Padé type approximants with preassigned zeros

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1 Introduction

In the present paper we study the convergence problem of Padé Type Approximants (PTAs) with partly preassigned zeros and/or poles. We concentrate our attention mainly to the case of approximants with preassigned zeros (partly or completely). As far as we know no general convergence results for such approximants have been obtained. We also state some open problems.

An argument to study PTAs is to enlarge the domain of approximation of a given function and sometimes even to accelerate it. Roughly speaking, if we choose the preassigned part of the approximants correctly, for example, fix zeros where the function is small, poles where the function is large, and poles and zeros at the same time near singularities, we may improve convergence of the approximants.

2 Definitions and main results

Throughout this paper $\Omega \subset \overline{\mathbf{C}}$ denotes a domain (open connected set) in the extended complex plane containing infinity: $\infty \in \Omega$; $g_\Omega(z)$ denotes the

Green function of Ω with pole at infinity; f is an analytic (holomorphic) function in Ω; $\sigma(P)$ denotes the normalized (unit) zero-counting measure of a polynomial P obtained by putting equal mass at each zero of P counting multiplicity.

Definition 1 *Let f be a function analytic at ∞, $f(\infty) \neq 0$ (for the case $f(\infty) = 0$ see Remark 1 below), and let $\tilde{P}_k, \tilde{Q}_\ell$ be (fixed) non-zero polynomials of degree k and ℓ, respectively. We say that a rational function $R_n = P_{n-k}\tilde{P}_k/Q_{n-\ell}\tilde{Q}_\ell$, where P_{n-k} and $Q_{n-\ell}$ are (free) non-zero polynomials of degree at most $n-k$ and $n-\ell$, respectively, determined from the interpolation condition*

$$f(z) - R_n(z) = \mathcal{O}(z^{-(2n+1-k-\ell)}) \quad (\text{as } z \to \infty), \tag{1}$$

is the PTA of order n to f with preassigned zeros at the zeros of $\tilde{P}_k$ and preassigned poles at the zeros of $\tilde{Q}_\ell$.

Formula (1) is written in non-linearized form. The definition could also be written in linearized form (multiplying (1) by $Q_{n-\ell}\tilde{Q}_\ell$) which has the advantage that the PTA always exists. However, in the case we shall consider the PTA exists also in non-linearized form.

Remark 1 *We should make one agreement about Definition 1. If the function f in the definition vanishes at ∞, the PTA R_n may not exist. This is the case, for example, when the numerator of R_n is completely fixed, $k = n$. If $f(\infty) = 0$ we always assume that the degree allowed of the free part of the numerator (i.e. $n-k$) is not less than the order of the zero of f at ∞. This condition allows the polynomial P_{n-k} to have degree less than $n-k$, which guarantees that R_n has a zero of the corresponding order at infinity. For example, if f has a zero of order d at infinity, then we may consider PTAs $R_n = \frac{P_d\tilde{P}_{n-d}}{Q_n}$, for which (1) reduces to*

$$f(z) - R_n(z) = \mathcal{O}(z^{-(2n+1-(n-d)-0)}) = \mathcal{O}(z^{-(n+1+d)}) \tag{1'}$$

and which in turn gives that $\deg P_d = 0$.

In (1) the polynomials P_{n-k} and $Q_{n-\ell}$ are determined by interpolation at infinity. An analogous definition of PTAs can be given by interpolation at any finite point of the complex plane, and also for non-diagonal approximants. When $k = \ell = 0$ Definition 1 reduces to the definition of the diagonal Padé approximant (PA) of degree n.

The rational functions in Definition 1 were introduced by C. Brezinski [5] (he called them Partial Padé Approximants). We note that our definition differs from the one in [5] in the case when the function vanishes at the point of interpolation (see Remark 1).

When $k = 0, \ell = n$, Definition 1 reduces to the definition of rational approximants with preassigned poles given by J.L. Walsh [9].

By analogy, in the case $k = n, \ell = 0$ when all zeros are preassigned and all poles are free the approximants can be called rational approximants with preassigned zeros.

In general we can not guarantee convergence of the approximants given by Definition 1. These are more general approximants than ordinary PAs, and for PAs we can not guarantee convergence in general except at the point of interpolation itself (see [8]). Below we consider some cases where we have convergence.

We start with the case $k = n, \ell = 0$. Before stating the general result (Theorem 2) about arbitrary analytic functions in Ω, we give a result (Theorem 1) about non-vanishing analytic functions. Remark 2 after Theorem 1 gives a motivation for Theorem 2.

