CHAPTER 4

GENERALIZED INDEX THEOREMS AND SPECIAL TOPICS

Introduction

This chapter is less detailed than the previous three as several lengthy calculations are omitted in the interests of brevity. In sections 4.1 through 4.6, we sketch the interrelations between the Atiyah-Patodi-Singer twisted index theorem, the Atiyah-Patodi-Singer index theorem for manifolds with boundary, and the Lefschetz fixed point formulas.

In section 4.1, we discuss the absolute and relative boundary conditions for the de Rham complex if the boundary is non-empty. We discuss Poincaré duality and the Hodge decomposition theorem in this context. The spin, signature, and Dolbeault complexes do not admit such local boundary conditions and the index theorem in this context looks quite different. In section 4.2, we prove the Gauss-Bonnet theorem for manifolds with boundary and identify the invariants arising from the heat equation with these integrands.

In section 4.3, we introduce the eta invariant as a measure of spectral asymmetry and establish the regularity at $s = 0$. We discuss without proof the Atiyah-Patodi-Singer index theorem for manifolds with boundary. The eta invariant enters as a new non-local ingredient which was missing in the Gauss-Bonnet theorem. In section 4.4, we review secondary characteristic classes and sketch the proof of the Atiyah-Patodi-Singer twisted index theorem with coefficients in a locally flat bundle. We discuss in some detail explicit examples on a 3-dimensional lens space.

In section 4.5, we turn to the Lefschetz fixed point formulas. We treat the case of isolated fixed points in complete detail in regard to the four classical elliptic complexes. We also return to the 3-dimensional examples discussed in section 4.4 to relate the Lefschetz fixed point formulas to the twisted index theorem using results of Donnelly. We discuss in some detail the Lefschetz fixed point formulas for the de Rham complex if the fixed point set is higher dimensional. There are similar results for both the spin and signature complexes which we have omitted for reasons of space. In section 4.6 we use these formulas for the eta invariant to compute the K-theory of spherical space forms.

In section 4.7, we turn to a completely new topic. In a lecture at M.I.T., Singer posed the question:

> *Suppose $P(G)$ is a scalar valued invariant of the metric so that $P(M) = \int_M P(G)\,\mathrm{dvol}$ is independent of the metric. Then is there a universal constant c so $P(M) = c\chi(M)$?*

The answer to this question (and to related questions involving form valued invariants) is yes. This leads immediately to information regarding the higher order terms in the expansion of the heat equation. In section 4.8, we use the functorial properties of the invariants to compute $a_n(x, P)$ for an arbitrary second order elliptic parital differential operator with leading symbol given by the metric tensor for $n = 0, 2, 4$. We list (without proof) the corresponding formula if $n = 6$. This leads to Patodi's formula for $a_n(x, \Delta_p^m)$ discussed in Theorem 4.8.18. In section 4.9 we discuss some results of Ikeda to give examples of spherical space forms which are isospectral but not diffeomorphic. We use the eta invariant to show these examples are not even equivariant cobordant.

The historical development of these ideas is quite complicated and in particular the material on Lefschetz fixed point formulas is due to a number of authors. We have included a brief historical survey at the beginning of section 4.6 to discuss this material.

4.1. The de Rham Complex for Manifolds with Boundary.

In section 1.9, we derived a formula for the index of a strongly elliptic boundary value problem. The de Rham complex admits suitable boundary conditions leading to the relative and absolute cohomolgy groups. It turns out that the other 3 classical elliptic complexes do *not* admit even a weaker condition of ellipticity.

Let $(d+\delta)\colon C^\infty(\Lambda(M)) \to C^\infty(\Lambda(M))$ be the de Rham complex. In the third chapter we assumed the dimension m of M to be even, but we place no such restriction on the parity here. M is assumed to be compact with smooth boundary dM. Near the boundary, we introduce coordinates (y,r) where $y=(y_1,\dots,y_{m-1})$ give local coordinates for dM and such that $M=\{\,x : r(x)\ge 0\,\}$. We further normalize the coordinates by assuming the curves $x(r)=(y_0,r)$ are unit speed geodesics perpendicular to dM for $r\in[0,\delta)$.

Near dM, we decompose any differential form $\theta\in\Lambda(M)$ as

$$\theta=\theta_1+\theta_2\wedge dr \qquad \text{where } \theta_i\in\Lambda(dM)$$

are tangential differential forms. We use this decomposition to define:

$$\alpha(\theta)=\theta_1-\theta_2\wedge dr.$$

α is self-adjoint and $\alpha^2=1$. We define the absolute and relative boundary conditions

$$B_a(\theta)=\theta_2 \qquad\text{and}\qquad B_r(\theta)=\theta_1.$$

We let B denote either B_a or B_r, then $B\colon\Lambda(M)|_{dM}\to\Lambda(dM)$. We note that B_a can be identified with orthogonal projection on the -1 eigenspace of α while B_r can be identified with orthogonal projection on the $+1$ eigenspace of α. There is a natural inclusion map $i\colon dM\to M$ and $B_r(\theta)=i^*(\theta)$ is just the pull-back of θ. The boundary condition B_r does not depend on the Riemannian metric chosen, while the boundary condition B_a does depend on the metric.

Lemma 4.1.1. *Let $B=B_a$ or B_r, then $(d+\delta,B)$ is self-adjoint and elliptic with respect to the cone $\mathbf{C}-\mathbf{R}_+-\mathbf{R}_-$.*

Proof: We choose a local orthonormal frame $\{e_0,\dots,e_{m-1}\}$ for T^*M near dM so that $e_0=dr$. Let

$$p_j=ie_j$$

act on $\Lambda(M)$ by Clifford multiplication. The p_j are self-adjoint and satisfy the commutation relation:

$$p_jp_k+p_kp_j=2\delta_{jk}.$$

The symbol of $(d+\delta)$ is given by:

$$p(x,\xi)=zp_0+\sum_{j=1}^{m-1}\zeta_j p_j.$$

As in Lemma 1.9.5, we define:

$$\tau(y,\zeta,\lambda)=ip_0\left(\sum_{j=1}^{m-1}\zeta_j p_j-\lambda\right),\quad (\zeta,\lambda)\neq(0,0)\in T^*(dM)\times\{\mathbf{C}-\mathbf{R}_+-\mathbf{R}_-\}.$$

We define the new matrices:

$$q_0=p_0 \qquad \text{and} \qquad q_j=ip_0p_j \qquad \text{for } 1\le j\le m-1$$

so that:

$$\tau(y,\zeta,\lambda)=-i\lambda q_0+\sum_{j=1}^{m-1}\zeta_j q_j.$$

The $\{q_j\}$ are self-adjoint and satisfy the commutation relations $q_jq_k+q_kq_j=2\delta_{jk}$. Consequently:

$$(y,\zeta,\lambda)^2=(|\zeta|^2-\lambda^2)I.$$

We have $(|\zeta|^2-\lambda^2)\in\mathbf{C}-\mathbf{R}_--0$ so we can choose $\mu^2=(|\zeta|^2-\lambda^2)$ with $\operatorname{Re}(\mu)>0$. Then τ is diagonalizable with eigenvalues $\pm\mu$ and $V_\pm(\tau)$ is the span of the eigenvectors of τ corresponding to the eigenvalue $\pm\mu$. (We set $V=\Lambda(M)=\Lambda(T^*M)$ to agree with the notation of section 9.1).

We defined $\alpha(\theta_1+\theta_2\wedge dr)=\theta_1-\theta_2\wedge dr$. Since Clifford multiplication by e_j for $1\le j\le m-1$ preserves $\Lambda(dM)$, the corresponding p_j commute with α. Since Clifford multiplication by dr interchanges the factors of $\Lambda(dM)\oplus\Lambda(dM)\wedge dr$, α anti-commutes with p_0. This implies α anti-commutes with all the q_j and consequently anti-commutes with τ. Thus the only common eigenvectors must belong to the eigenvalue 0. Since 0 is not an eigenvalue,

$$V_\pm(\alpha)\cap V_\pm(\tau)=\{0\}.$$

Since B_a and B_r are just orthogonal projection on the ∓1 eigenspaces of α, $\mathrm{N}(B_a)$ and $\mathrm{N}(B_r)$ are just the ±1 eigenspaces of α. Thus

$$B\colon V_\pm(\tau)\to\Lambda(dM)$$

is injective. Since $\dim(V_\pm(\tau))=\dim\Lambda(dM)=2^{m-1}$, this must be an isomorphism which proves the ellipticity. Since p_0 anti-commutes with α,

$$p_0\colon V_\pm(\alpha)\to V_\mp(\alpha)$$

so $(d+\delta)$ is self-adjoint with respect to either B_a or B_r by Lemma 1.9.5.

By Lemma 1.9.1, there is a spectral resolution for the operator $(d+\delta)_B$ in the form $\{\lambda_\nu, \phi_\nu\}_{\nu=1}^\infty$ where $(d+\delta)\phi_\nu = \lambda_\nu\phi_\nu$ and $B\phi_\nu = 0$. The $\phi_\nu \in C^\infty(\Lambda(M))$ and $|\lambda_\nu| \to \infty$. We set $\Delta = (d+\delta)^2 = d\delta + \delta d$ and define:

$$\begin{aligned}
\mathrm{N}((d+\delta)_B) &= \{\, \phi \in C^\infty(\Lambda(M)) : B\phi = (d+\delta)\phi = 0 \,\} \\
\mathrm{N}((d+\delta)_B)_j &= \{\, \phi \in C^\infty(\Lambda^j(M)) : B\phi = (d+\delta)\phi = 0 \,\} \\
\mathrm{N}(\Delta_B) &= \{\, \phi \in C^\infty(\Lambda(M)) : B\phi = B(d+\delta)\phi = \Delta\phi = 0 \,\} \\
\mathrm{N}(\Delta_B)_j &= \{\, \phi \in C^\infty(\Lambda^j(M)) : B\phi = B(d+\delta)\phi = \Delta\phi = 0 \,\}.
\end{aligned}$$

Lemma 4.1.2. *Let B denote either the relative or the absolute boundary conditions. Then*
(a) $\mathrm{N}((d+\delta)_B) = \mathrm{N}(\Delta_B)$.
(b) $\mathrm{N}((d+\delta)_B)_j = \mathrm{N}(\Delta_B)_j$.
(c) $\mathrm{N}((d+\delta)_B) = \bigoplus_j \mathrm{N}((d+\delta)_B)_j$.

Proof: Let $B\phi = B(d+\delta)\phi = \Delta\phi = 0$. Since $(d+\delta)_B$ is self-adjoint with respect to the boundary condition B, we can compute that $\Delta\phi \cdot \phi = (d+\delta)\phi \cdot (d+\delta)\phi = 0$ so $(d+\delta)\phi = 0$. This shows $\mathrm{N}(\Delta_B) \subset \mathrm{N}((d+\delta)_B)$ and $\mathrm{N}(\Delta_B)_j \subset \mathrm{N}((d+\delta)_B)_j$. The reverse inclusions are immediate which proves (a) and (b). It is also clear that $\mathrm{N}((d+\delta)_B)_j \subset \mathrm{N}((d+\delta)_B)$ for each j. Conversely, let $\theta \in \mathrm{N}((d+\delta)_B)$. We decompose $\theta = \theta_0 + \cdots + \theta_m$ into homogeneous pieces. Then $\Delta\theta = \sum_j \Delta\theta_j = 0$ implies $\Delta\theta_j = 0$ for each j. Therefore we must check $B\theta_j = B(d+\delta)\theta_j = 0$ since then $\theta_j \in \mathrm{N}(\Delta_B)_j = \mathrm{N}((d+\delta)_B)_j$ which will complete the proof.

Since B preserves the grading, $B\theta_j = 0$. Suppose $B = B_r$ is the relative boundary conditions so $B(\alpha_1 + \alpha_2 \wedge dr) = \alpha_1|_{dM}$. Then $Bd = dB$ so $B(d+\delta)\theta = dB(\theta) + B\delta(\theta) = B\delta(\theta) = 0$. Since B preserves the homogeneity, this implies $B\delta\theta_j = 0$ for each j. We observed $Bd\theta = dB\theta = 0$ so $Bd\theta_j = 0$ for each j as well. This completes the proof in this case. If $B = B_a$ is the absolute boundary condition, use a similar argument based on the identity $B\delta = \delta B$.

We illustrate this for $m = 1$ by considering $M = [0,1]$. We decompose $\theta = f_0 + f_1\,dx$ to decompose $C^\infty(\Lambda(M)) = C^\infty(M) \oplus C^\infty(M)\,dx$. It is immediate that:

$$(d+\delta)(f_0, f_1) = (-f_1', f_0')$$

so $(d+\delta)\theta = 0$ implies θ is constant. B_a corresponds to Dirichlet boundary conditions on f_1 while B_r corresponds to Dirichlet boundary conditions on f_0. Therefore:

$$\begin{aligned}
&H_a^0([0,1];\mathbf{C}) = \mathbf{C} \qquad && H_a^1([0,1];\mathbf{C}) = 0 \\
&H_r^0([0,1];\mathbf{C}) = 0 \qquad && H_r^1([0,1];\mathbf{C}) = \mathbf{C}.
\end{aligned}$$

A priori, the dimensions of the vector spaces $H^j_a(M;\mathbf{C})$ and $H^j_r(M;\mathbf{C})$ depend on the metric. It is possible, however, to get a more invariant definition which shows in fact they are independent of the metric. Lemma 1.9.1 shows these spaces are finite dimensional.

Let $d: C^\infty(\Lambda^j) \to C^\infty(\Lambda^{j+1})$ be the de Rham complex. The relative boundary conditions are independent of the metric and are d-invariant. Let $C^\infty_r(\Lambda^p) = \{\theta \in C^\infty\Lambda^p : B_r\theta = 0\}$. There is a chain complex $d: C^\infty_r(\Lambda^p) \to C^\infty_r(\Lambda^{p+1}) \to \cdots$. We define $H^p_r(M;\mathbf{C}) = (\ker d/\operatorname{image} d)_p$ on $C^\infty_r(\Lambda^p)$ to be the cohomology of this chain complex. The de Rham theorem for manifolds without boundary generalizes to identify these groups with the relative simplicial cohomology $H^p(M,dM;\mathbf{C})$. If $\theta \in \mathrm{N}((d+\delta)_{B_r})_p$ then $d\theta = B_r\theta = 0$ so $\theta_j \in H^p(M,dM;\mathbf{C})$. The Hodge decomposition theorem discussed in section 1.5 for manifolds without boundary generalizes to identify $H^p_r(M;\mathbf{C}) = \mathrm{N}(\Delta_{B_r})_p$. If we use absolute boundary conditions and the operator δ, then we can define $H^p_a(M;\mathbf{C}) = \mathrm{N}(\Delta_{B_a})_p = H^p(M;\mathbf{C})$. We summarize these results as follows:

Lemma 4.1.3. (Hodge decomposition theorem). *There are natural isomorphisms between the harmonic cohomology with absolute and relative boundary conditions and the simplicial cohomology groups of M:*

$$H^p_a(M;\mathbf{C}) = \mathrm{N}((d+\delta)_{B_a})_j \simeq H^p(M;\mathbf{C})$$

and

$$H^p_r(M;\mathbf{C}) = \mathrm{N}((d+\delta)_{B_r})_j \simeq H^p(M,dM;\mathbf{C}).$$

If M is oriented, we let $*$ be the Hodge operator $*: \Lambda^p \to \Lambda^{m-p}$. Since $*$ interchanges the decomposition $\Lambda(T^*dM) \oplus \Lambda(T^*dM) \wedge dr$, it anti-commutes with α and therefore $B_a(\theta) = 0$ if and only if $B_r(*\theta) = 0$. Since $d\theta = 0$ if and only if $\delta * \theta = 0$ and similarly $\delta\theta = 0$ if and only if $d * \theta = 0$, we conclude:

Lemma 4.1.4. *Let M be oriented and let $*$ be the Hodge operator. Then $*$ induces a map, called Poincaré duality,*

$$*: H^p(M;\mathbf{C}) \simeq H^p_a(M;\mathbf{C}) \stackrel{\simeq}{\to} H^{m-p}_r(M,\mathbf{C}) \simeq H^{m-p}(M,dM;\mathbf{C}).$$

We define the Euler-Poincaré characteristics by:

$$\chi(M) = \sum(-1)^p \dim H^p(M;\mathbf{C})$$
$$\chi(dM) = \sum(-1)^p \dim H^p(dM;\mathbf{C})$$
$$\chi(M,dM) = \sum(-1)^p \dim H^p(M,dM;\mathbf{C}).$$

The long exact sequence in cohomology:

$$\cdots H^p(dM;\mathbf{C}) \leftarrow H^p(M;\mathbf{C}) \leftarrow H^p(M,dM;\mathbf{C}) \leftarrow H^{p-1}(dM;\mathbf{C}) \cdots$$

shows that:

$$\chi(M) = \chi(dM) + \chi(M, dM).$$

If m is even, then $\chi(dM) = 0$ as dM is an odd dimensional manifold without boundary so $\chi(M) = \chi(M, dM)$. If m is odd and if M is orientable, then $\chi(M) = -\chi(M, dM)$ by Poincaré duality. $\chi(M)$ is the index of the de Rham complex with absolute boundary conditions; $\chi(M, dM)$ is the index of the de Rham complex with relative boundary conditions. By Lemma 1.9.3, there is a local formula for the Euler-Poincaré characteristic. Since $\chi(M) = -\chi(M, dM)$ if m is odd and M is orientable, by passing to the double cover if necessary we see $\chi(M) = -\chi(M, dM)$ in general if m is odd. This proves:

Lemma 4.1.5.
(a) If m is even, $\chi(M) = \chi(M, dM)$ and $\chi(dM) = 0$.
(b) If m is odd, $\chi(M) = -\chi(M, dM) = \frac{1}{2}\chi(dM)$.

In contrast to the situation of manifolds without boundary, if we pass to the category of manifolds with boundary, there exist non-zero index problems in all dimensions m.

In the next subsection, we will discuss the Gauss-Bonnet formula for manifolds with boundary. We conclude this subsection with a brief discussion of the more general ellipticity conditions considered by Atiyah and Bott. Let $Q: C^\infty(V_1) \to C^\infty(V_2)$ be an elliptic differential operator of order $d > 0$ on the interior—i.e., if $q(x, \xi)$ is the leading symbol of Q, then $q(x, \xi): V_1 \to V_2$ is an isomorphism for $\xi \neq 0$. Let $W_1 = V_1 \otimes 1^d\big|_{dM}$ be the bundle of Cauchy data. We assume $\dim W_1$ is even and let W_1' be a bundle over dM of dimension $\frac{1}{2}(\dim W_1)$. Let $B: C^\infty(W_1) \to C^\infty(W_1')$ be a tangential pseudo-differential operator. We consider the ODE

$$q(y, 0, \zeta, D_r)f = 0, \qquad \lim_{r\to\infty} f(r) = 0$$

and let $V_+(\tau)(\zeta)$ be the bundle of Cauchy data of solutions to this equation. We say that (Q, B) is elliptic with respect to the cone $\{0\}$ if for all $\zeta \neq 0$, the map:

$$\sigma_g(B)(y, \zeta): V_+(\tau)(\zeta) \to W_1'$$

is an isomorphism (i.e., we can find a unique solution to the ODE such that $\sigma_g(B)(y, \zeta)\underline{\gamma} f = f'$ is pre-assigned in W_1'). $V_+(\tau)$ is a sub-bundle of W_1 and is the span of the generalized eigenvectors corresponding to eigenvalues with positive real parts for a suitable endomorphism $\tau(\zeta)$ just as in the first order case. σ_g is the graded leading symbol as discussed in section 1.9.

This is a much weaker condition than the one we have been considering since the only complex value involved is $\lambda = 0$. We study the pair

$$(Q, B): C^\infty(V_1) \to C^\infty(V_2) \oplus C^\infty(W_1').$$

Under the assumption of elliptic with respect to the cone $\{0\}$, this operator is Fredholm in a suitable sense with closed range, finite dimensional null-space and cokernel. We let $\operatorname{index}(Q, B)$ be the index of this problem. The Atiyah-Bott theorem gives a formula for the index of this problem.

There exist elliptic complexes which do not admit boundary conditions satisfying even this weaker notion of ellipticity. Let $q(x, \xi)$ be a first order symbol and expand

$$q(y, 0, \xi, z) = q_0 z + \sum_{j=1}^{m-1} q_j \zeta_j .$$

As in Lemma 1.9.5 we define

$$\tau = i q_0^{-1} \sum_{j=1}^{m-1} q_j \zeta_j ;$$

the ellipticity condition on the interior shows τ has no purely imaginary eigenvalues for $\zeta \neq 0$. We let $V_\pm(\tau)(\zeta)$ be the sub-bundle of V corresponding to the span of the generalized eigenvectors of τ corresponding to eigenvalues with positive/negative real part. Then (Q, B) is elliptic if and only if

$$\sigma_g(B)(\zeta) \colon V_+(\tau)(\zeta) \to W_1'$$

is an isomorphism for all $\zeta \neq 0$.

Let $S(T^*(dM)) = \{\, \zeta \in T^*(dM) : |\zeta|^2 = 1 \,\}$ be the unit sphere bundle over dM. $V_\pm(\tau)$ define sub-bundles of V over $S(T^*(dM))$. The existence of an elliptic boundary condition implies these sub-bundles are trivial over the fiber spheres. We study the case in which $q^* q = |\zeta|^2 I$. In this case, $q_0^{-1} = q_0^*$. If we set $p_j = i q_0^{-1} q_j$, then these are self-adjoint and satisfy $p_j p_k + p_k p_j = 2\delta_{jk}$. If m is even, then the fiber spheres have dimension $m - 2$ which will be even. The bundles $V_\pm(\tau)$ were discussed in Lemma 2.1.5 and in particular are non-trivial if

$$\operatorname{Tr}(p_1, \dots, p_{m-1}) \neq 0.$$

For the spin, signature, and Dolbeault complexes, the symbol is given by Clifford multiplication and $p_1, \dots, p_{m-1}$ is multiplication by the orientation form (modulo some normalizing factor of i). Since the bundles involved were defined by the action of the orientation form being ± 1, this proves:

Lemma 4.1.6. *Let $Q \colon C^\infty(V_1) \to C^\infty(V_2)$ denote either the signature, the spin, or the Dolbeault complex. Then there does not exist a boundary condition B so that (Q, B) is elliptic with respect to the cone $\{0\}$.*

The difficulty comes, of course, in not permitting the target bundle W' to depend upon the variable ζ. In the first order case, there is a natural pseudo-differential operator $B(\zeta)$ with leading symbol given by projection on $V_+(\tau)(\zeta)$. This operator corresponds to global (as opposed to local) boundary conditions and leads to a well posed boundry value problem for the other three classical elliptic complexes. Because the boundary value problem is non-local, there is an additional non-local term which arises in the index theorem for these complexes. This is the eta invariant we will discuss later.

4.2. The Gauss-Bonnet Theorem For Manifolds with Boundary.

Let B denote either the absolute or relative boundary conditions for the operator $(d+\delta)$ discussed previously. We let $\chi(M)_B$ be either $\chi(M)$ or $\chi(M, dM)$ be the index of the de Rham complex with these boundary conditions. Let Δ_B^{even} and Δ_B^{odd} be the Laplacian on even/odd forms with the boundary conditions $B\theta = B(d+\delta)\theta = 0$. Let $a_n(x, d+\delta) = a_n(x, \Delta^{\text{even}}) - a_n(x, \Delta^{\text{odd}})$ be the invariants of the heat equation defined in the interior of M which were discussed in Lemma 1.7.4. On the boundary dM, let $a_n(y, d+\delta, B) = a_n(y, \Delta_B^{\text{even}}) - a_n(y, \Delta_B^{\text{odd}})$ be the invariants of the heat equation defined in Lemma 1.9.2. Then Lemma 1.9.3 implies:

$$\begin{aligned}\chi(M)_B &= \operatorname{Tr}\{\exp(-t\Delta_B^{\text{even}})\} - \operatorname{Tr}\exp\{(-t\Delta_B^{\text{odd}})\}\\ &\sim \sum_{n=0}^{\infty} t^{(n-m)/2} \int_M a_n(x, d+\delta)\,\text{dvol}(x)\\ &\qquad + \sum_{n=0}^{\infty} t^{(n-m+1)/2} \int_{dM} a_n(y, d+\delta, B)\,\text{dvol}(y).\end{aligned}$$

The interior invariants $a_n(x, d+\delta)$ do not depend on the boundary condition so we can apply Lemma 2.4.8 to conclude:

$$\begin{aligned} a_n(x, d+\delta) &= 0 && \text{if } n < m \text{ or if } m \text{ is odd}\\ a_m(x, d+\delta) &= E_m && \text{is the Euler intergrand if } m \text{ is even.}\end{aligned}$$

In this subsection we will prove the Gauss-Bonnet theorem for manifolds with boundary and identify the boundary integrands $a_n(y, d+\delta, B)$ for $n \le m-1$.

We let $\mathcal{P}$ be the algebra generated by the $\{g_{ij/\alpha}\}$ variables for $|\alpha| \neq 0$. We always normalize the coordinate system so $g_{ij}(X,G)(x_0) = \delta_{ij}$. We normalize the coordinate system $x = (y, r)$ near the boundary as discussed in section 4.1; this introduces some additional relations on the $g_{ij/\alpha}$ variables we shall discuss shortly. We let $P(Y,G)(y_0)$ be the evaluation of $P \in \mathcal{P}$ on a metric G and relative to the given coordinate system Y on dM. We say that P is invariant if $P(Y,G)(y_0) = P(\bar{Y},G)(y_0)$ for any two such coordinate systems Y and $\bar{Y}$. We introduce the same notion of homogeneity as that discussed in the second chapter and let $\mathcal{P}_{m,n}^b$ be the finite dimensional vector space of invariant polynomials which are homogeneous of order n on a manifold M of dimension m. The "b" stands for boundary and emphasizes that these are invariants only defined on dM; there is a natural inclusion $\mathcal{P}_{m,n} \to \mathcal{P}_{m,n}^b$; by restricting the admissible coordinate transformations we increase the space of invariants.

LEMMA 4.2.1. *If B denotes either absolute or relative boundary conditions, then $a_n(y, d+\delta, B)$ defines an element of $\mathcal{P}^b_{m,n}$.*

PROOF: By Lemma 1.9.2, $a_n(y, d+\delta, B)$ is given by a local formula in the jets of the metric which is invariant. Either by examining the analytic proof of Lemma 1.9.2 in a way similar to that used to prove Lemma 1.7.5 and 2.4.1 or by using dimensional analysis as was done in the proof of Lemma 2.4.4, we can show that a_n must be homogeneous of order n and polynomial in the jets of the metric.

Our normalizations impose some additional relations on the $g_{ij/\alpha}$ variables. By hypothesis, the curves (y_0, r) are unit speed geodesics perpendicular to dM at $r=0$. This is equivalent to assuming:

$$\nabla_N N = 0 \qquad \text{and} \qquad g_{jm}(y,0) = \delta_{jm}$$

where $N = \partial/\partial r$ is the inward unit normal. The computation of the Christoffel symbols of section 2.3 shows this is equivalent to assuming:

$$\Gamma_{mmj} = \tfrac{1}{2}(g_{mj/m} + g_{mj/m} - g_{mm/j}) = 0.$$

If we take $j=m$, this implies $g_{mm/m} = 0$. Since $g_{mm}(y,0) \equiv 1$, we conclude $g_{mm} \equiv 1$ so $g_{mm/\alpha} \equiv 0$. Thus $\Gamma_{mmj} = g_{mj/m} = 0$. As $g_{mj}(y,0) = \delta_{mj}$ we conclude $g_{mj} \equiv \delta_{mj}$ and therefore $g_{jm/\alpha} = 0$, $1 \le j \le m$. We can further normalize the coordinate system Y on dM by assuming $g_{jk/l}(Y,G)(y_0) = 0$ for $1 \le j,k,l \le m-1$. We eliminate all these variables from the algebra defining $\mathcal{P}$; the remaining variables are algebraically independent.

The only 1-jets of the metric which are left are the $\{g_{jk/m}\}$ variables for $1 \le j,k \le m-1$. The first step in Chapter 2 was to choose a coordinate system in which all the 1-jets of the metric vanish; this proved to be the critical obstruction to studying non-Kaehler holomorphic manifolds. It turns out that the $\{g_{jk/m}\}$ variables cannot be normalized to zero. They are tensorial and give essentially the components of the second fundamental form or shape operator.

Let $\{e_1, e_2\}$ be vector fields on M which are tangent to dM along dM. We define the shape operator:

$$S(e_1, e_2) = (\nabla_{e_1} e_2, N)$$

along dM. It is clear this expression is tensorial in e_1. We compute:

$$(\nabla_{e_1} e_2, N) - (\nabla_{e_2} e_1, N) = ([e_1, e_2], N).$$

Since e_1 and e_2 are tangent to dM along dM, $[e_1, e_2]$ is tangent to dM along dM and thus $([e_1, e_2], N) = 0$ along dM. This implies $S(e_1, e_2) =$

$S(e_2, e_1)$ is tensorial in e_2. The shape operator defines a bilinear map from $T(dM) \times T(dM) \to \mathbf{R}$. We compute

$$(\nabla_{\partial/\partial y_j} \partial/\partial y_k, N) = \Gamma_{jkm} = \tfrac{1}{2}(g_{jm/k} + g_{km/j} - g_{jk/m}) = -\tfrac{1}{2} g_{jk/m}.$$

We can construct a number of invariants as follows: let $\{e_j\}$ be a local orthonormal frame for $T(M)$ such that $e_m = N = \partial/\partial r$. Define:

$$\nabla e_j = \sum_{1 \le k \le m} \omega_{jk} e_k \qquad \text{for } \omega_{jk} \in T^*M \text{ and } \omega_{jk} + \omega_{kj} = 0$$

and

$$\Omega_{jk} = d\omega_{jk} - \sum_{1 \le \nu \le m} \omega_{j\nu} \wedge \omega_{\nu k}.$$

The ω_{jm} variables are tensorial as $\omega_{jm} = \sum_{k=1}^{m-1} S(e_j, e_k) \cdot e^k$. We define:

$$Q_{k,m} = c_{k,m} \sum \varepsilon(i_1, \dots, i_{m-1}) \Omega_{i_1 i_2} \wedge \cdots \wedge \Omega_{i_{2k-1}, i_{2k}} \\ \wedge \omega_{i_{2k+1}, m} \wedge \cdots \wedge \omega_{i_{m-1}, m} \in \Lambda^{m-1}$$

for

$$c_{k,m} = \frac{(-1)^k}{\pi^p k! 2^{k+p} \cdot 1 \cdot 3 \cdots (2p - 2k - 1)} \qquad \text{where } p = \left[\frac{m}{2}\right].$$

The sum defining Q_k is taken over all permutations of $m-1$ indices and defines an $m-1$ form over M. If m is even, we define:

$$E_m = \frac{(-1)^p}{\{2^m \pi^p p!\}} \sum \varepsilon(i_1, \dots, i_m) \Omega_{i_1 i_2} \wedge \cdots \wedge \Omega_{i_{m-1} i_m}$$

as the Euler form discussed in Chapter 2.

$Q_{k,m}$ and E_m are the SO-invariant forms on M. E_m is defined on all of M while $Q_{k,m}$ is only defined near the boundary. Chern noted that if m is even,

$$E_m = -d\biggl(\sum_k Q_{k,m}\biggr).$$

This can also be interpreted in terms of the transgression of Chapter 2. Let ∇_1 and ∇_2 be two Riemannian connections on TM. We defined an $m-1$ form $TE_m(\nabla_1, \nabla_2)$ so that

$$dTE_m(\nabla_1, \nabla_2) = E_m(\nabla_1) - E_m(\nabla_2).$$

Near dM, we split $T(M) = T(dM) \oplus 1$ as the orthogonal complement of the unit normal. We project the Levi-Civita connection on this decomposition, and let ∇_2 be the projected connection. ∇_2 is just the sum of the Levi-Civita connection of $T(dM)$ and the trivial connection on 1 and is flat in the normal direction. As ∇_2 is a direct sum connection, $E_m(\nabla_2) = 0$. $\nabla_1 - \nabla_2$ is essentially just the shape operator. $TE_m = -\sum Q_{k,m}$ and $dTE_m = E_m(\nabla_1) = E_m$. It is an easy exercise to work out the $Q_{k,m}$ using the methods of section 2 and thereby compute the normalizing constants given by Chern.

The Chern-Gauss-Bonnet theorem for manifolds with boundary in the oriented category becomes:

$$\chi(M) = \int_M E_m + \int_{dM} \sum_k Q_{k,m}.$$

In the unoriented category, we regard $E_m \operatorname{dvol}(x)$ as a measure on M and $\int Q_{k,m} \operatorname{dvol}(y)$ as a measure on dM. If m is odd, of course, $\chi(M) = \frac{1}{2}\chi(dM) = \frac{1}{2}\int_{dM} E_{m-1}$ so there is no difficulty with the Chern-Gauss-Bonnet theorem in this case.

We derive the Chern-Gauss-Bonnet theorem for manifolds with boundary from the theorem for manifolds without boundary. Suppose m is even and that the metric is product near the boundary. Let $\overline{M}$ be the double of M then $\chi(\overline{M}) = \int_{\overline{M}} E_m = 2\int_M E_m = 2\chi(M) - \chi(d\overline{M}) = 2\chi(M)$ so $\chi(M) = \int_M E_m$. If the metric is not product near the boundary, let $M' = dM \times [-1,0] \cup M$ be the manifold M with a collar sewed on. Let G_0 be the restriction of the metric on M to the boundary and let G_0' be the product metric on the collar $dM \times [-1,0]$. Using a partition of unity, extend the original metric on M to a new metric which agrees with G_0' near $dM \times \{-1\}$ which is the boundary of M'. Then:

$$\begin{aligned}\chi(M) = \chi(M') = \int_{M_1} E_m &= \int_M E_m - \int_{dM\times[-1,0]} d\Big(\sum_k Q_{k,m}\Big)\\ &= \int_M E_m + \int_{dM} \sum_k Q_{k,m}\end{aligned}$$

by Stoke's theorem; since the Q_k vanish identically near $dM \times \{-1\}$ there is no contribution from this component of the boundary of the collar (we change the sign since the orientation of dM as the boundary of M and as the boundary of $dM \times [-1,0]$ are opposite).

We now study the invariants of the heat equation. We impose no restrictions on the dimension m. If $M = S^1 \times M_1$ and if θ is the usual periodic parameter on S^1, then there is a natural involution on $\Lambda(T^*M)$ given by interchanging ψ with $d\theta \wedge \psi$ for $\psi \in \Lambda(M_1)$. This involution preserves the boundary conditions and the associated Laplacians, but changes the parity of the factors involved. This shows $a_n(y, d+\delta, B) = 0$ for such a product metric. We define:

$$r\colon \mathcal{P}^b_{m,n} \to \mathcal{P}^b_{m-1,n}$$

to be the dual of the map $M_1 \to S^1 \times M_1$. Then algebraically:

$$r(g_{ij/\alpha}) = \begin{cases} 0 & \text{if } \deg_1(g_{ij/\alpha}) \neq 0 \\ * & \text{if } \deg_1(g_{ij/\alpha}) = 0 \end{cases}$$

where "$*$" is simply a renumbering to shift all the indices down one. (At this stage, it is inconvenient to have used the last index to indicate the normal direction so that the first index must be used to denote the flat index; denoting the normal direction by the last index is sufficiently cannonical that we have not attempted to adopt a different convention despite the conflict with the notation of Chapter 2). This proves:

LEMMA 4.2.2. *Let B denote either the relative or absolute boundary conditions. Then $a_n(y, d+\delta, B) \in \mathcal{P}^b_{m,n}$. Furthermore, $r(a_n) = 0$ where $r\colon \mathcal{P}^b_{m,n} \to \mathcal{P}^b_{m-1,n}$ is the restriction map.*

We can now begin to identify $a_n(y, d+\delta, B)$ using the same techniques of invariance theory applied in the second chapter.

LEMMA 4.2.3.. *Let $P \in \mathcal{P}^b_{m,n}$. Suppose that $r(P) = 0$. Then:*
(a) $P = 0$ if $n < m-1$.
(b) If $n = m-1$, then P is a polynomial in the variables $\{g_{ij/m}, g_{ij/kl}\}$ for $1 \leq i,j,k,l \leq m-1$. Furthermore, $\deg_j(A) = 2$ for any monomial A of P and for $1 \leq j \leq m-1$.

