CHAPTER 3

THE INDEX THEOREM

Introduction

In this the third chapter, we complete the proof of the index theorem for the four classical elliptic complexes. We give a proof of the Aityah-Singer theorem in general based on the Chern isomorphism between K-theory and cohomology (which is not proved). Our approach is to use the results of the first chapter to show there exists a suitable formula with the appropriate functorial properties. The results of the second chapter imply it must be a characteristic class. The normalizing constants are then determined using the method of universal examples.

In section 3.1, we define the twisted signature complex and prove the Hirzebruch signature theorem. We shall postpone until section 3.4 the determination of all the normalizing constants if we take coefficients in an auxilary bundle. In section 3.2, we introduce spinors as a means of connecting the de Rham, signature and Dolbeault complexes. In section 3.3, we discuss the obstruction to putting a spin structure on a real vector bundle in terms of Stieffel-Whitney classes. We compute the characteristic classes of spin bundles.

In section 3.4, we discuss the spin complex and the $\hat{A}$ genus. In section 3.5, we use the spin complex together with the spin_c representation to discuss the Dolbeault complex and to prove the Riemann-Roch theorem for almost complex manifolds. In sections 3.6 and 3.7 we give another treatment of the Riemann-Roch theorem based on a direct approach for Kaehler manifolds. For Kaehler manifolds, the integrands arising from the heat equation can be studied directly using an invariant characterization of the Chern forms similar to that obtained for the Euler form. These two subsections may be deleted by a reader not interested in Kaehler geometry.

In section 3.8, we give the preliminaries we shall need to prove the Atiyah-Singer index theorem in general. The only technical tool we will use which we do not prove is the Chern isomorphism between rational cohomology and K-theory. We give a discussion of Bott periodicity using Clifford algebras. In section 3.9, we show that the index can be treated as a formula in rational K-theory. We use constructions based on Clifford algebras to determine the normalizing constants involved. For these two subsections, some familarity with K-theory is helpful, but not essential.

Theorems 3.1.4 and 3.6.10 were also derived by V. K. Patodi using a complicated cancellation argument as a replacement of the invariance theory presented in Chapter 2. A similar although less detailed discussion may also be found in the paper of Atiyah, Bott and Patodi.

3.1. The Hirzebruch Signature Formula.

The signature complex is best described using Clifford algebras as these provide a unified framework in which to discuss (and avoid) many of the $\pm$ signs which arise in dealing with the exterior algebra directly. The reader will note that we are choosing the opposite sign convention for our discussion of Clifford algebras from that adopted in the example of Lemma 2.1.5. This change in sign convention is caused by the $\sqrt{-1}$ present in discussing the symbol of a first order operator.

Let V be a real vector with a positive definite inner product. The Clifford algebra $\mathrm{CLIF}(V)$ is the universal algebra generated by V subject to the relations

$$v * v + (v, v) = 0 \text{ for } v \in V.$$

If the $\{e_i\}$ are an orthonormal basis for V and if $I = \{1 \le i_1 < \cdots < i_p \le \dim(V)\}$ then $e_i * e_j + e_j * e_i = -2\delta_{ij}$ is the Kronecker symbol and

$$e_I = e_{i_1} * \cdots * e_{i_p}$$

is an element of the Clifford algebra. $\mathrm{CLIF}(V)$ inherits a natural inner product from V and the $\{e_I\}$ form an orthonormal basis for V.

If $\Lambda(V)$ denotes the exterior algebra of V and if $\mathrm{END}(\Lambda(V))$ is the algebra of linear endomorphisms of $\Lambda(V)$, there is a natural representation of $\mathrm{CLIF}(V)$ into $\mathrm{END}(\Lambda(V))$ given by Clifford multiplication. Let $\mathrm{ext}\colon V \to \mathrm{END}(\Lambda(V))$ be exterior multiplication on the left and let $\mathrm{int}(v)$ be interior multiplication, the adjoint. For example:

$$\mathrm{ext}(e_1)(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \begin{cases} e_1 \wedge e_{i_1} \wedge \cdots \wedge e_{i_p} & \text{if } i_1 > 1 \\ 0 & \text{if } i_1 = 1 \end{cases}$$

$$\mathrm{int}(e_1)(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \begin{cases} e_{i_2} \wedge \cdots \wedge e_{i_p} & \text{if } i_1 = 1 \\ 0 & \text{if } i_1 > 1. \end{cases}$$

We define

$$c(v) = \mathrm{ext}(v) - \mathrm{int}(v)\colon V \to \mathrm{END}(\Lambda(V)).$$

It is immediate from the definition that

$$c(v)^2 = -(\mathrm{ext}(v)\,\mathrm{int}(v) + \mathrm{int}(v)\,\mathrm{ext}(v)) = -|v|^2 I$$

so that c extends to define an algebra morphism

$$c\colon \mathrm{CLIF}(V) \to \mathrm{END}(\Lambda(V)).$$

Furthermore:

$$c(e_I)1 = e_{i_1} \wedge \cdots \wedge e_{i_p}$$

so the map $w \mapsto c(w)1$ defines a vector space isomorphism (which is not, of course an algebra morphism) between $\mathrm{CLIF}(V)$ and $\Lambda(V)$. Relative to an orthonormal frame, we simply replace Clifford multiplication by exterior multiplication.

Since these constructions are all independent of the basis chosen, they extend to the case in which V is a real vector bundle over M with a fiber metric which is positive definite. We define $\mathrm{CLIF}(V)$, $\Lambda(V)$, and $c\colon \mathrm{CLIF}(V) \to \mathrm{END}(\Lambda(V))$ as above. Since we only want to deal with complex bundles, we tensor with $\mathbf{C}$ at the end to enable us to view these bundles as being complex. We emphasize, however, that the underlying constructions are all real.

Clifford algebras provide a convenient way to describe both the de Rham and the signature complexes. Let $(d+\delta)\colon C^\infty(\Lambda(T^*M)) \to C^\infty(\Lambda(T^*M))$ be exterior differentiation plus its adjoint as discussed earlier. The leading symbol of $(d+\delta)$ is $\sqrt{-1}(\mathrm{ext}(\xi) - \mathrm{int}(\xi)) = \sqrt{-1}c(\xi)$. We use the following diagram to define an operator A; let ∇ be covariant differentiation. Then:

$$A\colon C^\infty(\Lambda(T^*M)) \xrightarrow{\nabla} C^\infty(T^*M \otimes \Lambda(T^*M)) \xrightarrow{c} C^\infty(\Lambda(T^*M)).$$

A is invariantly defined. If $\{e_i\}$ is a local orthonormal frame for $T^*(M)$ which we identify with $T(M)$, then:

$$A(\omega) = \sum_i (\mathrm{ext}(e_i) - \mathrm{int}(e_i))\nabla_{e_i}(\omega).$$

Since the leading symbol of A is $\sqrt{-1}c(\xi)$, these two operators have the same leading symbol so $(d+\delta) - A = A_0$ is an invariantly defined 0^{th} order operator. Relative to a coordinate frame, we can express A_0 as a linear combination of the 1-jets of the metric with coefficients which are smooth in the $\{g_{ij}\}$ variables. Given any point x_0, we can always choose a frame so the $g_{ij/k}$ variables vanish at x_0 so $A_0(x_0) = 0$ so $A_0 \equiv 0$. This proves $A = (d+\delta)$ is defined by this diagram which gives a convenient way of describing the operator $(d+\delta)$ in terms of Clifford multiplication.

This trick will be useful in what follows. If A and B are natural first order differential operators with the same leading symbol, then $A = B$ since $A - B$ is a 0^{th} order operator which is linear in the 1-jets of the metric. This trick does not work in the holomorphic category unless we impose the additional hypothesis that M is Kaehler. This makes the study of the Riemann-Roch theorem more complicated as we shall see later since there are many natural operators with the same leading symbol.

We let $\alpha \in \mathrm{END}(\Lambda(T^*M))$ be defined by:

$$\alpha(\omega_p) = (-1)^p \omega_p \qquad \text{for } \omega_p \in \Lambda^p(T^*M).$$

It is immediate that

$$\operatorname{ext}(\xi)\alpha = -\alpha\operatorname{ext}(\xi) \qquad \text{and} \qquad \operatorname{int}(\xi)\alpha = -\alpha\operatorname{int}(\xi)$$

so that $-\alpha(d+\delta)\alpha$ and $(d+\delta)$ have the same leading symbol. This implies

$$\alpha(d+\delta) = -(d+\delta)\alpha.$$

We decompose $\Lambda(T^*M) = \Lambda^{\mathrm{e}}(T^*M) \oplus \Lambda^{\mathrm{o}}(T^*M)$ into the differential forms of even and odd degree. This decomposes $\Lambda(T^*M)$ into the ± 1 eigenspaces of α. Since $(d+\delta)$ anti-commutes with α, we decompose

$$(d+\delta)_{\mathrm{e,o}} : C^\infty(\Lambda^{\mathrm{e,o}}(T^*M)) \to C^\infty(\Lambda^{\mathrm{e,o}}(T^*M))$$

where the adjoint of $(d+\delta)_{\mathrm{e}}$ is $(d+\delta)_{\mathrm{o}}$. This is, of course, just the de Rham complex, and the index of this elliptic operator is $\chi(M)$.

