|_1 = M.\|f\|_{\mathcal{H}}$

(ii) Let $n \in \mathbb{N}^* \ , t_0, ..., t_n \in (0,1)$ s.t.: $t_0 = 0, \ t_n = 1 \ , t_{j+1} - t_j = \frac{1}{n} = h$.

Given $f \in \mathcal{H}$ we denote by $\sigma(.; f)$ the polynomial spline function, of degree one, on $\mathbb{R}$, s.t. :

$\tilde{\sigma}(.; f)$ is the solution of the problem :

$\mathcal{P} \ : \ Inf\{\|\tilde{g}\|_1 \ ; g \in \mathcal{H}, \ g(t_j) = f(t_j) \ , 0 \leq j \leq n\}$ and :

$\forall t \in \mathbb{R} \ , \sigma(t+1; f) = \sigma(t; f)$

It can be proved that : $\|\sigma(.; f) - f\|_{\mathcal{H}} \leq cst.h.\|f\|_{\mathcal{H}}$

Let $u_F : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{B}(\mathbb{R})$ s.t. :

$\forall t \in \mathbb{R}$, $\forall f \in \mathcal{H}$, $(u_F f)(t) = \sum_{p\in\mathbb{N}} \omega^{-p\delta} f(\omega^p t)$ $(with\ \omega > 1\ , 0 < \delta < 1)$

Then : $\|u_F\|_{\mathcal{B}} \leq \sum_{p\in\mathbb{N}} \omega^{-p\delta}\|f\|_{\mathcal{B}} \leq (1-\omega^{-\delta})^{-1}.M.\|f\|_{\mathcal{H}}$

$(\Rightarrow \|u_F\|_{\mathcal{L}(\mathcal{H},\mathcal{B})} \leq (1-\omega^{-\delta})^{-1})$

and : $\forall f \in \mathcal{H}, \|u_F(\sigma(.;f)-f)\|_{\mathcal{B}} \leq M_1(\omega,\delta).h.\|f\|_{\mathcal{H}}$.

8. Fractal variational problem .Fractal differential operator and Green function.

8.1 Let : $\mathcal{H} = \mathcal{H}^1(\mathbb{R})$ and :

$$\forall f, g \in \mathcal{H}\ , < f \mid g >= \int_{\mathbb{R}} (f.g + f'.g')dt,\ \|.\| = (< . \mid . >)^{\frac{1}{2}}.$$

Assume that $C^0(\mathbb{R})$ is endowed with its classical Frechet topology.

Then :$(\mathcal{H}, < . \mid . >) \in Hilb(C^0(\mathbb{R})) \bigcap Hilb(L^\infty(\mathbb{R}))$.

Let $\mu_0, \mu_1 \in \mathbb{R}_+^*$ and : $\forall f, g \in \mathcal{H}$, $\gamma(f,g) = \int_{\mathbb{R}} (\mu_0 f.g + \mu_1 f'.g')dt$

γ is bilinear,symmetric, continuous and coercive on $(\mathcal{H}, < . \mid . >)^2$.

Given $\rho \in L^2(\mathbb{R})$, let :

$\mathcal{P} : Find\ f \in \mathcal{H}\ s.t. : \forall g \in \mathcal{H},\ \gamma(f,g) = \int_{\mathbb{R}} \rho.g dt.$

$\mathcal{P}$ has a unique solution τ.

Moreover :$(\mathcal{H},\gamma) \in Hilb(L^\infty(\mathbb{R}))$ and $\forall s,t \in \mathbb{R},\ H(s,t) = \alpha.exp(-\beta \mid s-t \mid)$ α,β depend on μ_0 and μ_1.

If H is the Hilbertian Kernel of $(\mathcal{H},\gamma)$,then :

$\forall t \in \mathbb{R}$, $\tau(t) = \gamma(\tau, H(.,t)) = \int_{\mathbb{R}} \rho(s).H(s,t)ds.$

Denote by $< .,. >$ the duality bracket between $\mathcal{D}(\mathbb{R})$ and $\mathcal{D}'(\mathbb{R})$.