Theorem 1 *Let f be a non-vanishing analytic function in a regular domain Ω containing infinity. Let $\{\tilde{P}_n\}_1^\infty$ be a sequence of polynomials,* $\deg \tilde{P}_n = n$*, whose zeros have no limit point in Ω and satisfying* $\lim_{n\to\infty} |\tilde{P}_n(z)|^{\frac{1}{n}} = e^{g_\Omega(z)}$. *Then, for the PTAs $R_n = \tilde{P}_n/Q_n$ determined by Definition 1 we have*

$$\limsup_{n\to\infty} |(f - R_n)(z)|^{\frac{1}{n}} \leq e^{-g_\Omega(z)}, \tag{2}$$

uniformly on compact subsets of Ω.

If all zeros of all polynomials $\tilde{P}_n$ belong to $\partial\Omega$ then the limit condition on $\tilde{P}_n$ in Theorem 1 means that their zeros are asymptotically distributed uniformly with respect to the equilibrium measure of $\partial\Omega$.

Remark 2 *In Theorem 1 an additional condition, $f \neq 0$ in Ω, has been introduced compared to the analogous approximants with preassigned poles (see [9] or [3]). This condition allows us to consider the analytic function $1/f$ and apply a kind of duality property analogous to that for PAs. But this is not only a technical condition needed for the proof, this is a necessary assumption. In fact, suppose $f(z_0) = 0, z_0 \in \Omega$, and that (2) holds in Ω, and in particular on a small fixed circle centered at z_0 containing no zeros of f. Then by a well-known theorem from complex analysis (a consequence of Rouché's Theorem) we conclude that P_n must have a zero inside the circle for all sufficiently large n. But this contradicts the choice of P_n.*

For functions having zeros in Ω we must modify Theorem 1:

Theorem 2 *Let f be an analytic function in a regular domain Ω containing infinity and let $f(\infty) \neq 0$ (for the case $f(\infty) = 0$, see Remark 3). Let $\{\tilde{P}_n(z)\}_1^\infty$ be a sequence of polynomials, $\deg \tilde{P}_n = n$, such that $\tilde{P}_n = H_{n-m}U_m$, where H_{n-m} and U_m are polynomials of degree $n-m$ and m, respectively, such that $m = m(n)$ and $m(n)/n \to 0$ as $n \to \infty$; we suppose that U_m has zeros at the zeros of f and that the zeros of U_m exactly exhaust all the zeros of f (counting multiplicity), and for H_{n-m} we assume that $\lim_{n\to\infty} |H_{n-m}(z)|^{\frac{1}{n}} = e^{g_\Omega(z)}$ in Ω. Then we have the limit relation (2) for the PTAs $R_n = \tilde{P}_n/Q_n$ with preassigned zeros at the zeros of $\tilde{P}_n$ (and all poles free).*

Remark 3 *If f has a zero of degree d at infinity, then Theorem 2 remains valid if we consider the PTAs $R_n = \tilde{P}_n/Q_n$ in the sense of Remark 1; that is we assume that $\deg \tilde{P}_n = n - d$ (and $\deg H_{n-m} = n - m - d$) and that Q_n is determined from (1′).*

Now we turn to a special case of Markov functions

$$f(z) = \int_\Delta \frac{d\mu(t)}{z-t}, \tag{3}$$

where μ is a finite, positive measure on a compact interval $\Delta \subset \mathbf{R}$. In this case we may guarantee convergence of the PTAs even if some of their zeros are free (all poles are still free). We choose the preassigned zeros to have double multiplicity: let $\{L_k\}$ be a sequence of polynomials with all the zeros on Δ, $\deg L_k = k = k(n)$, and let $\tilde{P}_{2k} = L_k^2$. We suppose that either

$$\lim_{n\to\infty} \frac{k(n)}{n} = 0 \qquad \text{or} \qquad \lim_{n\to\infty} \sigma\big(L_{k(n)}\big) = \tau, \tag{4}$$

where $\sigma\big(L_{k(n)}\big)$ is the normalized (unit) zero-counting measure putting equal mass at each zero of $L_{k(n)}$, and τ is the equilibrium measure of Δ.

Theorem 3 *Let f be a Markov function (3) and R_{n+k} the PTAs given by Definition 1 with all poles free and partly preassigned zeros at the zeros of the polynomials $\tilde{P}_{2k} = L_k^2$, satisfying condition (4). Then*

$$\limsup_{n\to\infty} |f(z) - R_{n+k}(z)|^{\frac{1}{n}} \le e^{-2g_\Omega(z)} \tag{5}$$

uniformly on compact subsets of $\Omega = \overline{\mathbf{C}} \setminus \Delta$.