PROOF: As in the proof of Theorem 2.4.7, we shall count indices. Let $P \neq 0$ and let A be a monomial of P. Decompose A in the form:

$$A = g_{u_1v_1/\alpha_1} \dots g_{u_kv_k/\alpha_k} g_{i_1j_1/m} \dots g_{i_rj_r/m} \qquad \text{for } |\alpha_\nu| \geq 2.$$

(We have chosen our coordinate systems so the only non-zero 1-jets are the $g_{ij/m}$ variables.) Since $r(P) = 0$, $\deg_1(A) \neq 0$. Since P is invariant, $\deg_j(A) > 0$ is even for $1 \leq j \leq m-1$. This yields the inequalities:

$$2m-2 \leq \sum_{j \leq m-1} \deg_j(A) \qquad \text{and} \qquad r \leq \deg_m(A).$$

From this it follows that:

$$2m-2+r \leq \sum_j \deg_j(A) = 2r+2k+\sum_\nu |\alpha_\nu| + r = 2r+2k+n.$$

Since $|\alpha_\nu| \ge 2$ we conclude

$$2k \le \sum_\nu |\alpha_\nu| = n - r.$$

We combine these inequalities to conclude $2m-2+r \le 2n+r$ so $n \ge m-1$. This shows $P = 0$ if $n < m-1$ which proves (a). If $n = m-1$, all these inequalities must be equalities. $|\alpha_\nu| = 2$, $\deg_m(A) = r$, and $\deg_j(A) = 2$ for $1 \le j \le m-1$. Since the index m only appears in A in the $g_{ij/m}$ variables where $1 \le i, j \le m-1$, this completes the proof of (b).

Lemma 2.5.1 only used invariance under the group $\mathrm{SO}(2)$. Since P is invariant under the action of $\mathrm{SO}(m-1)$, we apply the argument used in the proof of Theorem 2.4.7 to choose a monomial A of P of the form:

$$A = g_{11/22} \cdots g_{2k-1,2k-1/kk}\, g_{k+1,k+1/m} \cdots g_{m-1,m-1/m}$$

where if $k = 0$ the terms of the first kind do not appear and if $k = m-1$, the terms of the second kind do not appear. Since $m-1 = 2k+r$, it is clear that $r \equiv m-1 \bmod 2$. We denote such a monomial by A_k. Since $P \ne 0$ implies $c(A_k, P) \ne 0$ for some k, we conclude the dimension of the space of such P is at most the cardinality of $\{A_k\} = \left[\frac{m+1}{2}\right]$. This proves:

Lemma 4.2.4. *Let $r\colon \mathcal{P}^b_{m,m-1} \to \mathcal{P}^b_{m-1,m-1}$ be the restriction map defined earlier. Then* $\dim \mathrm{N}(r) \le \left[\frac{m+1}{2}\right]$.

We can now show:

Lemma 4.2.5. *Let $P \in \mathcal{P}^b_{m,m-1}$ with $r(P) = 0$. Let $i\colon dM \to M$ be the inclusion map and $i^*\colon \Lambda^{m-1}(T^*M) \to \Lambda^{m-1}(T^*(dM))$ be the dual map. Let $*_{m-1}$ be the Hodge operator on the boundary so $*_{m-1}\colon \Lambda^{m-1}(T^*(dM)) \to \Lambda^0(T^*(dM))$. Let $\bar{Q}_{k,m} = *_{m-1}(i^* Q_{k,m})$. Then the $\{\bar{Q}_{k,m}\}$ form a basis for $\mathrm{N}(r)$ so we can express P as a linear combination of the $\bar{Q}_{k,m}$.*

Proof: It is clear $r(\bar{Q}_{k,m}) = 0$ and that these elements are linearly independent. We have $[(m+1)/2]$ such elements so by Lemma 4.2.4 they must be a basis for the kernel of r. (If we reverse the orientation we change both the sign of $*$ and Q so $\bar{Q}$ is a scalar invariant.)

We note that if m is odd, then $Q_{m-1,m} = c \cdot \sum \varepsilon(i_1, \dots, i_{m-1}) \Omega_{i_1 i_2} \wedge \cdots \wedge \Omega_{i_{m-1} i_{m-1}}$ is *not* the Euler form on the boundary since we are using the Levi-Civita connection on M and not the Levi-Civita connection on dM. However, E_{m-1} can be expressed in terms of the $\bar{Q}_{k,m}$ in this situation.

Before proceeding to discuss the heat equation, we need a uniqueness theorem:

LEMMA 4.2.6. *Let $P = \sum_k a_k Q_{k,m}$ be a linear combination of the $\{Q_{k,m}\}$. Suppose that $P \neq 0$. Then there exists a manifold M and a metric G so $\int_{dM} P(G)(y) \neq 0$.*

PROOF: By assumption not all the $a_k = 0$. Choose k maximal so $a_k \neq 0$. Let $n = m - 2k$ and let $M = S^{2k} \times D^{m-2k}$ with the standard metric. (If $m - 2k = 1$, we let $M = D^m$ and choose a metric which is product near S^{m-1}.) We let indices $1 \leq i \leq 2k$ index a frame for $T^*(S^{2k})$ and indices $2k+1 \leq u \leq m$ index a frame for $T^*(D^{m-2k})$. Since the metric is product, $\Omega_{iu} = \omega_{iu} = 0$ in this situation. Therefore $Q_{j,m} = 0$ if $j < k$. Since $a_j = 0$ for $j > k$ by assumption, we conclude $\int_{dM} P(G) = a_k \int_{dM} Q_{k,m}(G)$, so it suffices to show this integral is non-zero. This is immediate if $n = 1$ as $Q_{m-1,m} = E_{m-1}$ and $m - 1$ is even.

Let $n \geq 2$. We have $Q_{k,m}(G) = E_{2k}(G_1) \cdot Q_{0,m-2k}(G_2)$, since the metric is product. Since

$$\int_{S^{2k}} E_{2k} = \chi(S^{2k}) = 2 \neq 0$$

we must only show $\int_{dD^n} Q_{0,n}$ is non-zero for all $n > 1$. Let θ be a system of local coordinates on the unit sphere S^{n-1} and let r be the usual radial parameter. If ds_e^2 is the Euclidean metric and ds_θ^2 is the spherical metric, then

$$ds_e^2 = r^2\, ds_\theta^2 + dr \cdot dr.$$

From the description of the shape operator given previously we conclude that $S = -ds_\theta^2$. Let $\{e_1, \dots, e_{n-1}\}$ be a local oriented orthonormal frame for $T(S^n)$, then $\omega_{in} = -e^i$ and therefore $Q_{0,n} = c \cdot \mathrm{dvol}_{n-1}$ where c is a non-zero constant. This completes the proof.

We combine these results in the following Theorem:

THEOREM 4.2.7. (GAUSS-BONNET FORMULA FOR MANIFOLDS WITH BOUNDARY).
(a) Let the dimension m be even and let B denote either the relative or the absolute boundary conditions. Let

$$Q_{k,m} = c_{k,m} \sum \varepsilon(i_1, \dots, i_{m-1}) \Omega_{i_1,i_2} \wedge \dots \wedge \Omega_{i_{2k-1},i_{2k}} \wedge \omega_{i_{2k+1},m} \wedge \dots \wedge \omega_{i_{m-1},m}$$

for $c_{k,m} = (-1)^k / (\pi^{m/2} \cdot k! \cdot 2^{k+m/2} \cdot 1 \cdot 3 \cdots (m - 2k - 1))$. Let $\bar{Q}_{k,m} = *(Q_{k,m}|dM) \in \mathcal{P}^b_{m,m-1}$. Then:

(i) *$a_n(x, d + \delta) = 0$ for $n < m$ and $a_n(y, d + \delta, B) = 0$ for $n < m - 1$,*
(ii) *$a_m(x, d + \delta) = E_m$ is the Euler integrand,*
(iii) *$a_{m-1}(y, d + \delta, B) = \sum_k \bar{Q}_{k,m}$,*
(iv) *$\chi(M) = \chi(M, dM) = \int_M E_m\, \mathrm{dvol}(x) + \sum_k \int_{dM} \bar{Q}_{k,m}\, \mathrm{dvol}(y)$.*

(b) Let the dimension m be odd and let B_r be the relative and B_a the absolute boundary conditions. Then:

(i) $a_n(x, d+\delta) = 0$ *for all* n *and* $a_n(y, d+\delta, B_r) = a_n(y, d+\delta, B_a) = 0$ *for* $n < m-1$,

(ii) $a_{m-1}(y, d+\delta, B_a) = \frac{1}{2}E_{m-1}$ *and* $a_{m-1}(y, d+\delta, B_r) = -\frac{1}{2}E_{m-1}$,

(iii) $\chi(M) = -\chi(M, dM) = \frac{1}{2}\int_{dM} E_{m-1}\,\mathrm{dvol}(y) = \frac{1}{2}\chi(dM)$.

This follows immediately from our previous computations, and Lemmas 4.2.5 and 4.2.6.

The Atayah-Bott theorem gives a generalization of the Atiyah-Singer index theorem for index problems on manifolds with boundary. This theorem includes the Gauss-Bonnet theorem as a special case, but does not include the Atiyah-Patodi-Singer index theorem since the signature, spin, and Dolbeault complexes do not admit local boundary conditions of the form we have been discussing. We will discuss this in more detail in subsection 4.5.

4.3. The Regularity at $s = 0$ of the Eta Invariant.

In this section, we consider the eta invariant defined in section 1.10. This section will be devoted to proving eta is regular at $s = 0$. In the next section we will use this result to discuss the twisted index theorem using coefficients in a locally flat bundle. This invariant appears as a boundary correction term in the index theorem for manifolds with boundary.

We shall assume $P\colon C^\infty(V) \to C^\infty(V)$ is a self-adjoint elliptic pseudo-differential operator of order $d > 0$. We define

$$\eta(s, P) = \sum_{\lambda_i > 0} (\lambda_i)^{-s} - \sum_{\lambda_i < 0} (-\lambda_i)^{-s} \qquad \text{for } \operatorname{Re}(s) \gg 0$$

and use Theorem 1.10.3 to extend η meromorphically to the complex plane with isolated simple poles on the real axis. We define

$$R(P) = d \cdot \operatorname{Res}_{s=0} \eta(s, P).$$

We will show $R(P) = 0$ so η is regular at $s = 0$. The first step is to show:

Lemma 4.3.1. *Let P and Q be self-adjoint elliptic pseudo-differential operators of order $d > 0$.*
(a) $P \cdot (P^2)^v$ is a self-adjoint elliptic pseudo-differential operator for any v and if $2v + 1 > 0$, $R(P) = R(P \cdot (P^2)^v)$.
(b) $R(P \oplus Q) = R(P) + R(Q)$.
(c) There is a local formula $a(x, P)$ in the jets of the symbol of P up to order d so that $R(P) = \int_M a(x, P)\,|\operatorname{dvol}(x)|$.
(d) If P_t is a smooth 1-parameter family of such operators, then $R(P_t)$ is independent of the parameter t.
(e) If P is positive definite, then $R(P) = 0$.
(f) $R(-P) = -R(P)$.

Proof: We have the formal identity: $\eta(s, P \cdot (P^2)^v) = \eta((2v + 1)s, P)$. Since we normalized the residue by multiplying by the order of the operator, (a) holds. The fact that $P \cdot (P^2)^v$ is again a pseudo-differential operator follows from the work of Seeley. (b) is an immediate consequence of the definition. (c) and (d) were proved in Lemma 1.10.2 for differential operators. The extension to pseudo-differential operators again follows Seeley's work. If P is positive definite, then the zeta function and the eta function coincide. Since zeta is regular at the origin, (e) follows by Lemma 1.10.1. (f) is immediate from the definition.

We note this lemma continues to be true if we assume the weaker condition $\det(p(x, \xi) - it) \neq 0$ for $(\xi, t) \neq (0, 0) \in T^*M \times \mathbf{R}$.

We use Lemma 4.3.1 to interpret $R(P)$ as a map in K-theory. Let $S(T^*M)$ be the unit sphere bundle in T^*M. Let V be a smooth vector bundle over M equipped with a fiber inner product. Let $p\colon S(T^*M) \to$

$\mathrm{END}(V)$ be self-adjoint and elliptic; we assume $p(x,\xi) = p^*(x,\xi)$ and $\det p(x,\xi) \neq 0$ for $(x,\xi) \in S(T^*M)$. We fix the order of homogeneity $d > 0$ and let $p_d(x,\xi)$ be the extension of p to T^*M which is homogeneous of degree d. (In general, we must smooth out the extension near $\xi = 0$ to obtain a C^∞ extension, but we suppress such details in the interests of notational clarity.)

LEMMA 4.3.2. *Let $p\colon S(T^*M) \to \mathrm{END}(V)$ be self-adjoint and elliptic. Let $d > 0$ and let $P\colon C^\infty(V) \to C^\infty(V)$ have leading symbol p_d. Then $R(P)$ depends only on p and not on the order d nor the particular operator P.*

PROOF: Let P' have order d with the same leading symbol p_d. We form the elliptic family $P_t = tP' + (1-t)P$. By Lemma 4.3.1, $R(P_t)$ is independent of t so $R(P') = R(P)$. Given two different orders, let $(1+2v)d = d'$. Let $Q = P(P^2)^v$ then $R(Q) = R(P)$ by Lemma 4.3.1(a). The leading symbol of Q is $p(p^2)^v$. p^2 is positive definite and elliptic. We construct the homotopy of symbols $q_t(x,\xi) = p(x,\xi)(tp^2(x,\xi) + (1-t)|\xi|^2)$. This shows the symbol of Q restricted to $S(T^*M)$ is homotopic to the symbol of P restricted to $S(T^*M)$ where the homotopy remains within the class of self-adjoint elliptic symbols. Lemma 4.3.1 completes the proof.

We let $r(p) = R(P)$ for such an operator P. Lemma 4.3.1(d) shows $r(p)$ is a homotopy invariant of p. Let $\Pi_\pm(p)$ be the subspaces of V spanned by the eigenvectors of $p(x,\xi)$ corresponding to positive/negative eigenvalues. These bundles have constant rank and define smooth vector bundles over $S(T^*M)$ so $\Pi_+ \oplus \Pi_- = V$. In section 3.9, an essential step in proving the Atayah-Singer index theorem was to interpret the index as a map in K-theory. To show $R(P) = 0$, we must first interpret it as a map in K-theory. The natural space in which to work is $K(S(T^*M);\mathbf{Q})$ and not $K(\Sigma(T^*M);\mathbf{Q})$.

LEMMA 4.3.3. *Let G be an abelian group and let $R(P) \in G$ be defined for any self-adjoint elliptic pseudodifferential operator. Assume R satisfies properties (a), (b), (d), (e) and (f) [but not necessarily (c)] of Lemma 4.3.1. Then there exists a $\mathbf{Z}$-linear map $r\colon K(S(T^*M)) \to G$ so that:*

(a) $R(P) = r(\Pi_+(p))$,
(b) *If $\tau\colon S(T^*M) \to M$ is the natural projection, then $r(\tau^*V) = 0$ for all $V \in K(M)$ so that*

$$r\colon K(S(T^*M))/K(M) \to G.$$

Remark: We shall apply this lemma in several contexts later, so state it in somewhat greater generality than is needed here. If $G = \mathbf{R}$, we can extend r to a $\mathbf{Q}$ linear map

$$r\colon K(S(T^*M);\mathbf{Q}) \to \mathbf{R}.$$

PROOF: Let $p: S(T^*M) \to \mathrm{END}(V)$ be self-adjoint and elliptic. We define the bundles $\Pi_\pm(p)$ and let $\pi_\pm(x,\xi)$ denote orthogonal projection on $\Pi_\pm(p)(x,\xi)$. We let $p_0 = \pi_+ - \pi_-$ and define $p_t = tp + (1-t)p_0$ as a homotopy joining p and p_0. It is clear that the p_t are self-adjoint. Fix (x,ξ) and let $\{\lambda_i, v_i\}$ be a spectral resolution of the matrix $p(x,\xi)$. Then:

$$\begin{aligned} p(x,\xi)\Big(\sum c_i v_i\Big) &= \sum \lambda_i c_i v_i \\ p_0(x,\xi)\Big(\sum c_i v_i\Big) &= \sum \operatorname{sign}(\lambda_i) c_i v_i. \end{aligned}$$

Consequently

$$p_t\Big(\sum c_i v_i\Big) = \sum (t\lambda_i + (1-t)\operatorname{sign}(\lambda_i)) c_i v_i.$$

Since $t\lambda_i + (1-t)\operatorname{sign}(\lambda_i) \neq 0$ for $t \in [0,1]$, the family p_t is elliptic. We therefore assume henceforth that $p(x,\xi)^2 = I$ on $S(T^*M)$ and $\pi_\pm = \frac{1}{2}(1 \pm p)$.

We let k be large and choose $V \in \mathrm{Vect}_k(S(T^*M))$. Choose $W \in \mathrm{Vect}_k(S(T^*M))$ so $V \oplus W \simeq 1^{2k}$. We may choose the metric on 1^{2k} so this direct sum is orthogonal and we define $\pi_\pm$ to be orthogonal projection on V and W. We let $p = \pi_+ - \pi_-$ so $\Pi_+ = V$ and $\Pi_- = W$. We define $r(V) = r(p)$. We must show this is well defined on Vect_k. W is unique up to isomorphism but the isomorphism $V \oplus W = 1^{2k}$ is non-canonical. Let $u: V \to \overline{V}$ and $v: W \to \overline{W}$ be isomorphisms where we regard $\overline{V}$ and $\overline{W}$ as orthogonal complements of 1^{2k} (perhaps with another trivialization). We must show $r(p) = r(\bar{p})$. For $t \in [0,1]$ we let

$$\begin{aligned} V(t) &= \operatorname{span}\big(t \cdot v \oplus (1-t)\cdot u(v)\big)_{v \in V} \\ &\subseteq V \oplus W \oplus \overline{V} \oplus \overline{W} = 1^{2k} \oplus 1^{2k} = 1^{4k}. \end{aligned}$$

This is a smooth 1-parameter family of bundles connecting $V \oplus 0$ to $0 \oplus \overline{V}$ in 1^{4k}. This gives a smooth 1-parameter family of symbols $p(t)$ connecting $p \oplus (-1_{2k})$ to $(-1_{2k}) \oplus \bar{p}$. Thus $r(p) = r(p \oplus -1_{2k}) = r(p(t)) = r(-1_{2k} \oplus \bar{p}) = r(\bar{p})$ so this is in fact a well defined map $r: \mathrm{Vect}_k(S(T^*M)) \to G$. If V is the trivial bundle, then W is the trivial bundle so p decomposes as the direct sum of two self-adjoint matrices. The first is positive definite and the second negative definite so $r(p) = 0$ by Lemma 4.3.1(f). It is clear that $r(V_1 \oplus V_2) = r(V_1) + r(V_2)$ by Lemma 4.3.1(b) and consequently since $r(1) = 0$ we conclude r extends to an additive map from $\widetilde{K}(S(T^*M)) \to G$. We extend r to be zero on trivial bundles and thus $r: K(S(T^*M)) \to G$.

Suppose $V = \tau^* V_0$ for $V_0 \in \mathrm{Vect}_k(M)$. We choose $W_0 \in \mathrm{Vect}_k(M)$ so $V_0 \oplus W_0 \simeq 1^{2k}$. Then $p = p_+ \oplus p_-$ for $p_+: S(T^*M) \to \mathrm{END}(V_0, V_0)$

and $p_-: S(T^*M) \to \mathrm{END}(W_0, W_0)$. By Lemma 4.3.1, we conclude $r(p) = r(p_+) + r(p_-)$. Since p_+ is positive definite, $r(p_+) = 0$. Since p_- is negative definite, $r(p_-) = 0$. Thus $r(p) = 0$ and $r(\tau^* V_0) = 0$.

This establishes the existence of the map $r: K(S(T^*M)) \to G$ with the desired properties. We must now check $r(p) = r(\Pi_+)$ for general p. This follows by definition if V is a trivial bundle. For more general V, we first choose W so $V \oplus W = 1$ over M. We let $q = p \oplus 1$ on $V \oplus W$, then $r(q) = r(p) + r(1) = r(p)$. However, q acts on a trivial bundle so $r(q) = r(\Pi_+(q)) = r(\Pi_+(p) \oplus \tau^* W) = r(\Pi_+(p)) + r(\tau^* W) = r(\Pi_+(p))$ which completes the proof.

Of course, the bundles $\Pi_\pm(p)$ just measure the infinitesimal spectral asymmetry of P so it is not surprising that the bundles they represent in K-theory are related to the eta invariant. This construction is completely analogous to the construction given in section 3.9 which interpreted the index as a map in K-theory. We will return to this construction again in discussing the twisted index with coefficients in a locally flat bundle.

Such operators arise naturally from considering boundary value problems. Let M be a compact oriented Riemannian manifold of dimension $m = 2k - 1$ and let $N = M \times [0, 1)$; we let $n \in [0, 1)$ denote the normal parameter. Let

$$(d + \delta)_+ : C^\infty(\Lambda^+(T^*N)) \to C^\infty(\Lambda^-(T^*N))$$

be the operator of the signature complex. The leading symbol is given by Clifford multiplication. We can use $c(dn)$, where c is Clifford multiplication, to identify these two bundles over N. We express:

$$(d + \delta)_+ = c(dn)(\partial/\partial n + A).$$

The operator A is a tangential differential operator on $C^\infty(\Lambda^+(T^*N))$; it is called the tangential operator of the signature complex. Since we have $c(dn) * c(dn) = -1$, the symbol of A is $-ic(dn)c(\xi)$ for $\xi \in T^*(M)$. It is immediate that the leading symbol is self-adjoint and elliptic; since A is natural this implies A is a self-adjoint elliptic partial differential operator on M.

Let $\{e_1, \dots, e_m\}$ be a local oriented orthonormal frame for T^*M. Define:

$$\omega_m = i^k e_1 * \cdots * e_m \quad \text{and} \quad \omega_{m+1} = i^k(-dn) * e_1 * \cdots * e_m = -dn * \alpha_m$$

as the local orientations of M and N. ω_m is a central element of $\mathrm{CLIF}(M)$; $\omega_m^2 = \omega_{m+1}^2 = 1$. If $\phi \in \Lambda(M)$ define $\tau_\pm(\phi) = \phi \pm c(\omega_{m+1})\phi$. This gives an isomorphism $\tau_\pm\colon \Lambda(M) \to \Lambda^\pm(N)$. We compute:

$$\begin{aligned}
&(\tau_+)^{-1}\{-c(dn)c(\xi)\}\tau_+\phi \\
&\qquad = (\tau_+)^{-1}\{-c(dn)c(\xi)\phi - c(dn)c(\xi)c(-dn)c(\omega_m)\phi\} \\
&\qquad = (\tau_+)^{-1}\{-c(dn)c(\omega_m)c(\omega_m)c(\xi) + c(\omega_m)c(\xi)\phi\} \\
&\qquad = (\tau_+)^{-1}\{(c(\omega_{m+1}) + 1)c(\omega_m)c(\xi)\phi\} \\
&\qquad = c(\omega_m)c(\xi)\phi.
\end{aligned}$$

If we use τ_+ to regard A as an operator on $C^\infty(\Lambda(M))$ then this shows that A is given by the diagram:

$$C^\infty(\Lambda(M)) \xrightarrow{\nabla} C^\infty(T^*(M \otimes \Lambda(M)) \xrightarrow{c} C^\infty(\Lambda(M)) \xrightarrow{\omega} C^\infty(\Lambda(M))$$

where $\omega = \omega_m$. This commutes with the operator $(d+\delta)$; $A = \omega(d+\delta)$.

Both ω and $(d+\delta)$ reverse the parity so we could decompose $A = A^{\text{even}} \oplus A^{\text{odd}}$ acting on smooth forms of even and odd degree. We let $p = \sigma_L(A) = ic(\omega * \xi) = \sum_j \xi_j f_j$. The $\{f_j\}$ are self-adjoint matrices satisfying the commutation relation

$$f_j f_k + f_k f_j = 2\delta_{jk}.$$

We calculate:

$$f_1 \cdots f_m = i^m c(\omega^m * e_1 * \cdots * e_m) = i^{m-k} c(\omega^m * \omega) = i^{m-k} c(1)$$

so that if we integrate over a fiber sphere with the natural (not simplectic) orientation,

$$\begin{aligned}
\int_{S^{m-1}} ch(\Pi_+(p)) &= i^{k-1} 2^{1-k}\, \mathrm{Tr}(f_1 \cdots f_m) \\
&= i^{k-1} 2^{1-k} i^{m-k} 2^m = i^{k-1} 2^{1-k} i^{k-1} 2^{2k-1} \\
&= (-1)^{k-1} \cdot 2^k.
\end{aligned}$$

In particular, this is non-zero, so this cohomology class provides a cohomology extension to the fiber.

As a $H^*(M;\mathbf{Q})$ module, we can decompose $H^*(S(M);\mathbf{Q}) = H^*(M;\mathbf{Q}) \oplus xH^*(M;\mathbf{Q})$ where $x = ch(\Pi_+(p))$. If we twist the operator A by taking coefficients in an auxilary bundle V, then we generate $xH^*(M;\mathbf{Q})$. The same argument as that given in the proof of Lemma 3.9.8 permits us to interpret this in K-theory:

LEMMA 4.3.4. *Let M be a compact oriented Riemannian manifold of dimension $m = 2k-1$. Let A be the tangential operator of the signature complex on $M \times [0,1)$. If $\{e_j\}$ is an oriented local orthonormal basis for $T(M)$, let $\omega = i^k e_1 * \cdots * e_m$ be the orientation form acting by Clifford multiplication on the exterior algebra. $A = \omega(d+\delta)$ on $C^\infty(\Lambda(M))$. If the symbol of A is p,*

$$\int_{S^{m-1}} ch(\Pi_+(p)) = 2^k(-1)^{k-1}.$$

The natural map $K(M;\mathbf{Q}) \to K(S(T^*(M);\mathbf{Q})$ is injective and the group $K(S(T^*M);\mathbf{Q})/K(M;\mathbf{Q})$ is generated by the bundles $\{\Pi_+(A_V)\}$ as V runs over $K(M)$.

Remark: This operator can also be represented in terms of the Hodge operator. On $C^\infty(\Lambda^{2p})$ for example it is given by $i^k(-1)^{p+1}(*d - d*)$. We are using the entire tangential operator (and not just the part acting on even or odd forms). This will produce certain factors of 2 differing from the formulas of Atiyah-Patodi-Singer. If M admits a SPIN$_c$ structure, one can replace A by the tangential operator of the SPIN$_c$ complex; in this case the corresponding integrand just becomes $(-1)^{k-1}$.

We can use this representation to prove:

LEMMA 4.3.5. *Let $\dim M$ be odd and let $P: C^\infty(V) \to C^\infty(V)$ be a self-adjoint elliptic pseudo-differential operator of order $d > 0$. Then $R(P) = 0$—i.e., $\eta(s,P)$ is regular at $s = 0$.*

PROOF: We first suppose M is orientable. By Lemma 4.3.4, it suffices to prove $R(A_V) = 0$ since r is defined in K-theory and would then vanish on the generators. However, by Lemma 4.3.1(c), the residue is given by a local formula. The same analysis as that done for the heat equation shows this formula must be homogeneous of order m in the jets of the metric and of the connection on V. Therefore, it must be expressible in terms of Pontrjagin forms of TM and Chern forms of V by Theorem 2.6.1. As m is odd, this local formula vanishes and $R(A_V) = 0$. If M is not orientable, we pass to the oriented double cover. If $P: C^\infty(V) \to C^\infty(V)$ over M, we let $P': C^\infty(V') \to C^\infty(V')$ be the lift to the oriented double cover. Then $R(P) = \frac{1}{2}R(P')$. But $R(P') = 0$ since the double cover is oriented and thus $R(P) = 0$. This completes the proof.

This result is due to Atayah, Patodi, and Singer. The trick used in section 3.9 to change the parity of the dimension by taking products with a problem over the circle does not go through without change as we shall see. Before considering the even dimensional case, we must first prove a product formula.

LEMMA 4.3.6. *Let M_1 and M_2 be smooth manifolds. Let $P: C^\infty(V_1) \to C^\infty(V_2)$ be an elliptic complex over M_1. Let $Q: C^\infty(V) \to C^\infty(V)$ be a self-adjoint elliptic operator over M_2. We assume P and Q are differential operators of the same order and form:*

$$R = \begin{pmatrix} Q & P^* \\ P & -Q \end{pmatrix} \qquad \text{on } C^\infty(V_1 \otimes V \oplus V_2 \otimes V),$$

then $\eta(s, R) = \operatorname{index}(P) \cdot \eta(s, Q)$.

PROOF: This lemma gives the relationship between the index and the eta invariant which we will exploit henceforth. We perform a formal computation. Let $\{\lambda_\nu, \phi_\nu\}_{\nu=1}^\infty$ be a spectral resolution of the operator Q on $C^\infty(V)$. We let $\Delta = P^*P$ and decompose $C^\infty(V_1) = \mathrm{N}(\Delta) + R(\Delta)$. We let $\{\mu_j, \theta_j\}$ be a spectral resolution of Δ restricted to $\mathrm{N}(\Delta)^\perp = \mathrm{R}(\Delta)$. The μ_j are positive real numbers; $\{\mu_j, P\theta_j/\sqrt{\mu_j}\}$ form a spectral resolution of $\Delta' = PP^*$ on $\mathrm{N}(\Delta')^\perp = \mathrm{R}(\Delta') = \mathrm{R}(P)$.

We decompose $L^2(V_1 \otimes V) = \mathrm{N}(\Delta) \otimes L^2(V) \oplus \mathrm{R}(\Delta) \otimes L^2(V)$ and $L^2(V_2 \otimes V) = \mathrm{N}(\Delta') \otimes L^2(V) \oplus \mathrm{R}(\Delta') \otimes L^2(V)$. In $\mathrm{R}(\Delta) \otimes L^2(V) \oplus \mathrm{R}(\Delta') \otimes L^2(V)$ we study the two-dimensional subspace that is spanned by the elements: $\{\theta_j \otimes \phi_\nu, P\theta_j/\sqrt{\mu_j} \otimes \phi_\nu\}$. The direct sum of these subspaces as j, ν vary is $\mathrm{R}(\Delta) \otimes L^2(V) \oplus \mathrm{R}(\Delta') \otimes L^2(V)$. Each subspace is invariant under the operator R. If we decompose R relative to this basis, it is represented by the 2×2 matrix:

$$\begin{pmatrix} \lambda_\nu & \sqrt{\mu_j} \\ \sqrt{\mu_j} & -\lambda_\nu \end{pmatrix}.$$

This matrix has two eigenvalues with opposite signs: $\pm\sqrt{\lambda_\nu^2 + \mu_j}$. Since $\lambda_\nu > 0$ these eigenvalues are distinct and cancel in the sum defining eta. Therefore the only contribution to eta comes from $\mathrm{N}(\Delta) \otimes L^2(V)$ and $\mathrm{N}(\Delta') \otimes L^2(V)$. On the first subspace, R is $1 \otimes Q$. Each eigenvalue of Q is repeated $\dim \mathrm{N}(\Delta)$ times so the contribution to eta is $\dim \mathrm{N}(\Delta)\eta(s, Q)$. On the second subspace, R is $1 \otimes -Q$ and the contribution to eta is $-\dim \mathrm{N}(\Delta)\eta(s, Q)$. When we sum all these contributions, we conclude that:

$$\eta(s, R) = \dim \mathrm{N}(\Delta)\eta(s, Q) - \dim \mathrm{N}(\Delta')\eta(s, Q) = \operatorname{index}(P)\eta(s, Q).$$

Although this formal cancellation makes sense even if P and Q are not differential operators, R will not be a pseudo-differential operator if P and Q are pseudo-differential operators in general. This did not matter when we studied the index since the index was constant under approximations. The eta invariant is a more delicate invariant, however, so we cannot use the same trick. Since the index of any differential operator on the circle is zero, we cannot use Lemmas 4.3.5 and 4.3.6 directly to conclude $R(P)=0$ if m is even.

We recall the construction of the operator on $C^\infty(S^1)$ having index 1. We fix $a\in\mathbf{R}$ as a real constant and fix a positive order $d\in\mathbf{Z}$. We define:

$$\begin{aligned} Q_0 &= -i\partial/\partial\theta, \qquad Q_1(a)=(Q_0^{2d}+a^2)^{1/2d}\\ Q_2(a) &= \tfrac{1}{2}(e^{-i\theta}(Q_0+Q_1)+Q_0-Q_1)\\ Q(a) &= Q_2(a)\cdot(Q_1(a))^{d-1}.\end{aligned}$$

The same argument as that given in the proof of Lemma 3.9.4 shows these are pseudo-differential operators on $C^\infty(S^1)$. $Q_2(a)$ and $Q(a)$ are elliptic families. If $a=0$, then $Q_2(0)$ agrees with the operator of Lemma 3.9.4 so index $Q_2(0)=1$. Since the index is continuous under perturbation, $\operatorname{index} Q_2(a)=1$ for all values of a. $Q_1(a)$ is self-adjoint so its index is zero. Consequently $\operatorname{index} Q(a)=\operatorname{index} Q_2(a)+\operatorname{index} Q_1(a)=1$ for all a.

We let $P\colon C^\infty(V)\to C^\infty(V)$ be an elliptic self-adjoint partial differential operator of order $d>0$ over a manifold M. On $M\times S^1$ we define the operators:

$$\begin{aligned} Q_0 &= -i\partial/\partial\theta, \qquad Q_1=(Q_0^{2d}+P^2)^{1/2d}\\ Q_2 &= \tfrac{1}{2}(e^{-i\theta}(Q_0+Q_1)+Q_0-Q_1), \qquad Q=Q_2\cdot Q_1^{d-1}\end{aligned}$$

on $C^\infty(V)$. These are pseudo-differential operators over $M\times S^1$ since $Q_0^{2d}+P^2$ has positive definite leading order symbol. We define R by:

$$R=\begin{pmatrix} P & Q^*\\ Q & -P\end{pmatrix}\colon\ C^\infty(V\oplus V)\to C^\infty(V\oplus V).$$

This is a pseudo-differential operator of order d which is self-adjoint. We compute:

$$R^2=\begin{pmatrix} P^2+Q^*Q & 0\\ 0 & P^2+QQ^*\end{pmatrix}$$

so

$$\sigma_L(R^2)(\xi,z)=\begin{pmatrix} p(\xi)^2+q^*q(\xi,z) & 0\\ 0 & p(\xi)^2+qq^*(\xi,z)\end{pmatrix}$$

for $\xi\in T^*M$ and $z\in T^*S^1$. Suppose R is not elliptic so $\sigma_L(R)(\xi,z)v=0$ for some vector $v\in V\oplus V$. Then $\{\sigma_L(R)(\xi,z)\}^2v=0$. We decompose $v=v_1\oplus v_2$. We conclude:

$$(p(\xi)^2+q^*q(\xi,z))v_1=(p(\xi)^2+qq^*(\xi,z))v_2=0.$$

Using the condition of self-adjointness this implies:

$$p(\xi)v_1 = q(\xi, z)v_1 = p(\xi)v_2 = q^*(\xi, z)v_2 = 0.$$

We suppose $v \neq 0$ so not both v_1 and v_2 are zero. Since P is elliptic, this implies $\xi = 0$. However, for $\xi = 0$, the operators Q_1 and Q_2 agree with the operators of Lemma 3.9.4 and are elliptic. Therefore $q(\xi, z)v_1 = q^*(\xi, z)v_2 = 0$ implies $z = 0$. Therefore the operator R is elliptic.

Lemma 4.3.6 generalizes in this situation to become:

LEMMA 4.3.7. *Let P, Q, R be defined as above, then $\eta(s, R) = \eta(s, P)$.*

PROOF: We let $\{\lambda_\nu, \phi_\nu\}$ be a spectral resolution of P on $C^\infty(V)$ over M. This gives an orthogonal direct sum decomposition:

$$L^2(V) \text{ over } M \times S^1 = \bigoplus_\nu L^2(S^1) \otimes \phi_\nu.$$

Each of these spaces is invariant under both P and R. If R_ν denotes the restriction of R to this subspace, then

$$\eta(s, R) = \sum_\nu \eta(s, R_\nu).$$

On $L^2(\phi_\nu)$, P is just multiplication by the real eigenvalue λ_ν. If we replace P by λ_ν we replace Q by $Q(\lambda_\nu)$, so R_ν becomes:

$$R_\nu = \begin{pmatrix} \lambda_\nu & Q^*(\lambda_\nu) \\ Q(\lambda_\nu) & -\lambda_\nu \end{pmatrix}.$$

We now apply the argument given to prove Lemma 4.3.6 to conclude:

$$\eta(s, R_\nu) = \operatorname{sign}(\lambda_\nu)|\lambda_\nu|^{-s} \operatorname{index}(Q(\lambda_\nu)).$$

Since $\operatorname{index} Q(\lambda_\nu) = 1$, this shows $\eta(s, R) = \operatorname{sign}(\lambda_\nu)|\lambda_\nu|^{-s}$ and completes the proof.

We can now generalize Lemma 4.3.5 to all dimensions:

THEOREM 4.3.8. *Let $P\colon C^\infty(V) \to C^\infty(V)$ be a self-adjoint elliptic pseudo-differential operator of order $d > 0$. Then $R(P) = 0$—i.e., $\eta(s, P)$ is regular at $s = 0$.*

PROOF: This result follows from Lemma 4.3.5 if $\dim M = m$ is odd. If $\dim M = m$ is even and if P is a differential operator, then we form the pseudo-differential operator P over $M \times S^1$ with $\eta(s, R) = \eta(s, P)$. Then $\eta(s, R)$ is regular at $s = 0$ implies $\eta(s, P)$ is regular at $s = 0$. This proves

Theorem 4.3.8 for differential operators. Of course, if P is only pseudo-differential, then R need not be pseudo-differential so this construction does not work. We complete the proof of Theorem 4.3.8 by showing the partial differential operators of even order generate $K(S(T^*M);\mathbf{Q})/K(M;\mathbf{Q})$ if m is even. (We already know the operators of odd order generate if m is odd and if M is oriented).