$\forall t \in \mathbb{R}$, $\forall \phi \in \mathcal{D}(\mathbb{R})$,

$< \phi, \delta_t >= \phi(t) = \gamma(\phi, H(.,t)) = \int_{\mathbb{R}} (\mu_0\phi.H(.,t) + \mu_1\phi'.H'(.,t))dt$

$= \mu_0 < \phi, H(.,t) > +\mu_1 < \phi', H'(.,t) >=< \phi, \mu_0.H(.,t) - \mu_1.H''(.,t) >.$

We set : $L = -\mu_1.\frac{d^2}{dt^2} + \mu_0$.Then : $\forall t \in \mathbb{R}$, $LH(.,t) = \delta_t \ \in \mathcal{D}'(\mathbb{R})$

H is the Green function relating to the differential operator L .

Moreover :

(i)$\forall \phi, \psi \in \mathcal{D}(\mathbb{R})$, $< LH\phi, \psi >=< \phi, \psi >$

(ii)$\forall \phi \in \mathcal{D}(\mathbb{R})$, $< L\phi, \phi >\geq 0$ and :

L is a Hilbertian Kernel.

8.2 Let : $\forall t \in \mathbb{R}$, $\forall g \in C^0(\mathbb{R})$, $(u_F g)(t) = \sum_{k \in \mathbb{N}} 2^{-k-1} g(3^{2k} t)$

u_F is a linear map, w-continuous from $C^0(\mathbb{R})$ into $C^0(\mathbb{R})$
and $K = u_F.H.u_F^t$ is a Hilbertian fractal Kernel.

Let : $T : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ s.t.: $\forall g \in L^\infty(\mathbb{R})$, $Tg = \frac{1}{2} g(3^2.)$

Then : $\|T\|_{\mathcal{L}(L^\infty(\mathbb{R}), L^\infty(\mathbb{R}))} < \frac{1}{2}$ *and* $u_F = \frac{1}{2}(1-T)^{-1}$

We denote by $(\mathcal{K}, \gamma_{\mathcal{K}})$ the Hilbert space corresponding to K .

$\forall z_1, z_2 \in \mathcal{K},\ \gamma_{\mathcal{K}}(z_1, z_2) = \gamma(u_F^{-1} z_1, u_F^{-1} z_2)$ and $u_F \tau$ is the unique solution of :

$$\mathcal{P}_F \ : Inf_{z \in \mathcal{K}} \{ \tfrac{1}{2} \gamma_{\mathcal{K}}(z, z) - \int_{\mathbb{R}} \rho.(u_F^{-1} z).ds \}$$

Moreover :$\forall \phi, \psi \in \mathcal{D}(\mathbb{R}),\ u_F^{-1} \phi = 2\phi - \phi(9.) \in \mathcal{D}(\mathbb{R})$

$$u_F^t \psi = \sum_{k \in \mathbb{N}} 2^{-k-1}.3^{-2k}.\psi(3^{-2k}.)$$

and :$< (u_F^{-t})^{-1}.L.(u_F)^{-1}).(u_F.H.u_F^t)\phi, \psi > = < \phi, \psi >$

We can assert that :
$K = u_F.H.u_F^t$ is the *fractal Green Kernel* associated to the linear differential fractal operator :

$$M = (u_F^t)^{-1}.L.(u_F)^{-1})$$

$\forall \phi \in \mathcal{D}(\mathbb{R})$, $< M\phi, \phi > \geq 0$ and M is a fractal Hilbertian Kernel.

Let us calculate K and M.

$$\forall s, t \in \mathbb{R} , K(s,t) = \sum_{k,l \in \mathbb{N}} 2^{-(k+l)-2}.H(3^{2k}s, 3^{2k}t)$$

$$\forall \phi, \psi \in \mathcal{D}(\mathbb{R}) , < M\phi, \psi > = < L(u_F^{-1}\phi), (u_F^{-1}\psi) >$$

$$= \int_{\mathbb{R}} ((\mu_1.\frac{d^2}{dt^2} + \mu_0).(2\phi(t) - \phi(9t))).(2\psi(t) - \psi(9t))dt$$

Then :$\forall \phi \in \mathcal{D}(\mathcal{R})$,$\forall t \in \mathcal{R}$,

$$(M\phi)(t) = \mu_1(-\tfrac{2}{9}\phi''(\tfrac{t}{9}) + 13\phi''(t) - 162\phi''(9t)) + \mu_0(-\tfrac{2}{9}\phi(\tfrac{t}{9}) + \tfrac{37}{9}\phi(t) - 2\phi(9t))$$

References

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