If $\dim(M) = m$ is even and if M is oriented, there is another natural endomorphism $\tau \in \operatorname{END}(\Lambda(T^*M))$. It can be used to define an elliptic complex over M called the signature complex in just the same way that the de Rham complex was defined. Let $\operatorname{dvol} \in \Lambda^m(T^*M)$ be the volume form. If $\{e_i\}$ is an oriented local orthonormal frame for T^*M, then $\operatorname{dvol} = e_1 \wedge \cdots \wedge e_m$. We can also regard $\operatorname{dvol} = e_1 * \cdots * e_m \in \operatorname{CLIF}(T^*M)$ and we define

$$\tau = (\sqrt{-1})^{m/2} c(\operatorname{dvol}) = (\sqrt{-1})^{m/2} c(e_1)\dots c(e_m).$$

We compute:

$$\begin{aligned}
\tau^2 &= (-1)^{m/2} c(e_1 * \cdots * e_m * e_1 * \cdots * e_m) \\
&= (-1)^{m/2}(-1)^{m+(m-1)+\cdots+1} \\
&= (-1)^{(m+(m+1)m)/2} = 1.
\end{aligned}$$

Because m is even, $c(\xi)\tau = -\tau c(\xi)$ so τ anti-commutes with the symbol of $(d+\delta)$. If we decompose $\Lambda(T^*M) = \Lambda^+(T^*M) \oplus \Lambda^-(T^*M)$ into the ± 1 eigenvalues of τ, then $(d+\delta)$ decomposes to define:

$$(d+\delta)_\pm : C^\infty(\Lambda^\pm(T^*M)) \to C^\infty(\Lambda^\mp(T^*M))$$

where the adjoint of $(d+\delta)_+$ is $(d+\delta)_-$. We define:

$$\operatorname{signature}(M) = \operatorname{index}(d+\delta)_+$$

to be the signature of M. (This is also often refered to as the index of M, but we shall not use this notation as it might be a source of some confusion).

We decompose the Laplacian $\Delta = \Delta^+ \oplus \Delta^-$ so

$$\text{signature}(M) = \dim \mathrm{N}(\Delta^+) - \dim \mathrm{N}(\Delta^-).$$

Let $a_n^s(x, \mathrm{orn}) = a_n(x, \Delta^+) - a_n(x, \Delta^-)$ be the invariants of the heat equation; they depend on the orientation orn chosen. Although we have complexified the bundles $\Lambda(T^*M)$ and $\mathrm{CLIF}(T^*M)$, the operator $(d+\delta)$ is real. If $m \equiv 2$ (4), then τ is pure imaginary. Complex conjugation defines an isomorphism

$$\Lambda^+(T^*M) \overset{\approx}{\to} \Lambda^-(T^*M) \qquad \text{and} \qquad \Delta^+ \overset{\approx}{\to} \Delta^-.$$

This implies $\text{signature}(M) = 0$ and $a_n^s(x, \mathrm{orn}) = 0$ in this case. We can get a non-zero index if $m \equiv 2$ (4) if we take coefficients in some auxiliary bundle as we shall discuss shortly.

If $m \equiv 0$ (4), then τ is a real endomorphism. In general, we compute:

$$\begin{aligned} \tau(e_1 \wedge \cdots \wedge e_p) &= (\sqrt{-1})^{m/2} e_1 * \cdots * e_m * e_1 * \cdots * e_p \\ &= (\sqrt{-1})^{m/2} (-1)^{p(p-1)/2} e_{p+1} \wedge \cdots \wedge e_m. \end{aligned}$$

If "$*$" is the Hodge star operator discussed in the first sections, then

$$\tau_p = (\sqrt{-1})^{m/2 + p(p-1)} *_p$$

acting on p-forms. The spaces $\Lambda^p(T^*M) \oplus \Lambda^{m-p}(T^*M)$ are invariant under τ. If $p \neq m - p$ there is a natural isomorphism

$$\Lambda^p(T^*M) \overset{\approx}{\to} (\Lambda^p(T^*M) \oplus \Lambda^{m-p}(T^*M))^{\pm} \qquad \text{by} \quad \omega_p \mapsto \tfrac{1}{2}(\omega_p \pm \omega_p).$$

This induces a natural isomorphism from

$$\Delta_p \overset{\approx}{\to} \Delta^{\pm} \text{ on } (\Lambda^p(T^*M) \oplus \Lambda^{m-p}(T^*M))^{\pm}$$

so these terms all cancel off in the alternating sum and the only contribution is made in the middle dimension $p = m - p$.

If $m = 4k$ and $p = 2k$ then $\tau = *$. We decompose $\mathrm{N}(\Delta_p) = \mathrm{N}(\Delta_p^+) \oplus \mathrm{N}(\Delta_p^-)$ so $\text{signature}(M) = \dim \mathrm{N}(\Delta_p^+) - \dim \mathrm{N}(\Delta_p^-)$. There is a natural symmetric bilinear form on $H^{2k}(T^*M; \mathbf{C}) = \mathrm{N}(\Delta_p)$ defined by

$$I(\alpha_1, \alpha_2) = \int_M \alpha_1 \wedge \alpha_2.$$

If we use the de Rham isomorphism to identify de Rham and simplicial cohomology, then this bilinear form is just the evaluation of the cup product of two cohomology classes on the top dimensional cycle. This shows I can be defined in purely topological terms.

The index of a real quadratic form is just the number of $+1$ eigenvalues minus the number of -1 eigenvalues when it is diagonalized over $\mathbf{R}$. Since

$$I(\alpha,\beta)=\int_M \alpha\wedge\beta=\int_M(\alpha,*\beta)\,\mathrm{dvol}=(\alpha,*\beta)_{L^2},$$

we see

$$\begin{aligned}\mathrm{index}(I)&=\dim\{+1\text{ eigenspace of }*\text{ on }H^p\}\\&\qquad-\dim\{-1\text{ eigenspace of }*\text{ on }H^p\}\\&=\dim\mathrm{N}(\Delta^+)-\dim\mathrm{N}(\Delta^-)=\mathrm{signature}(M).\end{aligned}$$

This gives a purely topological definition of the signature of M in terms of cup product. We note that if we reverse the orientation of M, then the signature changes sign.

Example: Let $M=\mathbf{C}P_{2k}$ be complex projective space. Let $x\in H^2(M;\mathbf{C})$ be the generator. Since x^k is the generator of $H^{2k}(M;\mathbf{C})$ and since $x^k\wedge x^k=x^{2k}$ is the generator of $H^{4k}(M;\mathbf{C})$, we conclude that $*x^k=x^k$. $\dim\mathrm{N}(\Delta^+_{2k})=1$ and $\dim\mathrm{N}(\Delta^-_{2k})=0$ so $\mathrm{signature}(\mathbf{C}P_{2k})=1$.

An important tool in the study of the de Rham complex was its multiplicative properties under products. Let M_i be oriented even dimensional manifolds and let $M=M_1\times M_2$ with the induced orientation. Decompose:

$$\begin{aligned}\Lambda(T^*M)&=\Lambda(T^*M_1)\otimes\Lambda(T^*M_2)\\ \mathrm{CLIF}(T^*M)&=\mathrm{CLIF}(T^*M_1)\otimes\mathrm{CLIF}(T^*M_2)\end{aligned}$$

as graded non-commutative algebras—i.e.,

$$(\omega_1\otimes\omega_2)\circ(\omega_1'\otimes\omega_2')=(-1)^{\deg\omega_2\cdot\deg\omega_1'}(\omega_1\circ\omega_1')\otimes(\omega_2\circ\omega_2')$$

for $\circ=$ either $\wedge$ or $*$. Relative to this decomposition, we have:

$$\tau=\tau_1\otimes\tau_2\text{ where the }\tau_i\text{ commute.}$$

This implies that:

$$\begin{aligned}\Lambda^+(M)&=\Lambda^+(T^*M_1)\otimes\Lambda^+(T^*M_2)\ \oplus\ \Lambda^-(T^*M_1)\otimes\Lambda^-(T^*M_2)\\ \Lambda^-(M)&=\Lambda^-(T^*M_1)\otimes\Lambda^+(T^*M_2)\ \oplus\ \Lambda^+(T^*M_1)\otimes\Lambda^-(T^*M_2)\\ \mathrm{N}(\Delta^+)&=\mathrm{N}(\Delta_1^+)\otimes\mathrm{N}(\Delta_2^+)\ \oplus\ \mathrm{N}(\Delta_1^-)\otimes\mathrm{N}(\Delta_2^-)\\ \mathrm{N}(\Delta^-)&=\mathrm{N}(\Delta_1^-)\otimes\mathrm{N}(\Delta_2^+)\ \oplus\ \mathrm{N}(\Delta_1^+)\otimes\mathrm{N}(\Delta_2^-)\\ \mathrm{signature}(M)&=\mathrm{signature}(M_1)\,\mathrm{signature}(M_2).\end{aligned}$$

Example: Let ρ be a partition of $k = i_1 + \dots + i_j$ and $M_\rho^r = \mathbf{C}P_{2i_1} \times \dots \times \mathbf{C}P_{2i_j}$. Then signature$(M_\rho^r) = 1$. Therefore if L_k is the Hirzebruch L-polynomial,

$$\text{signature}(M_\rho^r) = \int_{M_\rho^p} L_k(T(M_\rho^r))$$

by Lemma 2.3.5.

We can now begin the proof of the Hirzebruch signature theorem. We shall use the same argument as we used to prove the Gauss-Bonnet theorem with suitable modifications. Let $m = 4k$ and let $a_n^s(x, \text{orn}) = a_n(x, \Delta^+) - a_n(x, \Delta^-)$ be the invariants of the heat equation. By Lemma 1.7.6:

$$\int_M a_n^s(x, \text{orn}) = \begin{cases} 0 & \text{if } n \neq m \\ \text{signature}(M) & \text{if } n = m \end{cases}$$

so this gives a local formula for signature(M). We can express τ functorially in terms of the metric tensor. We can find functorial local frames for $\Lambda^\pm$ relative to any oriented coordinate system in terms of the coordinate frames for $\Lambda(T^*M)$. Relative to such a frame, we express the symbol of $\Delta^\pm$ functorially in terms of the metric. The leading symbol is $|\xi|^2 I$; the first order symbol is linear in the 1-jets of the metric with coefficients which depend smoothly on the $\{g_{ij}\}$ variables; the 0^{th} order symbol is linear in the 2-jets of the metric and quadratic in the 1-jets of the metric with coefficients which depend smoothly on the $\{g_{ij}\}$ variables. By Lemma 2.4.2, we conclude $a_n^s(x, \text{orn})$ is homogeneous of order n in the jets of the metric.