Consider the involution $\xi \mapsto -\xi$ of the tangent space. This gives a natural $\mathbf{Z}_2$ action on $S(T^*M)$. Let $\pi\colon S(T^*M) \to S(T^*M)/\mathbf{Z}_2 = \mathbf{R}P(T^*M)$ be the natural projection on the quotient projective bundle. Since m is even, $m-1$ is odd and π^* defines an isomorphism between the cohomology of two fibers

$$\pi^*\colon H^*(\mathbf{R}P^{m-1};\mathbf{Q}) = \mathbf{Q}\oplus\mathbf{Q} \to H^*(S^{m-1};\mathbf{Q}) = \mathbf{Q}\oplus\mathbf{Q}.$$

The Kunneth formula and an appropriate Meyer-Vietoris sequence imply that

$$\pi^*\colon H^*(\mathbf{R}P(T^*M);\mathbf{Q}) \to H^*(S(T^*M);\mathbf{Q})$$

is an isomorphism in cohomology for the total spaces. We now use the Chern isomorphism between cohomology and K-theory to conclude there is an isomorphism in K-theory

$$\pi^*\colon K(\mathbf{R}P(T^*M);\mathbf{Q}) \simeq K(S(T^*M);\mathbf{Q}).$$

Let $\mathrm{S}_0(k)$ be the set of all $k\times k$ self-adjoint matrices A such that $A^2 = 1$ and $\mathrm{Tr}(A) = 0$. We noted in section 3.8 that if k is large, $\widetilde{K}(X) = [X, \mathrm{S}_0(2k)]$. Thus $\widetilde{K}(S(T^*M);\mathbf{Q}) = \widetilde{K}(\mathbf{R}P(T^*M);\mathbf{Q})$ is generated by maps $p\colon S(T^*M) \to \mathrm{S}_0(2k)$ such that $p(x,\xi) = p(x,-\xi)$. We can approximate any even map by an even polynomial using the Stone-Weierstrass theorem. Thus we may suppose p has the form:

$$p(x,\xi) = \sum_{\substack{|\alpha|\le n\\ |\alpha|\ \text{even}}} p_\alpha(x)\xi^\alpha\big|_{S(T^*M)},$$

where the $p_\alpha\colon M \to \mathrm{S}_0(2k)$ and where n is large. As α is even, we can replace ξ^α by $\xi^\alpha|\xi|^{\{n-|\alpha|\}/2}$ and still have a polynomial with the same values on $S(T^*M)$. We may therefore assume that p is a homogeneous even polynomial; this is the symbol of a partial differential operator which completes the proof.

If m is odd and if M is orientable, we constructed specific examples of operators generating $K(S(T^*M);\mathbf{Q})/K(M;\mathbf{Q})$ using the tangential operator of the signature complex with coefficients in an arbitrary coefficient bundle. If m is even, it is possible to construct explicit second order oprators generating this K-theory group. One can then prove directly that eta is regular at $s = 0$ for these operators as they are all "natural" in a certain suitable sense. This approach gives more information by explcitly exhibiting the generators; as the consturction is quite long and technical we have chosen to give an alternate argument based on K-theory and refer to (Gilkey, The residue of the global eta function at the origin) for details.

We have given a global proof of Theorem 4.3.8. In fact such a treatment is necessary since in general the local formulas giving the residue at $s = 0$ are non-zero. Let $P: C^\infty(V) \to C^\infty(V)$ be self-adjoint and elliptic. If $\{\lambda_\nu, \phi_\nu\}$ is a spectral resolution of P, we define:

$$\eta(s, P, x) = \sum_\nu \operatorname{sign}(\lambda_\nu)|\lambda_\nu|^{-s}(\phi_\nu, \phi_\nu)(x)$$

so that:

$$\eta(s, P) = \int_M \eta(s, P, x) \operatorname{dvol}(x).$$

Thus $\operatorname{Res}_{s=0} \eta(s, P, x) = a(P, x)$ is given by a local formula.

We present the following example (Gilkey, The residue of the local eta function at the origin) to show this local formula need not vanish identically in general.

Example 4.3.9. Let

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

be Clifford matrices acting on $\mathbf{C}^2$. Let T^m be the m-dimensional torus with periodic parameters $0 \le x_j \le 2$ for $1 \le j \le m$.
(a) Let $m = 3$ and let $b(x)$ be a real scalar. Let $P = i \sum_j e_j \partial/\partial x_j + b(x)I$. Then P is self-adjoint and elliptic and $a(x, P) = c\Delta b$ where $c \neq 0$ is some universal constant.
(b) Let $m = 2$ and let b_1 and b_2 be imaginary scalar functions. Let $P = e_1\partial^2/\partial x_1^2 + e_2\partial^2/\partial x_2^2 + 2e_3\partial^2/\partial x_1 \partial x_2 + b_1 e_1 + b_2 e_2$. Then P is self-adjoint and elliptic and $a(x, P) = c'(\partial b_1/\partial x_2 - \partial b_2/\partial x_1)$ where $c' \neq 0$ is some universal constant.
(c) By twisting this example with a non-trivial index problem and using Lemma 4.3.6, we can construct examples on T_m so that $a(x, P)$ does not vanish identically for any dimension $m \ge 2$.

The value of eta at the origin plays a central role in the Atiyah-Patodi-Singer index theorem for manifolds with boundary. In section 2, we discussed the transgression briefly. Let ∇_i be two connections on $T(N)$. We defined $TL_k(\nabla_1, \nabla_2)$ so that $dTL_k(\nabla_1, \nabla_2) = L_k(\nabla_1) - L_k(\nabla_2)$; this is a secondary characteristic class. Let ∇_1 be the Levi-Civita connection of N and near $M = dN$ let ∇_2 be the product connection arising from the product metric. As $L_k(\nabla_2) = 0$ we see $dTL_k(\nabla_1, \nabla_2) = 0$. This is the analogous term which appeared in the Gauss-Bonnet theorem; it can be computed in terms of the first and second fundamental forms. For example, if $\dim N = 4$

$$L_1(R) = \frac{-1}{24 \cdot \pi^2} \operatorname{Tr}(R \wedge R) \qquad TL_1(R, \omega) = \frac{-1}{24 \cdot 8 \cdot \pi^2} \operatorname{Tr}(R \wedge \omega)$$

where R is the curvature 2-form and ω is the second fundamental form.

Theorem 4.3.10 (Atiyah-Patodi-Singer index theorem for manifolds with boundary). *Let N be a $4k$ dimensional oriented compact Riemannian manifold with boundary M. Then:*

$$\operatorname{signature}(N) = \int_N L_k - \int_M TL_k - \tfrac{1}{2}\eta(0, A)$$

where A is the tangential operator of the signature complex discussed in Lemma 4.3.4.

In fact, the eta invariant more generally is the boundary correction term in the index theorem for manifolds with boundary. Let N be a compact Riemannian manifold with boundary M and let $P\colon C^\infty(V_1) \to C^\infty(V_2)$ be an elliptic first order differential complex over N. We take a metric which is product near the boundary and identify a neighborhood of M in N with $M \times [0, 1)$. We suppose P decomposes in the form $P = \sigma(dn)(\partial/\partial n + A)$ on this collared neighborhood where A is a self-adjoint elliptic first order operator over M. This is in fact the case for the signature, Dolbeault, or spin compexes. Let B be the spectral projection on the non-negative eigenvalues of A. Then:

Theorem 4.3.11. (The Atiyah-Patodi-Singer Index Theorem) . *Adopt the notation above. P with boundary condition B is an elliptic problem and*

$$\operatorname{index}(P, B) = \int_N \{a_n(x, P^*P) - a_n(x, PP^*)\} - \tfrac{1}{2}\{\eta(0, A) + \dim \mathrm{N}(A)\}.$$

In this expression, $n = \dim N$ and the invariants a_n are the invariants of the heat equation discussed previously.

Remark: If M is empty then this is nothing but the formula for $\operatorname{index}(P)$ discussed previously. If the symbol does not decompose in this product structure near the boundary of N, there are corresponding local boundary correction terms similar to the ones discussed previously. If one takes P to be the operator of the signature complex, then Theorem 4.3.10 can be derived from this more general result by suitably interpreting $\operatorname{index}(P, B) = \operatorname{signature}(\mathrm{N}) - \frac{1}{2} \dim \mathrm{N}(A)$.

4.4. The Eta Invariant with Coefficients In a Locally Flat Bundle.

The eta invariant plays a crucial role in the index theorem for manifolds with boundary. It is also possible to study the eta invariant with coefficients in a locally flat bundle to get a generalization of the Atiyah-Singer theorem.

Let $\rho\colon \pi_1(M) \to \mathrm{U}(k)$ be a unitary representation of the fundamental group. Let $\overline{M}$ be the universal cover of M and let $\overline{m} \to g\overline{m}$ for $\overline{m} \in \overline{M}$ and $g \in \pi_1(M)$ be the acion of the deck group. We define the bundle V_ρ over M by the identification:

$$V_\rho = \overline{M} \times \mathbf{C}^k \bmod (\overline{m}, z) = (g\overline{m}, \rho(g)z).$$

The transition functions of V_ρ are locally constant and V_ρ inherits a natural unitary structure and connection ∇_ρ with zero curvature. The holonomy of ∇_ρ is just the representation ρ. Conversely, given a unitary bundle with locally constant transition functions, we can construct the connection ∇ to be unitary with zero curvature and recover ρ as the holonomy of the connection. We assume ρ is unitary to work with self-adjoint operators, but all the constructions can be generalized to arbitrary representations in $\mathrm{GL}(k, \mathbf{C})$.

Let $P\colon C^\infty(V) \to C^\infty(V)$. Since the transition functions of V_ρ are locally constant, we can define P_ρ on $C^\infty(V \otimes V_\rho)$ uniquely using a partition of unity if P is a differential operator. If P is only pseudo-differential, P_ρ is well defined modulo infinitely smoothing terms. We define:

$$\begin{aligned}\tilde\eta(P) &= \tfrac{1}{2}\{\eta(P) + \dim \mathrm{N}(P)\} \mod \mathbf{Z}\\ \mathrm{ind}(\rho, P) &= \tilde\eta(P_\rho) - k\tilde\eta(P) \mod \mathbf{Z}.\end{aligned}$$

LEMMA 4.4.1. $\mathrm{ind}(\rho, P)$ *is a homotopy invariant of* P*. If we fix* ρ*, we can interpret this as a map*

$$\mathrm{ind}(\rho, *)\colon K(S(T^*M))/K(M) \to \mathbf{R} \bmod \mathbf{Z}$$

such that $\mathrm{ind}(\rho, P) = \mathrm{ind}(\rho, \Pi_+(\sigma_L(P)))$.

PROOF: We noted previously that $\tilde\eta$ was well defined in $\mathbf{R}$ mod $\mathbf{Z}$. Let $P(t)$ be a smooth 1-parameter family of such operators and let $P'(t) = \dfrac{d}{dt}P(t)$. In Theorem 1.10.2, we proved:

$$\frac{d}{dt}\tilde\eta(P_t) = \int_M a(x, P(t), P'(t))\ \mathrm{dvol}(x)$$

was given by a local formula. Let $P_k = P \otimes 1^k$ acting on $V \otimes 1^k$. This corresponds to the trivial representation of $\pi_1(M)$ in $\mathrm{U}(k)$. The operators

P_ρ and P_k are locally isomorphic modulo ∞ smoothing terms which don't affect the local invariant. Thus $a(x, P_k(t), P_k'(t)) - a(x, P_\rho(t), P_\rho'(t)) = 0$. This implies $\dfrac{d}{dt}\operatorname{ind}(\rho, P_t) = 0$ and completes the proof of homotopy invariance. If the leading symbol of P is definite, then the value $\eta(0, P)$ is given by a local formula so $\operatorname{ind}(\rho, *) = 0$ in this case, as the two local formulas cancel. This verifies properties (d) and (e) of Lemma 4.3.1; properties (a), (b) and (f) are immediate. We therefore apply Lemma 4.3.3 to regard

$$\operatorname{ind}(\rho, *)\colon K(S(T^*M))/K(M) \to \mathbf{R} \bmod \mathbf{Z}$$

which completes the proof.

In sections 4.5 and 4.6 we will adopt a slightly different notation for this invariant. Let G be a group and let $R(G)$ be the group representation ring generated by the unitary representations of G. Let $R_0(G)$ be the ideal of representations of virtual dimension 0. We extend $\tilde{\eta}\colon R(G) \to \mathbf{R} \bmod \mathbf{Z}$ to be a $\mathbf{Z}$-linear map. We let $\operatorname{ind}(\rho, P)$ denote the restriction to $R_0(G)$. If ρ is a representation of dimension j, then $\operatorname{ind}(\rho, P) = \operatorname{ind}(\rho - j \cdot 1, P) = \eta(P_\rho) - j\eta(P)$. It is convenient to use both notations and the context determines whether we are thinking of virtual representations of dimension 0 or the projection of an actual representation to $R_0(G)$. $\operatorname{ind}(\rho, P)$ is not topological in ρ, as the following example shows. We will discuss K-theory invariants arising from the ρ dependence in Lemma 4.6.5.

Example 4.4.2: Let $M = S^1$ be the circle with periodic parameter $0 \le \theta \le 2\pi$. Let $g(\theta) = e^{i\theta}$ be the generator of $\pi_1(M) \simeq \mathbf{Z}$. We let ε belong to $\mathbf{R}$ and define:

$$\rho_\varepsilon(g) = e^{2\pi i\varepsilon}$$

as a unitary representation of $\pi_1(M)$. The locally flat bundle V_ρ is topologically trivial since any complex bundle over S^1 is trivial. If we define a locally flat section to $S^1 \times \mathbf{C}$ by:

$$\vec{s}(\theta) = e^{i\varepsilon\theta}$$

then the holonomy defined by $\vec{s}$ gives the representation ρ_ε since

$$\vec{s}(2\pi) = e^{2\pi i\varepsilon}\,\vec{s}(0).$$

We let $P = -i\partial/\partial\theta$ on $C^\infty(S^1)$. Then

$$P_\varepsilon = P_{\rho_\varepsilon} = e^{+i\varepsilon\theta} P e^{-i\varepsilon\theta} = P - \varepsilon.$$

The spectrum of P is $\{n\}_{n\in\mathbf{Z}}$ so the spectrum of P_ε is $\{n-\varepsilon\}_{n\in\mathbf{Z}}$. Therefore:

$$\eta(s, P_\varepsilon) = \sum_{n-\varepsilon\neq 0} \operatorname{sign}(n-\varepsilon)|n-\varepsilon|^{-s}.$$

We differentiate with respect to ε to get:

$$\frac{d}{d\varepsilon}(s, P_\varepsilon) = s \sum_{n-\varepsilon \neq 0} |n-\varepsilon|^{-s-1}.$$

We evaluate at $s = 0$. Since the sum defining this shifted zeta function ranges over all integers, the pole at $s = 0$ has residue 2 so we conclude:

$$\frac{d}{d\varepsilon}\tilde{\eta}(P_\varepsilon) = 2 \cdot \frac{1}{2} = 1 \qquad \text{and} \qquad \operatorname{ind}(\rho_\varepsilon, P) = \int_0^\varepsilon 1\, d\varepsilon = \varepsilon.$$

We note that if we replace ε by $\varepsilon + j$ for $j \in \mathbf{Z}$, then the representation is unchanged and the spectrum of the operator P_ε is unchanged. Thus reduction mod $\mathbf{Z}$ is essential in making $\operatorname{ind}(\rho_\varepsilon, P)$ well defined in this context.

If V_ρ is locally flat, then the curvature of ∇_ρ is zero so $ch(V_\rho) = 0$. This implies V_ρ is a torsion element in K-theory so $V_\rho \otimes 1^n \simeq 1^{kn}$ for some integer n. We illustrate this with

Example 4.4.3: Let $M = \mathbf{R}P_3 = S^3/\mathbf{Z}_2 = \mathrm{SO}(3)$ be real projective space in dimension 3 so $\pi_1(M) = \mathbf{Z}_2$. Let $\rho\colon Z_2 \to \mathrm{U}(1)$ be the non-trivial representation with $\rho(g) = -1$ where g is the generator. Let $L = S^3 \times \mathbf{C}$ and identify $(x, z) = (-x, -z)$ to define a line bundle L_ρ over $\mathbf{R}P_3$ with holonomy ρ. We show L_ρ is non-trivial. Suppose the contrary, then L_ρ is trivial over $\mathbf{R}P_2$ as well. This shows there is a map $f\colon S^2 \to S^1$ with $-f(-x) = f(x)$. If we restrict f to the upper hemisphere of S^2, then $f\colon D^2_+ \to S^1$ satisfies $f(x) = -f(-x)$ on the boundary. Therefore f has odd degree. Since f extends to D_+, f must have zero degree. This contradiction establishes no such f exists and L_ρ is non-trivial.

The bundle $L_\rho \oplus L_\rho$ is $S^3 \times \mathbf{C}$ modulo the relation $(x, z) = (-x, -z)$. We let $g(x)\colon S^3 \to \mathrm{SU}(2)$ be the identity map:

$$g(x) = \begin{pmatrix} x_0 + ix_1 & -x_2 + ix_3 \\ x_2 + ix_3 & x_0 - ix_1 \end{pmatrix}$$

then $g(-x) = -g(x)$. Thus g descends to give a global frame on $L_\rho \oplus L_\rho$ so this bundle is topologically trivial and L_ρ represents a $\mathbf{Z}_2$ torsion class in $K(\mathbf{R}P_3)$. Since L_ρ is a line bundle and is not topologically trivial, $L_\rho - 1$ is a non-zero element of $\widetilde{K}(\mathbf{R}P_3)$. This construction generalizes to define L_ρ over $\mathbf{R}P_n$. We use the map $g\colon S^n \to \mathrm{U}(2^k)$ defined by Clifford algebras so $g(x) = -g(-x)$ to show $2^k L_\rho - 2^k = 0$ in $\widetilde{K}(\mathbf{R}P_n)$ where $k = [n/2]$; we refer to Lemma 3.8.9 for details.

We suppose henceforth in this section that the bundle V_ρ is topologically trivial and let $\vec{s}$ be a global frame for V_ρ. (In section 4.9 we will study the more general case). We can take $P_k = P \otimes 1$ relative to the frame $\vec{s}$ so that both P_k and P_ρ are defined on the same bundle with the same leading symbol. We form the 1-parameter family $tP_k + (1-t)P_\rho = P(t,\rho,\vec{s})$ and define:

$$\operatorname{ind}(\rho, P, \vec{s}) = \int_0^1 \frac{d}{dt}\tilde{\eta}(P(t,\rho,\vec{s}))\,dt = \int_M a(x,\rho,P,\vec{s})\,\mathrm{dvol}(x)$$

where

$$a(x,\rho,P,\vec{s}) = \int_0^1 a(x, P(t,\rho,\vec{s}), P'(t,\rho,\vec{s}))\,dt.$$

The choice of a global frame permits us to lift ind from $\mathbf{R} \bmod \mathbf{Z}$ to $\mathbf{R}$. If we take the operator and representation of example 4.4.2, then $\operatorname{ind}(\rho_\varepsilon, P, \vec{s}) = \varepsilon$ and thus in particular the lift depends on the global frame chosen (or equivalently on the particular presentation of the representation ρ on a trivial bundle). This also permits us to construct non-trivial real valued invariants even on simply connected manifolds by choosing suitable inequivalent global trivializations of V_ρ.

LEMMA 4.4.4.
(a) $\operatorname{ind}(\rho,P,\vec{s}) = \int_M a(x,\rho,P,\vec{s})\,\mathrm{dvol}(x)$ *is given by a local formula which depends on the connection* 1*-form of* ∇_ρ *relative to the global frame* $\vec{s}$ *and on the symbol of the operator* P.
(b) If $\vec{s}_t$ *is a smooth* 1*-parameter family of global sections, then* $\operatorname{ind}(\rho,P,\vec{s}_t)$ *is independent of the parameter* t.

PROOF: The first assertion follows from the definition of $a(x,\rho,P,\vec{s})$ given above and from the results of the first chapter. This shows $\operatorname{ind}(\rho,P,\vec{s}_t)$ varies continuously with t. Since its mod $\mathbf{Z}$ reduction is $\operatorname{ind}(\rho,P)$, this mod $\mathbf{Z}$ reduction is constant. This implies $\operatorname{ind}(\rho,P,\vec{s}_t)$ itself is constant.

We can use $\operatorname{ind}(\rho,P,\vec{s})$ to detect inequivalent trivializations of a bundle and thereby study the homotopy $[M, \mathrm{U}(k)]$ even if M is simply connected. This is related to spectral flow.

The secondary characteristic classes are cohomological invariants of the representation ρ. They are normally $\mathbf{R} \bmod \mathbf{Z}$ classes, but can be lifted to $\mathbf{R}$ and expressed in terms of local invariants if the bundle V_ρ is given a fixed trivialization. We first recall the definition of the Chern character. Let W be a smooth vector bundle with connection ∇. Relative to some local frame, we let ω be the connection 1-form and $\Omega = d\omega - \omega \wedge \omega$ be the curvature. The Chern character is given by $ch_k(\nabla) = \left(\frac{i}{2\pi}\right)^k \frac{1}{k!}\operatorname{Tr}(\Omega^k)$. This is a closed $2k$ form independent of the frame $\vec{s}$ chosen. If ∇_i are two connections for $i = 0,1$, we form $\nabla_t = t\nabla_1 + (1-t)\nabla_0$. If $\theta = \omega_1 - \omega_0$, then θ transforms like a tensor. If Ω_t is the curvature of the connection ∇_t, then:

$$ch_k(\nabla_1) - ch_k(\nabla_0) = \int_0^1 \frac{d}{dt}\,ch_k(\nabla_t)\,dt = d(\,Tch_k(\nabla_1,\nabla_0))$$

where the transgression Tch_k is defined by:

$$Tch_k(\nabla_1,\nabla_0)=\left(\frac{i}{2\pi}\right)^k\frac{1}{(k-1)!}\operatorname{Tr}\left\{\int_0^1\theta\Omega_t^{k-1}\,dt\right\}.$$

We refer to the second chapter for further details on this construction.

We apply this construction to the case in which both ∇_1 and ∇_0 have zero curvature. We choose a local frame so $\omega_0=0$. Then $\omega_1=\theta$ and $\Omega_1=d\theta-\theta\wedge\theta=0$. Consequently:

$$\omega_t=t\theta\qquad\text{and}\qquad\Omega_t=td\theta-t^2\theta\wedge\theta=(t-t^2)\theta\wedge\theta$$

so that:

$$Tch_k(\nabla_1,\nabla_0)=\left(\frac{i}{2\pi}\right)^k\frac{1}{(k-1)!}\int_0^1(t-t^2)^{k-1}\,dt\cdot\operatorname{Tr}(\theta^{2k-1}).$$

We integrate by parts to evaluate this coefficient:

$$\begin{aligned}\int_0^1(t-t^2)^{k-1}\,dt&=\int_0^1t^{k-1}(1-t)^{k-1}\,dt=\frac{k-1}{k}\int_0^1t^k(1-t)^{k-2}\,dt\\&=\frac{(k-1)!}{k\cdot(k+1)\cdots(2k-2)}\int_0^1t^{2k-2}\,dt\\&=\frac{(k-1)!\,(k-1)!}{(2k-1)!}.\end{aligned}$$

Therefore

$$Tch_k(\nabla_1,\nabla_0)=\left(\frac{i}{2\pi}\right)^k\frac{(k-1)!}{(2k-1)!}\cdot\operatorname{Tr}(\theta^{2k-1}).$$

We illustrate the use of secondary characteristic classes by giving another version of the Atiyah-Singer index theorem. Let $Q\colon C^\infty(1^k)\to C^\infty(1^k)$ be an elliptic complex. Let $\vec{s}_\pm$ be global frames on $\Pi_+(\Sigma q)$ over $D_\pm(T^*M)$ so that $\vec{s}_-=q^t(x,\xi)\vec{s}_+$ on $D_+(T^*M)\cap D_-(T^*M)=S(T^*M)$. (The clutching function is q; to agree with the notation adopted in the third section we express $\vec{s}_-=q^t\vec{s}_+$ as we think of q being a matrix acting on column vectors of $\mathbf{C}^k$. The action on the frame is therefore the transpose action). We choose connections $\nabla_\pm$ on $\Pi_+(\Sigma q)$ so $\nabla_\pm(\vec{s}_\pm)=0$ on $D_\pm(T^*M)$. Then:

$$\begin{aligned}\operatorname{index}(Q)&=(-1)^m\int_{\Sigma(T^*M)}Todd(M)\wedge ch(\nabla_-)\\&=(-1)^m\int_{D_+(M)}Todd(M)\wedge ch(\nabla_-).\end{aligned}$$

However, on D_+ we have $\Omega_+ = 0$ so we can replace $ch(\nabla_-)$ by $ch(\nabla_-) - ch(\nabla_+)$ without changing the value of the integral. $ch(\nabla_-) - ch(\nabla_+) = d\, Tch(\nabla_-, \nabla_+)$ so an application of Stokes theorem (together with a careful consideration of the orientations involved) yields:

$$\operatorname{index}(Q) = (-1)^m \int_{S(T^*M)} Todd(M) \wedge Tch(\nabla_-, \nabla_+).$$

Both connections have zero curvature near the equator $S(T^*M)$. The fibers over D_+ are glued to the fibers over D_- using the clutching function q. With the notational conventions we have established, if f_+ is a smooth section relative to the frame $\vec{s}_+$ then the corresponding representation is qf_+ relative to the frame $\vec{s}_-$. Therefore $\nabla_-(f_+) = q^{-1}dq \cdot f_+ + df_+$ and consequently

$$\nabla_- - \nabla_+ = \theta = q^{-1}dq.$$

(In obtaining the Maurer-Cartan form one must be careful which convention one uses—right versus left—and we confess to having used both conventions in the course of this book.) We use this to compute:

$$Tch_k(\nabla_-, \nabla_+) = \sum c_k \operatorname{Tr}((q^{-1}dq)^{2k-1})$$

for

$$c_k = \left(\frac{i}{2\pi}\right)^k \frac{(k-1)!}{(2k-1)!}.$$

Let $\theta = g^{-1}dg$ be the Maurer-Cartan form and let $Tch = \sum_k c_k \operatorname{Tr}(\theta^{2k-1})$. This defines an element of the odd cohomology of $\mathrm{GL}(\cdot, \mathbf{C})$ such that $Tch(\nabla_-, \nabla_+) = q^*(Tch)$. We summarize these computations as follows:

LEMMA 4.4.5. *Let $\theta = g^{-1}dg$ be the Maurer-Cartan form on the general linear group. Define:*

$$Tch = \sum_k \left(\frac{i}{2\pi}\right)^k \frac{(k-1)!}{(2k-1)!} \cdot \operatorname{Tr}(\theta^{2k-1})$$

as an element of the odd cohomology. If $Q\colon C^\infty(1^\cdot) \to C(1^\cdot)$ is an elliptic complex defined on the trivial bundle, let q be the symbol so $q\colon S(T^*M) \to GL(\cdot, \mathbf{C})$. Then $\operatorname{index}(Q) = (-1)^m \int_{S(T^*M)} Todd(M) \wedge q^*(Tch)$.

We compute explicitly the first few terms in the expansion:

$$Tch = \frac{i}{2\pi} \operatorname{Tr}(\theta) + \frac{-1}{24\pi^2} \operatorname{Tr}(\theta^3) + \frac{-i}{960\pi^3} \operatorname{Tr}(\theta^5) + \cdots.$$

In this version, the Atiyah-Singer theorem generalizes to the case of $dM \neq \emptyset$ as the Atiyah-Bott theorem. We shall discuss this in section 4.5.

We can now state the Atiyah-Patodi-Singer twisted index theorem:

Theorem 4.4.6. *Let $P: C^\infty(V) \rightarrow C^\infty(V)$ be an elliptic self-adjoint pseudo-differential operator of order $d > 0$. Let $\rho: \pi_1(M) \rightarrow \mathrm{U}(k)$ be a unitary representation of the fundamental group and assume the associated bundle V_ρ is toplogically trivial. Let $\vec{s}$ be a global frame for V_ρ and let $\nabla_0(\vec{s}) \equiv 0$ define the connection ∇_0. Let ∇_ρ be the connection defined by the representation ρ and let $\theta = \nabla_\rho(\vec{s})$ so that:*

$$Tch(\nabla_\rho, \nabla_0) = \sum_k \left(\frac{i}{2\pi}\right)^k \frac{(k-1)!}{(2k-1)!} \cdot \mathrm{Tr}(\theta^{2k-1}).$$

Then:

$$\mathrm{ind}(\rho, P, \vec{s}) = (-1)^m \int_{S(T^*M)} Todd(M) \wedge ch(\Pi_+ p) \wedge Tch(\nabla_\rho, \nabla_0).$$

$S(T^*M)$ is given the orientation induced by the simplectic orientation $dx_1 \wedge d\xi_1 \wedge \cdots \wedge dx_m \wedge d\xi_m$ on T^*M where we use the outward pointing normal so $N \wedge \omega_{2m-1} = \omega_{2m}$.

We postpone the proof of Theorem 4.4.6 for the moment to return to the examples considered previously. In example 4.4.2, we let $\rho = e^{2\pi i\varepsilon}$ on the generator of $\pi_1 S^1 \simeq \mathbf{Z}$. We let "1" denote the usual trivialization of the bundle $S^1 \times \mathbf{C}$ so that:

$$\nabla_\rho(1) = -i\varepsilon d\theta \qquad \text{and} \qquad Tch(\nabla_\rho, \nabla_0) = \varepsilon d\theta/2\pi.$$

The unit sphere bundle decomposes $S(T^*M) = S^1 \times \{1\} \cup S^1 \times \{-1\}$. The symbol of the operator P is multiplication by the dual variable ξ so $ch(\Pi_+ p) = 1$ on $S^1 \times \{1\}$ and $ch(\Pi_+ p) = 0$ on $S^1 \times \{-1\}$. The induced orientation on $S^1 \times \{1\}$ is $-d\theta$. Since $(-1)^m = -1$, we compute:

$$-\int_{S(T^*M)} Todd(M) \wedge ch(\Pi_+ p) \wedge Tch(\nabla_\rho, \nabla_0) = \int_0^{2\pi} \varepsilon\, d\theta/2\pi = \varepsilon.$$

This also gives an example in which Tch_1 is non-trivial.

In example 4.4.3, we took the non-trivial representation ρ of $\pi_1(\mathbf{R}P_3) = \mathbf{Z}_2$ to define a non-trivial complex line bundle V_ρ such that $V_\rho \oplus V_\rho = 1^2$. The appropriate generalization of this example is to 3-dimensional lens spaces and provides another application of Theorem 4.4.6:

Example 4.4.7: Let $m = 3$ and let n and q be relatively prime positive integers. Let $\lambda = e^{2\pi i/n}$ be a primitive n^{th} root of unity and let $\gamma = \mathrm{diag}(\lambda, \lambda^q)$ generate a cyclic subgroup Γ of $\mathrm{U}(2)$ of order n. If $\rho_s(\gamma) = \lambda^s$, then $\{\rho_s\}_{0 \le s < n}$ parametrize the irreducible representations of Γ. Γ acts without fixed points on the unit sphere S^3. Let $L(n, q) = S^3/\Gamma$ be the quotient manifold. As $\pi_1(S^3) = 0$, we conclude $\pi_1(L(n,q)) = \Gamma$. Let V_s be the line bundle corresponding to the representation ρ_s. It is defined from $S^3 \times \mathbf{C}$ by the equivalence relation $(z_1, z_2, w) = (\lambda z_1, \lambda^q z_2, \lambda^s w)$.

Exactly the same arguments (working mod n rather than mod 2) used to show V_1 is non-trivial over $\mathbf{R}P_3$ show V_s is non-trivial for $0 < s < n$. In fact $\widetilde{K}(L(n,q)) = \mathbf{Z}_n$ as we shall see later in Corollary 4.6.10 and the bundle $(V_1 - 1)$ generates the reduced K-theory group. A bundle $V_{s_1} \oplus \cdots \oplus V_{s_t}$ is topologically trivial if and only if $s_1 + \cdots + s_t \equiv 0\ (n)$.

The bundle $V_s \oplus V_{-s}$ is topologically trivial. We define:

$$g(z_1, z_2) = \begin{pmatrix} z_1^s & z_2^{sq'} \\ \bar{z}_2^{sq'} & -\bar{z}_1^s \end{pmatrix} \qquad \text{where } qq' \equiv 1\ (n).$$

It is immediate that:

$$g(\lambda z_1, \lambda^q z_2) = \begin{pmatrix} \lambda^s z_1^s & \lambda^s z_2^{sq'} \\ \lambda^{-s} \bar{z}_2^{sq'} & -\lambda^{-s} \bar{z}_1^s \end{pmatrix} = \gamma g(z_1, z_2)$$

so we can regard g as an equivariant frame to $V_s \otimes V_{-s}$. If $\{\nabla_s \oplus \nabla_{-s}\}$ denotes the connection induced by the locally flat structure and if $\nabla_0 \oplus \nabla_0$ denotes the connection defined by the new frame, then:

$$\theta = \{\nabla_s \oplus \nabla_{-s}\} - \{\nabla_0 \oplus \nabla_0\} = dg \cdot g^{-1}.$$

Suppose first $s = q = q' = 1$ so $g\colon S^3 \to \mathrm{SU}(2)$ is the identity map. $\mathrm{Tr}(\theta^3)$ is a right invariant 3-form and is therefore a constant multiple of the volume element of SU(2). We calculate at $z = (1, 0)$ in $\mathbf{C}^2$. Let

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

so that $g(z) = g(x) = x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3$. At $(1, 0)$ we have:

$$(dg \cdot g^{-1})^3 = \{(e_1\, dx_1 + e_2\, dx_2 + e_3\, dx_3) e_0\}^3.$$

e_0 commutes with e_1 and anti-commutes with e_2 and e_3 so that, as $e_0^2 = 1$,

$$\begin{aligned} (dg \cdot g^{-1})^3 &= (e_1\, dx_1 + e_2\, dx_2 + e_3\, dx_3)(e_1\, dx_1 - e_2\, dx_2 - e_3\, dx_3) \times \\ &\qquad (e_1\, dx_1 + e_2\, dx_2 + de_3\, dx_3) e_0 \\ &= (-e_1 e_2 e_3 + e_1 e_3 e_2 - e_2 e_1 e_3 - e_2 e_3 e_1 + e_3 e_1 e_2 + e_3 e_2 e_1) \times \\ &\qquad e_0\, dx_1 \wedge dx_2 \wedge dx_3 \\ &= -6 e_1 e_2 e_3 e_0 \cdot \mathrm{dvol} = -6 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathrm{dvol} \end{aligned}$$

so that $\mathrm{Tr}(\theta^3) = -12 \cdot \mathrm{dvol}$. Therefore:

$$\int_{S^3} Tch_2(\nabla_1 \oplus \nabla_{-1}, \nabla_0 \oplus \nabla_0) = \frac{12}{24\pi^2} \cdot \mathrm{volume}(S^3) = 1.$$

We can study other values of (s, q) by composing with the map $(z_1, z_2) \mapsto (z_1^s, z_2^{sq'})/|(z_1^s, z_2^{sq'}|$. This gives a map homotopic to the original map and doesn't change the integral. This is an s^2q'-to-one holomorophic map so the corresponding integral becomes s^2q'. If instead of integrating over S^3 we integrate over $L(n, q)$, we must divide the integral by n so that:

$$\int_{L(n,q)} Tch_2(\nabla_s \oplus \nabla_{-s}, \nabla_0 \oplus \nabla_0) = \frac{s^2q'}{n}.$$

Let P be the tangential operator of the signature complex. By Lemma 4.3.2, we have:

$$\int_{S^2} ch(\Pi_+(P)) = -4.$$

S^3 is parallelizable. The orientation of $S(T^*S^3)$ is $dx_1 \wedge dx_2 \wedge d\xi_2 \wedge dx_3 \wedge d\xi_3 = -dx_1 \wedge dx_2 \wedge dx_3 \wedge d\xi_2 \wedge d\xi_3$ so $S(T^*S^3) = -S^3 \times S^2$ given the usual orientation. Thus

$$\begin{aligned}(-1)^3 \int_{S(T^*S^3)} & Tch_2(\nabla_s \oplus \nabla_{-s}, \nabla_0 \oplus \nabla_0) \wedge ch(\Pi_+P) \\ &= (-1)(-1)(-1) \cdot 4 \cdot s^2q'/n = -4s^2q'/n.\end{aligned}$$

Consequently by Theorem 4.4.6, we conclude, since $Todd(S^3) = 1$,

$$\mathrm{ind}(\rho_s + \rho_{-s}, \rho_0 + \rho_0, P) = -4s^2q'/n$$

using the given framing.