We note here that the PTAs in Theorem 3 have larger order $(n+k)$ than the diagonal PAs of order n (which provide the same rate of convergence). However, to construct the PTAs we need the same number of coefficients of the function f ($2n$ coefficients for both type of approximants). Besides, the numerical calculation of PTAs is usually an easier procedure then the one of PAs.

Now we give a concrete example of PTAs with all zeros preassigned which shows that the correct choice of preassigned zeros can accelerate the convergence rate (compared to Taylor series):

Assertion: Let $R_n(z) = (z+n)^n/Q_{n-1}(z)$ be the PTA of order $(n, n-1)$ to the function $f(z) = e^z$ with all zeros preassigned at $z = -n$ (and all poles free), uniquely determined by the condition $e^z - R_n(z) = \mathcal{O}(z^n)$ (as $z \to 0$) (interpolation at zero). Then $R_n(z)$ converges to e^z, as $n \to \infty$, uniformly on compact subsets of $\mathbf{C}$ and the convergence is faster than for the sums of the Taylor series of the same order.

By using a "duality property" this result follows from Theorem 2 of [7], which asserts that PTAs $P_{n-1}(z)/(z-n)^n$ of order $(n-1, n)$ with all poles preassigned (at $z = n$) and all zeros free, determined by interpolation at zero, accelerate convergence of Taylor series. See [7] for the details.

A possible explanation of the phenomenom of convergence acceleration is that the function e^z increases to infinity along the positive real axis, and decreases to zero along the negative real axis, and by choosing zeros of PTAs on the negative real axis or poles on the positive real axis, we simulate the behaviour of e^z. The choice $z = -n$ for the zeros and $z = n$ for the poles of the corresponding approximants is probably optimal. As a comparison we remind of the fact that the ordinary diagonal PAs $[n/n]$ to e^z have all their zeros in the left half-plane, all poles in the right half-plane, and that they are contained in the annulus $\frac{2}{3}n < |z| < 2n$ (see [4] Theorems 5.7.4 and 5.7.5).

The discussion suggests the following problems.

Problem A. What can we say about convergence of PTAs of the form $R_{2n}(z) = P_n(z)(z+2n)^n/Q_n(z)(z-2n)^n$ to e^z, where P_n and Q_n are free polynomials of degree n determined from the interpolation condition: $e^z - R_{2n}(z) = \mathcal{O}(z^{2n+1})$, as $z \to 0$?

Note that in Problem A we have fixed zeros where e^z is small and poles where e^z is large. As a motivation for Problem A we made numerical experiments with the function e^z. We interpolated at zero and evaluated the approximation error at $z = 1$. Here is the result:

- The Taylor polynomial around zero of degree 40 gives 49 correct decimal digits at $z = 1$.
- The PTA $P_{40}(z)/(z-40)^{40}$ giving maximal interpolation at zero gives 55 correct decimal digits at $z = 1$.
- The PA $P_{20}(z)/Q_{20}(z)$ giving maximal interpolation at zero gives 60 correct decimal digits at $z = 1$.
- The PTA $P_{20}(z)(z+40)^{20}/Q_{20}(z)(z-40)^{20}$ giving maximal interpolation at zero gives 62 correct decimal digits at $z = 1$.

Problem B. The function $f(z) = \sin z$ is bounded on the real axis and converges to infinity along the imaginary axis. What can we say about the convergence of PTAs if we preassign zeros on the real axis and/or poles on the imaginary axis?

Questions analogous to those in Problem A and Problem B can be asked for some other functions as well.

Problem C. The functions in Problem A and B are entire functions and the separation of zeros and poles for PTAs were natural. But if we consider for example Markov functions (3), then in general they have singularities on the interval Δ. This suggests that we should fix some zeros and poles at the same time on Δ. In particular the following question can be asked: What can we say about convergence of PTAs $R_{2n}(z) = P_n(z)\tilde{P}_n(z)/Q_n(z)\tilde{Q}_n(z)$ to the function (3), where $\tilde{P}_n$ and $\tilde{Q}_n$ are fixed polynomials of degree n, whose zeros lie on Δ and have equilibrium asymptotic distribution, and P_n and Q_n are free polynomials of degree $n-1$ and n, respectively, determined by the condition $f(z) - R_{2n}(z) = \mathcal{O}(z^{-(2n+1)})$(as $z \to \infty$)?

To fix both zeros and poles near singularities of f (different from poles) may be a general rule for PTAs, since such a behaviour is observed by ordinary PAs.

3 Proofs

Theorem 1 is contained in Theorem 2.