It is worth noting that if we replace the metric G by $c^2 G$ for $c > 0$, then the spaces $\Lambda^\pm$ are not invariant. On Λ^p we have:

$$\tau(c^2 G)(\omega_p) = c^{2p-m} \tau(G)(\omega_p).$$

However, in the middle dimension we have τ is invariant as are the spaces $\Lambda_p^\pm$ for $2p = m$. Clearly $\Delta_p^\pm(c^2 G) = c^{-2} \Delta_p^\pm(G)$. Since $a_n^s(x, \text{orn})$ only depends on the middle dimension, this provides another proof that a_n^s is homogeneous of order n in the derivatives of the metric since

$$\begin{aligned} a_n^s(x, \text{orn})(c^2 G) &= a_n(x, c^{-2}\Delta_p^+) - a_n(x, c^{-2}\Delta_p^-) \\ &= c^{-n} a_n(x, \Delta_p^+) - c^{-n} a_n(x, \Delta_p^-) = c^{-n} a_n^s(x, \text{orn})(G). \end{aligned}$$

If we reverse the orientation, we interchange the roles of Δ^+ and Δ^- so a_n^s changes sign if we reverse the orientation. This implies a_n^s can be regarded as an invariantly defined m-form; $a_n^s(x) = a_n^s(x, \text{orn})\,\text{dvol} \in \mathcal{P}_{m,n,m}$.

Theorem 3.1.1. *Let* $a_n^s = \{a_n(x,\Delta^+) - a_n(x,\Delta^-)\}\,\mathrm{dvol} \in \mathcal{P}_{m,n,m}$ *then:*
(a) $a_n^s = 0$ *if either* $m \equiv 2\ (4)$ *or if* $n < m$.
(b) If $m = 4k$, *then* $a_{4k}^s = L_k$ *is the Hirzebruch polynomial so*

$$\mathrm{signature}(M) = \int_M L_k.$$

(Hirzebruch Signature Theorem).

Proof: We already noted that Δ^+ is naturally isomorphic to Δ^- if $m \equiv 2\ (4)$ so $a_n^s = 0$ in that case. Lemma 2.5.6 implies $a_n^s = 0$ for $n < m$. If $m = 4k$, then a_m^s is a characteristic form of $T(M)$ by Lemma 2.5.6. We know

$$\int_{M_\rho^r} a_m^s = \mathrm{signature}(M_\rho^r) = 1 = \int_{M_\rho^r} L_k$$

so Lemma 2.3.4 implies $a_m^s = L_k$. Since $\mathrm{signature}(M) = \int_M a_m^s$ for any manifold M, we conclude $\mathrm{signature}(M) = \int_M L_k$ in general which completes the proof of (b).

If $\omega \in \Lambda(T^*M)$, we define $\int_M \omega = \int_M \omega_m$ of the top degree form. With this notational convention, we can also express

$$\mathrm{signature}(M) = \int_M L$$

which is a common form in which the Hirzebruch signature theorem appears.

It is worth making a few remarks about the proof of this result. Just as in the case of the de Rham complex, the heat equation furnishes us with the a priori local formula for the signature of M. The invariance theory of the second chapter identifies this local formula as a characteristic class. We evaluate this local formula on a sufficient number of classifying examples to determine the normalizing constants to prove $a_m^s = L_k$.

There are a great many consequences of this theorem and of the Gauss-Bonnet theorem. We present just a few to illustrate some of the applications:

Corrolary 3.1.2.
(a) Let $F \to M_1 \to M_2$ *be a finite covering projection. Then* $\chi(M_1) = \chi(M_2)|F|$. *If* M_2 *is orientable, then* M_1 *is orientable and we give it the natural orientation inherited from* M_2. *Then* $\mathrm{signature}(M_1) = \mathrm{signature}(M_2)|F|$.
(b) If M_1 *and* M_2 *are manifolds of dimension* m, *we let* $M_1 \# M_2$ *be the connected sum. This is defined by punching out disks in both manifolds and gluing along the common resulting boundaries. Then* $\chi(M_1 \# M_2) + \chi(S^m) = \chi(M_1) + \chi(M_2)$. *If* M_1 *and* M_2 *are oriented by some orientation*

on $M_1 \# M_2$ *then* $\operatorname{signature}(M_1 \# M_2) = \operatorname{signature}(M_1) + \operatorname{signature}(M_2)$, *if* $m \equiv 0\ (4)$.

PROOF: (a) is an immediate consequence of the fact we have local formulas for the Euler characteristic and for the signature. To prove (b), we note that the two disks we are punching out glue together to form a sphere. We use the additivity of local formulas to prove the assertion about X. The second assertion follows similarly if we note $\operatorname{signature}(S^m) = 0$.

This corollary has topological consequences. Again, we present just one to illustrate the methods involved:

COROLLARY 3.1.3. *Let* $F \to \mathbf{C}P_{2j} \to M$ *be a finite covering. Then* $|F| = 1$ *and* $M = \mathbf{C}P_{2j}$.

PROOF: $\chi(\mathbf{C}P_{2j}) = 2j+1$ so as $2j+1 = |F|\chi(M)$, we conclude $|F|$ must be odd. Therefore, this covering projection is orientation preserving so M is orientable. The identity $1 = \operatorname{signature}(\mathbf{C}P_{2j}) = |F| \operatorname{signature}(M)$ implies $|F| = 1$ and completes the proof.

If $m \equiv 2\ (4)$, the signature complex does not give a non-trivial index. We twist by taking coefficients in an auxiliary complex vector bundle V to get a non-trivial index problem in any even dimension m.

Let V be a smooth complex vector bundle of dimension v equipped with a Riemannian connection ∇. We take the Levi-Civita connection on $T^*(M)$ and on $\Lambda(T^*M)$ and let ∇ be the tensor product connection on $\Lambda(T^*M) \otimes V$. We define the operator $(d+\delta)_V$ on $C^\infty(\Lambda(T^*M) \otimes V)$ using the diagram:

$$\begin{aligned}(d+\delta)_V\colon C^\infty(\Lambda(T^*M)\otimes V) &\xrightarrow{\nabla} C(T^*M \otimes \Lambda(T^*M) \otimes V)\\ &\xrightarrow{c\otimes 1} C^\infty(\Lambda(T^*M)\otimes V).\end{aligned}$$

We have already noted that if $V = 1$ is the trivial bundle with flat connection, then the resulting operator is $(d+\delta)$.

We define $\tau_V = \tau \otimes 1$, then a similar argument to that given for the signature complex shows $\tau_V^2 = 1$ and τ_V anti-commutes with $(d+\delta)_V$. The ± 1 eigenspaces of τ_V are $\Lambda^\pm(T^*M) \otimes V$ and the twisted signature complex is defined by the diagram:

$$(d+\delta)_V^\pm\colon C^\infty(\Lambda^\pm(T^*M)\otimes V) \to C^\infty(\Lambda^\mp(T^*M)\otimes V)$$

where as before $(d+\delta)_V^-$ is the adjoint of $(d+\delta)_V^+$. We let $\Delta_V^\pm$ be the associated Laplacians and define:

$$\begin{aligned}\operatorname{signature}(M,V) &= \operatorname{index}((d+\delta)_V^+) = \dim \mathrm{N}(\Delta_V^+) - \dim \mathrm{N}(\Delta_V^-)\\ a_n^s(x,V) &= \{a_n(x,\Delta_V^+) - a_n(x,\Delta_V^-)\}\,\mathrm{dvol} \in \Lambda^m\\ \int_M a_n^s(x,V) &= \operatorname{signature}(M,V).\end{aligned}$$

The invariance of the index under homotopies shows signature(M, V) is independent of the metric on M, of the fiber metric on V, and of the Riemannian connection on V. If we do *not* choose a Riemannian connection on V, we can still compute signature$(M, V) = \operatorname{index}((d+\delta)_V^+)$, but then $(d+\delta)_V^-$ is not the adjoint of $(d+\delta)_V^+$ and a_n^s is not an invariantly defined m form.

Relative to a functorial coordinate frame for $\Lambda^\pm$, the leading symbol of $\Delta^\pm$ is $|\xi|^2 I$. The first order symbol is linear in the 1-jets of the metric and the connection form on V. The 0^{th} order symbol is linear in the 2-jets of the metric and the connection form and quadratic in the 1-jets of the metric and the connection form. Thus $a_n^s(X, V) \in \mathcal{R}_{m,n,m,v}$. Theorem 2.6.1 implies $a_n^s = 0$ for $n < m$ while a_m^s is a characteristic form of $T(M)$ and of V.

If $m = 2$ and if $v = 1$, then $\mathcal{R}_{2,2,2,1}$ is one dimensional and is spanned by the first Chern class $c_1(V) = ch(V) = \frac{i}{2\pi}\Omega$. Consequently $a_2^s = cc_1$ in this case. We shall show later that this normalizing constant $c = 2$—i.e.,

Lemma 3.1.4. *Let $m = 2$ and let V be a line bundle over M_2. Then:*

$$\operatorname{signature}(M, V) = 2\int_M c_1(L).$$

We postpone the proof of this lemma until later in this chapter.

With this normalizing constant established, we can compute a formula for signature(M, V) in general:

Theorem 3.1.5. *Let L be the total L-polynomial and let $ch(V)$ be the Chern character. Then:*
(a) $a_n^s(x, V) = 0$ for $n < m$.
(b) $a_m^s(x, V) = \sum_{4s+2t=m} L_s(TM) \wedge 2^t ch_t(V)$ so that:

$$\operatorname{signature}(M, V) = \sum_{4s+2t=m} \int_M L_s(TM) \wedge 2^t ch_t(V).$$

The factors of 2^t are perhaps a bit mysterious at this point. They arise from the normalizing constant of Lemma 3.1.4 and will be explained when we discuss the spin and Dolbeault complexes.