The operator P splits into an operator on even and odd forms with equal eta invariants and a corresponding calculation shows that $\mathrm{ind}(\rho_s + \rho_{-s}, \rho_0 + \rho_0, P_{\mathrm{even}}) = -2s^2q'/n$. There is an orientation preserving isometry $T\colon L(p, q) \to L(p, q)$ defined by $T(z_1, z_2) = (\bar z_1, \bar z_2)$. It is clear T interchanges the roles of ρ_s and ρ_{-s} so that as $\mathbf{R}$ mod $\mathbf{Z}$-valued invariants, $\mathrm{ind}(\rho_s, P_{\mathrm{even}}) = \mathrm{ind}(\rho_{-s}, P_{\mathrm{even}})$ so that:

$$\begin{aligned}\mathrm{ind}(\rho_s, P) &= 2\,\mathrm{ind}(\rho_s, P_{\mathrm{even}}) = \mathrm{ind}(\rho_s, P_{\mathrm{even}}) + \mathrm{ind}(\rho_{-s}, P_{\mathrm{even}}) \\ &= -2s^2q'/n.\end{aligned}$$

This gives a formula in $\mathbf{R}$ mod $\mathbf{Z}$ for the index of a representation which need not be topologically trivial. We will return to this formula to discuss generalizations in Lemma 4.6.3 when discussing $\mathrm{ind}(*, *)$ for general spherical space forms in section 4.6.

We sketch the proof of Theorem 4.4.6. We shall omit many of the details in the interests of brevity. We refer to the papers of Atiyah, Patodi, and Singer for complete details. An elementary proof is contained in (Gilkey, The eta invariant and the secondary characteristic classes of locally flat bundles). Define:

$$\operatorname{ind}_1(\rho, P, s) = (-1)^m \int_{S(T^*M)} Todd(M) \wedge ch(\Pi_+ p) \wedge Tch(\nabla_\rho, \nabla_0)$$

then Lemmas 4.3.6 and 4.3.7 generalize immediately to:

LEMMA 4.4.8.
(a) Let M_1 and M_2 be smooth manifolds. Let $P: C^\infty(V_1) \to C^\infty(V_2)$ be an elliptic complex over M_1. Let $Q: C^\infty(V) \to C^\infty(V)$ be a self-adjoint elliptic operator over M_2. We assume P and Q are differential operators of the same order and form $R = \begin{pmatrix} Q & P^* \\ P & -Q \end{pmatrix}$ over $M = M_1 \times M_2$. Let ρ be a representation of $\pi_1(M_2)$. Decompose $\pi_1(M) = \pi_1(M_1) \oplus \pi_1(M_2)$ and extend ρ to act trivially on $\pi_1(M_1)$. Then we can identify V_ρ over M with the pull-back of V_ρ over M_2. Let s be a global trivialization of V_ρ then:

$$\begin{aligned} \operatorname{ind}(\rho, R, \vec{s}) &= \operatorname{index}(P) \operatorname{ind}(\rho, Q, \vec{s}) \\ \operatorname{ind}_1(\rho, R, \vec{s}) &= \operatorname{index}(P) \operatorname{ind}_1(\rho, Q, \vec{s}). \end{aligned}$$

(b) Let $M_1 = S^1$ be the circle and let (R, Q) be as defined in Lemma 4.3.7, then:

$$\begin{aligned} \operatorname{ind}(\rho, R, \vec{s}) &= \operatorname{ind}(\rho, P, \vec{s}) \\ \operatorname{ind}_1(\rho, R, \vec{s}) &= \operatorname{ind}_1(\rho, P, \vec{s}). \end{aligned}$$

PROOF: The assertions about ind follow directly from Lemmas 4.3.6 and 4.3.7. The assertions about ind_1 follow from Lemma 3.9.3(d) and from the Atiyah-Singer index theorem.

Lemma 4.4.8(c) lets us reduce the proof of Theorem 4.4.6 to the case $\dim M$ odd. Since both ind and ind_1 are given by local formulas, we may assume without loss of generality that M is also orientable. Using the same arguments as those given in subsection 4.3, we can interpret both ind and ind_1 as maps in K-theory once the representation ρ and the global frame $\vec{s}$ are fixed. Consequently, the same arguments as those given for the proof of Theorem 4.3.8 permit us to reduce the proof of Theorem 4.4.6 to the case in which $P = A_V$ is the operator discussed in Lemma 4.3.4.

$\operatorname{ind}(\rho, P, \vec{s})$ is given by a local formula. If we express everything with respect to the global frame $\vec{s}$, then $P_k = P \otimes 1$ and P_ρ is functorially expressible in terms of P and in terms of the connection 1-form $\omega = \nabla_\rho \vec{s}$;

$$\operatorname{ind}(\rho, P, \vec{s}) = \int_M a(x, G, \omega) \operatorname{dvol}(x).$$

The same arguments as those given in discussing the signature complex show $a(x,G,\omega)$ is homogeneous of order n in the jets of the metric and of the connection 1-form. The local invariant changes sign if the orientation is reversed and thus $a(x,G,\omega)\operatorname{dvol}(x)$ should be regarded as an m-form not as a measure. The additivity of η with respect to direct sums shows $a(x,G,\omega_1\oplus\omega_2)=a(x,G,\omega_1)+a(x,G,\omega_2)$ when we take the direct sum of representations. Arguments similar to those given in the second chapter (and which are worked out elsewhere) prove:

LEMMA 4.4.9. *Let $a(x,G,\omega)$ be an m-form valued invariant which is defined on Riemannian metrics and on 1-form valued tensors ω so that $d\omega-\omega\wedge\omega=0$. Suppose a is homogeneous of order m in the jets of the metric and the tensor ω and suppose that $a(x,G,\omega_1\oplus\omega_2)=a(x,G,\omega_1)+a(x,G,\omega_2)$. Then we can decompose:*

$$a(x,G,\omega)=\sum_\nu f_\nu(G)\wedge Tch_\nu(\omega)=\sum_\nu f_\nu(G)\wedge c_\nu\operatorname{Tr}(\omega^{2\nu-1})$$

where $f_\nu(G)$ are real characteristic forms of $T(M)$ of order $m+1-2\nu$.

We use the same argument as that given in the proof of the Atiyah-Singer index theorem to show there must exist a local formula for $\operatorname{ind}(\rho,P,s)$ which has the form:

$$\operatorname{ind}(\rho,P,s)=\int_{S(T^*M)}\sum_{4i+2j+2k=m+1} Td'_{i,m}(M)\wedge ch_j(\Pi_+p)\wedge Tch_k(\nabla_\rho,\nabla_0).$$

When the existence of such a formula is coupled with the product formula given in Lemma 4.4.8(a) and with the Atiyah-Singer index theorem, we deduce that the formula must actually have the form:

$$\operatorname{ind}(\rho,P,s)=\int_{S(T^*M)}\sum_{j+2k=m+1}\big(Todd(M)\wedge ch(\Pi_+p)\big)_j\wedge c(k)\;Tch_k(\nabla_\rho,\nabla_0)$$

where $c(k)$ is some universal constant which remains to be determined.

If we take an Abelian representation, all the Tch_k vanish for $k>1$. We already verified that the constant $c(1)=1$ by checking the operator of example 4.4.2 on the circle. The fact that the other normalizing constants are also 1 follows from a detailed consideration of the asymptotics of the heat equation which arise; we refer to (Gilkey, The eta invariant and the secondary characteristic classes of locally flat bundles) for further details regarding this verification. Alternatively, the Atiyah-Patodi-Singer index theorem for manifolds with boundary can be used to check these normalizing constants; we refer to (Atiyah, Patodi, Singer: Spectral asymmetry and Riemannian geometry I–III) for details.

The Atiyah-Singer index theorem is a formula on $K(\Sigma(T^*M))$. The Atiyah-Patodi-Singer twisted index theorem can be regarded as a formula on $K(S(T^*M)) = K^1(\Sigma(T^*M))$. These two formulas are to be regarded as suspensions of each other and are linked by Bott periodicity in a purely formal sense which we shall not make explicit.

If V_ρ is not topologically trivial, there is no local formula for $\operatorname{ind}(\rho, P)$ in general. It is possible to calculate this using the Lefschetz fixed point formulas as we will discuss later.

We conclude this section by discussing the generalization of Theorem 4.4.6 to the case of manifolds with boundary. It is a fairly straightforward computation using the methods of section 3.9 to show:

LEMMA 4.4.9. *We adopt the notation of Theorem 4.4.6. Then for any* $k \geq 0$,

$$\operatorname{ind}(\rho, P, \vec{s}) = (-1)^m \int_{\Sigma^{2k}(T^*M)} Todd(M) \wedge ch(\Pi_+(\Sigma^{2k}(p))) \wedge Tch(\nabla_\rho, \nabla_0).$$

Remark: This shows that we can stabilize by suspending as often as we please. The index can be computed as an integral over $\Sigma^{2k+1}(T^*M)$ while the twisted index is an integral over $\Sigma^{2k}(T^*M)$. These two formulas are at least formally speaking the suspensions of each other.

It turns out that the formula of Theorem 4.4.6 does *not* generalize directly to the case of manifolds with boundary, while the formula of Lemma 4.4.9 with $k = 1$ does generalize. Let M be a compact manifold with boundary dM. We choose a Riemannian metric on M which is product near dM. Let $P: C^\infty(V) \to C^\infty(V)$ be a first order partial differential operator with leading symbol p which is formally self-adjoint. We suppose $p(x,\xi)^2 = |\xi|^2 \cdot 1_V$ so that p is defined by Clifford matrices. If we decompose $p(x,\xi) = \sum_j p_j(x)\xi_j$ relative to a local orthonormal frame for $T^*(M)$, then the p_j are self-adjoint and satisfy the relations $p_j p_k + p_k p_j = 2\delta_{jk}$.

Near the boundary we decompose $T^*(M) = T^*(dM) \oplus 1$ into tangential and normal directions. We let $\xi = (\zeta, z)$ for $\zeta \in T^*(dM)$ reflect this decomposition. Decompose $p(x,\xi) = \sum_{1 \leq j \leq m-1} p_j(x)\xi_j + p_m z$. Let t be a real parameter and define:

$$\tau(x,\zeta,t) = i \cdot p_m \left\{ \sum_{1 \leq j \leq m-1} p_j(x)\zeta_j - it \right\}$$

as the endomorphism defined in Chapter 1. We suppose given a self-adjoint endomorphism q of V which anti-commutes with τ. Let B denote the orthogonal projection $\frac{1}{2}(1+q)$ on the $+1$ eigenspace of q. (P, B) is a self-adjoint elliptic boundary value problem. The results proved for manifolds without boundary extend to this case to become:

THEOREM 4.4.10. *Let (P, B) be an elliptic first order boundary value problem. We assume P is self-adjoint and $\sigma_L(p)^2 = |\xi|^2 \cdot I_V$. We assume $B = \frac{1}{2}(1+q)$ where q anti-commutes with $\tau = i \cdot p_m \left(\sum_{1\le j\le m-1} p_j\zeta_j - it\right)$. Let $\{\lambda_\nu\}_{\nu=1}^\infty$ denote the spectrum of the operator P_B. Define:*

$$\eta(s, P, B) = \sum_\nu \operatorname{sign}(\lambda_\nu)|\lambda_\nu|^{-s}.$$

(a) $\eta(s, P, B)$ is well defined and holomorphic for $\operatorname{Re}(s) \gg 0$.
(b) $\eta(s, P, B)$ admits a meromorphic extension to $\mathbf{C}$ with isolated simple poles at $s = (n - m)/2$ for $n = 0, 1, 2, \dots$. The residue of η at such a simple pole is given by integrating a local formula $a_n(x, P)$ over M and a local formula $a_{n-1}(x, P, B)$ over dM.
(c) The value $s = 0$ is a regular value.
(d) If (P_u, B_u) is a smooth 1-parameter family of such operators, then the derivative $\frac{d}{du}\eta(0, P_u, B_u)$ is given by a local formula.

Remark: This theorem holds in much greater generality than we are stating it; we restrict to operators and boundary conditions given by Clifford matrices to simplify the discussion. The reader should consult (Gilkey-Smith) for details on the general case.

Such boundary conditions always exist if M is orientable and m is even; they may not exist if m is odd. For m even and M orientable, we can take $q = p_1 \dots p_{m-1}$. If m is odd, the obstruction is $\operatorname{Tr}(p_1 \dots p_m)$ as discussed earlier. This theorem permits us to define $\operatorname{ind}(\rho, P, B)$ if ρ is a unitary representation of $\pi_1(M)$ just as in the case where dM is empty. In $\mathbf{R} \bmod \mathbf{Z}$, it is a homotopy invariant of (P, B). If the bundle V_ρ is topologically trivial, we can define

$$\operatorname{ind}(\rho, P, B, \vec{s})$$

as a real-valued invariant which is given by a local formula integrated over M and dM.

The boundary condition can be used to define a homotopy of Σp to an elliptic symbol which doesn't depend upon the tangential fiber coordinates.

Homotopy 4.4.11: Let (P, B) be as in Theorem 4.4.10 with symbols given by Clifford matrices. Let u be an auxilary parameter and define:

$$\tau(x, \zeta, t, u) = \cos\left(\tfrac{\pi}{2} \cdot u\right)\tau(x, \zeta, t) + \sin\left(\tfrac{\pi}{2} \cdot u\right) q \qquad \text{for } u \in [-1, 0].$$

It is immediate that $\tau(x, \zeta, t, 0) = \tau(x, \zeta, t)$ and $\tau(x, \zeta, t, -1) = -q$. As τ and q are self-adjoint and anti-commute, this is a homotopy through self-adjoint matrices with eigenvalues $\pm\left\{\cos^2\left(\frac{\pi}{2} \cdot u\right)(|\zeta|^2 + t^2) + \sin^2\left(\frac{\pi}{2} \cdot u\right)\right\}^{1/2}$.

Therefore $\tau(x,\zeta,t,u)-iz$ is a non-singular elliptic symbol for $(\zeta,t,z)\neq(0,0,0)$. We have

$$\Sigma p(x,\zeta,z,t)=p(x,\zeta,z)-it=ip_m\cdot\{\tau(x,\zeta,t)-iz\}$$

so we define the homotopy

$$\Sigma p(x,\zeta,z,t,u)=ip_m\cdot\{\tau(x,\zeta,t,u)-iz\}\qquad\text{for } u\in[-1,0].$$

We identifiy a neighborhood of dM in M with $dM\times[0,\delta)$. We sew on a collared neighborhood $dM\times[-1,0]$ to define $\widetilde{M}$. We use the homotopy just defined to define an elliptic symbol $(\Sigma p)_B$ on $\Sigma(T^*\widetilde{M})$ which does not depend upon the tangential fiber coordinates ζ on the boundary $dM\times\{-1\}$. We use the collaring to construct a diffeomorophism of M and $\widetilde{M}$ to regard $(\Sigma p)_B$ on $\Sigma(T^*M)$ where this elliptic symbol is independent of the tangential fiber coordinates on the boundary. We call this construction Homotopy 4.4.11.

We can now state the generalization of Theorem 4.4.6 to manifolds with boundary:

THEOREM 4.4.12. *We adopt the notation of Theorem 4.4.10 and let (P,B) be an elliptic self-adjoint first order boundary value problem with the symbols given by Clifford matrices. Let ρ be a representation of $\pi_1(M)$. Suppose V_ρ is topologically trivial and let $\vec{s}$ be a global frame. Then*

$$\operatorname{ind}(\rho,P,B,\vec{s})=$$

$$(-1)^m\int_{\Sigma^{2k}(T^*M)}Todd(M)\wedge ch(\Pi_+(\Sigma^{2k-1}((\Sigma p)_B)))\wedge Tch(\nabla_\rho,\nabla_0),$$

for any $k\geq 1$. Here $(\Sigma p)_B$ is the symbol on $\Sigma(T^*M)$ defined by Homotopy 4.4.11 so that it is an elliptic symbol independent of the tangential fiber variables ζ on the boundary.

Remark: This theorem is in fact true in much greater generality. It is true under the much weaker assumption that P is a first order formally self-adjoint elliptic differential operator and that (P,B) is self-adjoint and strongly elliptic in the sense discussed in Chapter 1. The relevant homotopy is more complicated to discuss and we refer to (Gilkey-Smith) for both details on this generalization and also for the proof of this theorem.

We conclude by stating the Atiyah-Bott formula in this framework. Let $P\colon C^\infty(V_1)\to C^\infty(V_2)$ be a first order elliptic operator and let B be an elliptic boundary value problem. The boundary value problem gives a homotopy of Σp through *self-adjoint* elliptic symbols to a symbol independent of the tangential fiber variables. We call the new symbol $\{\Sigma p\}_B$. The Atiyah-Bott formula is

$$\operatorname{index}(P,B)=(-1)^m\int_{\Sigma(T^*M)}Todd(M)\wedge ch(\Pi_+((\Sigma p)_B)).$$

4.5. Lefschetz Fixed Point Formulas.

In section 1.8, we discussed the Lefschetz fixed point formulas using heat equation methods. In this section, we shall derive the classical Lefschetz fixed point formulas for a non-degenerate smooth map with isolated fixed points for the four classical elliptic complexes. We shall also discuss the case of higher dimensional fixed point sets for the de Rham complex. The corresponding analysis for the signature and spin complexes is much more difficult and is beyond the scope of this book, and we refer to (Gilkey, Lefschetz fixed point formulas and the heat equation) for further details. We will conclude by discussing the theorem of Donnelly relating Lefschetz fixed point formulas to the eta invariant.

A number of authors have worked on proving these formulas. Kotake in 1969 discussed the case of isolated fixed points. In 1975, Lee extended the results of Seeley to yield the results of section 1.8 giving a heat equation approach to the Lefschetz fixed point formulas in general. We also derived these results independently not being aware of Lee's work. Donnelly in 1976 derived some of the results concerning the existence of the asymptotic expansion if the map concerned was an isometry. In 1978 he extended his results to manifolds with boundary.

During the period 1970 to 1976, Patodi had been working on generalizing his results concerning the index theorem to Lefschetz fixed point formulas, but his illness and untimely death in December 1976 prevented him from publishing the details of his work on the G-signature theorem. Donnelly completed Patodi's work and joint papers by Patodi and Donnelly contain these results. In 1976, Kawasaki gave a proof of the G-signature theorem in his thesis on V-manifolds. We also derived all the results of this section independently at the same time. In a sense, the Lefschetz fixed point formulas should have been derived by heat equation methods at the same time as the index theorem was proved by heat equation methods in 1972 and it remains a historical accident that this was not done. The problem was long over-due for solution and it is not surprising that it was solved simultaneously by a number of people.

We first assume $T\colon M \to M$ is an isometry.

Lemma 4.5.1. *If T is an isometry, then the fixed point set of T consists of the disjoint union of a finite number of totally geodesic submanifolds $N_1, \dots$. If N is one component of the fixed point set, the normal bundle ν is the orthogonal complement of $T(N)$ in $T(M)|_N$. ν is invariant under dT and $\det(I - dT_\nu) > 0$ so T is non-degenerate.*

Proof: Let $a > 0$ be the injectivity radius of M so that if $\operatorname{dist}(x, y) < a$, then there exists a unique shortest geodesic γ joining x to y in M. If $T(x) = x$ and $T(y) = y$, then $T\gamma$ is another shortest geodesic joining x to y so $T\gamma = \gamma$ is fixed pointwise and γ is contained in the fixed point set. This shows the fixed point set is totally geodesic. Fix $T(x) = x$ and decompose $T(M)_x = V_1 \oplus \nu$ where $V_1 = \{\, v \in T(M)_x : dT(x)v = v \,\}$. Since $dT(x)$ is orthogonal, both V_1 and ν are invariant subspaces and dT_ν is an orthogonal matrix with no eigenvalue 1. Therefore $\det(I - dT_\nu) > 0$. Let γ be a geodesic starting at x so $\gamma'(0) \in V_1$. Then $T\gamma$ is a geodesic starting at x with $T\gamma'(0) = dT(x)\gamma'(0) = \gamma'(0)$ so $T\gamma = \gamma$ pointwise and $\exp_x(V_1)$ parametrizes the fixed point set near x. This completes the proof.

We let M be oriented and of even dimension $m = 2n$. Let $(d+\delta)\colon C^\infty(\Lambda^+) \to C^\infty(\Lambda^-)$ be the operator of the signature complex. We let $H^\pm(M;\mathbf{C}) = \mathrm{N}(\Delta_\pm)$ on $C^\infty(\Lambda^\pm)$ so that $\operatorname{signature}(M) = \dim H^+ - \dim H^-$. If T is an orientation preserving isometry, then $T^*d = dT^*$ and $T^* * = *T^*$ where "$*$" denotes the Hodge operator. Therefore T^* induces maps $T^\pm$ on $H^\pm(M;\mathbf{C})$. We define:

$$L(T)_{\text{signature}} = \operatorname{Tr}(T^+) - \operatorname{Tr}(T^-).$$

Let $A = dT \in \mathrm{SO}(m)$ at an isolated fixed point. Define

$$\operatorname{defect}(A, \text{signature}) = \{\operatorname{Tr}(\Lambda^+(A)) - \operatorname{Tr}(\Lambda^-(A))\}/\det(I - A)$$

as the contribution from Lemma 1.8.3. We wish to calculate this characteristic polynomial. If $m = 2$, let $\{e_1, e_2\}$ be an oriented orthonormal basis for $T(M) = T^*(M)$ such that

$$Ae_1 = (\cos\theta)e_1 + (\sin\theta)e_2 \qquad \text{and} \qquad Ae_2 = (\cos\theta)e_2 - (\sin\theta)e_1.$$

The representation spaces are defined by:

$$\Lambda^+ = \operatorname{span}\{1 + ie_1e_2, e_1 + ie_2\} \qquad \text{and} \qquad \Lambda^- = \operatorname{span}\{1 - ie_1e_2, e_1 - ie_2\}$$

so that:

$$\operatorname{Tr}(\Lambda^+(A)) = 1 + e^{-i\theta}, \quad \operatorname{Tr}(\Lambda^-(A)) = 1 - e^{-i\theta}, \quad \det(I - A) = 2 - 2\cos\theta$$

and consequently:

$$\operatorname{defect}(A, \text{signature}) = (-2i\cos\theta)/(2 - 2\cos\theta) = -i\cot(\theta/2).$$

More generally let $m = 2n$ and decompose $A = dT$ into a product of mutually orthogonal and commuting rotations through angles θ_j corresponding to complex eigenvalues $\lambda_j = e^{i\theta_j}$ for $1 \le j \le n$. The multiplicative nature of the signature complex then yields the defect formula:

$$\operatorname{defect}(A, \text{signature}) = \prod_{j=1}^{n}\{-i\cot(\theta_j/2)\} = \prod_{j=1}^{n}\frac{\lambda_j + 1}{\lambda_j - 1}.$$

This is well defined since the condition that the fixed point be isolated is just $0 < \theta_j < 2\pi$ or equivalently $\lambda_j \neq 1$.

There are similar characteristic polynomials for the other classical elliptic complexes. For the de Rham complex, we noted in Chapter 1 the corresponding contribution to be $\pm 1 = \operatorname{sign}(\det(I - A))$. If M is spin, let $A\colon C^\infty(\Delta^+) \to C^\infty(\Delta^-)$ be the spin complex. If T is an isometry which can be lifted to a spin isometry, we can define $L(T)_{\text{spin}}$ to be the Lefschetz number of T relative to the spin complex. This lifts A from $\mathrm{SO}(m)$ to $\mathrm{SPIN}(m)$. We define:

$$\operatorname{defect}(A, \text{spin}) = \{\operatorname{Tr}(\Delta^+(A) - \operatorname{Tr}(\Delta^-(A))\}/\det(I - A)$$

and use Lemma 3.2.5 to calculate if $m = 2$ that:

$$\begin{aligned}\operatorname{defect}(A, \text{spin}) &= (e^{-i\theta/2} - e^{i\theta/2})/(2 - 2\cos\theta)\\ &= -i\sin(\theta/2)/(1 - \cos\theta) = -\frac{i}{2}\operatorname{cosec}(\theta/2)\\ &= \left\{\sqrt{\bar\lambda} - \sqrt{\lambda}\right\}/(2 - \lambda - \bar\lambda) = \sqrt{\lambda}/(\lambda - 1).\end{aligned}$$

Using the multiplicative nature we get a similar product formula in general.

Finally, let M be a holomorphic manifold and let $T\colon M \to M$ be a holomorphic map. Then T and $\bar\partial$ commute. We let $L(T)_{\text{Dolbeault}} = \sum_q (-1)^q \operatorname{Tr}(T^* \text{ on } H^{0,q})$ be the Lefschetz number of the Dolbeault complex. Just because T has isolated fixed points does not imply that it is non-degenerate; the map $z \mapsto z + 1$ defines a map on the Riemann sphere S^2 which has a single isolated degenerate fixed point at ∞. We suppose $A = L(T) \in \mathrm{U}(\frac{m}{2})$ is in fact non-degenerate and define

$$\operatorname{defect}(A, \text{Dolbeault}) = \{\operatorname{Tr}(\Lambda^{0,\text{even}}(A)) - \operatorname{Tr}(\Lambda^{0,\text{odd}}(A))\}\det(I - A_{\text{real}}).$$

If $m = 2$, it is easy to calculate

$$\begin{aligned}\operatorname{defect}(A, \text{Dolbeault}) &= (1 - e^{i\theta})/(2 - 2\cos\theta) = (1 - \lambda)/(2 - \lambda - \bar\lambda)\\ &= \lambda/(\lambda - 1)\end{aligned}$$

with a similar multiplicative formula for $m > 2$. We combine these result with Lemma 1.8.3 to derive the classical Lefschetz fixed point formulas:

THEOREM 4.5.2. *Let $T\colon M \to M$ be a non-degenerate smooth map with isolated fixed points at $F(T) = \{x_1, \dots, x_r\}$. Then*
(a) $L(T)_{\text{de Rham}} = \sum_j \operatorname{sign}(\det(I - dT))(x_j)$.
(b) Suppose T is an orientation preserving isometry. Define:

$$\operatorname{defect}(A, \text{signature}) = \prod_j \{-i \cdot \cot\theta_j\} = \prod_j \frac{\lambda_j + 1}{\lambda_j - 1},$$

then $L(T)_{\text{signature}} = \sum_j \operatorname{defect}(dT(x_j), \text{signature})$.
(c) Suppose T *is an isometry preserving a spin structure. Define:*

$$\operatorname{defect}(A, \text{spin}) = \prod_j \left\{ -\frac{i}{2} \operatorname{cosec}(\theta_j/2) \right\} = \prod_j \frac{\sqrt{\lambda_j}}{\lambda_j - 1},$$

then $L(T)_{\text{spin}} = \sum_j \operatorname{defect}(dT(x_j), \text{spin})$.
(d) Suppose T *is holomorphic. Define:*

$$\operatorname{defect}(A, \text{Dolbeault}) = \prod_j \frac{\lambda_j}{\lambda_j - 1},$$

then $L(T)_{\text{Dolbeault}} = \sum_j \operatorname{defect}(dT(x_j), \text{Dolbeault})$.

We proved (a) in Chapter 1. (b)–(d) follow from the calculations we have just given together with Lemma 1.8.3. In defining the defect, the rotation angles of A are $\{\theta_j\}$ so $Ae_1 = (\cos\theta)e_1 + (\sin\theta)e_2$ and $Ae_2 = (-\sin\theta)e_1 + (\cos\theta)e_2$. The corresponding complex eigenvalues are $\{\lambda_j = e^{i\theta_j}\}$ where $1 \le j \le m/2$. The formula in (a) does not depend on the orientation. The formula in (b) depends on the orientation, but not on a unitary structure. The formula in (c) depends on the particular lift of A to spin. The formula in (d) depends on the unitary structure. The formulas (b)–(d) are all for even dimensional manifolds while (a) does not depend on the parity of the dimension. There is an elliptic complex called the PIN_c complex defined over non-orientable odd dimensional manifolds which also has an interesting Lefschetz number relative to an orientation reversing isometry.

Using Lemmas 1.8.1 and 1.8.2 it is possible to get a local formula for the Lefschetz number concentrated on the fixed pont set even if T has higher dimensional fixed point sets. A careful analysis of this situation leads to the G-signature theorem in full generality. We refer to the appropriate papers of (Gilkey, Kawasaki, and Donnelly) for details regarding the signature and spin complexes. We shall discuss the case of the de Rham complex in some detail.

The interesting thing about the Dolbeault complex is that Theorem 4.5.2 does *not* generalize to yield a corresponding formula in the case of higher dimensional fixed point sets in terms of characteristic classes. So far, it has not proven possible to identify the invariants of the heat equation with generalized cohomology classes in this case. The Atiyah-Singer index theorem in its full generality also does not yield such a formula. If one assumes that T is an isometry of a Kaehler metric, then the desired result follows by passing first to the SPIN_c complex as was done in Chapter 3. However, in the general case, no heat equation proof of a suitable generalization is yet known. We remark that Toledo and Tong do have a formula for $L(T)_{\text{Dolbeault}}$ in this case in terms of characteristic classes, but their method of proof is quite different.

Before proceeding to discuss the de Rham complex in some detail, we pause to give another example:

Example 4.5.3: Let T_2 be the 2-dimensional torus $S^1 \times S^1$ with usual periodic parameters $0 \le x, y \le 1$. Let $T(x,y) = (-y,x)$ be a rotation through 90°. Then
(a) $L(T)_{\text{de Rham}} = 2$.
(b) $L(T)_{\text{signature}} = 2i$.
(c) $L(T)_{\text{spin}} = 2i/\sqrt{2}$.
(d) $L(T)_{\text{Dolbeault}} = 1 - i$.

PROOF: We use Theorem 4.5.2 and notice that there are two fixed points at $(0,0)$ and $(1/2,1/2)$. We could also compute directly using the indicated action on the cohomology groups. This shows that although the signature of a manifold is always zero if $m \equiv 2\ (4)$, there do exist non-trivial $L(T)_{\text{signature}}$ in these dimensions. Of course, there are many other examples.

We now consider the Lefschetz fixed point formula for the de Rham complex if the fixed point set is higher dimensional. Let $T: M \to M$ be smooth and non-degenerate. We assume for the sake of simplicity for the moment that the fixed point set of T has only a single component N of dimension n. The general case will be derived by summing over the components of the fixed point set. Let ν be the sub-bundle of $T(M)|_N$ spanned by the generalized eigenvectors of dT corresponding to eigenvalues other than 1. We choose the metric on M so the decomposition $T(M)|_N = T(N) \oplus \nu$ is orthogonal. We further normalize the choice of metric by assuming that N is a totally goedesic submanifold.

Let $a_k(x,T,\Delta_p)$ denote the invariants of the heat equation discussed in Lemma 1.8.1 so that

$$\operatorname{Tr}(T^* e^{-t\Delta_p}) \sim \sum_{k=0}^{\infty} t^{(k-n)/2} \int_N a_k(x,T,\Delta_p)\, \operatorname{dvol}(x).$$

Let $a_k(x,T)_{\text{de Rham}} = \sum (-1)^p a_k(x,T,\Delta_p)$, then Lemma 1.8.2 implies:

$$L(T)_{\text{de Rham}} \sim \sum_{k=0}^{\infty} t^{(k-n)/2} \int_N a_k(x,T)_{\text{de Rham}}\, \operatorname{dvol}(x).$$

Thus:

$$\int_N a_k(x,T)_{\text{de Rham}}\, \operatorname{dvol}(x) = \begin{cases} 0 & \text{if } k \ne \dim N \\ L(T)_{\text{de Rham}} & \text{if } k = \dim N. \end{cases}$$

We choose coordinates $x = (x_1, \dots, x_n)$ on N. We extend these coordinates to a system of coordinates $z = (x, y)$ for M near N where $y = (y_1, \dots, y_{m-n})$. We adopt the notational convention:

indices $1 \le i, j, k \le m$	index a frame for $T(M)\vert_N$
$1 \le a, b, c \le n$	index a frame for $T(N)$
$n < u, v, w \le m$	index a frame for ν.

If α is a multi-index, we deompose $\alpha = (\alpha_N, \alpha_\nu)$ into tangential and normal components.

We let $g_{ij/\alpha}$ denote the jets of the metric tensor. Let $T = (T_1, \dots, T_m)$ denote the components of the map T relative to the coordinate system Z. Let $T_{i/\beta}$ denote the jets of the map T; the $T_{i/j}$ variables are tensorial and are just the components of the Jacobian. Define:

$$\operatorname{ord}(g_{ij/\alpha}) = |\alpha|, \qquad \operatorname{ord}(T_{i/\beta}) = |\beta| - 1, \qquad \deg_v(T_{i/\beta}) = \delta_{i,v} + \beta(v).$$

Let $\mathcal{T}$ be the polynomial algebra in the formal variables $\{g_{ij/\alpha}, T_{i/\beta}\}$ for $|\alpha| > 0$, $|\beta| > 1$ with coefficients $c(g_{ij}, T_{i/j})|\det(I - dT_\nu)|^{-1}$ where the $c(g_{ij}, T_{i/j})$ are smooth in the g_{ij} and $T_{i/j}$ variables. The results of section 1.8 imply $a_k(x, T)_{\text{de Rham}} \in \mathcal{T}$. If Z is a coordinate system, if G is a metric, and if T is the germ of a non-degenerate smooth map we can evaluate $p(Z, T, G)(x)$ for $p \in \mathcal{T}$ and $x \in N$. p is said to be invariant if $p(Z, T, G)(x) = p(Z', T, G)(x)$ for any two coordinate systems Z, Z' of this form. We let $\mathcal{T}_{m,n}$ be the sub-algebra of $\mathcal{T}$ of all invariant polynomials and we let $\mathcal{T}_{m,n,k}$ be the sub-space of all invariant polynomials which are homogeneous of degree k using the grading defined above. It is not difficult to show there is a direct sum decomposition $\mathcal{T}_{m,n} = \bigoplus_k \mathcal{T}_{m,n,k}$ as a graded algebra.

LEMMA 4.5.4. $a_k(x, T)_{\text{de Rham}} \in \mathcal{T}_{m,n,k}$.

PROOF: The polynomial dependence upon the jets involved together with the form of the coefficients follows from Lemma 1.8.1. Since $a_k(x, T)_{\text{de Rham}}$ does not depend upon the coordinate system chosen, it is invariant. We check the homogeneity using dimensional analysis as per usual. If we replace the metric by a new metric $c^2 G$, then a_k becomes $c^{-k} a_k$. We replace the coordinate system Z by $Z' = cZ$ to replace $g_{ij/\alpha}$ by $c^{-|\alpha|} g_{ij/\alpha}$ and $T_{i/\beta}$ by $c^{-|\beta|+1} T_{i/\beta}$. This completes the proof. The only feature different from the analysis of section 2.4 is the transformation rule for the variables $T_{i/\beta}$.

The invariants a_k are multiplicative.

LEMMA 4.5.5. *Let $T' \colon M' \to M'$ be non-degenerate with fixed submanifold N'. Define $M = S^1 \times M'$, $N = S^1 \times N'$, and $T = I \times T'$. Then*

$T\colon M \to M$ is non-degenerate with fixed submanifold N. We give M the product metric. Then $a_k(x,T)_{\text{de Rham}} = 0$.

PROOF: We may decompose

$$\Lambda^p(M) = \Lambda^0(S^1)\otimes\Lambda^p(M') \;\oplus\; \Lambda^1(S^1)\otimes\Lambda^{p-1}(M') \simeq \Lambda^p(M')\oplus\Lambda^{p-1}(M').$$

This decomposition is preserved by the map T. Under this decomposition, the Laplacian Δ_p^M splits into $\Delta'_p \oplus \Delta''_p$. The natural bundle isomorphism identifies $\Delta''_p = \Delta'_{p-1}$. Since $a_k(x,T,\Delta_p^M) = a_k(x,T,\Delta'_p) + a_k(x,T,\Delta''_p) = a_k(x,T,\Delta'_p) + a_k(x,T,\Delta'_{p-1})$, the alternating sum defining $a_k(x,T)_{\text{de Rham}}$ yields zero in this example which completes the proof.