Proof of theorem 2. We give the proof in the general case of Remark 3. From (1′) we get by dividing both sides by $f\tilde{P}_{n-d}/Q_n$ (this function has a zero of order $2d$ at infinity)

$$\frac{1}{f(z)} - \frac{Q_n(z)}{\tilde{P}_{n-d}(z)} = \mathcal{O}\left(z^{-(n+1+d-2d)}\right) = \mathcal{O}\left(z^{-(n+1-d)}\right).$$

This shows that $\frac{Q_n}{\tilde{P}_{n-d}}$ is the PTA with preassigned denominator $\tilde{P}_{n-d}$ (and all poles free) to the function $1/f(z)$, which is analytic or has a pole at infinity (see [1] for the definition of PTAs in this case). From Theorem 3 of [1] it follows that

$$\limsup_{n\to\infty} \left| \frac{1}{f(z)} - \frac{Q_n(z)}{\tilde{P}_{n-d}(z)} \right|^{\frac{1}{n}} \leq e^{-g_\Omega(z)} \tag{6}$$

uniformly on any compact subset $F \subset \Omega$ not containing zeros of f. In particular, the functions $\left|\frac{Q_n}{\tilde{P}_{n-d}}\right|$ are uniformly (in n) bounded from below on F. Multiplying (6) by $\left|f(z) \cdot \frac{\tilde{P}_{n-d}}{Q_n}(z)\right|$ we get the limit relation (2) on F.

Now let z_0 be a zero of f, $f(z_0) = 0$, and let D be a closed disk centered at z_0 and containing no zeros of f except z_0. From the arguments above the limit relation (2) follows uniformly on ∂D (boundary of D). By our choice the polynomials $\tilde{P}_{n-d}$ have asymptotically the same number of zeros on D as f. From the argument principle and the uniform convergence on ∂D it now follows that the polynomials Q_n have asymptotically no zeros on D. Consequently, for large n the function $f(z) - R_n(z)$ is analytic and $|f(z) - R_n(z)|^{\frac{1}{n}}$ subharmonic on D. The right-hand side of (2) is harmonic on D (as composition of harmonic and analytic functions). Now, using the principle of domination for harmonic and subharmonic functions, the limit relation (2), which we have uniformly on ∂D, can be extended to D. Finally, since any compact subset of Ω contains only a finite number of zeros of f the arguments above give (2) in full generality.

Proof of theorem 3. The function $-1/f$ can be written in the form $-1/f = H_1 + f^*$, where H_1 is a polynomial of the first degree and f^* a Markov function with support Δ (see [6]). Now, from Theorem 2' of [2], it follows that there exist polynomials P_{n-1-k} of degree $n-1-k$, and polynomials A_{n-2+k} of degree $n-2+k$, such that

$$R^*_{n+k-1}(z) := f^*(z) - \frac{A_{n-2+k}(z)}{P_{n-1-k}(z)\tilde{P}_{2k}(z)} = \mathcal{O}(z^{-(2n-1)}) \tag{7}$$

and

$$\limsup_{n\to\infty} |R^*_{n+k-1}(z)|^{\frac{1}{n}} \le e^{-2g_\Omega(z)}. \tag{8}$$

From (7) we get that

$$R^*_{n+k-1}(z) = \frac{-1}{f(z)} - \left(H_1(z) + \frac{A_{n-2+k}(z)}{P_{n-1-k}(z)\tilde{P}_{2k}(z)}\right) = \tag{9}$$
$$= \frac{-1}{f(z)} - \frac{Q_{n+k}(z)}{P_{n-1-k}(z)\tilde{P}_{2k}(z)} = \mathcal{O}\left(z^{-(2n-2)}\right),$$

where Q_{n+k} is a polynomial of degree $n+k$. Since $|1/f(z)| \neq 0$ in Ω (9) and (8) give that the rational functions $r^*_{n+k} := Q_{n+k}/P_{n-1-k} \cdot \tilde{P}_{2k}$ are asymptotically bounded from below on compact subsets of Ω. Consequently, the functions $1/r^*_{n+k}$ are (asymptocially) analytic and bounded from above. In addition $1/r^*_{n+k}$ has a zero of order 1 at infinity. Consequently, the functions f/r^*_{n+k} are analytic and bounded from above asymptotically on compact subsets of Ω and they have a zero of order 2 at infinity. Multiplying (9) by f/r^*_{n+k} we get

$$-f(z) - \frac{P_{n-1-k}(z)\tilde{P}_{2k}(z)}{Q_{n+k}(z)} = \mathcal{O}(z^{-(2n+1)}),$$

which shows that the rational functions $-P_{n-1-k}\tilde{P}_{2k}/Q_{n+k}$ are the PTAs R_{n+k} to f of Theorem 3. By inserting (9) in (8) and multiplying by $|f/r^*_{n+k}|$ we get (5).

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