Proof: We have already proved (a). We know $a_m^s(x, V)$ is a characteristic form which integrates to signature(M, V) so it suffices to verify the formula of (b). If V_1 and V_2 are bundles, we let $V = V_1 \oplus V_2$ with the direct sum connection. Since $\Delta_V^\pm = \Delta_{V_1}^\pm \oplus \Delta_{V_2}^\pm$ we conclude

$$\operatorname{signature}(M, V_1 \oplus V_2) = \operatorname{signature}(M, V_1) + \operatorname{signature}(M, V_2).$$

Since the integrals are additive, we apply the uniqueness of Lemma 2.6.3 to conclude the local formulas must be additive. This also follows from Lemma 1.7.5 so that:

$$a_n^s(x, V_1 \oplus V_2) = a_n^s(x, V_1) + a_n^s(x, V_2).$$

Let $\{P_\rho\}_{|\rho|=s}$ be the basis for $\mathcal{P}_{m,4s,4s}$ and expand:

$$a_m^s(x, V) = \sum_{4|\rho|+2t=m} P_\rho \wedge Q_{m,t,v,\rho} \qquad \text{for} \quad Q_{m,t,v,\rho} \in \mathcal{Q}_{m,2t,2t,v}$$

a characteristic form of V. Then the additivity under direct sum implies:

$$Q_{m,t,v,\rho}(V_1 \oplus V_2) = Q_{m,t,v_1,\rho}(V_1) + Q_{m,t,v_2,\rho}(V_2).$$

If $v = 1$, then $Q_{m,t,1,\rho}(V_1) = c \cdot c_1(V)^t$ since $\mathcal{Q}_{m,2t,2t,1}$ is one dimensional. If A is diagonal matrix, then the additivity implies:

$$Q_{m,t,v,\rho}(A) = Q_{m,t,v,\rho}(\lambda) = c \cdot \sum_j \lambda_j^t = c \cdot ch_t(A).$$

Since Q is determined by its values on diagonal matrices, we conclude:

$$Q_{m,t,v,\rho}(V) = c(m,t,\rho) ch_t(V)$$

where the normalizing constant does not depend on the dimension v. Therefore, we expand a_m^s in terms of $ch_t(V)$ to express:

$$a_m^s(x, V) = \sum_{4s+2t=m} P_{m,s} \wedge 2^t ch_t(V) \qquad \text{for } P_{m,s} \in \mathcal{P}_{m,4s,4s}.$$

We complete the proof of the theorem by identifying $P_{m,s} = L_s$; we have reduced the proof of the theorem to the case $v = 1$.

We proceed by induction on m; Lemma 3.1.4 establishes this theorem if $m = 2$. Suppose $m \equiv 0$ (4). If we take V to be the trivial bundle, then if $4k = m$,

$$a_m^s(x, 1) = L_k = P_{m,k}$$

follows from Theorem 3.1.1. We may therefore assume $4s < m$ in computing $P_{m,s}$. Let $M = M_1 \times S^2$ and let $V = V_1 \otimes V_2$ where V_1 is a line bundle over M_1 and where V_2 is a line bundle over S^2 so $\int_{S^2} c_1(V_2) = 1$. (We constructed such a line bundle in section 2.1 using Clifford matrices). We take the product connection on $V_1 \otimes V_2$ and decompose:

$$\Lambda^+(V) = \Lambda^+(V_1) \otimes \Lambda^+(V_2) \;\oplus\; \Lambda^-(V_1) \otimes \Lambda^-(V_2)$$
$$\Lambda^-(V) = \Lambda^-(V_1) \otimes \Lambda^+(V_2) \;\oplus\; \Lambda^+(V_1) \otimes \Lambda^-(V_2).$$

A similar decomposition of the Laplacians yields:

$$\begin{aligned}\operatorname{signature}(M,V) &= \operatorname{signature}(M_1,V_1)\operatorname{signature}(M_2,V_2)\\ &= 2\operatorname{signature}(M_1,V_1)\end{aligned}$$

by Lemma 3.1.4. Since the signatures are multiplicative, the local formulas are multiplicative by the uniqueness assertion of Lemma 2.6.3. This also follows by using Lemma 1.7.5 that

$$a_m^s(x,V) = \sum_{p+q=m} a_p^s(x_1,V_1)a_q^s(x_2,V_2)$$

and the fact $a_p = 0$ for $p < m_1$ and $a_q = 0$ for $q < m_2$. Thus we conclude:

$$a_m^s(x,V) = a_{m_1}^s(x_1,V_1)a_{m_2}^s(x_2,V_2)$$

where, of course, $m_2 = 2$ and $m_1 = m - 2$.

We use the identity:

$$ch(V_1 \otimes V_2) = ch(V_1)ch(V_2)$$

to conclude therefore:

$$\begin{aligned}&\operatorname{signature}(M_1,V_1)\\ &\qquad = \frac{1}{2}\operatorname{signature}(M,V)\\ &\qquad = \frac{1}{2}\left\{\sum_{4s+2t=m-2}\int_{M_I} P_{m,s}\wedge 2^t ch_t(V_1)\right\}\int_{M_2} 2ch_1(V_2)\\ &\qquad = \sum_{4s+2t=m-2}\int_{M_1} P_{m,s}\wedge 2^t ch_t(V_1).\end{aligned}$$

We apply the uniqueness assertion of Lemma 2.6.3 to conclude $P_{m,s} = P_{m-2,s}$ for $4s \le m-2$. Since by induction, $P_{m-2,s} = L_s$ this completes the proof of the theorem.

We note that the formula is non-zero, so Lemma 2.6.3 implies that in any even dimension m, there always exist (M,V) so $\operatorname{signature}(M,V) \ne 0$. In fact, much more is true. Given any orientable manifold M, we can find V over M so $\operatorname{signature}(M,V) \ne 0$ if $\dim(M)$ is even. Since the proof of this assertion relies on the fact $ch\colon K(M)\otimes \mathbf{Q} \overset{\simeq}{\to} H^{2*}(M;\mathbf{Q})$ we postpone a discussion of this fact until we discuss the index theorem in general.

3.2. Spinors and their Representations.

To define the signature complex, we needed to orient the manifold M. For any Riemannian manifold, by restricting to local orthonormal frames, we can always assume the transition functions of $T(M)$ are maps $g_{\alpha\beta}: U_\alpha \cap U_\beta \to \mathrm{O}(m)$. If M is oriented, by restricting to local orthonormal frames, we can choose the transition functions of $T(M)$ to lie in $\mathrm{SO}(m)$ and to reduce the structure group from $\mathrm{GL}(m,R)$ to $\mathrm{SO}(m)$. The signature complex results from the represntation $\Lambda^\pm$ of $\mathrm{SO}(m)$; it cannot be defined in terms of $\mathrm{GL}(m,R)$ or $\mathrm{O}(m)$. By contrast, the de Rham complex results from the representation $\Lambda^{\mathrm{e,o}}$ which is a representation of $\mathrm{GL}(m,R)$ so the de Rham complex is defined for non-orientable manifolds.

To define the spin complex, which is in some sense a more fundamental elliptic complex than is either the de Rham or signature complex, we will have to lift the transition functions from $\mathrm{SO}(m)$ to $\mathrm{SPIN}(m)$. Just as every manifold is not orientable, in a similar fashion there is an obstruction to defining a spin structure.

If $m \geq 3$, then $\pi_1(\mathrm{SO}(m)) = \mathbf{Z}_2$. Abstractly, we define $\mathrm{SPIN}(m)$ to be the universal cover of $\mathrm{SO}(m)$. (If $m = 2$, then $\mathrm{SO}(2) = S^1$ and we let $\mathrm{SPIN}(2) = S^1$ with the natural double cover $\mathbf{Z}_2 \to \mathrm{SPIN}(2) \to \mathrm{SO}(2)$ given by $\theta \mapsto 2\theta$). To discuss the representations of $\mathrm{SPIN}(m)$, it is convenient to obtain a more concrete representation of $\mathrm{SPIN}(m)$ in terms of Clifford algebras.

Let V be a real vector space of dimension $v \equiv 0\ (2)$. Let

$$\bigotimes V = \mathbf{R} \oplus V \oplus (V \otimes V) \oplus \cdots \oplus \bigotimes{}^k V \oplus \cdots$$

be the complete tensor algebra of V. We assume V is equipped with a symmetric positive definite bilinear form. Let I be the two-sided ideal of $\bigotimes V$ generated by $\{v \otimes v + |v|^2\}_{v \in V}$, then the real Clifford algebra $\mathrm{CLIF}(V) = \bigotimes V \bmod I$. (Of course, we will always construct the corresponding complex Clifford algebra by tensoring $\mathrm{CLIF}(V)$ with the complex numbers). There is a natural transpose defined on $\bigotimes V$ by:

$$(v_1 \otimes \cdots \otimes v_k)^t = v_k \otimes \cdots \otimes v_1.$$

Since this preserves the ideal I, it extends to $\mathrm{CLIF}(V)$. If $\{e_1, \dots, e_v\}$ are the orthonormal basis for V, then $e_i * e_j + e_j * e_i = -2\delta_{ij}$ and

$$(e_{i_1} * \cdots * e_{i_p})^t = (-1)^{p(p-1)/2} e_{i_1} * \cdots * e_{i_p}.$$

If V is oriented, we let $\{e_i\}$ be an oriented orthonormal basis and define:

$$\tau = (\sqrt{-1})^{v/2} e_1 * \cdots * e_v.$$

We already computed that:

$$\begin{aligned}\tau^2 &= (-1)^{v/2} e_1 * \dots * e_v * e_1 * \dots e_v \\ &= (-1)^{v(v-1)/2} e_1 * \dots * e_v * e_1 * \dots * e_v \\ &= e_1 * \dots * e_v * e_v * \dots * e_1 = (-1)^v = 1.\end{aligned}$$

We let $\mathrm{SPIN}(V)$ be the set of all $w \in \mathrm{CLIF}(V)$ such that w can be decomposed as a formal product $w = v_1 * \dots * v_{2j}$ for some j where the $v_i \in V$ are elements of length 1. It is clear that w^t is such an element and that $ww^t = 1$ so that $\mathrm{SPIN}(V)$ forms a group under Clifford multiplication. We define

$$\rho(w)x = wxw^t \qquad \text{for } x \in \mathrm{CLIF}, w \in \mathrm{CLIF}(V).$$

For example, if $v_1 = e_1$ is the first element of our orthonormal basis, then:

$$e_1 e_i e_1 = \begin{cases} -e_1 & i = 1 \\ e_i & i \neq 1. \end{cases}$$

The natural inclusion of V in $\bigotimes V$ induces an inclusion of V in $\mathrm{CLIF}(V)$. $\rho(e_1)$ preserves V and is reflection in the hyperplane defined by e_1. If $w \in \mathrm{SPIN}(V)$, then $\rho(w)\colon V \to V$ is a product of an even number of hyperplane reflections. It is therefore in $\mathrm{SO}(V)$ so

$$\rho\colon \mathrm{SPIN}(V) \to \mathrm{SO}(V)$$

is a group homomorphism. Since any orthogonal transformation of determinant one can be decomposed as a product of an even number of hyperplane reflections, ρ is subjective.