We normalize the coordinate system Z by requiring that $g_{ij}(x_0) = \delta_{ij}$. There are further normalizations we shall discuss shortly. There is a natural restriction map $r\colon \mathcal{T}_{m,n,k} \to \mathcal{T}_{m-1,n-1,k}$ defined algebraically by:

$$r(g_{ij/\alpha}) = \begin{cases} 0 & \text{if } \deg_1(g_{ij/\alpha}) \neq 0 \\ g'_{ij/\alpha} & \text{if } \deg_1(g_{ij/\alpha}) = 0 \end{cases}$$

$$r(T_{i/\beta}) = \begin{cases} 0 & \text{if } \deg_1(T_{i/\beta}) \neq 0 \\ T'_{i/\beta} & \text{if } \deg_1(T_{i/\beta}) = 0. \end{cases}$$

In this expression, we let g' and T' denote the renumbered indices to refer to a manifold of one lower dimension. (It is inconvenient to have used the last $m-n$ indices for the normal bundle at this point, but again this is the cannonical convention which we have chosen not to change). In particular we note that Lemma 4.5.5 implies $r(a_k(x,T)_{\text{de Rham}}) = 0$. Theorem 2.4.7 generalizes to this setting as:

LEMMA 4.5.6. *Let $p \in \mathcal{T}_{m,n,k}$ be such that $r(p) = 0$.*
(a) If k is odd or if $k < n$ then $p = 0$.
(b) If $k = n$ is even we let E_n be the Euler form of the metric on N. Then $a_n(x,T)_{\text{de Rham}} = |\det(I - dT_\nu)|^{-1} f(dT_\nu)E_n$ for some $\mathrm{GL}(m-n)$ invariant smooth function $f(*)$.

We postpone the proof of this lemma for the moment to complete our discussion of the Lefschetz fixed point formula for the de Rham complex.

THEOREM 4.5.7. *Let $T\colon M \to M$ be non-degenerate with fixed point set consisting of the disjoint union of the submanifolds $N_1, N_2, \dots, N_r$. Let N denote one component of the fixed point set of dimension n and let $a_k(x,T)_{\text{de Rham}}$ be the invariant of the heat equation. Then:*
(a) $a_k(x,T)_{\text{de Rham}} = 0$ for $k < n$ or if k is odd.
(b) If $k = n$ is even, then $a_n(x,T)_{\text{de Rham}} = \mathrm{sign}(\det(I - dT_\nu))E_n$.
(c) $L(T)_{\text{de Rham}} = \sum_\nu \mathrm{sign}(\det(I - dT))\chi(N_\nu)$ (Classical Lefschetz fixed point formula).

PROOF: (a) follows directly from Lemmas 4.5.4–4.5.7. We also conclude that if $k = n$ is even, then $a_n(x,T)_{\text{de Rham}} = h(dT_\nu)E_n$ for some invariant function h. We know $h(dT_\nu) = \operatorname{sign}(\det(I - dT))$ if $n = m$ by Theorem 1.8.4. The multiplicative nature of the de Rham complex with respect to products establishes this formula in general which proves (b). Since

$$L(T)_{\text{de Rham}} = \sum_\mu \operatorname{sign}(\det(I - dT_\nu)) \int_{N_\mu} E_{n_\mu} \operatorname{dvol}(x_\mu)$$

by Theorem 1.8.2, (c) follows from the Gauss-Bonnet theorem already established. This completes the proof of Theorem 4.5.7.

Before beginning the proof of Lemma 4.5.6, we must further normalize the coordinates being considered. We have assumed that the metric chosen makes N a totally geodesic submanifold. (In fact, this assumption is inessential, and the theorem is true in greater generality. The proof, however, is more complicated in this case). This implies that we can normalize the coordinate system chosen so that $g_{ij/k}(x_0) = 0$ for $1 \le i, j, k \le m$. (If the submanifold is not totally geodesic, then the second fundamental form enters in exactly the same manner as it did for the Gauss-Bonnet theorem for manifolds with boundary).

By hypothesis, we decomposed $T(M)|_N = T(N) \oplus \nu$ and decomposed $dT = I \oplus dT_\nu$. This implies that the Jacobian matrix $T_{i/j}$ satisfies:

$$T_{a/b} = \delta_{a/b}, \qquad T_{a/u} = T_{u/a} = 0 \text{ along } N.$$

Consequently:

$$T_{a/\beta_N \beta_\nu} = 0 \text{ for } |\beta_\nu| < 2 \qquad \text{and} \qquad T_{u/\beta_N \beta_\nu} = 0 \text{ for } |\beta_\nu| = 0.$$

Consequently, for the non-zero $T_{i/\beta}$ variables:

$$\sum_{a \le n} \deg_a(T_{i/\beta}) \le \operatorname{ord}(T_{i/\beta}) = |\beta| - 1.$$

Let p satisfy the hypothesis of Lemma 4.5.6. Let $p \ne 0$ and let A be a monomial of p. We decompose:

$$A = f(dT_\nu) \cdot A_1 \cdot A_2 \quad \text{where} \quad \begin{cases} A_1 = g_{i_1 j_1/\alpha_1} \dots g_{i_r j_r/\alpha_r} & \text{for } |\alpha_\nu| \ge 2 \\ A_2 = A_{k_1/\beta_1} \dots A_{k_s/\beta_s} & \text{for } |\beta_\nu| \ge 2. \end{cases}$$

Since $r(p) = 0$, $\deg_1 A \ne 0$. Since p is invariant under orientation reversing changes of coordinates on N, $\deg_1 A$ must be even. Since p is invariant under coordinate permutations on N, $\deg_a A \ge 2$ for $1 \le a \le n$. We now count indices:

$$\begin{aligned} 2n &\le \sum_{1 \le a \le n} \deg_a(A) = \sum_{1 \le a \le n} \deg_a(A_1) + \sum_{1 \le a \le n} \deg_a(A_2) \\ &\le 2r + \operatorname{ord}(A_1) + \operatorname{ord}(A_2) \le 2 \operatorname{ord}(A_1) + 2 \operatorname{ord}(A_2) = 2 \operatorname{ord}(A). \end{aligned}$$

This shows $p = 0$ if $\operatorname{ord}(A) < n$.

We now study the limiting case $\operatorname{ord}(A) = n$. All these inequalities must have been equalities. This implies in particular that $\operatorname{ord}(A_2) = 2\operatorname{ord}(A_2)$ so $\operatorname{ord}(A_2) = 0$ and A_2 does not appear. Furthermore, $4r = \operatorname{ord}(A_1)$ implies A_1 is a polynomial in the 2-jets of the metric. Finally, $\sum_{1\le a\le n} \deg_a(A_1) = 4r$ implies that A_1 only depends upon the $\{g_{ab/cd}\}$ variables. Since $p \neq 0$ implies $n = 2r$, we conclude $p = 0$ if $k = n$ is odd. This completes the proof of (a).

To prove (b), we know that p is a polynomial in the $\{g_{ab/cd}\}$ variables with coefficients which depend on the $\{T_{u/v}\} = dT_\nu$ variables. Exactly the same arguments as used in section 2.5 to prove Theorem 2.4.7 now show that p has the desired form. This completes the proof; the normal and tangential indices decouple completely.

The Lefschetz fixed point formulas have been generalized by (Donnelly, The eta invariant of G-spaces) to the case of manifolds with boundary. We briefly summarize his work. Let N be a compact Riemannian manifold with boundary M. We assume the metric is product near the boundary so a neighborhood of M in N has the form $M \times [0, \varepsilon)$. Let n be the normal parameter. Let G be a finite group acting on N by isometries; G must preserve M as a set. We suppose G has no fixed points on M and only isolated fixed points in the interior of N for $g \neq I$. Let $F(g)$ denote the set of fixed points for $g \neq I$. Let $\overline{M} = M/G$ be the resulting quotient manifold. It bounds a V-manifold in the sense of Kawasaki, but does not necessarily bound a smooth manifold as N/G need not be a manifold.

Let $P\colon C^\infty(V_1) \to C^\infty(V_2)$ be an elliptic first order differential complex over N. Near the boundary, we assume P has the form $P = p(dn)(\partial/\partial n + A)$ where A is a self-adjoint elliptic first order differential operator over M whose coefficients are independent of the normal parameter. We also assume the G action on N extends to an action on this elliptic complex. Then $gA = Ag$ for all $g \in G$. Decompose $L^2(V_1|_M) = \bigoplus_\lambda E(\lambda)$ into the finite dimensional eigenspaces of A. Then g induces a representation on each $E(\lambda)$ and we define:

$$\eta(s, A, g) = \sum_\lambda \operatorname{sign}(\lambda) \cdot |\lambda|^{-s} \operatorname{Tr}(g \text{ on } E(\lambda))$$

as the equivariant version of the eta invariant. This series converges absolutely for $\operatorname{Re}(s) \gg 0$ and has a meromorphic extension to $\mathbf{C}$. It is easy to see using the methods previously developed that this extension is regular for all values of s since g has no fixed points on M for $g \neq I$. If $g = I$, this is just the eta invariant previously defined.

Let B be orthonormal projection on the non-negative spectrum of A. This defines a non-local elliptic boundary value problem for the operator

P. Since g commutes with the operator A, it commutes with the boundary conditions. Let $L(P,B,g)$ be the Lefschetz number of this problem. Theorems 4.3.11 and 1.8.3 generalize to this setting to become:

THEOREM 4.5.8 (DONNELLY). *Let N be a compact Riemannian manifold with boundary M. Let the metric on N be product near M. Let $P: C^\infty(V_1) \rightarrow C^\infty(V_2)$ be a first order elliptic differential complex over N. Assume near M that P has the form $P = p(dn)(\partial/\partial n + A)$ where A is a self-adjoint elliptic tangential operator on $C^\infty(V_1)$ over the boundary M. Let $L^2(V_1|_M) = \bigoplus_\lambda E(\lambda)$ be a spectral resolution of A and let B be orthonormal projection on the non-negative spectrum of A. (P,B) is an elliptic boundary value problem. Assume given an isometry $g: N \rightarrow N$ with isolated fixed points $x_1, \dots, x_r$ in the interior of N. Assume given an action g on V_i so $gP = Pg$. Then $gA = Ag$ as well. Define:*

$$\eta(s,A,g) = \sum_\lambda \operatorname{sign}(\lambda)|\lambda|^{-s} \cdot \operatorname{Tr}(g \text{ on } E(\lambda)).$$

This converges for $\operatorname{Re}(s) \gg 0$ and extends to an entire function of s. Let $L(P,B,g)$ denote the Lefschetz number of g on this elliptic complex and let

$$\operatorname{defect}(P,g)(x_i) = \{(\operatorname{Tr}(g \text{ on } V_1) - \operatorname{Tr}(g \text{ on } V_2))/\det(I - dT)\}(x_i)$$

then:

$$L(P,B,g) = \left\{\sum_i \operatorname{defect}(P,G)(x_i)\right\} - \frac{1}{2}\{\eta(0,A,g) + \operatorname{Tr}(g \text{ on } \operatorname{N}(A))\}.$$

Remark: Donnelly's theorem holds in greater generality as one does not need to assume the fixed points are isolated and we refer to (Donnelly, The eta invariant of G-spaces) for details.

We use this theorem to compute the eta invariant on the quotient manifold $\overline{M} = M/G$. Equivariant eigensections for A over M correspond to the eigensections of $\bar{A}$ over $\overline{M}$. Then:

$$\begin{aligned}
\tilde{\eta}(\bar{A}) &= \frac{1}{2}\{\eta(0,\bar{A}) + \dim \operatorname{N}(\bar{A})\} \\
&= \frac{1}{|G|} \cdot \frac{1}{2} \cdot \sum_{g \in G} \{\eta(0,A,g) + \operatorname{Tr}(g \text{ on } \operatorname{N}(A))\} \\
&= \frac{1}{|G|} \sum_{g \in G} \{-L(P,B,g)\} + \frac{1}{|G|} \int_N (a_n(x,P^*P) - a_n(x,PP^*))\,dx \\
&\qquad + \frac{1}{|G|} \sum_{\substack{g \in G \\ g \neq I}} \sum_{x \in F(g)} \operatorname{defect}(P,g)(x).
\end{aligned}$$

The first sum over the group gives the equivariant index of P with the given boundary condition. This is an integer and vanishes in $\mathbf{R}$ mod $\mathbf{Z}$. The second contribution arises from Theorem 4.3.11 for $g = I$. The final contribution arises from Theorem 4.5.8 for $g \neq I$ and we sum over the fixed points of g (which may be different for different group elements).

If we suppose $\dim M$ is even, then $\dim N$ is odd so $a_n(x, P^*P) - a_n(x, PP^*) = 0$. This gives a formula for $\eta(\bar{A})$ over $\overline{M}$ in terms of the fixed point data on N. By replacing V_i by $V_i \otimes 1^k$ and letting G act by a representation ρ of G in $\mathrm{U}(k)$, we obtain a formula for $\eta(\bar{A}_\rho)$.

If $\dim M$ is odd and $\dim N$ is even, the local interior formula for $\operatorname{index}(P, B)$ given by the heat equation need not vanish identically. If we twist by a virtual representation ρ, we alter the defect formulas by multiplying by $\operatorname{Tr}(\rho(g))$. The contribution from Theorem 4.3.11 is multiplied by $\operatorname{Tr}(\rho(1)) = \dim \rho$. Consequently this term disappears if ρ is a representation of virtual dimension 0. This proves:

Theorem 4.5.9. *Let N be a compact Riemannian manifold with boundary M. Let the metric on N be product near M. Let $P\colon C^\infty(V_1) \to C^\infty(V_2)$ be a first order elliptic differential complex over N. Assume near M that P has the form $P = p(dn)(\partial/\partial + A)$ where A is a self-adjoint elliptic tangential operator on $C^\infty(V_1)$ over M. Let G be a finite group acting by isometries on N. Assume for $g \neq I$ that g has only isolated fixed points in the interior of N, and let $F(g)$ denote the fixed point set. Assume given an action on V_i so $gP = Pg$ and $gA = Ag$. Let $\overline{M} = M/G$ be the quotient manifold and $\bar{A}$ the induced self-adjoint elliptic operator on $C^\infty(\bar{V}_1 = V_1/G)$ over $\overline{M}$. Let $\rho \in R(G)$ be a virtual representation. If $\dim M$ is odd, we assume $\dim \rho = 0$. Then:*

$$\tilde{\eta}(\bar{A}_\rho) = \frac{1}{|G|} \sum_{\substack{g \in G \\ g \neq I}} \sum_{s \in F(g)} \operatorname{Tr}(\rho(g)) \operatorname{defect}(P, g)(x) \quad \bmod \mathbf{Z}$$

for

$$\operatorname{defect}(P, g)(x) = \{(\operatorname{Tr}(g \text{ on } V_1) - \operatorname{Tr}(g \text{ on } V_2)) / \det(I - dg)\}(x).$$

We shall use this result in the next section and also in section 4.9 to discuss the eta invariant for sherical space forms.

4.6. The Eta Invariant and the K-Theory of Spherical Space Forms.

So far, we have used K-theory as a tool to prove theorems in analysis. K-theory and the Chern isomorphism have played an important role in our discussion of both the index and the twisted index theorems as well as in the regularity of eta at the origin. We now reverse the process and will use analysis as a tool to compute K-theory. We shall discuss the K-theory of spherical space forms using the eta invariant to detect the relevant torsion.

In section 4.5, Corollary 4.5.9, we discussed the equivariant computation of the eta invariant. We apply this to spherical space forms as follows. Let G be a finite group and let $\tau: G \rightarrow \mathrm{U}(l)$ be a fixed point free representation. We suppose $\det(I - \tau(g)) \neq 0$ for $g \neq I$. Such a representation is necessarily faithful; the existence of such a representation places severe restrictions on the group G. In particular, all the Sylow subgroups for odd primes must be cyclic and the Sylow subgroup for the prime 2 is either cyclic or generalized quaternionic. These groups have all been classified by (Wolf, Spaces of Constant Curvature) and we refer to this work for further details on the subject.

$\tau(G)$ acts without fixed points on the unit sphere S^{2l-1} in $\mathbf{C}^l$. Let $M = M(\tau) = S^{2l-1}/\tau(G)$. We suppose $l > 1$ so, since S^{2l-1} is simply connected, τ induces an isomorphism between G and $\pi_1(M)$. M inherits a natural orientation and Riemannian metric. It also inherits a natural Cauchy-Riemann and SPIN_c structure. (M is not necessarily a spin manifold). The metric has constant positive sectional curvature. Such a manifold is called a spherical space form; all odd dimensional compact manifolds without boundary admitting metrics of constant positive sectional curvature arise in this way. The only even dimensional spherical space forms are the sphere S^{2l} and the projective space $\mathbf{R}P^{2l}$. We concentrate for the moment on the odd dimensional case; we will return to consider $\mathbf{R}P^{2l}$ later in this section.

We have the geometrical argument:

$$T(S^{2l-1}) \oplus 1 = T(\mathrm{R}^{2l})|_{S^{2l-1}} = S^{2l-1} \times \mathrm{R}^{2l} = S^{2l-1} \times \mathbf{C}^l$$

is the trivial complex bundle of dimension l. The defining representation τ is unitary and acts naturally on this bundle. If V_τ is the locally flat complex bundle over $M(\tau)$ defined by the representation of $\pi_1(M(\tau)) = G$, then this argument shows

$$(V_\tau)_{\mathrm{real}} = T(M(\tau)) \oplus 1;$$

this is, of course, the Cauchy-Riemann structure refered to previously. In particular V_τ admits a nowhere vanishing section so we can split $V_\tau = V_1 \oplus 1$ where V_1 is an orthogonal complement of the trivial bundle corresponding to the invariant normal section of $T(\mathrm{R}^{2l})|_{S^{2l-1}}$. Therefore

$$\sum_\nu (-1)^\nu \Lambda^\nu(V_\tau) = \sum_\nu (-1)^\nu \Lambda^\nu(V_1 \oplus 1) = \sum_\nu (-1)^\nu \{\Lambda^\nu(V_1) \oplus \Lambda^{\nu-1}(V_1)\} = 0$$

in $K(M)$. This bundle corresponds to the virtual representation $\alpha = \sum_\nu (-1)^\nu \Lambda^\nu(\tau) \in R_0(G)$. This proves:

LEMMA 4.6.1. *Let τ be a fixed point free representation of a finite group G in $\mathrm{U}(l)$ and let $M(\tau) = S^{2l-1}/\tau(G)$. Let $\alpha = \sum_\nu (-1)^\nu \Lambda^\nu(\tau) \in R_0(G)$. Then:*

$$T(M(\tau)) \oplus 1 = (V_\tau)_{\text{real}} \qquad \text{and} \qquad V_\alpha = 0 \text{ in } \widetilde{K}(M(\tau)).$$

The sphere bounds a disk D^{2l} in C^l. The metric is not product near the boundary of course. By making a radial change of metric, we can put a metric on D^{2l} agreeing with the standard metric at the origin and with a product metric near the boundary so that the action of $\mathrm{O}(2l)$ continues to be an action by isometries. The transition functions of V_τ are unitary so $T(M(\tau))$ inherits a natural SPIN_c structure. Let $*$ = signature or Dolbeault and let A_* be the tangential operator of the appropriate elliptic complex over the disk. $\tau(G)$ acts on D^{2l} and the action extends to an action on both the signature and Dolbeault complexes. There is a single fixed point at the origin of the disk. Let $\operatorname{defect}(\tau(g), *)$ denote the appropriate term from the Lefschetz fixed point formulas. Let $\{\lambda_\nu\}$ denote the complex eigenvalues of $\tau(g)$, and let $\tau(g)_r$ denote the corresponding element of $\mathrm{SO}(2l)$. It follows from section 4.5 that:

$$\begin{aligned}
\operatorname{defect}(\tau(g), \text{signature}) &= \prod_\nu \frac{\lambda_\nu + 1}{\lambda_\nu - 1}, \\
\operatorname{defect}(\tau(g), \text{Dolbeault}) &= \frac{\det(\tau(g))}{\det(\tau(g) - I)} = \prod_\nu \frac{\lambda_\nu}{\lambda_\nu - 1}.
\end{aligned}$$

We apply Corollary 4.5.9 to this situation to compute:

4.6.2. *Let $\tau\colon G \to \mathrm{U}(l)$ be a fixed point free representation of a finite group. Let $M(\tau) = S^{2l-1}/\tau(G)$ be a spherical space form. Let $*$ = signature or Dolbeault and let A_* be the tangential operator of the appropriate elliptic complex. Let $\rho \in R_0(G)$ be a virtual representation of dimension 0. Then:*

$$\tilde\eta((A_*)_\rho) = \frac{1}{|G|} \sum_{\substack{g \in G \\ g \neq I}} \operatorname{Tr}(\rho(g)) \operatorname{defect}(\tau(g), *) \qquad \text{in } \mathbf{R} \bmod \mathbf{Z}.$$

Remark: A priori, this identity is in $\mathbf{R}$ mod $\mathbf{Z}$. It is not difficult to show that this generalized Dedekind sum is always $\mathbf{Q}$ mod $\mathbf{Z}$ valued and that $|G|^l \tilde\eta \in \mathbf{Z}$ so one has good control on the denominators involved.

The perhaps somewhat surprising fact is that this invariant is polynomial. Suppose $G = \mathbf{Z}_n$ is cyclic. Let $x = (x_1, \dots, x_l)$ be a collection of indeterminates. Let Td and L be the Todd and Hirzebruch polynomials discussed previously. We define:

$$\begin{aligned} Td_0(\vec{x}) &= 1 & L_0(\vec{x}) &= 1 \\ Td_1(\vec{x}) &= \frac{1}{2}\sum x_j & L_1(\vec{x}) &= 0 \\ Td_2(\vec{x}) &= \frac{1}{12}\Big\{\sum_{j<k} x_j x_k + \Big(\sum_j x_j\Big)^2\Big\} & L_2(\vec{x}) &= \frac{1}{3}\sum x_i^2 \end{aligned}$$

where we have renumbered the L_k polynomials to be homogeneous of degree k. Let s be another parameter which represents the first Chern class of a line bundle. The integrands of the index formula are given by:

$$\begin{aligned} P_l(s;\vec{x};\text{signature}) &= \sum_{j+k=l} s^j L_k(\vec{x}) 2^j/j! \\ P_l(s;\vec{x};\text{Dolbeault}) &= \sum_{j+k=l} s^j \; Td_k(\vec{x})/j!\,. \end{aligned}$$

Let $\mu(l)$ denote the least common denominator of these rational polynomials.

We identify $\mathbf{Z}_n$ with the group of n^{th} roots of unity in $\mathbf{C}$. Let $\rho_s(\lambda) = \lambda^s$ for $0 \le s < n$ parameterize the irreducible representations. If $\vec{q} = (q_1, \dots, q_l)$ is a collection of integers coprime to n, let $\tau = \rho_{q_1} \oplus \cdots \oplus \rho_{q_l}$ so $\tau(\lambda) = \operatorname{diag}(\lambda^{q_1}, \dots, \lambda^{q_l})$. This is a fixed pont free representation; up to unitary equivalence any fixed point free representation has this form. Let $L(n;\vec{q}) = M(\tau) = S^{2l-1}/\tau(\mathbf{Z}_n)$ be the corresponding spherical space form; this is called a lens space.

LEMMA 4.6.3. *Let $M = L(n;\vec{q})$ be a lens space of dimension $2l-1$. Let $e \in \mathbf{Z}$ satisfy $eq_1 \dots q_l \equiv 1 \bmod n \cdot \mu(l)$. Let $* =$ signature or Dolbeault and let A_* be the tangential operator of the corresponding elliptic complex. Let $P_l(s;x;*)$ denote the corresponding rational polynomial as defined above. Then*

$$\operatorname{ind}(\rho_s - \rho_0, A_*) = -\frac{e}{n}\big\{P_l(s;n,\vec{q};*) - P_l(0;n,\vec{q};*)\big\} \bmod \mathbf{Z}.$$

Remark: If M admits a spin structure, there is a corresponding formula for the tangential operator of the spin complex. This illustrates the close relationship between the Lefschetz fixed point formulas, the Atiyah-Singer index theorem, and the eta invariant, as it ties together all these elements. If $l = 2$ so $M = L(n;1,q)$ then

$$\operatorname{ind}(\rho_s - \rho_0, A_{\text{signature}}) \equiv -\frac{q'}{n}\cdot(2s)^2\cdot\frac{1}{2} \equiv \frac{-q'\cdot 2s^2}{n}$$

which is the formula obatined previously in section 4.4.

PROOF: We refer to (Gilkey, The eta invariant and the K-theory of odd dimensional spherical space forms) for the proof; it is a simple residue calculation using the results of Hirzebruch-Zagier.

This is a very computable invariant and can be calculated using a computer from either Lemma 4.6.2 or Lemma 4.6.3. Although Lemma 4.6.3 at first sight only applies to cyclic groups, it is not difficult to use the Brauer induction formula and some elementary results concerning these groups to obtain similar formulas for arbitrary finite groups admitting fixed point free representations.

Let M be a compact manifold without boundary with fundamental group G. If ρ is a unitary representation of G and if P is a self-adjoint elliptic differential operator, we have defined the invariant $\tilde\eta(P_\rho)$ as an $\mathbf{R}$ mod $\mathbf{Z}$ valued invariant. (In fact this invariant can be defined for arbitrary representations and for elliptic pseudo-differential operators with leading symbol having no purely imaginary eigenvalues on $S(T^*M)$ and most of what we will say will go over to this more general case. As we are only interested in finite groups it suffices to work in this more restricted category).

Let $R(G)$ be the group representation ring of unitary virtual representations of G and let $R_0(G)$ be the ideal of virtual representations of virtual dimension 0. The map $\rho \mapsto V_\rho$ defines a ring homomorphism from $R(G)$ to $K(M)$ and $R_0(G)$ to $\widetilde K(M)$. We shall denote the images by $K_{\text{flat}}(M)$ and $\widetilde K_{\text{flat}}(M)$; these are the rings generated by virtual bundles admitting locally flat structures, or equivalently by virtual bundles with constant transition functions. Let P be a self-adjoint and elliptic differential oprator. The map $\rho \mapsto \tilde\eta(A_\rho)$ is additive with respect to direct sums and extends to a map $R(G) \to \mathbf{R}$ mod $\mathbf{Z}$ as already noted. We let $\operatorname{ind}(\rho, P)$ be the map form $R_0(G)$ to $\mathbf{R}$ mod $\mathbf{Z}$. This involves a slight change of notation from section 4.4; if ρ is a representation of G, then

$$\operatorname{ind}(\rho - \dim(\rho)\cdot 1, P)$$

denotes the invariant previously defined by $\operatorname{ind}(\rho, P)$. This invariant is constant under deformations of P within this class; the Atiyah-Patodi-Singer index theorem for manifolds with boundary (Theorem 4.3.11) implies it is also an equivariant cobordism invariant. We summarize its relevant properties:

LEMMA 4.6.4. *Let M be a compact smooth manifold without boundary. Let P be a self-adjoint elliptic differential operator over M and let $\rho \in R_0(\pi_1(M))$.*
(a) Let $P(a)$ be a smooth 1-parameter family of such operators, then $\operatorname{ind}(\rho, P(a))$ is independent of a in $\mathbf{R}$ mod $\mathbf{Z}$. If G is finite, this is $\mathbf{Q}$ mod $\mathbf{Z}$ valued.

(b) Let P be a first order operator. Suppose there exists a compact manifold N with $dN = M$. Suppose there is an elliptic complex $Q: C^\infty(V_1) \to C^\infty(V_2)$ over N so P is the tangential part of Q. Suppose the virtual bundle V_ρ can be extended as a locally flat bundle over N. Then $\operatorname{ind}(\rho, P) = 0$.

PROOF: The first assertion of (a) follows from Lemma 4.4.1. The locally flat bundle V_ρ is rationally trivial so by multipying by a suitable integer we can actually assume V_ρ corresponds to a flat structure on the difference of trivial bundles. The index is therefore given by a local formula. If we lift to the universal cover, we multiply this local formula by $|G|$. On the universal cover, the index vanishes identically as $\pi_1 = 0$. Thus an integer multiple of the index is 0 in $\mathbf{R}/\mathbf{Z}$ so the index is in $\mathbf{Q}/\mathbf{Z}$ which proves (a). To prove (b) we take the operator Q with coefficients in V_ρ. The local formula of the heat equation is just multiplied by the scaling constant $\dim(\rho) = 0$ since V_ρ is locally flat over N. Therefore Theorem 4.3.11 yields the identity $\operatorname{index}(Q, B, \text{coeff in } V_\rho) = 0 - \operatorname{ind}(\rho, P)$. As the index is always an integer, this proves (b). We will use (b) in section 4.9 to discuss isospectral manifolds which are not diffeomorphic.

Examle 4.4.2 shows the index is *not* an invariant in K-theory. We get K-theory invariants as follows:

LEMMA 4.6.5. *Let M be a compact manifold without boundary. Let P be an elliptic self-adjoint pseudo-differential operator. Let $\rho_\nu \in R_0(\pi_1(M))$ and define the associative symmetric bilinear form on $R_0 \otimes R_0$ by:*

$$\operatorname{ind}(\rho_1, \rho_2, P) = \operatorname{ind}(\rho_1 \otimes \rho_2, P).$$

This takes values in $\mathbf{Q} \bmod \mathbf{Z}$ *and extends to an associative symmetric bilinear form* $\operatorname{ind}(*, *, P): \widetilde{K}_{\text{flat}}(M) \otimes \widetilde{K}_{\text{flat}}(M) \to \mathbf{Q} \bmod \mathbf{Z}$. *If we consider the dependence upon P, then we get a trilinear map*

$$\operatorname{ind}: \widetilde{K}_{\text{flat}}(M) \otimes \widetilde{K}_{\text{flat}}(M) \otimes K(S(T^*M))/K(M) \to \mathbf{Q} \bmod \mathbf{Z}.$$

PROOF: The interpretation of the dependence in P as a map in K-theory follows from 4.3.3 and is therefore omitted. Any virtual bundle admitting a locally flat structure has vanishing rational Chern character and must be a torsion class. Once we have proved the map extends to K-theory, it will follow it must be $\mathbf{Q} \bmod \mathbf{Z}$ valued. We suppose given representations ρ_1, $\hat\rho_1$, ρ_2 and a bundle isomorphism $V_{\rho_1} = V_{\hat\rho_1}$. Let $j = \dim(\rho_1)$ and $k = \dim(\rho_2)$. If we can show

$$\operatorname{ind}((\rho_1 - j) \otimes (\rho_2 - k), P) = \operatorname{ind}((\hat\rho_1 - j) \otimes (\rho_2 - k), P)$$

then the form will extend to $\widetilde{K}_{\text{flat}}(M) \otimes \widetilde{K}_{\text{flat}}(M)$ and the lemma will be proved.

We calculate that:

$$\begin{aligned}
&\operatorname{ind}((\rho_1 - j)\otimes(\rho_2 - k), P)\\
&\qquad = \tilde\eta(P_{\rho_1\otimes\rho_2}) + j\cdot k\cdot\tilde\eta(P) - j\cdot\tilde\eta(P_{\rho_2}) - k\cdot\tilde\eta(P_{\rho_1})\\
&\qquad = \tilde\eta\{(P_{\rho_1})_{\rho_2}\} - k\cdot\tilde\eta(P_{\rho_1}) - j\cdot\{\tilde\eta(P_{\rho_2}) - k\cdot\tilde\eta(P)\}\\
&\qquad = \operatorname{ind}(\rho_2, P_{\rho_1}) - j\cdot\operatorname{ind}(\rho_2, P).
\end{aligned}$$

By hypothesis the bundles defined by ρ_1 and $\hat\rho_1$ are isomorphic. Thus the two operators P_{ρ_1} and $P_{\hat\rho_1}$ are homotopic since they have the same leading symbol. Therefore $\operatorname{ind}(\rho_2, P_{\rho_1}) = \operatorname{ind}(\rho_2, P_{\hat\rho_1})$ which completes the proof. We remark that this bilinear form is also associative with respect to multiplication by $R(G)$ and $K_{\mathrm{flat}}(M)$.

We will use this lemma to study the K-theory of spherical space forms.

LEMMA 4.6.6. *Let τ be a fixed point free representation of a finite group G. Define*

$$\operatorname{ind}_\tau(\rho_1, \rho_2) = \frac{1}{|G|}\sum_{\substack{g\in G\\ g\neq I}} \operatorname{Tr}(\rho_1\otimes\rho_2)(g)\frac{\det(\tau(g))}{\det(\tau(g)-I)}.$$

Let $\alpha = \sum_\nu (-1)^\nu\Lambda^\nu(\tau) \in R_0(G)$. Let $\rho_1 \in R_0(G)$ and suppose $\operatorname{ind}_\tau(\rho_1,\rho_2) = 0$ in $\mathbf{Q} \bmod \mathbf{Z}$ for all $\rho_2 \in R_0(G)$. Then $\rho_2 \in \alpha R(G)$.

PROOF: The virtual representation α is given by the defining relation that $\operatorname{Tr}(\alpha(g)) = \det(I - \tau(g))$. $\det(\tau)$ defines a 1-dimensional representation of G; as this is an invertible element of $R(G)$ we see that the hypothesis implies

$$\frac{1}{|G|}\sum_{\substack{g\in G\\ g\neq I}} \operatorname{Tr}(\rho_1\otimes\rho_2)/\det(I-\tau(g)) \in \mathbf{Z} \qquad \text{for all } \rho_2 \in R_0(G).$$

If f and $\tilde f$ are any two class functions on G, we define the symmetric inner product $(f,\tilde f) = \frac{1}{|G|}\sum_{g\in G} f(g)\tilde f(g)$. The orthogonality relations show that f is a virtual character if and only if $(f, \operatorname{Tr}(\rho)) \in \mathbf{Z}$ for all $\rho \in R(\mathbf{Z})$. We define:

$$\begin{aligned}
f(g) &= \operatorname{Tr}(\rho_1(g))/\det(I-\tau(g)) && \text{for } g \neq I\\
f(g) &= -\sum_{\substack{h\in G\\ h\neq I}} \operatorname{Tr}(\rho_1(h))/\det(I-\tau(h)) && \text{if } g = I.
\end{aligned}$$

Then $(f,1) = 0$ by definition. As $(f,\rho_2) \in \mathbf{Z}$ by hypothesis for $\rho_2 \in R_0(G)$ we see $(f,\rho_2) \in \mathbf{Z}$ for all $\rho_2 \in R(G)$ so f is a virtual character. We let $\operatorname{Tr}(\rho)(g) = f(g)$. The defining equation implies:

$$\operatorname{Tr}(\rho\otimes\alpha)(g) = \operatorname{Tr}(\rho_1(g)) \qquad \text{for all } g \in G.$$

This implies $\rho_1 = \rho \otimes \alpha$ and completes the proof.

We can now compute the K-theory of odd dimensional spherical space forms.

THEOREM 4.6.7. *Let $\tau: G \to \mathrm{U}(l)$ be a fixed point free representation of a finite group G. Let $M(\tau) = S^{2l-1}/\tau(G)$ be a spherical space form. Suppose $l > 1$. Let $\alpha = \sum_\nu (-1)^\nu \Lambda^\nu(\tau) \in R_0(G)$. Then $\widetilde{K}_{\text{flat}}(M) = R_0(G)/\alpha R(G)$ and*

$$\operatorname{ind}(*, *, A_{\text{Dolbeault}}): \widetilde{K}_{\text{flat}}(M) \otimes \widetilde{K}_{\text{flat}}(M) \to \mathbf{Q} \bmod \mathbf{Z}$$

is a non-singular bilinear form.

Remark: It is a well known topological fact that for such spaces $\widetilde{K} = \widetilde{K}_{\text{flat}}$ so we are actually computing the reduced K-theory of odd dimensional spherical space forms (which was first computed by Atiyah). This particular proof gives much more information than just the isomorphism and we will draw some corollaries of the proof.

PROOF: We have a surjective map $R_0(G) \to \widetilde{K}_{\text{flat}}(M)$. By Lemma 4.6.1 we have $\alpha \mapsto 0$ so $\alpha R(G)$ is in the kernel of the natural map. Conversely, suppose $V_\rho = 0$ in $\widetilde{K}$. By Lemma 4.6.5 we have $\operatorname{ind}(\rho, \rho_1, A_{\text{Dolbeault}}) = 0$ for all $\rho_1 \in R_0(G)$. Lemma 4.6.2 lets us identify this invariant with $\operatorname{ind}_\tau(\rho, \rho_1)$. Lemma 4.6.6 lets us conclude $\rho \in \alpha R(G)$. This shows the kernel of this map is precisely $\alpha R(G)$ which gives the desired isomorphism. Furthermore, $\rho \in \ker(\operatorname{ind}_\tau(\rho, *))$ if and only if $\rho \in \alpha R(G)$ if and only if $V_\rho = 0$ so the bilinear form is non-singular on $\widetilde{K}$.

It is possible to prove a number of other results about the K-theory ring using purely group theoretic methods; the existence of such a non-singular associative symmetric $\mathbf{Q}$ mod $\mathbf{Z}$ form is an essential ingredient.

COROLLARY 4.6.8. *Adopt the notation of Theorem 4.6.7.*
(a) $\widetilde{K}(M)$ only depends on (G, l) as a ring and not upon the particular τ chosen.
(b) The index of nilpotency for this ring is at most l—i.e., if $\rho_\nu \in R_0(G)$ then $\prod_{1 \le \nu \le l} \rho_\nu \in \alpha R_0(G)$ so the product of l virtual bundles of $\widetilde{K}(M)$ always gives 0 in $\widetilde{K}(M)$.
(c) Let $V \in \widetilde{K}(M)$. Then $V = 0$ if and only if $\pi^*(V) = 0$ for all possible covering projections $\pi: L(n; \vec{q}) \to M$ by lens spaces.

There is, of course, a great deal known concerning these rings and we refer to (N. Mahammed, K-theorie des formes spheriques) for further details.

If $G = \mathbf{Z}_2$, then the resulting space is $\mathbf{R}P^{2l-1}$ which is projective space. There are two inequivalent unitary irreducible representations ρ_0, ρ_1 of G. Let $x = \rho_1 - \rho_0$ generate $R_0(\mathbf{Z}_2) = \mathbf{Z}$; the ring structure is given by

$x^2 = -2x$. Let A be the tangential operator of the Dolbeault complex:

$$\begin{aligned}\operatorname{ind}(x, A) &= \frac{1}{2}\operatorname{Tr}(x(-1)) \cdot \det(-I_l)/\det((-I)_l - I_l)\\ &= -2^{-l}\end{aligned}$$

using Lemma 4.6.2. Therefore $\operatorname{ind}(x, x, A) = 2^{-l+1}$ which implies that $\widetilde{K}(\mathbf{R}P^{2l-1}) = \mathbf{Z}/2^{l-1}\mathbf{Z}$. Let $L = V_{\rho_1}$ be the tautological bundle over $\mathbf{R}P^{2l-1}$. It is $S^{2l-1} \times \mathbf{C}$ modulo the relation$(x, z) = (-x, -z)$. Using Clifford matrices we can construct a map $e\colon S^{2l-1} \to \mathrm{U}(2^{l-1})$ such that $e(-x) = -e(x)$. This gives an equivariant trivialization of $S^{2l-1} \times \mathbf{C}^{2^{l-1}}$ which descends to give a trivialization of $2^{l-1} \cdot L$. This shows explicitly that $2^{l-1}(L-1) = 0$ in $K(\mathbf{R}P^{2l-1})$; the eta invariant is used to show no lower power suffices. This proves:

COROLLARY 4.6.9. *Let $M = \mathbf{R}P^{2l-1}$ then $\widetilde{K}(M) \simeq \mathbf{Z}/2^{l-1}\mathbf{Z}$ where the ring structure is $x^2 = -2x$.*

Let $l = 2$ and let $G = \mathbf{Z}_n$ be cyclic. Lemma 4.6.3 shows

$$\operatorname{ind}(\rho_s - \rho_0, A_{\text{Dolbeault}}) = \frac{-q'}{n} \cdot \frac{s^2}{2}.$$

Let $x = \rho_1 - \rho_0$ so $x^2 = \rho_2 - \rho_0 - 2(\rho_1 - \rho_0)$ and

$$\operatorname{ind}(x, x, A_{\text{Dolbeault}}) = \frac{-q'}{n} \cdot \frac{4-2}{2}$$

is a generator of $\mathbf{Z}[\frac{1}{n}]$ mod $\mathbf{Z}$. As $x^2 = 0$ in $\widetilde{K}$ and as $x \cdot R(G) = R_0(G)$ we see:

COROLLARY 4.6.10. *Let $M = L(n; 1, q)$ be a lens space of dimension 3. $\widetilde{K}(M) = \mathbf{Z}_n$ with trivial ring structure.*

We have computed the K-theory for the odd dimensional spherical space forms. $\widetilde{K}(S^{2l}) = \mathbf{Z}$ and we gave a generator in terms of Clifford algebras in Chapter 3. To complete the discussion, it suffices to consider even dimensional real projective space $M = \mathbf{R}P^{2l}$. As $\widetilde{H}^{\text{even}}(M; \mathbf{Q}) = 0$, $\widetilde{K}$ is pure torsion by the Chern isomorphism. Again, it is known that the flat and regular K-theory coincide. Let $x = L - 1 = V_{\rho_1} - V_{\rho_0}$. This is the restriction of an element of $\widetilde{K}(\mathbf{R}P^{2l+1})$ so $2^l x = 0$ by Corollary 4.6.9. It is immediate that $x^2 = -2x$. We show $\widetilde{K}(\mathbf{R}P^{2l+1}) = \mathbf{Z}/2^l\mathbf{Z}$ by giving a surjective map to a group of order 2^l.

We construct an elliptic complex Q over the disk D^{2l+1}. Let $\{e_0, \dots, e_{2l}\}$ be a collection of $2^l \times 2^l$ skew-adjoint matrices so $e_j e_k + e_k e_j = -2\delta_{jk}$. Up to unitary equivalence, the only invariant of such a collection is $\operatorname{Tr}(e_0 \dots e_{2l}) =$

$\pm(2i)^l$. There are two inequivalent collections; the other is obtained by taking $\{-e_0, \dots, -e_{2l}\}$. Let Q be the operator $\sum_j e_j \partial/\partial x_j$ acting to map $Q: C^\infty(V_1) \to C^\infty(V_2)$ where V_i are trivial bundles of dimension 2^l over the disk. Let $g(x) = -x$ be the antipodal map. Let g act by $+1$ on V_1 and by -1 on V_2, then $gQ = Qg$ so this is an equivariant action. Let A be the tangential operator of this complex on S^{2l} and $\bar{A}$ the corresponding operator on $S^{2l}/\mathbf{Z}_2 = M$. $\bar{A}$ is a self-adjoint elliptic first order operator on $C^\infty(1^{2^l})$. If we replace Q by $-Q$, the tangential operator is unchanged so $\bar{A}$ is invariantly defined independent of the choice of the $\{e_j\}$. (In fact, $\bar{A}$ is the tangential operator of the PIN_c complex.)

The dimension is even so we can apply Corollary 4.5.9 to conclude

$$\tilde{\eta}(\bar{A}) = \frac{1}{2} \cdot \det(I_{2l+1} - (-I_{2l+1}))^{-1}\{2^l - (-2^l)\} = 2^{-l-1}.$$

If we interchange the roles of V_1 and V_2, we change the sign of the eta invariant. This is equivalent to taking coefficients in the bundle $L = V_{\rho_1}$. Let $x = L - 1$ then $\operatorname{ind}(\rho_1 - \rho_0, \bar{A}) = -2^{-l-1} - 2^{-l-1} = -2^{-l}$.

The even dimensional rational cohomology of $S(T^*M)$ is generated by H^0 and thus $K(S(T^*M))/K(M) \otimes \mathbf{Q} = 0$. Suppose $2^{l-1} \cdot x = 0$ in $\widetilde{K}(M)$. Then there would exist a local formula for $2^{l-1} \operatorname{ind}(\rho_1 - \rho_0, *)$ so we could lift this invariant from $\mathbf{Q}$ mod $\mathbf{Z}$ to $\mathbf{Q}$. As this invariant is defined on the torsion group $K(S(T^*M))/K(M)$ it would have to vanish. As $2^{l-1} \operatorname{ind}(\rho_1 - \rho_0, \bar{A}) = -\frac{1}{2}$ does not vanish, we conclude $2^{l-1} \cdot x$ is non-zero in K-theory as desired. This proves:

COROLLARY 4.6.11. $\widetilde{K}(\mathbf{R}P^{2l}) \simeq \mathbf{Z}/2^l\mathbf{Z}$. *If $x = L - 1$ is the generator, then $x^2 = -2x$.*

We can squeeze a bit more out of this construction to compute the K-theory of the unit sphere bundle $K(S(T^*M))$ where M is a spherical space form. First suppose $\dim M = 2l-1$ is odd so $M = S^{2l-1}/\tau(G)$ where τ is a unitary representation. Then M has a Cauchy-Riemann structure and we can decompose $T^*(M) = 1 \oplus V$ where V admits a complex structure; $T(M) \oplus 1 = (V_\tau)_{\mathrm{real}}$. Thus $S(T^*M)$ has a non-vanishing section and the exact sequence $0 \to K(M) \to K(S(T^*M)) \to K(S(T^*M))/K(M) \to 0$ splits. The usual clutching function construction permits us to identify $K(S(T^*M))/K(M)$ with $K(V)$. As V admits a complex structure, the Thom isomorphism identifies $K(V) = x \cdot K(M)$ where x is the Thom class. This gives the structure $K(S(T^*M)) = K(M) \oplus xK(M)$. The bundle x over $S(T^*M)$ can be taken to be $\Pi_+(p)$ where p is the symbol of the tangential operator of the Dolbeault complex. The index form can be regarded as a pairing $\widetilde{K}(M) \otimes x \cdot \widetilde{K}(M) \to \mathbf{Q}$ mod $\mathbf{Z}$ which is non-degenerate.

It is more difficult to analyse the even dimensional case. We wish to compute $K(S(T^*\mathbf{R}P^{2l}))$. $2^l \operatorname{ind}(\rho_1 - \rho_0, *)$ is given by a local formula since $2^l x = 0$. Thus this invariant must be zero as $K(S(T^*M))/K(M)$ is torsion. We have constructed an operator $\bar{A}$ so $\operatorname{ind}(\rho_1 - \rho_0, \bar{A}) = -2^{-l}$ and thus $\operatorname{ind}(\rho_1 - \rho_0, *): K(S(T^*M))/K(M) \to \mathbf{Z}[2^{-l}]/\mathbf{Z}$ is surjective with the given range. This is a cyclic group of order 2^l so equivalently

$$\operatorname{ind}(\rho_1 - \rho_0, *): K(S(T^*M))/K(M) \to \mathbf{Z}/2^l\mathbf{Z} \to 0.$$

Victor Snaith (private communication) has shown us that the existence of such a sequence together with the Hirzebruch spectral sequence in K-theory shows

$$0 \to K(M) \to K(S(T^*M)) \text{ is exact and } |K(S(T^*M))/K(M)| = 2^l$$

so that $\operatorname{index}(\rho_1 - \rho_0, *)$ becomes part of a short exact sequence:

$$0 \to K(M) \to K(S(T^*M)) \to \mathbf{Z}/2^l\mathbf{Z} \to 0.$$

To compute the structure of $K(S(T^*M)) = \mathbf{Z} \oplus \widetilde{K}(S(T^*M))$, we must determine the group extension.

Let $x = L - 1 = V_{\rho_1} - V_{\rho_0}$ generate $\widetilde{K}(M)$. Let $y = \Pi_+(\sigma_L \bar{A})) - 2^{l-1} \cdot 1$, then this exact sequence together with the computation $\operatorname{ind}(\rho_0 - \rho_1, \bar{A}) = 2^{-l}$ shows $\widetilde{K}(S(T^*M))$ is generated by x and y. We know $2^l x = 0$ and that $|\widetilde{K}(S(T^*M))| = 4^l$; to determine the additive structure of the group, we must find the order of y. Consider the de Rham complex $(d+\delta): C^\infty(\Lambda^{\text{even}}(D)) \to C^\infty(\Lambda^{\text{odd}}(D))$ over the disk. The antipodal map acts by $+1$ on Λ^{e} and by -1 on Λ^{o}. Let $\bar{A}_1$ be the tangential operator of this complex. We decompose $(d + \delta)$ into 2^l operators each of which is isomorphic to $\pm Q$. This indeterminacy does not affect the tangential operator and thus $\bar{A}_1 = 2^l \bar{A}$.

The symbol of $\bar{A}_1$ on $\Lambda^{\text{even}}(D)$ is $-ic(dn)c(\zeta)$ for $\zeta \in T^*(M)$. Let $\tau\{\theta_{\text{even}} + \theta_{\text{odd}}\} = \theta_{\text{even}} + c(dn)\theta_{\text{odd}}$ provide an isomorphism between $\Lambda(M)$ and $\Lambda^{\text{even}}(D|_M)$. We may regard $\bar{A}_1$ as an operator on $C^\infty(\Lambda(M))$ with symbol $\bar{a}_1$ given by:

$$\begin{aligned}
\bar{a}_1(x,\zeta)(\theta_{\text{even}} + \theta_{\text{odd}}) &= \{\tau^{-1} \cdot -ic(dn)c(\zeta) \cdot \tau\}\{\theta_{\text{even}} + \theta_{\text{odd}}\} \\
&= \{\tau^{-1} \cdot -ic(dn)c(\zeta)\}\{\theta_{\text{even}} + c(dn)\theta_{\text{odd}}\} \\
&= \tau^{-1}\{-ic(dn)c(\zeta)\theta_{\text{even}} - ic(\zeta)\theta_{\text{odd}}\} \\
&= -ic(\zeta)\{\theta_{\text{even}} + \theta_{\text{odd}}\}
\end{aligned}$$

so that $\bar{A}_1 = -(d + \delta)$ on $C^\infty(\Lambda(M))$. Let $\varepsilon(\theta_{\text{odd}}) = \theta_{\text{odd}} - ic(\zeta)\theta_{\text{odd}}$ provide an isomorphism between $\Lambda^{\text{odd}}(M)$ and $\Pi_+(\bar{a}_1)$. This shows:

$$2^l \cdot y = \Lambda^{\text{odd}}(M) - 2^{2l-1} \cdot 1.$$

Let $\gamma(W) = W - \dim(W)\cdot 1$ be the natural projection of $K(M)$ on $\widetilde{K}(M)$. We wish to compute $\gamma(\Lambda^{\text{odd}}(M))$. As complex vector bundles we have $T^*(M)\oplus 1 = (2l+1)\cdot L$ so that we have the relation $\Lambda^j(T^*M)\oplus\Lambda^{j-1}(T^*M) = \binom{2l+1}{j}\cdot L^j$. This yields the identities:

$$\begin{aligned}
&\gamma(\Lambda^j(T^*M)) + \gamma(\Lambda^{j-1}(T^*M)) = 0 && \text{if } j \text{ is even}\\
&\gamma(\Lambda^j(T^*M)) + \gamma(\Lambda^{j-1}(T^*M)) = \binom{2l+1}{j}\cdot\gamma(L) && \text{if } j \text{ is odd.}
\end{aligned}$$

Thus $\gamma(\Lambda^j(T^*M)) = \gamma(\Lambda^{j-2}(T^*M)) + \binom{2l+1}{j}\gamma(L)$ if j is odd. This leads to the identity:

$$\begin{aligned}
\gamma(\Lambda^{2j+1}(T^*M)) &= \left\{\tbinom{2l+1}{2j+1} + \tbinom{2l+1}{2j-1} + \cdots + \tbinom{2l+1}{1}\right\}\cdot\gamma(L)\\
\gamma(\Lambda^{\text{odd}}(T^*M)) &= \left\{l\tbinom{2l+1}{1} + (l-1)\tbinom{2l+1}{3} + \cdots + \tbinom{2l+1}{2l-1}\right\}\cdot\gamma(L).
\end{aligned}$$

We complete this calculation by evaluating this coefficient. Let

$$\begin{aligned}
f(t) &= \frac{1}{2}\big((t+1)^{2l+1} - (t-1)^{2l+1}\big)\\
&= t^{2l}\tbinom{2l+1}{1} + t^{2l-2}\tbinom{2l+1}{3} + \cdots + t^2\tbinom{2l+1}{2l+1} + 1,\\
f'(t) &= \frac{1}{2}(2l+1)\big((t+1)^{2l} - (t-1)^{2l}\big)\\
&= 2\cdot\left\{t^{2l-1}\cdot l\cdot\tbinom{2l+1}{1} + t^{2l-3}\cdot(l-1)\cdot\tbinom{2l+1}{3} + \cdots + t\cdot 1\cdot\tbinom{2l+1}{2l-1}\right\}.
\end{aligned}$$

We evaluate at $t=1$ to conclude:

$$\begin{aligned}
l\tbinom{2l+1}{1} &+ (l-1)\tbinom{2l+1}{3} + \cdots + \tbinom{2l+1}{2l-1}\\
&= \tfrac{1}{2}f'(1) = \tfrac{1}{4}(2l+1)2^{2l} = (2l+1)4^{l-1}.
\end{aligned}$$

Therefore:

$$2^l\cdot y = (2l+1)\cdot 4^{l-1}(L-1).$$

If $l=1$, this gives the relation $2y = 3x = x$ so $\widetilde{K}(S(T^*M)) = \mathbf{Z}_4$. In fact, $S(T^*M) = S^3/\mathbf{Z}_4$ is a lens space so this calculation agrees with Corollary 4.6.10. If $l>1$, then $2^l \mid 4^{l-1}$ so $2^l\cdot y = 0$. From this it follows $\widetilde{K}(S(T^*M)) = \mathbf{Z}/2^l\mathbf{Z}\oplus\mathbf{Z}/2^l\mathbf{Z}$ and the short exact sequence actually splits in this case. This proves:

THEOREM 4.6.12 (V. SNAITH). *Let $X = S(T^*(\mathbf{R}P^{2l}))$ be the unit tangent bundle over even dimenional real projective space. If $l=1$ then $K(X) = \mathbf{Z}\oplus\mathbf{Z}_4$. Otherwise $K(X) = \mathbf{Z}\oplus\mathbf{Z}/2^l\mathbf{Z}\oplus\mathbf{Z}/2^l\mathbf{Z}$. The map*

$\mathrm{ind}(*,*)$ *gives a perfect pairing* $K(\mathbf{R}P^{2l}) \otimes K(X)/K(\mathbf{R}P^{2l}) \to \mathbf{Q} \bmod \mathbf{Z}$. *The generators of* $K(X)$ *are* $\{1, x, y\}$ *for* $x = L - 1$ *and* $y = \gamma\Pi_+(\bar{a})$.

Remark: This gives the additive structure. We have $x^2 = -2x$. We can calculate $x \cdot y$ geometrically. Let $p(u) = i \sum v_j \cdot e_j$. We regard $p\colon 1^v \to L^v$ for $v = 2^l$ over $\mathbf{R}P_2l$. Let $\bar{a}_L = a \otimes I$ on L^v. Then $pa_Lp = \bar{a}_L$ so $\Pi_\pm(\bar{a}) \otimes L = \Pi_\mp(\bar{a})$. Therefore:

$$\begin{aligned}(L-1) \otimes \left(\Pi_+(\bar{a}) - 1^{2^{l-1}}\right) &= -2^{l-1}(L-1) + (L \otimes \Pi_+(\bar{a}) - \Pi_+(\bar{a}))\\ &= 2^{l-1}(L-1) + \Pi_-(\bar{a}) - \Pi_+(\bar{a})\\ &= 2^{l-1}(L-1) + \Pi_+(\bar{a}) + \Pi_-(\bar{a}) - 2\Pi_+(\bar{a})\\ &= 2^{l-1} \cdot x - 2y\end{aligned}$$

so that $x \cdot y = 2^{l-1} \cdot x - 2y$. This gives at least part of the ring structure; we do not know a similar simple geometric argument to compute $y \cdot y$.

4.7. Singer's Conjecture for the Euler Form.

In this section, we will study a partial converse to the Gauss-Bonnet theorem as well as other index theorems. This will lead to information regarding the higher order terms which appear in the heat equation. In a lecture at M.I.T., I. M. Singer proposed the following question:

Suppose that $P(G)$ is a scalar valued invariant of the metric such that $P(M) = \int_M P(G)\,\mathrm{dvol}$ is independent of the metric. Then is there some universal constant c so that $P(M) = c\chi(M)$?

Put another way, the Gauss-Bonnet theorem gives a local formula for a topological invariant (the Euler characteristic). Is this the only theorem of its kind? The answer to this question is yes and the result is due to E. Miller who settled the conjecture using topological means. We also settled the question in the affirmative using local geometry independently and we would like to present at least some the the ideas involved. If the invariant is allowed to depend upon the orientation of the manifold, then the characteristic numbers also enter as we shall see later.

We let $P(g_{ij/\alpha})$ be a polynomial invariant of the metric; real analytic or smooth invariants can be handled similarly. We suppose $P(M) = \int_M P(G)\,\mathrm{dvol}$ is independent of the particular metric chosen on M.

LEMMA 4.7.1. *Let P be a polynomial invariant of the metric tensor with coefficients which depend smoothly on the g_{ij} variables. Suppose $P(M)$ is independent of the metric chosen. We decompose $P = \sum P_n$ for $P_n \in \mathcal{P}_{m,n}$ homogeneous of order n in the metric. Then $P_n(M)$ is independent of the metric chosen separately for each n; $P_n(M) = 0$ for $n \neq m$.*

PROOF: This lets us reduce the questions involved to the homogeneous case. If we replace the metric G by c^2G then $P_n(c^2G) = c^{-n}P_n(G)$ by Lemma 2.4.4. Therefore $\int_M P(c^2G)\,\mathrm{dvol}(c^2G) = \sum_n c^{m-n}\int_M P_n(G)$. Since this is independent of the constant c, $P_n(M) = 0$ for $n \leq m$ and $P_m(M) = P(M)$ which completes the proof.

If Q is 1-form valued, we let $P = \operatorname{div} Q$ be scalar valued. It is clear that $\int_M \operatorname{div} Q(G)\,\mathrm{dvol}(G) = 0$ so $P(M) = 0$ in this case. The following gives a partial converse:

LEMMA 4.7.2. *Let $P \in \mathcal{P}_{m,n}$ for $n \neq m$ satisfy $P(M) = \int_M P(G)\,\mathrm{dvol}$ is independent of the metric G. Then there exists $Q \in \mathcal{P}_{m,n-1,1}$ so that $P = \operatorname{div} Q$.*

PROOF: Since $n \neq m$, $P(M) = 0$. Let $f(x)$ be a real valued function on M and let G_t be the metric $e^{tf(x)}G$. If $n = 0$, then P is constant so $P = 0$ and the lemma is immediate. We assume $n > 0$ henceforth. We let

$P(t) = P(e^{tf(x)}G)$ and compute:

$$\frac{d}{dt}\big(P(e^{tf(x)}G)\,\mathrm{dvol}(e^{tf(x)}G)\big) = \frac{d}{dt}\big(P(e^{tf(x)}G)e^{tmf(x)/2}\big)\,\mathrm{dvol}(G)$$
$$= Q(G,f)\,\mathrm{dvol}(G).$$

$Q(f,G)$ is a certain expression which is linear in the derivatives of the scaling function f. We let $f_{;i_1\dots i_p}$ denote the multiple covariant derivatives of f and let $f_{:i_1\dots i_p}$ denote the symmetrized covariant derivatives of f. We can express

$$Q(f,G) = \sum Q_{i_1\dots i_p} f_{:i_1\dots i_p}$$

where the sum ranges over symmetric tensors of length less than n. We formally integrate by parts to express:

$$Q(f,G) = \mathrm{div}\, R(f,G) + \sum (-1)^p Q_{i_1\dots i_p : i_1\dots i_p} f.$$

By integrating over the structure group $\mathrm{O}(m)$ we can ensure that this process is invariantly defined. If $S(G) = \sum (-1)^p Q_{i_1\dots i_p : i_1 \dots i_p}$ then the identity:

$$0 = \int_M Q(f,G)\,\mathrm{dvol}(G) = \int_M S(G) f\,\mathrm{dvol}(G)$$

is true for every real valued function f. This implies $S(G) = 0$ so $Q(f,G) = \mathrm{div}\, R(f,G)$. We set $f = 1$. Since

$$e^{tm/2}P(e^tG) = e^{(m-n)t/2}P(G),$$

we conclude $Q(1,G) = \frac{m-n}{2}P(G)$ so $P(G) = \frac{2}{m-n}\,\mathrm{div}\, R(1,G)$ which completes the proof.

There is a corresponding lemma for form valued invariants. The proof is somewhat more complicated and we refer to (Gilkey, Smooth invariants of a Riemannian manifold) for further details:

LEMMA 4.7.3.
(a) Let $P \in \mathcal{P}_{m,n,p}$. *We assume* $n \neq p$ *and* $dP = 0$. *If* $p = m$, *we assume* $\int_M P(G)$ *is independent of* G *for every* G *on* M. *Then there exists* $Q \in \mathcal{P}_{m,n-1,p-1}$ *so that* $dQ = P$.
(b) Let $P \in \mathcal{P}_{m,n,p}$. *We assume* $n \neq m - p$ *and* $\delta P = 0$. *If* $p = 0$ *we assume* $\int_M P(G)\,\mathrm{dvol}$ *is independent of* G *for every* G *in* M. *Then there exists* $Q \in \mathcal{P}_{m,n-1,p+1}$ *so that* $\delta Q = P$.

Remark: (a) and (b) are in a sense dual if one works with $\mathrm{SO}(m)$ invariance and not just $O(m)$ invariance. We can use this Lemma together with the results of Chapter 2 to answer the generalized Singer's conjecture:

Theorem 4.7.4.
(a) Let P be a scalar valued invariant so that $P(M) = \int_M P(G)\,\mathrm{dvol}$ is independent of the particular metric chosen. Then we can decompose $P = c \cdot E_m + \operatorname{div} Q$ where E_m is the Euler form and where Q is a 1-form valued invariant. This implies $P(M) = c\chi(M)$.
(b) Let P be a p-form valued invariant so that $dP = 0$. If $p = m$, we assume $P(M) = \int_M P(G)$ is independent of the particular metric chosen. Then we can decompose $P = R + dQ$. Q is $p-1$ form valued and R is a Pontrjagin form.

Proof: We decompose $P = \sum P_j$ into terms which are homogeneous of order j. Then each P_j satisfies the hypothesis of Theorem 4.7.4 separately so we may assume without loss of generality that P is homogeneous of order n. Let P be as in (a). If $n \neq m$, then $P = \operatorname{div} Q$ be Lemma 4.7.2. If $n = m$, we let $P_1 = r(P) \in \mathcal{P}_{m-1,m}$. It is immediate $\int_{M_1} P_1(G_1)\,\mathrm{dvol}(G_1) = \frac{1}{2\pi}\int_{S^1\times M_1} P(1 \times G_1)$ is independent of the metric G_1 so P_1 satisfies the hypothesis of (a) as well. Since $m-1 \neq m$ we conclude $P_1 = \operatorname{div} Q_1$ for $Q_1 \in \mathcal{P}_{m-1,m-1,1}$. Since r is surjective, we can choose $Q \in \mathcal{P}_{m,m-1,1}$ so $r(Q) = Q_1$. Therefore $r(P - \operatorname{div} Q) = P_1 - \operatorname{div} Q_1 = 0$ so by Theorem 2.4.7, $P - \operatorname{div} Q = cE_m$ for some constant c which completes the proof. (The fact that $r : \mathcal{P}_{m,n,p} \to \mathcal{P}_{m-1,n,p} \to 0$ is, of course, a consequence of H. Weyl's theorem so we are using the full force of this theorem at this point for the first time).

If P is p-form valued, the proof is even easier. We decompose P into homogeneous terms and observe each term satisfies the hypothesis separately. If P is homogeneous of degree $n \neq p$ then $P = dQ$ be Lemma 4.7.3. If P is homogeneous of degree $n = p$, then P is a Pontrjagin form by Lemma 2.5.6 which completes the proof in this case.

The situation in the complex catagory is not as satisfactory.

Theorem 4.7.5. *Let M be a holomorphic manifold and let P be a scalar valued invariant of the metric. Assume $P(M) = \int_M P(G)\,\mathrm{dvol}$ is independent of the metric G. Then we can express $P = R + \operatorname{div} Q + \mathcal{E}$. Q is a 1-form valued invariant and $P = *R'$ where R' is a Chern form. The additional error term $\mathcal{E}$ satisfies the conditions: $r(\mathcal{E}) = 0$ and $\mathcal{E}$ vanishes for Kaehler metric. Therefore $P(M)$ is a characteristic number if M admits a Kaehler metric.*

The additional error term arises because the axiomatic characterization of the Chern forms given in Chapter 3 were only valid for Kaehler metrics. $\mathcal{E}$ in general involves the torsion of the holomorphic connection and to show $\operatorname{div} Q' = \mathcal{E}$ for some Q' is an open problem. Using the work of E. Miller, it does follow that $\int \mathcal{E}\,\mathrm{dvol} = 0$ but the situation is not yet completely resolved.

We can use these results to obtain further information regarding the higher terms in the heat expansion:

Theorem 4.7.6.
(a) Let $a_n(x, d+\delta)$ denote the invariants of the heat equation for the de Rham complex. Then

(i) $a_n(x, d+\delta) = 0$ *if m or n is odd or if $n < m$,*
(ii) $a_m(x, d+\delta) = E_m$ *is the Euler form,*
(iii) *If m is even and if $n > m$, then $a_n(x, d+\delta) \not\equiv 0$ in general. However, there does exist a 1-form valued invariant $q_{m,n}$ so $a_n = \operatorname{div} q_{m,n}$ and $r(q_{m,n}) = 0$.*

(b) Let $a_n^{\text{sign}}(x, V)$ be the invariants of the heat equation for the signature complex with coefficients in a bundle V. Then

(i) $a_n^{\text{sign}}(x, V) = 0$ *for $n < m$ or n odd,*
(ii) $a_m^{\text{sign}}(x, V)$ *is the integrand of the Atiyah-Singer index theorem,*
(iii) $a_n^{\text{sign}}(x, V) = 0$ *for n even and $n < m$. However, there exists an $m-1$ form valued invariant $q_{m,n}^{\text{sign}}(x, V)$ so that $a_n^{\text{sign}} = d(q_{m,n}^{\text{sign}})$.*

A similar result holds for the invariants of the spin complex.

(c) Let $a_n^{\text{Dolbeault}}(x, V)$ denote the invariants of the heat equation for the Dolbeault complex with coefficients in V. We do not assume that the metric in question is Kaehler. Then:

(i) $a_n^{\text{Dolbeault}}(x, V) = \operatorname{div} Q_{m,n}$ *where $Q_{m,n}$ is a 1-form valued invariant for $n \ne m$,*
(ii) $a_m^{\text{Dolbeault}}(x, V) = \operatorname{div} Q_{m,n} +$ *the integrand of the Riemann-Roch theorem.*

The proof of all these results relies heavily on Lemma 4.7.3 and 4.7.2 and will be omitted. The results for the Dolbeault complex are somewhat more difficult to obtain and involve a use of the SPIN_c complex.

We return to the study of the de Rham complex. These arguments are due to L. Willis. Let $m = 2n$ and let $a_{m+2}(x, d+\delta)$ be the next term above the Euler form. Then $a_{m+2} = \operatorname{div} Q_{m+1}$ where $Q_{m+1} \in \mathcal{P}_{m,m+1,1}$ satisfies $r(Q_{m+1}) = 0$. We wish to compute a_{m+2}. The first step is:

Lemma 4.7.7. *Let m be even and let $Q \in \mathcal{P}_{m,m+1,1}$ be 1-form valued. Suppose $r(Q) = 0$. Then Q is a linear combination of dE_m and Φ_m defined by:*

$$\Phi_m = \sum_{k,\rho,\tau} \operatorname{sign}(\rho)\operatorname{sign}(\tau) \cdot \left\{(-8\pi)^{m/2}\left(\frac{m}{2}-1\right)!\right\}^{-1}$$
$$\times R_{\rho(1)\rho(2)\tau(1)k;k} R_{\rho(3)\rho(4)\tau(3)\tau(4)} \dots R_{\rho(m-1)\rho(m)\tau(m-1)\tau(m)}\, e^{\tau(2)} \in \Lambda^1$$

where $\{e^1, \dots, e^m\}$ are a local orthonormal frame for T^*M.

Proof: We let Ae^k be a monomial of P. We express A in the form:

$$A = g_{i_1j_1/\alpha_1} \cdots g_{i_rj_r/\alpha_r}.$$

By Lemma 2.5.5, we could choose a monomial B of P with $\deg_k(B) = 0$ for $k > 2\ell(A) = 2r$. Since $r(Q) = 0$, we conclude $2r \geq m$. On the other hand, $2r \leq \sum |\alpha_\nu| = m + 1$ so we see

$$m \leq 2r \leq m + 1.$$

Since m is even, we conclude $2r = m$. This implies one of the $|\alpha_\nu| = 3$ while all the other $|\alpha_\nu| = 2$. We choose the notation so $|\alpha_1| = 3$ and $|\alpha_\nu| = 2$ for $\nu > 1$. By Lemma 2.5.1, we may choose A in the form:

$$A = g_{ij/111} g_{i_2j_2/k_2l_2} \cdots g_{i_rj_r/k_rl_r}.$$

We first suppose $\deg_1(A) = 3$. This implies A appears in the expression Ae^1 in P so $\deg_j(A) \geq 2$ is even for $j > 1$. We estimate $2m + 1 = 4r + 1 = \sum_j \deg_j(A) = 3 + \sum_{j>1} \deg_j(A) \geq 3 + 2(m - 1) = 2m + 1$ to show $\deg_j(A) = 2$ for $j > 1$. If $m = 2$, then $A = g_{22/111}$ which shows $\dim \mathrm{N}(r) = 1$ and Q is a multiple of dE_2. We therefore assume $m > 2$. Since $\deg_j(A) = 2$ for $j > 1$, we can apply the arguments used to prove Theorem 2.4.7 to the indices $j > 1$ to show that

$$g_{22/111} g_{33/44} \cdots g_{m-1/mm} e^1$$

is a monomial of P.

Next we suppose $\deg_1(A) > 3$. If $\deg_1(A)$ is odd, then $\deg_j(A) \geq 2$ is even for $j \geq 2$ implies $2m+1 = 4r+1 = \sum_j \deg_j(A) \geq 5+2(m-1) = 2m+3$ which is false. Therefore $\deg_1(A)$ is even. We choose the notation in this case so Ae^2 appears in P and therefore $\deg_2(A)$ is odd and $\deg_j(A) \geq 2$ for $j > 2$. This implies $2m + 1 = 4r + 1 = \deg_j(A) \geq 4 + 1 + \sum_{j>2} \deg_j(A) \geq 5 + 2(m - 2) = 2m + 1$. Since all the inequalities must be equalities we conclude

$$\deg_1(A) = 4, \quad \deg_2(A) = 1, \quad \deg_3(A) = 2 \text{ for } j > 2.$$

We apply the arguments of the second chapter to choose A of this form so that every index $j > 2$ which does not touch either the index 1 or the index 2 touches itself. We choose A so the number of indices which touch themselves in A is maximal. Suppose the index 2 touches some other index than the index 1. If the index 2 touches the index 3, then the index 3 cannot touch itself in A. An argument using the fact $\deg_2(A) = 1$ and using the arguments of the second chapter shows this would contradict the maximality of A and thus the index 1 must touch the index 2 in A. We use non-linear changes of coordinates to show $g_{12/111}$ cannot divide A. Using non-linear changes of coordinates to raplace $g_{44/12}$ by $g_{12/44}$ if necessary, we conclude A has the form

$$A = g_{33/111} g_{12/44} g_{55/66} \cdots g_{m-1,m-1/mm}\,.$$

This shows that if $m \geq 4$, then $\dim \mathrm{N}(r\colon \mathcal{P}_{m,m+1,1} \to \mathcal{P}_{m-1,m+1,1}) \leq 2$. Since these two invariants are linearly independent for $m \geq 4$ this completes the proof of the lemma.

We apply the lemma and decompose

$$a_{m+2}(x, d+\delta) = c_1(m)(E_m)_{;kk} + c_2(m)\operatorname{div}\Phi_m.$$

The $c_1(m)$ and $c_2(m)$ are universal constants defending only on the dimension m. We consider a product manifold of the form $M = S^2 \times M^{m-2}$ to define a map:

$$r_{(2)}\colon \mathcal{P}_{m,n,0} \to \bigoplus_{q \leq n} \mathcal{P}_{m-2,q,0}\,.$$

This restriction map does not preserve the grading since by throwing derivatives to the metric over S^2, we can lower the order of the invariant involved.

Let $x_1 \in S^2$ and $x_2 \in M^{m-2}$. The multiplicative nature of the de Rham complex implies:

$$a_n(x, d+\delta) = \sum_{j+k=m} a_j(x_1, d+\delta)_{S^2}\, a_k(x_2, d+\delta)_{M^{m-2}}\,.$$

However, $a_j(x_1, d+\delta)$ is a constant since S^2 is a homogeneous space. The relations $\int_{S^2} a_j(x_1, d+\delta) = 2\delta_{j,2}$ implies therefore

$$a_n(x, d+\delta) = \frac{1}{2\pi} a_{n-2}(x_2, d+\delta)$$

so that

$$r_{(2)}\, a^m_{m+2} = \frac{1}{2\pi} a^{m-2}_m.$$

Since $r_{(2)}(E_m)_{;kk} = \frac{1}{2\pi}(E_{m-2})_{;kk}$ and $r_{(2)}\operatorname{div}\Phi_m = \frac{1}{2\pi}\operatorname{div}\Phi_{m-2}$, we conclude that in fact the universal constants c_1 and c_2 do not depend upon m. (If $m = 2$, these two invariants are not linearly independent so we adjust $c_1(2) = c_1$ and $c_2(2) = c_2$). It is not difficult to use Theorem 4.8.16 which will be discussed in the next section to compute that if $m = 4$, then:

$$a_6(x, d+\delta) = \frac{1}{12}(E_m)_{;kk} - \frac{1}{6}\operatorname{div}\Phi_m.$$

We omit the details of the verification. This proves:

THEOREM 4.7.8. *Let m be even and let $a_{m+2}(x, d+\delta)$ denote the invariant of the Heat equation for the de Rham complex. Then:*

$$a_{m+2} = \frac{1}{12}(E_m)_{;kk} - \frac{1}{6}\operatorname{div}\Phi_m.$$

We gave a different proof of this result in (Gilkey, Curvature and the heat equation for the de Rham complex). This proof, due to Willis, is somewhat simpler in that it uses Singer's conjecture to simplify the invariance theory involved.