If $w \in \mathrm{CLIF}(V)$ is such that

$$wvw^t = v \text{ all } v \in V \text{ and } ww^t = 1$$

then $wv = vw$ for all $v \in V$ so w must be in the center of $\mathrm{CLIF}(V)$.

Lemma 3.2.1. *If* $\dim V = v$ *is even, then the center of* $\mathrm{CLIF}(V)$ *is one dimensional and consists of the scalars.*

Proof: Let $\{e_i\}$ be an orthonormal basis for V and let $\{e_I\}$ be the corresponding orthonormal basis for $\mathrm{CLIF}(V)$. We compute:

$$e_i * e_{i_1} * \dots * e_{i_p} * e_i = e_{i_1} * \dots * e_{i_p} \qquad \text{for } p \geq 1,$$

if (a) p is even and i is one of the i_j or (b) p is odd and i is not one of the i_j. Thus given I we can choose i so $e_i * e_I = -e_I * e_i$. Thus

$e_i * (\sum_I c_I e_I) = \sum_I c_I e_I * e_i$ for all i implies $c_I = 0$ for $|I| > 0$ which completes the proof.

(We note that this lemma fails if $\dim V$ is odd since the center in that case consists of the elements $a + b e_1 * \cdots * e_v$ and the center is two dimensional).

If $\rho(w) = 1$ and $w \in \mathrm{SPIN}(V)$, this implies w is scalar so $w = \pm 1$. By considering the arc in $\mathrm{SPIN}(V)$ given by $w(\theta) = ((\cos\theta)e_1 + (\sin\theta)e_2) * (-(\cos\theta)e_1 + (\sin\theta)e_2)$, we note $w(0) = 1$ and $w(\frac{\pi}{2}) = -1$ so $\mathrm{SPIN}(V)$ is connected. This proves we have an exact sequence of groups in the form:

$$\mathbf{Z}_2 \to \mathrm{SPIN}(V) \to \mathrm{SO}(V)$$

and shows that $\mathrm{SPIN}(V)$ is the universal cover of $\mathrm{SO}(V)$ for $v > 2$.

We note that is it possible to define $\mathrm{SPIN}(V)$ using Clifford algebras even if v is odd. Since the center of $\mathrm{CLIF}(V)$ is two dimensional for v odd, more care must be used with the relevant signs which arise. As we shall not need that case, we refer to Atiyah-Bott-Shapiro (see bibliography) for further details.

$\mathrm{SPIN}(V)$ acts on $\mathrm{CLIF}(V)$ from the left. This is an orthogonal action.

Lemma 3.2.2. *Let* $\dim V = 2v_1$. *We complexify* $\mathrm{CLIF}(V)$. *As a left* $\mathrm{SPIN}(V)$ *module, this is not irreducible. We can decompose* $\mathrm{CLIF}(V)$ *as a direct sum of left* $\mathrm{SPIN}(V)$ *modules in the form*

$$\mathrm{CLIF}(V) = 2^{v_1}\Delta.$$

This representation is called the spin representation. It is not irreducible but further decomposes in the form

$$\Delta = \Delta^+ \oplus \Delta^-.$$

If we orient V *and let* τ *be the orientation form discussed earlier, then left multiplication by* τ *is* ± 1 *on* $\Delta^\pm$ *so these are inequivalent representations. They are irreducible and act on a representation space of dimension* $2^{v_1 - 1}$ *and are called the half-spin representations.*

Proof: Fix an oriented orthonormal basis $\{e_i\}$ for V and define:

$$\alpha_1 = \sqrt{-1}\, e_1 e_2, \quad \alpha_2 = \sqrt{-1}\, e_3 e_4, \quad \ldots, \quad \alpha_{v_1} = \sqrt{-1}\, e_{v-1} e_v$$

as elements of $\mathrm{CLIF}(V)$. It is immediate that $\tau = \alpha_1 \ldots \alpha_{v_1}$ and:

$$\alpha_i^2 = 1 \qquad \text{and} \qquad \alpha_i \alpha_j = \alpha_j \alpha_i.$$

We let the $\{\alpha_i\}$ act on $\mathrm{CLIF}(V)$ from the right and decompose $\mathrm{CLIF}(V)$ into the 2^{v_1} simultaneous eigenspaces of this action. Since right and left multiplication commute, each eigenspace is invariant as a left $\mathrm{SPIN}(V)$

module. Each eigenspace corresponds to one of the 2^{v_1} possible sequences of $+$ and $-$ signs. Let ε be such a string and let Δ_ε be the corresponding representation space.

Since $e_1\alpha_1 = -\alpha_1 e_1$ and $e_1\alpha_i = \alpha_i e_1$ for $i > 1$, multiplication on the right by e_1 transforms $\Delta_\varepsilon \to \Delta_{\varepsilon'}$, where $\varepsilon(1) = -\varepsilon'(1)$ and $\varepsilon(i) = \varepsilon'(i)$ for $i > 1$. Since this map commutes with left multiplication by $\mathrm{SPIN}(V)$, we see these two representations are isomorphic. A similar argument on the other indices shows all the representation spaces Δ_ε are equivalent so we may decompose $\mathrm{CLIF}(V) = 2^{v_1}\Delta$ as a direct sum of 2^{v_1} equivalent representation spaces Δ. Since $\dim(CLIF(V)) = 2^v = 4^{v_1}$, the representation space Δ has dimension 2^{v_1}. We decompose Δ into ± 1 eigenspaces under the action of τ to define $\Delta^\pm$. Since $e_1 * \cdots * e_v$ is in the center of $\mathrm{SPIN}(V)$, these spaces are invariant under the left action of $\mathrm{SPIN}(V)$. We note that elements of V anti-commute with τ so Clifford multiplication on the left defines a map:

$$cl\colon V \otimes \Delta^\pm \to \Delta^\mp$$

so both of the half-spin representations have the same dimension. They are clearly inequivalent. We leave the proof that they are irreducible to the reader as we shall not need this fact.

We shall use the notation $\Delta^\pm$ to denote both the representations and the corresponding representation spaces. Form the construction we gave, it is clear the $\mathrm{CLIF}(V)$ acts on the left to preserve the space Δ so Δ is a representation space for the left action by the whole Clifford algebra. Since $v\tau = -\tau v$ for $v \in V$, Clifford multiplication on the left by an element of V interchanges Δ^+ and Δ^-.

Lemma 3.2.3. *There is a natural map given by Clifford multiplication of $V \otimes \Delta^\pm \to \Delta^\mp$. This map induces a map on the representations involved: $\rho \otimes \Delta^\pm \mapsto \Delta^\mp$. If this map is denoted by $v * w$ then $v * v * w = -|v|^2 w$.*

Proof: We already checked the map on the spaces. We check:

$$wvw^t \otimes wx \mapsto wvw^t wx = wvx$$

to see that the map preserves the relevant representation. We emphasize that $\Delta^\pm$ are complex representations since we must complexify $\mathrm{CLIF}(V)$ to define these representation spaces ($ww^t = 1$ for $w \in \mathrm{SPIN}$).

There is a natural map $\rho\colon \mathrm{SPIN}(V) \to \mathrm{SO}(V)$. There are natural representations Λ, $\Lambda^\pm$, $\Lambda^{e,o}$ of $\mathrm{SO}(V)$ on the subspaces of $\Lambda(V) = \mathrm{CLIF}(V)$. We use ρ to extend these representations of $\mathrm{SPIN}(V)$ as well. They are related to the half-spin representations as follows:

Lemma 3.2.4.
(a) $\Lambda = \Delta \otimes \Delta$.
(b) $(\Lambda^+ - \Lambda^-) = (\Delta^+ - \Delta^-) \otimes (\Delta^+ + \Delta^-)$.

(c) $(\Lambda^e - \Lambda^o) = (\Delta^+ - \Delta^-) \otimes (\Delta^+ - \Delta^-)(-1)^{v/2}$.
These identities are to be understood formally (in the sense of K-theory). They are shorthand for the identities:

$$\Lambda^+ = (\Delta^+ \otimes \Delta^+) \oplus (\Delta^+ \otimes \Delta^-) \quad \text{and} \quad \Delta^- = (\Delta^- \otimes \Delta^+) \oplus (\Delta^- \otimes \Delta^-)$$

and so forth.

Proof: If we let $\mathrm{SPIN}(V)$ act on $\mathrm{CLIF}(V)$ by right multiplication by $w^t = w^{-1}$, then this representation is equivalent to left multiplication so we get the same decomposition of $\mathrm{CLIF}(V) = 2^{v_1}\Delta$ as right spin modules. Since $\rho(w)x = wxw^t$, it is clear that $\Lambda = \Delta \otimes \Delta$ where one factor is viewed as a left and the other as a right spin module. Since $\Lambda^\pm$ is the decomposition of $\mathrm{SPIN}(V)$ under the action of τ from the left, (b) is immediate. (c) follows similarly once the appropriate signs are taken into consideration; $w \in \mathrm{CLIF}(V)^{\text{even}}$ if and only if

$$\tau w \tau^t = (-1)^{v/2} w$$

which proves (c).