There are many other results concerning the invariants which appear in the heat equation for the de Rham complex. Gunther and Schimming have given various shuffle formulas which generalize the alternating sum defined previously. The combinatorial complexities are somewhat involved so we shall simply give an example of the formulas which can be derived.

THEOREM 4.7.9. *Let m be odd. Then*

$$\sum(-1)^p(m-p)a_n(x, \Delta_p^m) = \begin{cases} 0 & \text{if } n < m-1 \\ E_{m-1} & \text{if } n = m-1. \end{cases}$$

PROOF: We remark that there are similar formulas giving the various Killing curvatures E_k for $k < m$ in all dimensions. Since $r\colon \mathcal{P}_{m,n} \rightarrow \mathcal{P}_{m-1,n}$ for $n < m$ is an isomorphism, it suffices to prove this formula under restriction. Since $r(a_n(x, \Delta_p^m)) = a_n(x, \Delta_p^{m-1}) + a_n(x, \Delta_{p-1}^{m-1})$ we must study

$$\sum(-1)^p(m-p)(a_n(x, \Delta_p^{m-1}) + a_n(x, \Delta_{p-1}^{m-1})) = (-1)^p a_n(x, \Delta_p^{m-1})$$

and apply Theorem 2.4.8.

We remark that all the shuffle formulas of Gunther and Schimming (including Theorem 4.7.9) have natural analogues for the Dolbeault complex for a Kaehler metric and the proofs are essentially the same and rely on Theorem 3.6.9 and 3.4.10.

4.8. Local Formulas for the Invariants Of the Heat Equation.

In this subsection, we will compute $a_n(x, \Delta_p^m)$ for $n = 0, 1, 2$. In principle, the combinatorial formulas from the first chapter could be used in this calculation. In practice, however, these formulas rapidly become much too complicated for practical use so we shall use instead some of the functorial properties of the invariants involved.

Let $P\colon C^\infty(V) \to C^\infty(V)$ be a second order elliptic operator with leading symbol given by the metric tensor. This means we can express:

$$P = -\left(\sum_{i,j} g^{ij}\partial^2/\partial x_i \partial x_j + \sum_k A_k \partial/\partial x_k + B\right)$$

where the A_k, B are endomorphisms of the bundle V and where the leading term is scalar. This category of differential operators includes all the Laplacians we have been considering previously. Our first task is to get a more invariant formulation for such an operator.

Let ∇ be a connection on V and let $E \in C^\infty(\mathrm{END}(V))$ be an endomorphism of V. We use the Levi-Civita connection on M and the connection ∇ on V to extend ∇ to tensors of all orders. We define the differential operator P_∇ by the diagram:

$$P_\nabla\colon C^\infty(V) \xrightarrow{\nabla} C^\infty(T^*M \otimes V) \xrightarrow{\nabla} C^\infty(T^*M \otimes T^*M \otimes V) \xrightarrow{-g\otimes 1} C^\infty(V).$$

Relative to a local orthonormal frame for T^*M, we can express

$$P_\nabla(f) = -f_{;ii}$$

so this is the trace of second covariant differentiation. We define

$$P(\nabla, E) = P_\nabla - E$$

and our first result is:

Lemma 4.8.1. *Let $P\colon C^\infty(V) \to C^\infty(V)$ be a second order operator with leading symbol given by the metric tensor. There exists a unique connection on V and a unique endomorphism so $P = P(\nabla, E)$.*

Proof: We let indices i, j, k index a coordinate frame for $T(M)$. We shall not introduce indices to index a frame for V but shall always use matrix notation. The Christoffel symbols ${\Gamma_{ij}}^k = -{\Gamma_i}^k{}_j$ of the Levi-Civita connection are given by:

$$\begin{aligned} {\Gamma_{ij}}^k &= \tfrac{1}{2} g^{kl}(g_{il/j} + g_{jl/i} - g_{ij/l}) \\ \nabla_{\partial/\partial x_i}(\partial/\partial x_j) &= {\Gamma_{ij}}^k(\partial/\partial x_k) \\ \nabla_{\partial/\partial x_i}(dx_j) &= {\Gamma_i}^j{}_k(dx_k) \end{aligned}$$

where we sum over repeated indices. We let $\vec{s}$ be a local frame for V and let ω_i be the components of the connection 1-form ω so that:

$$\nabla(f\cdot\vec{s}) = dx^i \otimes (\partial f/\partial x_i + \omega_i(f))\cdot\vec{s}.$$

With this notational convention, the curvature is given by:

$$\Omega_{ij} = \omega_{j/i} - \omega_{i/j} + \omega_i\omega_j - \omega_j\omega_i \qquad (\Omega = d\omega + \omega\wedge\omega).$$

We now compute:

$$\begin{aligned}\nabla^2(f\cdot\vec{s}) = dx^j \otimes dx^i \otimes (&\partial^2 f/\partial x_i\partial x_j + \omega_j\partial f/\partial x_i + \Gamma_j{}^k{}_i\partial f/\partial x_k \\ &+ \omega_i\partial f/\partial x_j + \omega_{i/j}f + \omega_j\omega_i f + \Gamma_j{}^k{}_i\omega_k f)\cdot\vec{s}\end{aligned}$$

from which it follows that:

$$\begin{aligned}P_\nabla(f\cdot\vec{s}) = -\{&g^{ij}\partial f/\partial x_i\partial x_j + (2g^{ij}\omega_i - g^{jk}\Gamma_{jk}{}^i)\partial f/\partial x_i \\ &+ (g^{ij}\omega_{i/j} + g^{ij}\omega_i\omega_j - g^{ij}\Gamma_{ij}{}^k\omega_k)f\}\cdot\vec{s}\end{aligned}$$

We use this identity to compute:

$$(P - P_\nabla)(f\cdot\vec{s}) = -\{(A_i - 2g^{ij}\omega_j + g^{jk}\Gamma_{jk}{}^i)\partial f/\partial x_i + (*)f\}\cdot\vec{s}$$

where we have omitted the 0^{th} order terms. Therefore $(P - P_\nabla)$ is a 0^{th} order operator if and only if

$$A_i - 2g^{ij}\omega_j + g^{jk}\Gamma_{jk}{}^i = 0 \quad \text{or equivalently} \quad \omega_i = \tfrac{1}{2}(g_{ij}A_j + g_{ij}g^{kl}\Gamma_{kl}{}^j).$$

This shows that the $\{\omega_i\}$ are uniquely determined by the condition that $(P - P_\nabla)$ is a 0^{th} order operator and specificies the connection ∇ uniquely.

We define:

$$E = P_\nabla - P \quad \text{so} \quad E = B - g^{ij}\omega_{i/j} - g^{ij}\omega_i\omega_j + g^{ij}\omega_k\Gamma_{ij}{}^k.$$

We fix this connection ∇ and endomorphism E determined by P. We summarize these formulas as follows:

COROLLARY 4.8.2. *If (∇, E) are determined by the second order operator $P = -(g^{ij}\partial^2 f/\partial x_i\partial x_j + A_i\partial j/\partial x_i + Bf)\cdot\vec{s}$, then*

$$\begin{aligned}\omega_i &= \tfrac{1}{2}(g_{ij}A_j + g_{ij}g^{kl}\Gamma_{kl}{}^j) \\ E &= B - g^{ij}\omega_{i/j} - g^{ij}\omega_i\omega_j + g^{ij}\omega_k\Gamma_{ij}{}^k.\end{aligned}$$

We digress briefly to express the Laplacian Δ_p in this form. If ∇ is the Levi-Civita connection acting on p-forms, it is clear that $\Delta_p - P_\nabla$ is a first order operator whose leading symbol is linear in the 1-jets of the metric. Since it is invariant, the leading symbol vanishes so $\Delta_p - P_\nabla$ is a 0^{th} order and the Levi-Civita connection is in fact the connection determined by the operator Δ_p. We must now compute the curvature term. The operator $(d+\delta)$ is defined by Clifford multiplication:

$$(d+\delta)\colon C^\infty(\Lambda(T^*M)) \to C^\infty(T^*M \otimes \Lambda(T^*M)) \xrightarrow{\text{Clifford multiplication}} C^\infty(\Lambda(T^*M)).$$

If we expand $\theta = f_I\,dx_I$ and $\nabla\theta = dx_i \otimes (\partial f_I/\partial x_i + \Gamma_{iI}{}^J f_I)\,dx_J$ then it is immediate that if "$*$" denotes Clifford multiplication,

$$\begin{aligned}(d+\delta)(f_I\,dx_I) &= (f_{I/i}\,dx_i * dx_I) + (f_I\Gamma_{iIJ}\,dx_i * dx_J)\\ \Delta(f_I dx_I) &= (f_{I/ij}\,dx_j * dx_i * dx_I) + (f_I\Gamma_{iIJ/j}\,dx_j * dx_i * dx_J) + \cdots\end{aligned}$$

where we have omitted terms involving the 1-jets of the metric. Similary, we compute:

$$P_\nabla(f_I\,dx_I) = -g^{ij} f_{I/ij}\,dx_I - g^{ij}\Gamma_{iIJ/j} f_I\,dx_J + \cdots .$$

We now fix a point x_0 of M and let X be a system of geodesic polar coordinates centered at x_0. Then $\Gamma_{iIJ/i} = 0$ and $\Gamma_{iIJ/j} = \frac{1}{2}R_{jiIJ}$ gives the curvature tensor at x_0. Using the identities $dx_i * dx_j + dx_j * dx_i = -2\delta_{ij}$ we see $f_{I/ij}\,dx_j * dx_i = -f_{I/ii}$ and consequently:

$$(P_\nabla - \Delta)(f_I\,dx_I) = -\tfrac{1}{2}R_{ijIJ} f_I\,dx_i * dx_j * dx_J = \sum_{i<j} R_{ijIJ} f_I\,dx_j * dx_i * dx_J.$$

This identity holds true at x_0 in geodesic polar coordinates. Since both sides are tensorial, it holds in general which proves:

LEMMA 4.8.3. *Let Δ_p be the Laplacian acting on p-forms and let R_{ijIJ} be the curvature tensor of the Levi-Civita connection. Then*

$$(P_\nabla - \Delta_p)(f_I\,dx_I) = \sum_{i<j} R_{ijIJ} f_I\,dx_j * dx_i * dx_J.$$

We now return to the problem of computing the invariants $a_n(x,P)$.

LEMMA 4.8.4. *Let P be an operator as in Lemma 4.8.1. Then $a_0(x,P) = (4\pi)^{-m/2}\dim V$.*

PROOF: We first consider the operator $P = -\partial^2/\partial\theta^2$ on the unit circle $[0,2\pi]$. The eigenvalues of P are $\{n^2\}_{n\in\mathbf{Z}}$. Since a_0 is homogeneous of order 0 in the jets of the symbol, $a_0(\theta, P) = a_0(P)/\operatorname{vol}(M)$ is constant. We compute:

$$\sum_n e^{-tn^2} \sim t^{-1/2} \int_M a_0(\theta, P)\, d\theta = t^{-1/2}(2\pi)a_0(\theta, P).$$

However, the Riemann sums approximating the integral show that

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-tx^2}\, dx = \lim_{t \to 0} \sum_n e^{-(\sqrt{t}\, n)^2} \sqrt{t}$$

from which it follows immediately that $a_0(\theta, P) = \sqrt{\pi}/(2\pi) = (4\pi)^{-1/2}$. More generally, by taking the direct sum of these operators acting on $C^\infty(S^1 \times \mathbf{C}^k)$ we conclude $a_0(\theta, P) = (4\pi)^{-1/2} \dim V$ which completes the proof if $m = 1$.

More generally, $a_0(x, P)$ is homogeneous of order 0 in the jets of the symbol so $a_0(x, P)$ is a constant which only depends on the dimension of the manifold and the dimension of the vector bundle. Using the additivity of Lemma 1.7.5, we conclude $a_0(x, P)$ must have the form:

$$a_0(x, P) = c(m) \cdot \dim V.$$

We now let $M = S^1 \times \cdots \times S^1$, the flat m-torus, and let $\Delta = -\sum_\nu \partial^2/\partial\theta_\nu^2$. The product formula of Lemma 1.7.5(b) implies that:

$$a_0(x, \Delta) = \prod_\nu a_0(\theta_\nu, -\partial^2/\partial\theta_\nu^2) = (4\pi)^{-m/2}$$

which completes the proof of the lemma.

The functorial properties of Lemma 1.7.5 were essential to the proof of Lemma 4.8.4. We will continue to exploit this functoriality in computing a_2 and a_4. (In principal one could also compute a_6 in this way, but the calculations become of formidable difficulty and will be omitted).

It is convenient to work with more tensorial objects than with the jets of the symbol of P. We let $\dim V = k$ and $\dim M = m$. We introduce formal variables $\{ R_{i_1 i_2 i_3 i_4 ;\ldots}, \Omega_{i_1 i_2 ;\ldots}, E_{;\ldots} \}$ for the covariant derivatives of the curvature tensor of the Levi-Civita connection, of the curvature tensor of ∇, and of the covariant derivatives of the endomorphism E. We let $\mathcal{S}$ be the non-commutative algebra generated by these variables. Since there are relations among these variables, $\mathcal{S}$ isn't free. If $S \in \mathcal{S}$ and if e is a local orthonormal frame for $T^*(M)$, we define $S(P)(e)(x) = S(G, \nabla, E)(e)(x)(\mathrm{END}(V))$ by evaluation. We say S is invariant if $S(P) = S(P)(a)$ is independent of the orthonormal frame e chosen for $T(M)$. We define:

$$\begin{aligned} \operatorname{ord}(R_{i_1 i_2 i_4 i_4 ; i_5 \ldots i_k}) &= k + 2 \\ \operatorname{ord}(\Omega_{i_1 i_2 ; i_3 \ldots i_k}) &= k \\ \operatorname{ord}(E_{; i_1 \ldots i_k}) &= k + 2 \end{aligned}$$

to be the degree of homogeneity in the jets of the symbol of P, and let $\mathcal{S}_{m,n,k}$ be the finite dimensional subspace of S consisting of the invariant polynomials which are homogeneous of order n.

If we apply H. Weyl's theorem to this situation and apply the symmetries involved, it is not difficult to show:

LEMMA 4.8.5.
(a) $\mathcal{S}_{m,2,k}$ *is spanned by the two polynomials* $R_{ijij}I$, E.
(b) $\mathcal{S}_{m,4,k}$ *is spanned by the eight polynomials*

$$\begin{matrix} R_{ijij;kk}I, & R_{ijij}R_{klkl}I, & R_{ijik}R_{ljlk}I, & R_{ijkl}R_{ijkl}I, \\ E^2, & E_{;ii}, & R_{ijij}E, & \Omega_{ij}\Omega_{ij}. \end{matrix}$$

We omit the proof in the interests of brevity; the corresponding spanning set for $\mathcal{S}_{m,6,k}$ involves 46 polynomials.

The spaces $\mathcal{S}_{m,n,k}$ are related to the invariants $a_n(x,P)$ of the heat equation as follows:

LEMMA 4.8.6. *Let* (m,n,k) *be given and let* P *satisfy the hypothesis of Lemma 4.8.1. Then there exists* $S_{m,n,k} \in \mathcal{S}_{m,n,k}$ *so that* $a_n(x,P) = \operatorname{Tr}(S_{m,n,k})$.

PROOF: We fix a point $x_0 \in M$. We choose geodesic polar coordinates centred at x_0. In such a coordinate system, all the jets of the metric at x_0 can be computed in terms of the $R_{i_1i_2i_3i_4;\dots}$ variables. We fix a frame s_0 for the fiber V_0 over x_0 and extend s_0 by parallel translation along all the geodesic rays from x_0 to get a frame near x_0. Then all the derivatives of the connection 1-form at x_0 can be expressed in terms of the $R_{\dots}$ and $\Omega_{\dots}$ variables at x_0. We can solve the relations of Corollary 4.8.2 to express the jets of the symbol of P in terms of the jets of the metric, the jets of the connection 1-form, and the jets of the endomorphism E. These jets can all be expressed in terms of the variables in $\mathcal{S}$ at x_0 so any invariant endomorphism which is homogeneous of order n belongs to $\mathcal{S}_{m,n,k}$. In Lemma 1.7.5, we showed that $a_n(x,P) = \operatorname{Tr}(e_n(x,P))$ was the trace of an invariant endomorphism and this completes the proof; we set $S_{m,n,k} = e_n(x,P)$.

We use Lemmas 4.8.5 and 4.8.6 to expand $a_n(x,P)$. We regard scalar invariants of the metric as acting on V by scalar multiplication; alternatively, such an invariant R_{ijij} could be replaced by $R_{ijij}I_V$.

LEMMA 4.8.7. *Let* $m = \dim M$ *and let* $k = \dim V$. *Then there exist universal constants* $c_i(m,k)$ *so that if* P *is as in Lemma 4.8.1,*
(a) $a_2(x,P) = (4\pi)^{-m/2}\operatorname{Tr}(c_1(m,k)R_{ijij} + c_2(m,k)E)$,

(b)

$$\begin{aligned}a_4(x,P) = (4\pi)^{-m/2}\,\mathrm{Tr}(&c_3(m,k)R_{ijij;kk} + c_4(m,k)R_{ijij}R_{klkl}\\ &+ c_5(m,k)R_{ijik}R_{ljlk} + c_6(m,k)R_{ijkl}R_{ijkl}\\ &+ c_7(m,k)E_{;ii} + c_8(m,k)E^2\\ &+ c_9(m,k)ER_{ijij} + c_{10}(m,k)\Omega_{ij}\Omega_{ij}).\end{aligned}$$

This is an important simplification because it shows in particular that terms such as $\mathrm{Tr}(E)^2$ do not appear in a_4.

The first observation we shall need is the following:

Lemma 4.8.8. *The constants $c_i(m,k)$ of Lemma 4.8.7 can be chosen to be independent of the dimension m and the fiber dimension k.*

Proof: The leading symbol of P is scalar. The analysis of Chapter 1 in this case immediately leads to a combinatorial formula for the coefficients in terms of certain trignometric integrals $\int \xi^\alpha e^{-|\xi|^2}\,d\xi$ and the fiber dimension does not enter. Alternatively, we could use the additivity of $e_n(x,P_1 \oplus P_2) = e_n(x,P_1) \oplus e_n(x,P_2)$ of Lemma 1.7.5 to conclude the formulas involved must be independent of the dimension k. We may therefore write $c_i(m,k) = c_i(m)$.

There is a natural restriction map $r\colon \mathcal{S}_{m,n,k} \to \mathcal{S}_{m-1,n,k}$ defined by restricting to operators of the form $P = P_1 \otimes 1 + I_V \otimes (-\partial^2/\partial\theta^2)$ over $M = M_1 \times S^1$. Algebraically, we simply set to zero any variables involving the last index. The multiplicative property of Lemma 1.7.5 implies

$$r(S_{m,n,k}) = \sum_{p+q=n} S_{m-1,p,k}(P_1) \otimes S_{1,q,1}(-\partial^2/\partial\theta^2).$$

Since all the jets of the symbol of $-\partial^2/\partial\theta^2$ vanish for $q > 0$, $S_{1,q,1} = 0$ for $q > 0$ and $a_0 = (4\pi)^{-1/2}$ by Lemma 4.8.4. Therefore

$$r(S_{m,n,k}) = (4\pi)^{-1/2} S_{m-1,n,k}\,.$$

Since we have included the normalizing constant $(4\pi)^{-m/2}$ in our definition, the constants are independent of the dimension m for $m \geq 4$. If $m = 1, 2, 3$, then the invariants of Lemma 4.8.6 are not linearly independent so we choose the constants to agree with $c_i(m,k)$ in these cases.

We remark that if P is a higher order operator with leading symbol given by a power of the metric tensor, then there a similar theory expressing a_n in terms of invariant tensorial expressions. However, in this case, the coefficients depend upon the dimension m in a much more fundamental way than simply $(4\pi)^{-m/2}$ and we refer to (Gilkey, the spectral geometry of the higher order Laplacian) for further details.

Since the coefficients do not depend on (m,k), we drop the somewhat cumbersome notation $S_{m,n,k}$ and return to the notation $e_n(x,P)$ discussed in the first chapter so $\mathrm{Tr}(e_n(x,P)) = a_n(x,P)$. We use the properties of the exponential function to compute:

LEMMA 4.8.9. *Using the notation of Lemma 4.8.7,*

$$c_2 = 1, \quad c_8 = \frac{1}{2}, \quad c_9 = c_1.$$

PROOF: Let P be as in Lemma 4.8.1 and let a be a real constant. We construct the operator $P_a = P - a$. The metric and connection are unchanged; we must replace E by $E + a$. Since $e^{-t(P-a)} = e^{-tP}e^{ta}$ we conclude:

$$e_n(x, P - a) = \sum_{p+q=n} e_p(x, P)a^q/q!$$

by comparing terms in the asymptotic expansion. We shall ignore factors of $(4\pi)^{-m/2}$ henceforth for notational convenience. Then:

$$e_2(x, P - a) = e_2(x, P) + c_2 a = e_2(x, P) + e_0(x, P)a = e_2(x, P) + a.$$

This implies that $c_2 = 1$ as claimed. Next, we have

$$\begin{aligned} e_4(x, P - a) &= e_4(x, P) + c_8 a^2 + 2c_8 aE + c_9 aR_{ijij} \\ &= e_4(x, P) + e_2(x, P)a + e_0(x, P)a^2/2 \\ &= e_4(x, P) + (c_1 R_{ijij} + c_2 E)a + a^2/2 \end{aligned}$$

which implies $c_8 = 1/2$ and $c_1 = c_9$ as claimed.

We now use some recursion relations derived in (Gilkey, Recursion relations and the asymptotic behavior of the eigenvalues of the Laplacian). To illustrate these, we first suppose $m = 1$. We consider two operators:

$$A = \partial/\partial x + b, \quad A^* = -\partial/\partial x + b$$

where b is a real scalar function. This gives rise to operators:

$$\begin{aligned} P_1 &= A^*A = -(\partial^2/\partial x^2 + (b' - b^2)) \\ P_2 &= AA^* = -(\partial^2/\partial x^2 + (-b' - b^2)) \end{aligned}$$

acting on $C^\infty(S^1)$. The metric and connection defined by these operators is flat. $E(P_1) = b' - b^2$ and $E(P_2) = -b' - b^2$.

LEMMA 4.8.10.

$$(n - 1)\big(e_n(x, P_1) - e_n(x, P_2)\big) = \partial/\partial x\{\partial/\partial x + 2b\}e_{n-2}(x, P_1).$$

PROOF: Let $\{\lambda_\nu, \theta_\nu\}$ be a complete spectral resolution of P_1. We ignore any possible zero spectrum since it won't contribute to the series we shall be constructing. Then $\{\lambda_\nu, A\theta_\nu/\sqrt{\lambda_\nu}\}$ is a complete spectral resolution of P_2. We compute:

$$\begin{aligned}
&\frac{d}{dt}\{K(t,x,x,P_1) - K(t,x,x,P_2)\} \\
&\qquad = \sum e^{-t\lambda_\nu}\{-\lambda_\nu\theta_\nu\theta_\nu + A\theta_\nu A\theta_\nu\} \\
&\qquad = \sum e^{-t\lambda_\nu}\{-P_1\theta_\nu\cdot\theta_\nu + A\theta_\nu A\theta_\nu\} \\
&\qquad = \sum e^{-t\lambda_\nu}\{\theta_\nu''\theta_\nu + (b'-b^2)\theta_\nu^2 + \theta_\nu'\theta_\nu' + 2b\theta_\nu'\theta_\nu + b^2\theta_\nu^2\} \\
&\qquad = \sum e^{-t\lambda_\nu}\left(\tfrac{1}{2}\partial/\partial x\right)(\partial/\partial x + 2b)\{\theta_\nu^2\} \\
&\qquad = \tfrac{1}{2}\partial/\partial x(\partial/\partial x + 2b)K(t,x,x,P_1).
\end{aligned}$$

We equate terms in the asymptotic expansions

$$\begin{aligned}
&\sum t^{(n-3)/2}\frac{n-1}{2}(e_n(x,P_1) - e_n(x,P_2)) \\
&\qquad\qquad \sim \frac{1}{2}\sum t^{(n-1)/2}\partial/\partial x(\partial/\partial x + 2b)e_n(x,P_1)
\end{aligned}$$

to complete the proof of the lemma.

We apply this lemma to compute the coefficient c_7. If $n = 4$, then we conclude:

$$\begin{aligned}
e_4(x,P_1) &= c_7(b'-b^2)'' + c_8(b'-b^2)^2 = c_7b''' + \text{ lower order terms} \\
e_4(x,P_2) &= c_7(-b'-b^2)'' + c_8(-b'-b^2)^2 = -c_7b''' + \text{ lower order terms} \\
e_2(x,P_1) &= b' - b^2
\end{aligned}$$

so that:

$$\begin{aligned}
3(e_4(x,P_1) - e_4(x,P_2)) &= 6c_7b''' + \text{ lower order terms} \\
\partial/\partial x(\partial/\partial x + 2b)(b'-b^2) &= b''' + \text{ lower order terms}
\end{aligned}$$

from which it follows that $c_7 = 1/6$. It is also convenient at this stage to obtain information about e_6. If we let $e_6 = cE''''+$ lower order terms then we express:

$$\begin{aligned}
e_6(x,P_1) - e_6(x,P_2) &= 2cb^{(5)} \\
\partial/\partial x(\partial/\partial x + 2b)(e_4) &= c_7b^{(5)}
\end{aligned}$$

from which it follows that the constant c is $(4\pi)^{-1/2}\cdot c_7/10 = (4\pi)^{-1/2}/60$. We summarize these results as follows:

LEMMA 4.8.11. *We can expand a_n in the form:*
(a) $a_2(x,P) = (4\pi)^{-m/2}\operatorname{Tr}(c_1 R_{ijij} + E)$.
(b)

$$\begin{aligned} a_4(x,P) = (4\pi)^{-m/2}\operatorname{Tr}(c_3 R_{ijij;kk} + c_4 R_{ijij}R_{klkl} + c_5 R_{ijik}R_{ljlk} \\ + c_6 R_{ijkl}R_{ijkl} + E_{;kk}/6 + E^2/2 \\ + c_1 R_{ijij}E + c_{10}\Omega_{ij}\Omega_{ij}). \end{aligned}$$

(c)

$$\begin{aligned} a_6(x,P) = (4\pi)^{-m/2}\operatorname{Tr}(E_{;kkll}/60 \\ + c_{11}R_{ijij;kkll} + \text{lower order terms}). \end{aligned}$$

PROOF: (a) and (b) follow immediately from the computations previously performed. To prove (c) we argue as in the proof of 4.8.7 to show $a_6(x,P) = (4\pi)^{-m/2}\operatorname{Tr}(cE_{;kk} + c_{11}R_{ijij;kk} +$ lower order terms) and then use the evaluation of c given above.

We can use a similar recursion relation if $m = 2$ to obtain further information regarding these coefficients. We consider the de Rham complex, then:

LEMMA 4.8.12. *If $m = 2$ and Δ_p is the Laplacian on $C^\infty(\Lambda^p(T^*M))$, then:*

$$\frac{(n-2)}{2}\{a_n(x,\Delta_0) - a_n(x,\Delta_1) + a_n(x,\Delta_2)\} = a_{n-2}(x,\Delta_0)_{;kk}.$$

PROOF: This recursion relationship is due to McKean and Singer. Since the invariants are local, we may assume M is orientable and $a_n(x,\Delta_0) = a_n(x,\Delta_2)$. Let $\{\lambda_\nu, \theta_\nu\}$ be a spectral resolution for the non-zero spectrum of Δ_0 then $\{\lambda_\nu, *\theta_\nu\}$ is a spectral resolution for the non-zero spectrum of Δ_2 and $\{\lambda_\nu, d\theta_\nu/\sqrt{\lambda_\nu}, \delta * \theta_\nu/\sqrt{\lambda_\nu}\}$ is a spectral resolution for the non-zero spectrum of Δ_1. Therefore:

$$\begin{aligned} &\frac{d}{dt}\big(K(t,x,x,\Delta_0) - K(t,x,x,\Delta_1) + K(t,x,x,\Delta_2)\big) \\ &\qquad = \sum e^{-t\lambda_\nu}(-2\lambda_\nu\theta_\nu\theta_\nu + d\theta_\nu\cdot d\theta_\nu + \delta*\theta_\nu\cdot\delta*\theta_\nu) \\ &\qquad = \sum e^{-t\lambda_\nu}(-2\Delta_0\theta_\nu\cdot\theta_\nu + 2d\theta_\nu\cdot d\theta_\nu) = K(t,x,x,\Delta_0)_{;kk} \end{aligned}$$

from which the desired identity follows.

Before using this identity, we must obtain some additional information about Δ_1.

Lemma 4.8.13. *Let m be arbitrary and let $\rho_{ij} = -R_{ijik}$ be the Ricci tensor. Then $\Delta\theta = -\theta_{;kk} + \rho(\theta)$ for $\theta \in C^\infty(\Lambda^1)$ and where $\rho(\theta)_i = \rho_{ij}\theta_j$. Thus $E(\Delta_2) = -\rho$.*

Proof: We apply Lemma 4.8.3 to conclude

$$E(\theta) = \frac{1}{2} R_{abij}\theta_i e_b * e_a * e_j$$

from which the desired result follows using the Bianchi identities.

We can now check at least some of these formulas. If $m = 2$, then $E(\Delta_1) = R_{1212} I$ and $E(\Delta_0) = E(\Delta_2) = 0$. Therefore:

$$\begin{aligned} a_2(x,\Delta_0) - a_2(x,\Delta_1) + a_2(x,\Delta_2) &= (4\pi)^{-1}\{(1-2+1)c_1 R_{ijij} - 2R_{1212}\} \\ &= -R_{1212}/2\pi \end{aligned}$$

which is, in fact, the integrand of the Gauss-Bonnet theorem. Next, we compute, supressing the factor of $(4\pi)^{-1}$:

$$\begin{aligned} &a_4(x,\Delta_0) - a_4(x,\Delta_1) + a_4(x,\Delta_2) \\ &\qquad = (c_3 R_{ijij;kk} + c_4 R_{ijij}R_{klkl} + c_5 R_{ijik}R_{ljlk} \\ &\qquad\qquad + c_6 R_{ijkl}R_{ijkl})(1-2+1) \\ &\qquad\qquad + \frac{1}{6}(-2)(R_{1212;kk}) - \frac{2}{2}R_{1212}R_{1212} \\ &\qquad\qquad - 4c_1 R_{1212}R_{1212} - c_{10}\operatorname{Tr}(\Omega_{ij}\Omega_{ij}) \\ &\qquad = -\frac{1}{3}(R_{1212;kk}) - (1 + 4c_1 - 4c_{10})(R_{1212})^2 \\ &a_2(x,\Delta_0)_{;kk} = 2c_1 R_{1212;kk} \end{aligned}$$

so that Lemma 4.2.11 applied to the case $n = 4$ implies:

$$a_4(x,\Delta_0) - a_4(x,\Delta_1) + a_4(x,\Delta_2) = a_2(x,\Delta_0)_{;kk}$$

from which we derive the identities:

$$c_1 = -\frac{1}{6} \qquad \text{and} \qquad 1 + 4c_1 - 4c_{10} = 0$$

from which it follows that $c_{10} = 1/12$. We also consider a_6 and Lemma 4.8.11(c)

$$\begin{aligned} a_6(x,\Delta_0) - a_6(x,\Delta_1) + a_6(x,\Delta_2) &= c_{11} R_{ijij;kkll}(1-2+1) - \frac{2}{60} R_{1212;kkll} \\ &\qquad + \text{ lower order terms} \\ a_4(x,\Delta_0)_{;kk} &= 2c_3 R_{1212;jjkk} + \text{ lower order terms} \end{aligned}$$

so Lemma 4.2.12 implies that $2(-2/60) = 2c_3$ so that $c_3 = -1/30$.

This leaves only the constants c_4, c_5, and c_6 undetermined. We let $M = M_1 \times M_2$ be the product manifold with $\Delta_0 = \Delta_0^1 + \Delta_0^2$. Then the product formulas of Lemma 1.7.5 imply:

$$a_4(\Delta_0, x) = a_4(\Delta_0^1, x_1) + a_4(\Delta_0^2, x_2) + a_2(\Delta_0^1, x_1)a_2(\Delta_0^2, x_2).$$

The only term in the expression for a_4 giving rise to cross terms involving derivatives of both metrics is $c_4 R_{ijij} R_{klkl}$. Consequently, $2c_4 = c_1^2 = 1/36$ so $c_4 = 1/72$. We summarize these computations as follows:

LEMMA 4.8.14. *We can expand α_n in the form:*
(a) $a_2(x, P) = (4\pi)^{-m/2} \operatorname{Tr}(-R_{kjkj} + 6E)/6$.
(b)

$$\begin{aligned} a_4(x, P) = \frac{(4\pi)^{-m/2}}{360} & \operatorname{Tr}(-12R_{ijij;kk} + 5R_{ijij}R_{klkl} + c_5 R_{ijik}R_{ljlk} \\ & + c_6 R_{ijkl}R_{ijkl} - 60ER_{ijij} + 180E^2 + 60E_{;kk} + 30\Omega_{ij}\Omega_{ij}). \end{aligned}$$

We have changed the notation slightly to introduce the common denominator 360.

We must compute the universal constants c_5 and c_6 to complete our determination of the formula a_4. We generalize the recursion relations of Lemmas 4.8.10 and 4.8.12 to arbitrary dimensions as follows. Let $M = T_m$ be the m-dimensional torus with usual periodic parameters $0 \le x_i \le 2$ for $i = 1, \dots, m$. We let $\{e_i\}$ be a collection of Clifford matrices so $e_i e_j + e_j e_i = 2\delta_{ij}$. Let $h(x)$ be a real-valued function on T_m and let the metric be given by:

$$ds^2 = e^{-h}(dx_1^2 + \cdots + dx_m^2), \qquad \mathrm{dvol} = e^{-hm/2} dx_1 \dots dx_m.$$

We let the operator A and A^* be defined by:

$$\begin{aligned} A &= e^{mh/4} \sum e_j \frac{\partial}{\partial x_j} e^{(2-m)h/4} \\ A^* &= e^{(2+m)h/4} \sum e_j \frac{\partial}{\partial x_j} e^{-mh/4} \end{aligned}$$

and define:

$$\begin{aligned} P_1 = A^*A &= -e^{(2+m)h/4} \sum_j \frac{\partial^2}{\partial x_j^2} e^{(2-m)h/4} \\ &= -e^h \Big\{ \sum \frac{\partial^2}{\partial x_j^2} + \frac{1}{2}(2-m)h_{/j} \frac{\partial}{\partial x_j} \\ &\qquad + \frac{1}{16}(4(2-m)h_{/jj} + (2-m)^2 h_{/j}h_{/j}) \Big\} \\ P_2 = AA^* &= e^{mh/4} \sum_j e_j \frac{\partial}{\partial x_j} e^h \sum_k e_k \frac{\partial}{\partial x_k} e^{-mh/4}. \end{aligned}$$

LEMMA 4.8.15. *With the notation given above,*

$$(n-m)\{a_n(x,P_1)-a_n(x,P_2)\} = e^{hm/2}\frac{\partial^2}{\partial x_k^2}e^{h(2-m)/2}\,a_{n-2}(x,P).$$

PROOF: We let P_0 be the scalar operator

$$-e^{(2+m)h/4}\sum_j \frac{\partial^2}{\partial x_j^2}e^{(2-m)h/4}.$$

If the representation space on which the e_j act has dimension u, then $P_1 = P_0 \otimes I_u$. We let v_s be a basis for this representation space. Let $\{\lambda_\nu,\theta_\nu\}$ be a spectral resolution for the operator P_0 then $\{\lambda_\nu,\theta_\nu\otimes v_s\}$ is a spectral resolution of P_1. We compute:

$$\begin{aligned}
&\frac{d}{dt}\big(\operatorname{Tr}K(t,x,x,P_1)-\operatorname{Tr}(t,x,x,P_2)\big)\\
&\qquad=\sum_{\nu,s}e^{-t\lambda_\nu}\{-(P_1\theta_\nu\otimes u_s,\theta_\nu\otimes u_s)+(A\theta_\nu\otimes u_s,A\theta_\nu\otimes u_s)\}
\end{aligned}$$

where (,) denotes the natural inner product $(u_s,u_{s'})=\delta_{s,s'}$. The e_j are self-adjoint matrices. We use the identity $(e_ju_s,e_ku_s)=\delta_{jk}$ to compute:

$$\begin{aligned}
&=\sum_\nu v\cdot e^{-t\lambda_\nu}e^h\bigg\{\theta_{\nu/kk}\theta_\nu+\frac12(2-m)h_{/k}\theta_{\nu/k}\theta_\nu+\frac14(2-m)h_{/kk}\theta_\nu\theta_\nu\\
&\qquad+\frac1{16}(2-m)^2h_{/k}h_{/k}\theta_\nu\theta_\nu+\theta_{\nu/k}\theta_{\nu/k}+\frac12(2-m)h_{/k}\theta_{\nu/k}\theta_\nu\\
&\qquad\quad+\frac1{16}(2-m)^2h_{/k}h_{/k}\theta_\nu\theta_\nu\bigg\}\\
&=\sum_\nu v\cdot e^{-t\lambda_\nu}e^{hm/2}\frac12\sum_j\frac{\partial^2}{\partial x_j^2}e^{h(2-m)/2}\,(\theta_\nu\theta_\nu)\\
&=\frac12e^{hm/2}\sum_j\frac{\partial^2}{\partial x_j^2}e^{h(2-m)/2}\operatorname{Tr}K(t,x,x,P_1).
\end{aligned}$$

We compare coefficients of t in the two asymptotic expansions to complete the proof of the lemma.

We apply Lemma 4.8.15 to the special case $n=m=6$. This implies that

$$\sum_j\frac{\partial^2}{\partial x_j^2}e^{-2h}a_4(x,P_1)=0.$$

Since $a_x(x,P_1)$ is a formal polynomial in the jets of h with coefficients which are smooth functions of h, and since a_4 is homogeneous of order 4, this identity for all h implies $a_4(x,P_1)=0$. This implies $a_4(x,P_0)=\frac{1}{v}a_4(x,P_1)=0$.

Using the formulas of lemmas 4.8.1 and 4.8.2 it is an easy exercise to calculate $R_{ijkl}=0$ if all 4 indices are distinct. The non-zero curvatures are given by:

$$\begin{aligned}
R_{iji}{}^k &= -\tfrac{1}{4}(2h_{/jk}+h_{/j}h_{/k}) \text{ if } j\neq k && \text{(don't sum over } i)\\
R_{iji}{}^j &= -\tfrac{1}{4}(2h_{/ii}+2h_{/jj}-\sum_{k\neq i,k\neq j} h_{/k}h_{/k} && \text{(don't sum over } i,j)\\
\Omega_{ij} &= 0\\
E(P_0) &= e^h\sum_k(-h_{/kk}+h_{/k}h_{/k}).
\end{aligned}$$

When we contract the curvature tensor to form scalar invariants, we must include the metric tensor since it is not diagonal. This implies:

$$\tau=\sum_{i,j}g^{ii}R_{iji}{}^j=e^h\sum_k(-5h_{/kk}+5h_{/k}h_{/k})=5E(P_1)$$

which implies the helpful identities:

$$-12\tau_{;kk}+60E_{;kk}=0 \qquad\text{and}\qquad 5R_{ijij}R_{klkl}-60R_{ijij}E+180E^2=5E^2$$

so that we conclude:

$$5E^2+c_5R_{ijik}R_{ljlk}+c_6R_{ijkl}R_{ijkl}=0.$$

We expand:

$$\begin{aligned}
R_{ijik}R_{ljlk} &= e^{2h}\left(\tfrac{15}{2}h_{/11}h_{/11}+8h_{/12}h_{/12}+\text{ other terms}\right)\\
R_{ijkl}R_{ijkl} &= e^{2h}\left(5h_{/11}h_{/11}+8h_{/12}h_{/12}+\text{ other terms}\right)\\
E^2 &= e^{2h}\left(h_{/11}h_{/11}+0\cdot h_{/12}h_{/12}+\text{ other terms}\right)
\end{aligned}$$

so we conclude finally:

$$15c_5+10c_6+10=0 \qquad\text{and}\qquad c_5+c_6=0.$$

We solve these equations to conclude $c_5=-2$ and $c_6=2$ which proves finally:

THEOREM 4.8.16. *Let P be a second order differential operator with leading symbol given by the metric tensor. Let $P=P_\nabla-E$ be decomposed as in 4.8.1. Let $a_n(x,P)$ be the invariants of the heat equation discussed in Chapter 1.*

(a) $a_0(x,P) = (4\pi)^{-m/2}\operatorname{Tr}(I)$.
(b) $a_2(x,P) = (4\pi)^{-m/2}\operatorname{Tr}(-R_{ijij}+6E)/6$.
(c)

$$\begin{aligned}a_4(x,P) = \frac{(4\pi)^{-m/2}}{360}\times&\\ \operatorname{Tr}(-12R_{ijij;kk} &+ 5R_{ijij}R_{klkl} - 2R_{ijik}R_{ljlk} + 2R_{ijkl}R_{ijkl}\\ &- 60R_{ijij}E + 180E^2 + 60E_{;kk} + 30\Omega_{ij}\Omega_{ij}).\end{aligned}$$

(d)

$$\begin{aligned}a_6(x,P) &= (4\pi)^{-m/2}\times\\ \operatorname{Tr}\Big\{&\frac{1}{7!}\big(-18R_{ijij;kkll} + 17R_{ijij;k}R_{ulul;k} - 2R_{ijik;l}R_{ujuk;l}\\ &- 4R_{ijik;l}R_{ujul;k} + 9R_{ijku;l}R_{ijku;l} + 28R_{ijij}R_{kuku;ll}\\ &- 8R_{ijik}R_{ujuk;ll} + 24R_{ijik}R_{ujul;kl} + 12R_{ijkl}R_{ijkl;uu}\big)\\ &+\frac{1}{9\cdot 7!}\big(-35R_{ijij}R_{klkl}R_{pqpq} + 42R_{ijij}R_{klkp}R_{qlqp}\\ &- 42R_{ijij}R_{klpq}R_{klpq} + 208R_{ijik}R_{julu}R_{kplp}\\ &- 192R_{ijik}R_{uplp}R_{jukl} + 48R_{ijik}R_{julp}R_{kulp}\\ &- 44R_{ijku}R_{ijlp}R_{kulp} - 80R_{ijku}R_{ilkp}R_{jlup}\big)\\ &+\frac{1}{360}\big(8\Omega_{ij;k}\Omega_{ij;k} + 2\Omega_{ij;j}\Omega_{ik;k} + 12\Omega_{ij}\Omega_{ij;kk} - 12\Omega_{ij}\Omega_{jk}\Omega_{ki}\\ &- 6R_{ijkl}\Omega_{ij}\Omega_{kl} + 4R_{ijik}\Omega_{jl}\Omega_{kl} - 5R_{ijij}\Omega_{kl}\Omega_{kl}\big)\\ &+\frac{1}{360}\big(6E_{;iijj} + 60EE_{;ii} + 30E_{;i}E_{;i} + 60E^3 + 30E\Omega_{ij}\Omega_{ij}\\ &- 10R_{ijij}E_{;kk} - 4R_{ijik}E_{;jk} - 12R_{ijij;k}E_{;k} - 30R_{ijij}E^2\\ &- 12R_{ijij;kk}E + 5R_{ijij}R_{klkl}E\\ &- 2R_{ijik}R_{ijkl}E + 2R_{ijkl}R_{ijkl}E\big).\Big\}\end{aligned}$$

PROOF: We have derived (a)–(c) explicitly. We refer to (Gilkey, The spectral geometry of a Riemannian manifold) for the proof of (d) as it is quite long and complicated. We remark that our sign convention is that $R_{1212} = -1$ on the sphere of radius 1 in R^3.

We now begin our computation of $a_n(x,\Delta_p^m)$ for $n = 0,2,4$.

LEMMA 4.8.17. *Let $m = 4$, and let Δ_p be the Laplacian on p-forms. Decompose*

$$\begin{aligned} a_2(x,\Delta_p) &= (4\pi)^{-2}\cdot c_0(p)\cdot R_{ijij}/6 \\ a_4(x,\Delta_p) &= (4\pi)^{-2}\{c_1(p)R_{ijij;kk} + c_2(p)R_{ijij}R_{klkl} \\ &\qquad c_3(p)R_{ijik}R_{ljlk} + c_4(p)R_{ijkl}R_{ijkl}\}/360. \end{aligned}$$

Then the c_i are given by the following table:

	c_0	c_1	c_2	c_3	c_4
$p=0$	-1	-12	5	-2	2
$p=1$	2	12	-40	172	-22
$p=2$	6	48	90	-372	132
$p=3$	2	12	-40	172	-22
$p=4$	-1	-12	5	-2	2
$\sum(-1)^p$	0	0	180	-720	180

PROOF: By Poincare duality, $a_n(x,\Delta_p) = a_n(x,\Delta_{4-p})$ so we need only check the first three rows as the corresponding formulas for Δ_3 and Δ_4 will follow. In dimension 4 the formula for the Euler form is $(4\pi)^{-2}(R_{ijij}R_{klkl} - 4R_{ijik}R_{ljlk} + R_{ijkl}R_{ijkl})/2$ so that the last line follows from Theorem 2.4.8.

If $p = 0$, then $E = \Omega = 0$ so the first line follows from Theorem 4.8.15. If $p = 1$, then $E = -\rho_{ij} = R_{ikij}$ is the Ricci tensor by Lemma 4.8.13. Therefore:

$$\begin{aligned} \operatorname{Tr}(-12R_{ijij;kk} + 60E_{;kk}) &= -48R_{ijij;kk} + 60R_{ijij;kk} \\ &= 12R_{ijij;kk} \\ \operatorname{Tr}(5R_{ijij}R_{klkl} - 60R_{ijij}E) &= 20R_{ijij}R_{klkl} - 60R_{ijij}R_{klkl} \\ &= -40R_{ijij}R_{klkl} \\ \operatorname{Tr}(-2R_{ijik}R_{lklk} + 180E^2) &= -8R_{ijik}R_{ljlk} + 180R_{ijik}R_{ljlk} \\ &= 172R_{ijik}R_{ljlk} \\ \operatorname{Tr}(2R_{ijkl}R_{ijkl} + 30\Omega_{ij}\Omega_{ij}) &= 8R_{ijkl}R_{ijkl} - 30R_{ijkl}R_{ijkl} \\ &= -22R_{ijkl}R_{ijkl} \end{aligned}$$

which completes the proof of the second line. Thus the only unknown is $c_k(2)$. This is computed from the alternating sum and completes the proof.

More generally, we let $m > 4$. Let $M = M_4 \times T_{m-4}$ be a product manifold, then this defines a restriction map $r_{m-4}\colon \mathcal{P}_{m,n} \to \mathcal{P}_{4,n}$ which is an isomorphism for $n \le 4$. Using the multiplication properties given in Lemma 1.7.5, it follows:

$$a_n(x, \Delta_p^m) = \sum_{\substack{i+j=n \\ p_1+p_2=p}} a_i(x, \Delta_{p_1}^4) a_j(x, \Delta_{p_2}^{m-4}).$$

On the flat torus, all the invariants vanish for $j > 0$. By Lemma 4.8.4 $a_0(x, \Delta_{p_2}^{m-4}) = (4\pi)^{-m/2}\binom{m-4}{p_2}$ so that:

$$a_n(x, \Delta_u^4) = \sum_{u+v=p} (4\pi)^{(m-4)/2} \binom{m-4}{v} a_n(x, \Delta_u^4) \qquad \text{for } n \le 4$$

where $a_n(x, \Delta_u^4)$ is given by Lemma 4.8.16. If we expand this as a polynomial in m for small values of u we conclude:

$$\begin{aligned} a_2(x, \Delta_1^m) &= \frac{(4\pi)^{-m/2}}{6}(6-m)R_{ijij} \\ a_4(x, \Delta_1^m) &= \frac{(4\pi)^{-m/2}}{360}\{(60-12m)R_{ijij;kk} + (5m-60)R_{ijij}R_{klkl} \\ &\qquad + (180-2m)R_{ijik}R_{ljlk} + (2m-30)R_{ijkl}R_{ijkl}\} \end{aligned}$$

and similarly for $a_2(x, \Delta_2^m)$ and $a_4(x, \Delta_2^m)$. In this form, the formulas also hold true for $m = 2, 3$. We summarize our conclusions in the following theorem:

Theorem 4.8.18. *Let Δ_p^m denote the Laplacian acting on the space of smooth p-forms on an m-dimensional manifold. We let R_{ijkl} denote the curvature tensor with the sign convention that $R_{1212} = -1$ on the sphere of radius 1 in $\mathbf{R}^3$. Then:*
(a) $a_0(x, \Delta_p^m) = (4\pi)^{-m/2}\binom{m}{p}$.
(b) $a_2(x, \Delta_p^m) = \dfrac{(4\pi)^{-m/2}}{6}\left\{\binom{m-2}{p-2} + \binom{m-2}{p} - 4\binom{m-2}{p-1}\right\}(-R_{ijij})$.
(c) Let

$$\begin{aligned} a_4(x, \Delta_p^m) = \frac{(4\pi)^{-m/2}}{360}\big(&c_1(m,p)R_{ijij;kk} + c_2(m,p)R_{ijij}R_{klkl} \\ &+ c_3(m,p)R_{ijik}R_{ljlk} + c_4(m,p)R_{ijkl}R_{ijkl}\big). \end{aligned}$$

Then for $m \ge 4$ the coefficients are:

$$\begin{aligned} c_1(m,p) &= -12\left[\binom{m-4}{p} + \binom{m-4}{p-4}\right] + 12\left[\binom{m-4}{p-1} + \binom{m-4}{p-3}\right] + 48\binom{m-4}{p-2} \\ c_2(m,p) &= 5\left[\binom{m-4}{p} + \binom{m-4}{p-4}\right] - 40\left[\binom{m-4}{p-1} + \binom{m-4}{p-3}\right] + 90\binom{m-4}{p-2} \\ c_3(m,p) &= -2\left[\binom{m-4}{p} + \binom{m-4}{p-4}\right] + 172\left[\binom{m-4}{p-1} + \binom{m-4}{p-3}\right] - 372\binom{m-4}{p-2} \\ c_4(m,p) &= 2\left[\binom{m-4}{p} + \binom{m-4}{p-4}\right] - 22\left[\binom{m-4}{p-1} + \binom{m-4}{p-3}\right] + 132\binom{m-4}{p-2}. \end{aligned}$$

(d)

$$
\begin{aligned}
a_0(x, \Delta_0^m) &= (4\pi)^{-m/2}\\
a_2(x, \Delta_0^m) &= \frac{(4\pi)^{-m/2}}{6}(-R_{ijij})\\
a_4(x, \Delta_0^m) &= \frac{(4\pi)^{-m/2}}{360}(-12R_{ijij;kk} + 5R_{ijij}R_{klkl} - 2R_{ijik}R_{ljlk}\\
&\qquad + 2R_{ijkl}R_{ijkl})
\end{aligned}
$$

(e)

$$
\begin{aligned}
a_0(x, \Delta_1^m) &= (4\pi)^{-m/2}\\
a_2(x, \Delta_1^m) &= \frac{(4\pi)^{-m/2}}{6}(6-m)R_{ijij}\\
a_4(x, \Delta_1^m) &= \frac{(4\pi)^{-m/2}}{360}\{(60-12m)R_{ijij;kk} + (5m-60)R_{ijij}R_{klkl}\\
&\qquad + (180-2m)R_{ijik}R_{ljlk}\\
&\qquad\quad + (2m-30)R_{ijkl}R_{ijkl})\}
\end{aligned}
$$

(f)

$$
\begin{aligned}
a_0(x, \Delta_2^m) &= \frac{(4\pi)^{-m/2}}{2}m(m-1)\\
a_2(x, \Delta_2^m) &= \frac{(4\pi)^{-m/2}}{12}(-m^2+13m-24)R_{ijij}\\
a_4(x, \Delta_2^m) &= \frac{(4\pi)^{-m/2}}{720}\{(-12m^2+108m-144)R_{ijijk;kk}\\
&\qquad + (5m^2-115m+560)R_{ijij}R_{klkl}\\
&\qquad\quad + (-2m^2+358m-2144)R_{ijik}R_{ljlk}\\
&\qquad\qquad + (2m^2-58m+464)R_{ijkl}R_{ijkl}\}
\end{aligned}
$$

These results are, of course, not new. They were first derived by Patodi. We could apply similar calculations to determine a_0, a_2, and a_4 for any operator which is natural in the sense of Epstein and Stredder. In particular, the Dirac operator can be handled in this way.

In principal, we could also use these formulas to compute a_6, but the lower order terms become extremely complicated. It is not too terribly difficult, however, to use these formulas to compute the terms in a_6 which involve the 6 jets of the metric and which are bilinear in the 4 and 2 jets of the metric. This would complete the proof of the result concerning a_6 if $m = 4$ discussed in section 4.7.

4.9. Spectral Geometry.

Let M be a compact Riemannian manifold without boundary and let $\operatorname{spec}(M,\Delta)$ denote the spectrum of the scalar Laplacian where each eigenvalue is repeated according to its multiplicity. Two manifolds M and $\overline{M}$ are said to be *isospectral* if $\operatorname{spec}(M,\Delta) = \operatorname{spec}(\overline{M},\Delta)$. The leading term in the asymptotic expansion of the heat equation is $(4\pi)^{-m/2} \cdot \operatorname{vol}(M) \cdot t^{-m/2}$ so that if M and $\overline{M}$ are isospectral,

$$\dim M = \dim \overline{M} \qquad \text{and} \qquad \operatorname{volume}(M) = \operatorname{volume}(\overline{M})$$

so these two quantitites are spectral invariants. If $P \in \mathcal{P}_{m,n,0}$ is an invariant polynomial, we define $P(M) = \int_M P(G)|\operatorname{dvol}|$. (This depends on the metric in general.) Theorem 4.8.18 then implies that $R_{ijij}(M)$ is a spectral invasriant since this appears with a non-zero coefficient in the asymptotic expansion of the heat equation.

The scalar Laplacian is not the only natural differential operator to study. (We use the word “natural” in the technical sense of Epstein and Stredder in this context.) Two Riemannian manifolds M and $\overline{M}$ are said to be *strongly isospectral* if $\operatorname{spec}(M,P) = \operatorname{spec}(\overline{M},P)$ for all natural operators P. Many of the global geometry properties of the manifold are reflected by their spectral geometry. Patodi, for example, proved:

THEOREM 4.9.1 (PATODI). *Let* $\operatorname{spec}(M,\Delta_p) = \operatorname{spec}(\overline{M},\Delta_p)$ *for* $p = 0,1,2$. *Then:*
(a)

$$\begin{aligned}
&\dim M = \dim \overline{M}, && \operatorname{volume}(M) = \operatorname{volume}(\overline{M}),\\
&R_{ijij}(M) = R_{ijij}(\overline{M}), && R_{ijij}R_{klkl}(M) = R_{ijij}R_{klkl}(\overline{M}),\\
&R_{ijik}R_{ljlk}(M) = R_{ijik}R_{ljlk}(\overline{M}), && R_{ijkl}R_{ijkl}(M) = R_{ijkl}R_{ijkl}(\overline{M}).
\end{aligned}$$

(b) If M has constant scalar curvature c, then so does $\overline{M}$.
(c) If M is Einstein, then so is $\overline{M}$.
(d) If M has constant sectional curvature c, then so does $\overline{M}$.

PROOF: The first three identities of (a) have already been derived. If $m \geq 4$, the remaining 3 integral invariants are independent. We know:

$$\begin{aligned}
a_4(\Delta_p^m) = c_2(m,p)R_{ijij}R_{klkl} + c_3(m,p)R_{ijik}R_{ljlk}(M)\\
+ c_4(m,p)R_{ijkl}R_{ijkl}(M).
\end{aligned}$$

As $p = 0,1,2$ the coefficients form a 3×3 matrix. If we can show the matrix has rank 3, we can solve for the integral invariants in terms of the spectral invariants to prove (a). Let $\nu = m - 4$ and $c = (4\pi)^{-\nu/2}$. then our computations in section 4.8 show

$$\begin{aligned}
c_j(m,0) &= c \cdot c_j(4,0)\\
c_j(m,1) &= c \cdot \nu \cdot c_j(4,0) + c \cdot c_j(4,1)\\
c_j(m,2) &= c \cdot \nu \cdot (\nu - 1)/2 \cdot c_j(4,0) + c \cdot \nu \cdot c_j(4,1) + c \cdot c_j(4,2)
\end{aligned}$$

so that the matrix for $m > 4$ is obtained from the matrix for $m = 4$ by elementary row operations. Thus it suffices to consider the case $m = 4$. By 4.8.17, the matrix there (modulo a non-zero normalizing constant) is:

$$\begin{pmatrix} 5 & -2 & 2 \\ -40 & 172 & -22 \\ 90 & -372 & 132 \end{pmatrix}.$$

The determinant of this matrix is non-zero from which (a) follows. The case $m = 2$ and $m = 3$ can be checked directly using 4.8.18.

To prove (b), we note that M has constant scalar curvature c if and only if $(2c + R_{ijij})^2(M) = 0$ which by (a) is a spectral invariant. (c) and (d) are similar.

From (d) follows immediately the corollary:

COROLLARY 4.9.2. *Let M and $\overline{M}$ be strongly isospectral. If M is isometric to the standard sphere of radius r, then so is $\overline{M}$.*

PROOF: If M is a compact manifold with sectional curvature $1/r$, then the universal cover of M is the sphere of radius r. If $\mathrm{vol}(M)$ and $\mathrm{vol}(S(r))$ agree, then M and the sphere are isometric.

There are a number of results which link the spectral and the global geometry of a manifold. We list two of these results below:

THEOREM 4.9.3. *Let M and $\overline{M}$ be strongly isospectral manifolds. Then:*
(a) If M is a local symmetric space (i.e., $\nabla R = 0$), then so is $\overline{M}$.
(b) If the Ricci tensor of M is parallel (i.e., $\nabla\rho = 0$), then the Ricci tensor of $\overline{M}$ is parallel. In this instance, the eigenvalues of ρ do not depend upon the particular point of the manifold and they are the same for M and $\overline{M}$.

Although we have chosen to work in the real category, there are also isospectral results available in the holomorphic category:

THEOREM 4.9.4. *Let M and $\overline{M}$ be holomorphic manifolds and suppose* $\mathrm{spec}(M, \Delta_{p,q}) = \mathrm{spec}(\overline{M}, \Delta_{p,q})$ *for all (p, q).*
(a) If M is Kaehler, then so is $\overline{M}$.
(b) If M is $\mathbf{C}P_n$, then so is $\overline{M}$.

At this stage, the natural question to ask is whether or not the spectral geometry completely determines M. This question was phrased by Kac in the form: *Can you hear the shape of a drum?* It is clear that if there exists an isometry between two manifolds, then they are strongly isospectral. That the converse need not hold was shown by Milnor, who gave examples of isospectral tori which were not isometric. In 1978 Vigneras gave examples of isospectral manifolds of constant negative curvature which are not isometric. (One doesn't know yet if they are strongly isospectral.) If the dimension is at least 3, then the manifolds have different fundamental groups, so are not homotopic. The fundamental groups in question are all infinite and the calculations involve some fairly deep results in quaternion algebras.

In 1983 Ikeda constructed examples of spherical space forms which were strongly isospectral but not isometric. As de Rham had shown that diffeomorphic spherical space forms are isometric, these examples are not diffeomorphic. Unlike Vigneras' examples, Ikeda's examples involve finite fundamental groups and are rather easily studied. In the remainder of this section, we will present an example in dimension 9 due to Ikeda illustrating this phenomenon. These examples occur much more generally, but this example is particularly easy to study. We refer to (Gilkey, On spherical space forms with meta-cyclic fundamental group which are isospectral but not equivariant cobordant) for more details.

Let G be the group of order 275 generated by two elements A, B subject to the relations:

$$A^{11} = B^{25} = 1 \qquad \text{and} \qquad BAB^{-1} = A^3.$$

We note that $3^5 \equiv 1 \mod 11$. This group is the semi-direct product $\mathbf{Z}_{11} \propto \mathbf{Z}_{25}$. The center of G is generated by B^5 and the subgroup generated by A is a normal subgroup. We have short exact sequences:

$$0 \to \mathbf{Z}_{11} \to G \to \mathbf{Z}_{25} \to 0 \qquad \text{and} \qquad 0 \to \mathbf{Z}_{55} \to G \to \mathbf{Z}_5 \to 0.$$

We can obtain an explicit realization of G as a subgroup of $\mathrm{U}(5)$ as follows. Let $\alpha = e^{2\pi i/11}$ and $\beta = e^{2\pi i/25}$ be primitive roots of unity. Let $\{e_j\}$ be the standard basis for $\mathbf{C}^5$ and let $\sigma \in \mathrm{U}(5)$ be the permutation matrix $\sigma(e_j) = e_{j-1}$ where the index j is regarded as defined mod 5. Define a representation:

$$\pi_k(A) = \mathrm{diag}(\alpha, \alpha^3, \alpha^9, \alpha^5, \alpha^4) \qquad \text{and} \qquad \pi_k(B) = \beta^k \cdot \sigma.$$

It is immediate that $\pi_k(A)^{11} = \pi_k(B)^{25} = 1$ and it is an easy computation that $\pi_k(B)\pi_k(A)\pi_k(B)^{-1} = \pi_k(A)^3$ so this extends to a representation of G for $k = 1, 2, 3, 4$. (If H is the subgroup generated by $\{A, B^5\}$, we let $\rho(A) = \alpha$ and $\rho(B^5) = \beta^{5k}$ be a unitary representation of H. The representation $\pi_k = \rho^G$ is the induced representation.)

In fact these representations are fixed point free:

Lemma 4.9.5. *Let G be the group of order 275 generated by $\{A,B\}$ with the relations $A^{11}=B^{25}=1$ and $BAB^{-1}=A^3$. The center of G is generated by B^5; the subgroup generated by A is normal. Let $\alpha=e^{2\pi i/11}$ and $\beta=e^{2\pi i/25}$ be primitive roots of unity. Define representations of G in $\mathrm{U}(5)$ for $1\le k\le 4$ by: $\pi_k(A)=\operatorname{diag}(\alpha,\alpha^3,\alpha^9,\alpha^5,\alpha^4)$ and $\pi_k(B)=\beta^k\cdot\sigma$ where σ is the permutation matrix defined by $\sigma(e_i)=e_{i-1}$.*
(a) Enumerate the elements of G in the form A^aB^b for $0\le a<11$ and $0\le b<25$. If $(5,b)=1$, then A^aB^b is conjugate to B^b.
(b) The representations π_k are fixed point free.
(c) The eigenvalues of $\pi_k(A^aB^b)$ and $\pi_1(A^aB^{kb})$ are the same so these two matrices are conjugate in $\mathrm{U}(5)$.

Proof: $A^{-j}B^bA^j=A^{j(3^b-1)}B^b$. If $(5,b)=1$, then 3^b-1 is coprime to 11 so we can solve the congruence $j(3^b-1)\equiv a \mod 11$ to prove (a). Suppose $(5,b)=1$ so the eigenvalues of $\pi_k(A^aB^b)$ and $\pi_k(B^b)$ coincide. Let $\{\varepsilon_1,\dots,\varepsilon_5\}$ be the 5^{th} roots of unity; these are the eigenvalues of σ and of σ^b. Thus the eigenvalues of $\pi_k(B^b)$ are $\{\beta^{kb}\varepsilon_1,\dots,\beta^{kb}\varepsilon_5\}$ and are primitive 25^{th} roots of unity. Thus $\det(\pi_k(B^b)-I)\ne 0$ and $\pi_k(B^b)$ has the same eigenvalues as $\pi_1(B^{kb})$. To complete the proof, we conside an element A^aB^{5b}. π_k is diagonal with eigenvalues $\beta^{5kb}\{\alpha^a,\alpha^{3a},\alpha^{9a},\alpha^{5a},\alpha^{4a}\}$. If $(a,11)=1$ and $(b,5)=1$ these are all primitive 55^{th} roots of unity; if $(a,11)=1$ and $(b,5)=5$ these are all primitive 11^{th} roots of unity; if $(a,11)=11$ and $(b,5)=1$ these are all primitive 5^{th} roots of unity. This shows $\pi_k(A^aB^{5b})$ is fixed point free and has the same eigenvalues as $\pi_1(A^aB^{5kb})$ which completes the proof.

We form the manifolds $M_k=S^9/\pi_k(G)$ with fundamental group G. These are all spherical space forms which inherit a natural orientation and metric from S^9 as discussed previously.

Lemma 4.9.6. *Adopt the notation of Lemma 4.9.5. Let $M=S^9/\pi_k(G)$ be spherical space forms. Then M_1, M_2, M_3 and M_4 are all strongly isospectral.*

Proof: Let P be a self-adjoint elliptic differential operator which is natural in the category of oriented Riemannian manifolds. Let P_0 denote this operator on S^9, and let P_k denote the corresponding operator on M_k. (For example, we could take P to be the Laplacian on p-forms or to be the tangential operator of the signature complex). Let $\lambda\in\mathbf{R}$ and let $E_0(\lambda)$ and $E_k(\lambda)$ denote the eigenspaces of P_0 and P_k. We must show $\dim E_k(\lambda)$ is independent of k for $1\le k\le 4$. The unitary group acts on S^9 by orientation preserving isometries. The assumption of naturality lets us extend this to an action we shall denote by $e_0(\lambda)$ on $E_0(\lambda)$. Again, the assumption of naturality implies the eigenspace $E_k(\lambda)$ is just the subspace of $E_0(\lambda)$ invariant under the action of $e_0(\lambda)(\pi_k(G))$. We can calculate the dimension of this invariant subspace by:

$$\begin{aligned}\dim e_k(\lambda) &= \frac{1}{|G|}\sum_{g\in G}\operatorname{Tr}\{e_0(\lambda)(\pi_k(g)\}\\ &= \frac{1}{275}\sum_{a,b}\operatorname{Tr}\{e_0(\lambda)(\pi_k(A^aB^b)\}.\end{aligned}$$

We apply Lemma 4.9.5 to conclude $\pi_k(A^aB^b)$ is conjugate to $\pi_1(A^aB^{kb})$ in U(5) so the two traces are the same and

$$\begin{aligned}\dim e_k(\lambda) &= \frac{1}{275}\sum_{a,b}\operatorname{Tr}\{e_0(\lambda)\pi_1(A^aB^{kb})\}\\ &= \frac{1}{275}\sum_{a,b}\operatorname{Tr}\{e_0(\lambda)\pi_1(A^aB^b)\} = \dim e_1(\lambda)\end{aligned}$$

since we are just reparameterizing the group. This completes the proof.

Let $\rho \in R_0(\mathbf{Z}_5)$. We regard $\rho \in R_0(G)$ by defining $\rho(A^aB^b) = \rho(b)$. This is nothing but the pull-back of ρ using the natural map $0 \to \mathbf{Z}_{55} \to G \to \mathbf{Z}_5 \to 0$.

Lemma 4.9.7. *Let $\rho \in R_0(\mathbf{Z}_5)$ and let G be as in Lemma 4.9.5. Let $\psi: G \to G$ be a group automorphism. Then $\rho\cdot\psi = \rho$ so this representation of G is independent of the marking chosen.*

Proof: The Sylow 11-subgroup is normal and hence unique. Thus $\psi(A) = A^a$ for some $(a, 11) = 1$ as A generates the Sylow 11-subgroup. Let $\psi(B) = A^cB^d$ and compute:

$$\psi(A^3) = A^{3a} = \psi(B)\psi(A)\psi(B)^{-1} = A^cB^dA^aB^{-d}A^{-c} = A^{a\cdot 3^d}.$$

Since $(a, 11) = 1$, $3^d \equiv 1\ (11)$. This implies $d \equiv 1\ (5)$. Therefore $\psi(A^uB^v) = A^*B^{dv}$ so that $\rho\psi(A^uB^v) = \rho(dv) = \rho(v)$ as $\rho \in R_0(\mathbf{Z}_5)$ which completes the proof.

These representations are canonical; they do not depend on the marking of the fundamental group. This defines a virtual locally flat bundle V_ρ over each of the M_k. Let P be the tangential operator of the signature complex; $\operatorname{ind}(\rho, \text{signature}, M_k)$ is an oriented diffemorphism of M_k. In fact more is true. There is a canonical $\mathbf{Z}_5$ bundle over M_k corresponding to the sequence $G \to \mathbf{Z}_5 \to 0$ which by Lemma 4.9.7 is independent of the particular isomorphism of $\pi_1(M_k)$ with G chosen. Lemma 4.6.4(b) shows this is a $\mathbf{Z}_5$-cobordism invariant. We apply Lemma 4.6.3 and calculate for $\rho \in R_0(\mathbf{Z}_5)$ that:

$$\operatorname{ind}(\rho, \text{signature}, M_k) = \frac{1}{275}\cdot{\sum_{a,b}}'\operatorname{Tr}(\rho(b))\cdot\operatorname{defect}(\pi_k(A^aB^b), \text{signature}).$$

$\sum'$ denotes the sum over $0 \le a < 11$, $0 \le b < 25$, $(a, b) \ne (0, 0)$. Since this is an element of $R_0(\mathbf{Z}_5)$, we may suppose $(5, b) = 1$. As $A^a B^b$ is conjugate to B^b by 4.9.4, we can group the 11 equal terms together to see

$$\operatorname{ind}(\rho, \text{signature}, M_k) = \frac{1}{25} \sum_b{}' \operatorname{Tr}(\rho(b)) \cdot \operatorname{defect}(\pi_k(B^b), \text{signature})$$

where we sum over $0 \le b < 25$ and $(5, b) = 1$.

If $(5, b) = 1$, then the eigenvalues of $\pi_k(B^b)$ are $\{\beta^{bk}\varepsilon_1, \dots, \beta^{bk}\varepsilon_5\}$. Thus

$$\begin{aligned} \operatorname{defect}(\pi_k(B^b), \text{signature}) &= \prod_\nu \frac{\beta^{bk}\varepsilon_\nu + 1}{\beta^{bk}\varepsilon_\nu - 1} \\ &= \frac{\beta^{5bk} + 1}{\beta^{5bk} - 1} \end{aligned}$$

since the product ranges over the primitive 5^{th} roots of unity. Let $\gamma = \beta^5 = e^{2\pi i/5}$, then we conclude

$$\begin{aligned} \operatorname{ind}(\rho, \text{signature}, M_k) &= \frac{1}{25} \cdot \sum_b{}' \operatorname{Tr}(\rho(b)) \cdot \frac{\gamma^{kb} + 1}{\gamma^{kb} - 1} \\ &= \frac{1}{5} \cdot \sum_{0<b<5} \operatorname{Tr}(\rho(b)) \cdot \frac{\gamma^{kb} + 1}{\gamma^{kb} - 1}, \end{aligned}$$

if we group equal terms together.

We now calculate for the specific example $\rho = \rho_1 - \rho_0$:

$$\begin{aligned} \operatorname{ind}(\rho_1 - \rho_0, \text{signature}, M_k) &= \frac{1}{5} \sum_{b=1}^{4} (\gamma^b - 1) \cdot \frac{\gamma^{kb} + 1}{\gamma^{kb} - 1} \\ &= \frac{1}{5} \sum_{b=1}^{4} (\gamma^{b\bar{k}} - 1) \cdot \frac{\gamma^b + 1}{\gamma^b - 1} \end{aligned}$$

if we let $k\bar{k} \equiv 1$ (5). We perform the indicated division; $(x^{\bar{k}} - 1)/(x - 1) = x^{\bar{k}-1} + \dots + 1$ so we obtain

$$= \frac{1}{5} \sum_{b=1}^{4} (\gamma^{b\bar{k}-b} + \gamma^{b\bar{k}-2b} + \dots + 1)(\gamma^b + 1).$$

This expression is well defined even if $b = 0$. If we sum over the entire group, we get an integer by the orthogonality relations. The value at 0 is $+2\bar{k}/5$ and therefore $\operatorname{ind}(\rho_1 - \rho_0, \text{signature}, M_k) = -2\bar{k}/5 \bmod \mathbf{Z}$.

We choose the orientation arising from the given orientation on S^9. If we reverse the orientation, we change the sign of the tangential operator of the signature complex which changes the sign of this invariant.

THEOREM 4.9.8. *Let $M_k = S^9/\pi_k(G)$ with the notation of Lemma 4.9.5. These four manifolds are all strongly isospectral for $k = 1, 2, 3, 4$. There is a natural $\mathbf{Z}_5$ bundle over each M_k.* $\operatorname{ind}(\rho_1 - \rho_0, \text{signature}, M_k) = -2\bar{k}/5$ mod $\mathbf{Z}$ *where $k\bar{k} \equiv 1$ (5). As these 5 values are all different, these 4 manifolds are* not *$\mathbf{Z}_5$ oriented equivariant cobordant. Thus in particular there is no orientation preserving diffeomorphism between any two of these manifolds. $M_1 = -M_4$ and $M_2 = -M_3$. There is no diffeomorphism between M_1 and M_2 so these are different topological types.*

PROOF: If we replace β by $\bar{\beta}$ we replace π_k by π_{5-k} up to unitary equivalence. The map $z \mapsto \bar{z}$ reverses the orientation of $\mathbf{C}^5$ and thus $M_k = -M_{5-k}$ as oriented Riemannian manifolds. This change just alters the sign of $\operatorname{ind}(*, \text{signature}, *)$. Thus the given calculation shows $M_1 \neq \pm M_2$. The statement about oriented equivariant cobordism follows from section 4.6. (In particular, these manifolds are not oriented G-cobordant either.) This gives an example of strongly isospectral manifolds which are of different topological types.

Remark: These examples, of course, generalize; we have chosen to work with a particular example in dimension 9 to simplify the calculations involved.