It is helpful to illustrate this for the case $\dim V = 2$. Let $\{e_1, e_2\}$ be an oriented orthonormal basis for V. We compute that:

$$\begin{aligned}\big((\cos\alpha)e_1 + (\sin\alpha)e_2\big)\big((\cos\beta)e_1 + (\sin\beta)e_2\big)& \\ = \big(-\cos\alpha\cos\beta - \sin\alpha\sin\beta\big)& \\ + \big(\cos\alpha\sin\beta - \sin\alpha\cos\beta\big)e_1e_2&\end{aligned}$$

so elements of the form $\cos\gamma + (\sin\gamma)e_1e_2$ belong to $\mathrm{SPIN}(V)$. We compute:

$$\big(\cos\alpha + (\sin\alpha)e_1e_2\big)\big(\cos\beta + (\sin\beta)e_1e_2\big) = \cos(\alpha+\beta) + \sin(\alpha+\beta)e_1e_2$$

so $\mathrm{spin}(V)$ is the set of all elements of this form and is naturally isomorphic to the circle $S^1 = [0, 2\pi]$ with the endpoints identified. We compute that

$$\begin{aligned}\rho(w)(e_1) &= \big(\cos\theta + (\sin\theta)e_1e_2\big)e_1\big(\cos\theta + (\sin\theta)e_2e_1\big) \\ &= \big(\cos^2\theta - \sin^2\theta\big)e_1 + 2(\cos\theta\sin\theta)e_2 \\ &= \cos(2\theta)e_1 + \sin(2\theta)e_2 \\ \rho(w)(e_2) &= \cos(2\theta)e_2 - \sin(2\theta)e_1\end{aligned}$$

so the map $\rho\colon S^1 \to S^1$ is the double cover $\theta \mapsto 2\theta$.

We construct the one dimensional subspaces V_i generated by the elements:

$$\begin{array}{llll} v_1 = 1+\tau, & v_2 = 1-\tau, & v_3 = (1+\tau)e_1, & v_4 = (1-\tau)e_1, \\ \tau v_1 = v_1, & \tau v_2 = -v_2, & \tau v_3 = v_3, & \tau v_4 = -v_4, \\ v_1\tau = v_1, & v_2\tau = -v_2, & v_3\tau = -v_3, & v_4\tau = v_4. \end{array}$$

When we decompose $\mathrm{CLIF}(V)$ under the left action of $\mathrm{SPIN}(V)$,

$$V_1 \simeq V_3 \simeq \Delta^+ \qquad \text{and} \qquad V_2 \simeq V_4 \simeq \Delta^-.$$

When we decompose $\mathrm{CLIF}(V)$ under the right action of $\mathrm{SPIN}(V)$, and replace τ by $-\tau = \tau^t$ acting on the right:

$$V_2 \simeq V_3 \simeq \Delta^+ \qquad \text{and} \qquad V_1 \simeq V_4 \simeq \Delta^-.$$

From this it follows that as $\mathrm{SO}(V)$ modules we have:

$$V_1 = \Delta^+ \otimes \Delta^-, \quad V_2 = \Delta^- \otimes \Delta^+, \quad V_3 = \Delta^+ \otimes \Delta^+, \quad V_4 = \Delta^- \otimes \Delta^-$$

from which it is immediate that:

$$\begin{aligned}
\Lambda^{\mathrm{e}} &= V_1 \oplus V_2 = \Delta^+ \otimes \Delta^- \ \oplus\ \Delta^- \otimes \Delta^+ \\
\Lambda^{\mathrm{o}} &= V_3 \oplus V_4 = \Delta^+ \otimes \Delta^+ \ \oplus\ \Delta^- \otimes \Delta^- \\
\Lambda^{+} &= V_1 \oplus V_3 = \Delta^+ \otimes \Delta^- \ \oplus\ \Delta^+ \otimes \Delta^+ \\
\Lambda^{-} &= V_2 \oplus V_4 = \Delta^- \otimes \Delta^+ \ \oplus\ \Delta^- \otimes \Delta^-.
\end{aligned}$$

If V is a one-dimensional complex vector space, we let V_r be the underlying real vector space. This defines a natural inclusion of $\mathrm{U}(1) \to \mathrm{SO}(2)$. If we let $J : V \to V$ be complex multiplication by $\sqrt{-1}$ then $J(e_1) = e_2$ and $J(e_2) = -e_1$ so J is equivalent to Clifford multiplication by e_1e_2 on the left. We define a complex linear map from V to $\Delta^- \otimes \Delta^-$ by:

$$T(v) = v - iJ(v) \in V_4$$

by computing $T(e_1) = (e_1 - ie_2) = (1 - ie_1e_2)(e_1)$ and $T(e_2) = e_2 + ie_1 = (i)(e_1 - ie_2)$.

LEMMA 3.2.5. *Let* $\mathrm{U}(1)$ *be identified with* $\mathrm{SO}(2)$ *in the usual manner. If* V *is the underlying complex* 1*-dimensional space corresponding to* V_r *then* $V \simeq \Delta^- \otimes \Delta^-$ *and* $V^* \simeq \Delta^+ \otimes \Delta^+$ *as representation spaces of* $\mathrm{SPIN}(2)$.

PROOF: We have already verified the first assertion. The second follows from the fact that V^* was made into a complex space using the map $-J$ instead of J on V_r. This takes us into V_3 instead of into V_4. It is also clear that $\Delta^- = (\Delta^+)^*$ if $\dim V = 2$. Of course, all these statements are to be interpreted as statements about representations since they are trivial as statements about vector spaces (since any vector spaces of the same dimension are isomorphic).

3.3. Spin Structures on Vector Bundles.

We wish to apply the constructions of 3.2 to vector bundles over manifolds. We first review some facts regarding principal bundles and Stieffel-Whitney classes which we shall need.

Principal bundles are an extremely convenient bookkeeping device. If G is a Lie group, a principal G-bundle is a fiber space $\pi: P_G \to M$ with fiber G such that the transition functions are elements of G acting on G by left multiplication in the group. Since left and right multiplication commute, we can define a right action of G on P_G which is fiber preserving. For example, let $\mathrm{SO}(2k)$ and $\mathrm{SPIN}(2k)$ be the groups defined by $\mathbf{R}^{2k}$ with the cannonical inner product. Let V be an oriented Riemannian vector bundle of dimension $2k$ over M and let P_{SO} be the bundle of oriented frames of V. P_{SO} is an $\mathrm{SO}(2k)$ bundle and the natural action of $\mathrm{SO}(2k)$ from the right which sends an oriented orthonormal frame $s = (s_1, \dots, s_{2k})$ to $s \cdot g = (s'_1, \dots, s'_{2k})$ is defined by:

$$s'_i = s_1 g_{1,i} + \cdots + s_{2k} g_{2k,i}.$$

The fiber of P_{SO} is $\mathrm{SO}(V_x)$ where V_x is the fiber of V over the point x. This isomorphism is not natural but depends upon the choice of a basis.

It is possible to define the theory of characteristic classes using principal bundles rather than vector bundles. In this approach, a connection is a splitting of $T(P_G)$ into vertical and horizontal subspaces in an equivariant fashion. The curvature becomes a Lie algebra valued endomorphism of $T(P_G)$ which is equivariant under the right action of the group. We refer to Euguchi, Gilkey, Hanson for further details.

Let $\{U_\alpha\}$ be a cover of M so V is trivial over U_α and let $\vec{s}_\alpha$ be local oriented orthonormal frames over U_α. On the overlap, we express $\vec{s}_\alpha = g_{\alpha\beta}\vec{s}_\beta$ where $g_{\alpha\beta}: U_\alpha \cap U_\beta \to \mathrm{SO}(2k)$. These satisfy the cocycle condition:

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = I \qquad \text{and} \qquad g_{\alpha\alpha} = I.$$

The principal bundle P_{SO} of oriented orthonormal frames has transition functions $g_{\alpha\beta}$ acting on $\mathrm{SO}(2k)$ from the left.

A spin structure on V is a lifting of the transition functions to $\mathrm{SPIN}(2k)$ preserving the cocyle condition. If the lifting is denoted by g', then we assume:

$$\rho(g'_{\alpha\beta}) = g_{\alpha\beta}\,, \qquad g'_{\alpha\beta} g'_{\beta\gamma} g'_{\gamma\alpha} = I\,, \qquad \text{and} \qquad g'_{\alpha\alpha} = I.$$

This is equivalent to constructing a principal $\mathrm{SPIN}(2k)$ bundle P_{SPIN} together with a double covering map $\rho: P_{\mathrm{SPIN}} \to P_{\mathrm{SO}}$ which preserves the group action—i.e.,

$$\rho(x \cdot g') = \rho(x) \cdot \rho(g').$$

The transition functions of P_{SPIN} are just the $g'_{\alpha\beta}$ acting on the left.

Attempting to find a spin structure on V is equivalent to taking a square root in a certain sense which we will make clear later. There is an obstruction to defining a spin structure which is similar to that to defining an orientation on V. These obstructions are $\mathbf{Z}_2$ characteristic classes called the Stieffel-Whitney classes and are most easily defined in terms of Čech cohomology. To understand these obstructions better, we review the construction briefly.

We fix a Riemannian structure on $T(M)$ to define a notion of distance. Geodesics on M are curves which locally minimize distance. A set U is said to be geodesically convex if (a) given $x, y \in U$ there exists a unique geodesic in M joining x to y with $d(x,y) = \mathrm{length}(\gamma)$ and (b) if γ is any such geodesic then γ is actually contained in U. It is immediate that the intersection of geodesically convex sets is again geodesically convex and that every geodesically convex set is contractible.

It is a basic theorem of Riemannian geometry that there exist open covers of M by geodesically convex sets. A cover $\{U_\alpha\}$ is said to be *simple* if the intersection of any number of sets of the cover is either empty or is contractible. A cover of M by open geodesically convex sets is a simple cover.

We fix such a simple cover hence forth. Since U_α is contractible, any vector bundle over M is trivial over U_α. Let $\mathbf{Z}_2$ be the multiplicative group $\{\pm 1\}$. A Čech j-cochain is a function $f(\alpha_0, \ldots, \alpha_j) \in \mathbf{Z}_2$ defined for $j+1$-tuples of indices where $U_{\alpha_0} \cap \cdots \cap U_{\alpha_j} \neq \emptyset$ which is totally symmetric—i.e.,

$$f(\alpha_{\sigma(0)}, \ldots, \alpha_{\sigma(j)}) = f(\alpha_0, \ldots, \alpha_j)$$

for any permutation σ. If $C^j(M;\mathbf{Z}_2)$ denotes the multiplicative group of all such functions, the coboundary $\delta\colon C^j(M,\mathbf{Z}_2) \to C^{j+1}(M;\mathbf{Z}_2)$ is defined by:

$$(\delta f)(\alpha_0, \ldots, \alpha_{j+1}) = \prod_{i=0}^{j+1} f(\alpha_0, \ldots, \widehat{\alpha_i}, \ldots, \alpha_{j+1}).$$

The multiplicative identity of $C^j(M;\mathbf{Z}_2)$ is the function 1 and it is an easy combinatorial exercise that $\delta^2 f = 1$. For example, if $f_0 \in C^0(M;\mathbf{Z}_2)$ and if $f_1 \in C^1(M;\mathbf{Z}_2)$, then:

$$\begin{aligned} \delta(f_0)(\alpha_0, \alpha_1) &= f(\alpha_1) f(\alpha_0) \\ \delta(f_1)(\alpha_0, \alpha_1, \alpha_2) &= f(\alpha_1, \alpha_2) f(\alpha_0, \alpha_1) f(\alpha_o, \alpha_2). \end{aligned}$$

$\mathbf{Z}_2$ is a particularly simple coefficient group to work with since every element is its own inverse; in defining the Čech cohomology with coefficients in other abelian groups, more care must be taken with the signs which arise.

Let $H^j(M;\mathbf{Z}_2) = \mathrm{N}(\delta_j)/\,\mathrm{R}(\delta_{j-1})$ be the cohomology group; it is an easy exercise to show these groups are independent of the particular simple cover chosen. There is a ring structure on $H^*(M;\mathbf{Z}_2)$, but we shall only use the "additive" group structure (which we shall write multiplicatively).

Let V be a real vector bundle, not necessarily orientable. Since the U_α are contractible, V is trivial over the U and we can find a local orthonormal frame $\vec{s}_\alpha$ for V over U_α. We let $\vec{s}_\alpha = g_{\alpha\beta}\vec{s}_\beta$ and define the 1-cochain:

$$f(\alpha\beta) = \det(g_{\alpha\beta}) = \pm 1.$$

This is well defined since $U_\alpha \cap U_\beta$ is contractible and hence connected. Since $f(\alpha,\beta) = f(\beta,\alpha)$, this defines an element of $C^1(M;\mathbf{Z}_2)$. Since the $\{g_{\alpha\beta}\}$ satisfy the cocycle condition, we compute:

$$\delta f(\alpha,\beta,\gamma) = \det(g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}) = \det(I) = 1$$

so $\delta(f) = 1$ and f defines an element of $H^1(M;\mathbf{Z}_2)$. If we replace $\vec{s}_\alpha$ by $\vec{s}\,'_\alpha = h_\alpha \vec{s}_\alpha$ the new transition functions become $g'_{\alpha\beta} = h_\alpha g_{\alpha\beta} h_\beta^{-1}$ so if $f_0(\alpha) = \det(h_\alpha)$,

$$f'(\alpha,\beta) = \det(h_\alpha g_{\alpha\beta} h_\beta^{-1}) = \det(h_\alpha) f(\alpha,\beta)\det(h_\beta) = \delta(f_0)f$$

and f changes by a coboundary. This proves the element in cohomology defined by f is independent of the particular frame chosen and we shall denote this element by $w_1(V) \in H^1(M;\mathbf{Z}_2)$.

If V is orientable, we can choose frames so $\det(g_{\alpha\beta}) = 1$ and thus $w_1(V) = 1$ represents the trivial element in cohomology. Conversely, if $w_1(V)$ is trivial, then $f = \delta f_0$. If we choose h_α so $\det(h_\alpha) = f_0(\alpha)$, then the new frames $\vec{s}\,'_\alpha = h_\alpha \vec{s}_\alpha$ will have transition functions with $\det(g'_{\alpha\beta}) = 1$ and define an orientation of V. Thus V is orientable if and only if $w_1(V)$ is trivial and $w_1(V)$, which is called the first Stieffel-Whitney class, measures the obstruction to orientability.

If V is orientable, we restrict henceforth to oriented frames. Let $\dim V = 2k$ be even and let $g_{\alpha\beta} \in \mathrm{SO}(2k)$ be the transition functions. We choose any lifting $\tilde{g}_{\alpha\beta}$ to $\mathrm{SPIN}(2k)$ so that:

$$\rho(\tilde{g}_{\alpha\beta}) = g_{\alpha\beta} \qquad \text{and} \qquad \tilde{g}_{\alpha\beta}\tilde{g}_{\beta\alpha} = I;$$

since the U_α are contractible such lifts always exist. We have $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = I$ so $\rho(\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha}) = I$ and hence $\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha} = \pm I = f(\alpha,\beta,\gamma)I$ where $f(\alpha,\beta,\gamma) \in \mathbf{Z}_2$. V admits a spin structure if and only if we can choose the lifting so $f(\alpha,\beta,\gamma) = 1$. It is an easy combinatorial exercise to show that f is symmetric and that $\delta f = 0$. Furthermore, if we change the choice of the $\vec{s}_\alpha$ or change the choice of lifts, then f changes by a coboundary. This implies f defines an element $w_2(V) \in H^2(M; \mathbf{Z}_2)$ independent of the choices made. w_2 is called the second Stieffel-Whitney class and is trivial if and only if V admits a spin structure.

We suppose $w_1(V) = w_2(V) = 1$ are trivial so V is orientable and admits a spin structure. Just as there are two orientations which can be chosen on V, there can be several possible inequivalent spin structures (i.e., several non-isomorphic principal bundles P_{SPIN}). It is not difficult to see that inequivalent spin structures are parametrized by representations of the fundamental group $\pi_1(M) \to \mathbf{Z}_2$ just as inequivalent orientations are parametrized by maps of the components of M into $\mathbf{Z}_2$.

To illustrate the existence and non-existence of spin structures on bundles, we take $M = S^2$. $S^2 = \mathbf{C}P_1$ is the Riemann sphere. There is a natural projection from $\mathbf{C}^2 - 0$ to S^2 given by sending $(x, w) \mapsto z/w$; S^2 is obtained by identifying $(z, w) = (\lambda z, \lambda w)$ for $(z, w) \neq (0, 0)$ and $\lambda \neq 0$. There are two natural charts for S^2:

$$U_1 = \{z : |z| \leq 1\} \qquad \text{and} \qquad U_2 = \{z : |z| \geq 1\} \cup \{\infty\}$$

where $w = 1/z$ gives coordinates on U_2.

$\mathbf{C}P_1$ is the set of lines in $\mathbf{C}^2$; we let L be the natural line bundle over S^2; this is also refered to as the tautological line bundle. We have natural sections to L over U_i defined by:

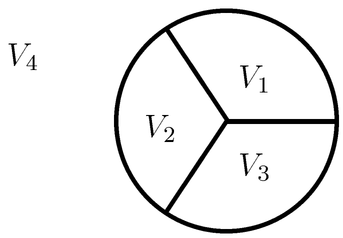
$$s_1 = (z, 1) \text{ over } U_1 \qquad \text{and} \qquad s_2 = (1, w) \text{ over } U_2.$$

these are related by the transition function:

$$s_1 = z s_2$$

so on $U_1 \cap U_2$ we have $g_{12} = e^{i\theta}$. The double cover $\text{SPIN}(2) \to \text{SO}(2)$ is defined by $\theta \mapsto 2\theta$ so this bundle does not have a spin structure since this transition function cannot be lifted to SPIN(2).

This cover is not a simple cover of S^2 so we construct the cover given in the following



(where we should "fatten up" the picture to have an open cover). If we have sections $\vec{s}_i$ to L over V_i, then we can choose the transition functions so $\vec{s}_j = e^{i\theta}\vec{s}_4$ for $1 \le j \le 3$ and $\vec{s}_1 = \vec{s}_2 = \vec{s}_3$. We let

$$\begin{array}{rll} 0 \le \theta \le 2\pi/3 & \text{parametrize} & V_1 \cap V_4 \\ 2\pi/3 \le \theta \le 4\pi/3 & \text{parametrize} & V_2 \cap V_4 \\ 4\pi/3 \le \theta \le 2\pi & \text{parametrize} & V_3 \cap V_4 \end{array}$$

and define:

$$\tilde{g}_{14} = e^{i\theta/2}, \quad \tilde{g}_{24} = e^{i\theta/2}, \quad \tilde{g}_{34} = e^{i\theta/2}$$

and all the $\tilde{g}_{jk} = 1$ for $1 \le j,\, k \le 3$ then we compute:

$$\tilde{g}_{13}\tilde{g}_{34}\tilde{g}_{41} = e^0 e^{i\pi} = -1$$

so the cochain defining w_2 satisfies:

$$f(1,3,4) = -1 \quad \text{with} \quad f(i,j,k) = 1 \quad \text{if } i \text{ or } j \text{ or } k\text{=2.}$$

We compute this is non-trivial in cohomology as follows: suppose $f = \delta f_1$. Then:

$$\begin{aligned} 1 &= f(1,2,3) = f_1(1,2)f_1(1,3)f_1(2,3) \\ 1 &= f(1,2,4) = f_1(1,2)f_1(1,4)f_1(2,4) \\ 1 &= f(2,3,4) = f_1(2,3)f_1(2,4)f_1(3,4) \end{aligned}$$

and multiplying together:

$$\begin{aligned} 1 &= f_1(1,2)^2 f_1(2,3)^2 f_1(2,4)^2 f_1(1,4)f_1(1,3)f_1(3,4) \\ &= f_1(1,4)f_1(1,3)f_1(3,4) = f(1,3,4) = -1 \end{aligned}$$

which is a contradiction. Thus we have computed combinatorially that $w_2 \ne 1$ is non-trivial.

Next we let $V = T(M)$ be the real tangent bundle. We identify U(1) with SO(2) to identify $T(M)$ with $T_C(M)$. Since $w = 1/z$ we have:

$$\frac{d}{dz} = \frac{dw}{dz}\frac{d}{dw} = -z^{-2}\frac{d}{dw}$$

so the transition function on the overlap is $-e^{-2i\theta}$. The minus sign can be eliminated by using $-\dfrac{d}{dw}$ instead of $\dfrac{d}{dw}$ as a section over U_2 to make the transition function be $e^{-2i\theta}$. Since the double cover of SPIN(2) $\to$ SO(2) is given by $\theta \mapsto 2\theta$, this transition function lifts and $T(M)$ has a spin structure. Since S^2 is simply connected, the spin structure is unique. This also proves $T_C(M) = L^* \otimes L^*$ since the two bundles have the same transition functions.

There is a natural inclusion of $\mathrm{SPIN}(V)$ and $\mathrm{SPIN}(W)$ into subgroups of $\mathrm{SPIN}(V \oplus W)$ which commute. This induces a map from $\mathrm{SPIN}(V) \times \mathrm{SPIN}(W) \to \mathrm{SPIN}(V \oplus W)$. Using this map, it is easy to compute $w_2(V \oplus W) = w_2(V)w_2(W)$. To define $w_2(V)$, we needed to choose a fixed orientation of V. It is not difficult to show that $w_2(V)$ is independent of the orientation chosen and of the fiber metric on V. (Our definition does depend upon the fact that V is orientable. Although it is possible to define $w_2(V)$ more generally, the identity $w_2(V \oplus W) = w_2(V)w_2(W)$ fails if V and W are not orientable in general). We summarize the properties of w_2 we have derived:

Lemma 3.3.1. *Let V be a real oriented vector bundle and let $w_2(V) \in H^2(M; \mathbf{Z}_2)$ be the second Stieffel-Whitney class. Then:*
(a) $w_2(V) = 1$ represents the trivial cohomology class if and only if V admits a spin structure.
(b) $w_2(V \oplus W) = w_2(V)w_2(W)$.
(c) If L is the tautological bundle over S^2 then $w_2(L)$ is non-trivial.

We emphasize that (b) is written in multiplicative notation since we have chosen the multiplicative version of $\mathbf{Z}_2$. w_2 is also functorial under pull-backs, but as we have not defined the Čech cohomology as a functor, we shall not discuss this property. We also note that the Stieffel-Whitney classes can be defined in general; $w(V) = 1 + w_1(V) + \cdots \in H^*(M; \mathbf{Z}_2)$ is defined for any real vector bundle and has many of the same properties that the Chern class has for complex bundles.

We can use Lemma 3.3.1 to obtain some other results on the existence of spin structures:

Lemma 3.3.2.
(a) If $M = S^m$ then any bundle over S^m admits a spin structure for $m \neq 2$.
(b) If $M = \mathbf{C}P_k$ and if L is the tautological bundle over M, then L does not admit a spin structure.
(c) If $M = \mathbf{C}P_k$ and if $V = T(M)$ is the tangent space, then V admits a spin structure if and only if k is odd.
(d) If $M = \mathbf{Q}P_k$ is quaternionic projective space, then any bundle over M admits a spin structure.

Proof: $H^2(M; \mathbf{Z}_2) = \{1\}$ in (a) and (d) so w_2 must represent the trivial element. To prove (b), we suppose L admits a spin structure. S^2 is embedded in $\mathbf{C}P_k$ for $k \geq 2$ and the restriction of L to S^2 is the tautological bundle over S^2. This would imply L admits a spin structure over S^2 which is false. Finally, we use the representation:

$$T_c(\mathbf{C}P_j) \oplus 1 = L^* \oplus \cdots \oplus L^*$$

so that

$$T(\mathbf{C}P_j) \oplus 1^2 = \underbrace{(L_r^*) \oplus \cdots \oplus (L_r^*)}_{j+1 \text{ times}}$$

where L_r^* denotes the real vector bundle obtained from L^* by forgetting the complex structure. Then $w_2(L_r^*) = w_2(L_r)$ since these bundles are isomorphic as real bundles. Furthermore:

$$w_2(T(\mathbf{C}P_j)) = w_2(L_r)^{j+1}$$

and this is the trivial class if $j+1$ is even (i.e., j is odd) while it is $w_2(L_r)$ and non-trivial if $j+1$ is odd (i.e., j is even).

Let M be a manifold and let V be a real vector bundle over M. If V admits a spin structure, we shall say V is spin and not worry for the moment about the lack of uniqueness of the spin structure. This will not become important until we discuss Lefschetz fixed point formulas.

If V is spin, we define the bundles $\Delta^\pm(V)$ to have transition functions $\Delta^\pm(\tilde{g}_{\alpha\beta})$ where we apply the representation $\Delta^\pm$ to the lifted transition functions. Alternatively, if P_{SPIN} is the principal spin bundle defining the spin structure, we define:

$$\Delta^\pm(V) = P_{\text{SPIN}} \otimes_{\Delta^\pm} \Delta^\pm$$

where the tensor product across a representation is defined to be $P_{\text{SPIN}} \times \Delta^\pm$ module the identification $(p \cdot g) \times z = p \times (\Delta^\pm(g)z)$ for $p \in P_{\text{SPIN}}$, $g \in P_{\text{SPIN}}(2k)$, $z \in \Delta^\pm$. (The slight confusion of notation is caused by our convention of using the same symbol for the representation and the representation space.)

Let $b_\pm$ be a fixed unitary frame for $\Delta^\pm$. If $\vec{s}$ is a local oriented orthonormal frame for V, we let $\tilde{s}_i$ be the two lifts of $\vec{s} \in P_{\text{SO}}$ to P_{SPIN}. $\tilde{s}_1 = -\tilde{s}_2$ but there is no natural way to distinguish these lifts, although the pair is cannonically defined. If ∇ is a Riemannian connection on V, let $\nabla_{\vec{s}} = \omega\vec{s}$ be the connection 1-form. ω is a skew symmetric matrix of 1-forms and is a 1-form valued element of the Lie algebra of $\text{SO}(2k)$. Since the Lie algebra of $\text{SO}(2k)$ and $\text{SPIN}(2k)$ coincide, we can also regard ω as an element which is 1-form valued of the Lie algebra of $\text{SPIN}(2k)$ and let $\Delta^\pm(\omega)$ act on $\Delta^\pm(V)$. We define bases $\tilde{s}_i \otimes b_\pm$ for $\Delta^\pm(V)$ and define:

$$\nabla(\tilde{s}_i \otimes b_\pm) = \tilde{s}_i \otimes \Delta^\pm(\omega) b_\pm$$

to define a natural connection on $\Delta^\pm(V)$. Since the same connection is defined whether $\tilde{s}_1$ or $\tilde{s}_2 = -\tilde{s}_1$ is chosen, the $\mathbf{Z}_2$ ambiguity is irrelevant and ∇ is well defined.

Lemma 3.2.4 and 3.2.5 extend to:

Lemma 3.3.3. *Let V be a real oriented Riemannian vector bundle of even fiber dimension v. Suppose V admits a spin structure and let $\Delta^{\pm}(V)$ be the half-spin bundles. Then:*
(a) $\Lambda(V) \simeq \Delta(V) \otimes \Delta(V)$.
(b) $(\Lambda^+ - \Lambda^-)(V) \simeq (\Delta^+ - \Delta^-) \otimes (\Delta^+ - \Delta^-)(V)$.
(c) $(\Lambda^{\mathrm{e}} - \Lambda^{\mathrm{o}})(V) \simeq (-1)^{v/2}(\Delta^+ - \Delta^-) \otimes (\Delta^+ - \Delta^-)(V)$.
(d) If $v = 2$ and if V is the underlying real bundle of a complex bundle V_c then $V_c \simeq \Delta^- \otimes \Delta^-$ and $V_c^* \simeq \Delta^+ \otimes \Delta^+$.
(e) If ∇ is a Riemannian connection on V and if we extend ∇ to $\Delta^{\pm}(V)$, then the isomorphisms given are unitary isomorphisms which preserve ∇.

Spinors are multiplicative with respect to products. It is immediate from the definitions we have given that:

Lemma 3.3.4. *Let V_i be real oriented Riemannian vector bundles of even fiber dimensions v_i with given spin structures. Let $V_1 \oplus V_2$ have the natural orientation and spin structure. Then:*

$$(\Delta^+ - \Delta^-)(V_1 \oplus V_2) \simeq \{(\Delta^+ - \Delta^-)(V_1)\} \otimes \{(\Delta^+ - \Delta^-)(V_2)\}.$$

If ∇_i are Riemannian connections on V_i and if we define the natural induced connections on these bundles, then the isomorphism is unitary and preserves the connections.

A spin structure always exists locally since the obstruction to a spin structure is global. Given any real oriented Riemannian bundle V of even fiber dimension with a fixed Riemannian connection ∇, we define $\Lambda(V)$ and $\Delta(V) = \Delta^+(V) \oplus \Delta^-(V)$. With the natural metrics and connections, we relate the connection 1-forms and curvatures of these 3 bundles:

Lemma 3.3.5. *Let $\{e_i\}$ be a local oriented orthonormal frame for V. We let $\nabla e_j = \omega_{jk} s_k$ represent the connection 1-form and $\Omega e_j = \Omega_{jk} e_k$ be the curvature matrix for $\Omega_{jk} = d\omega_{jk} - \omega_{jl} \wedge \omega_{lk}$. Let Λe and Δe denote the natural orthonormal frames on $\Lambda(V)$ and $\Delta(V)$. Relative to these frames we compute the connection 1-forms and curvature matrices of $\Lambda(V)$ and $\Delta(V)$ by:*

$$\text{(a)} \qquad \omega_\Lambda = \omega_{jk}\,\mathrm{ext}(e_k)\,\mathrm{int}(e_j) \qquad \Omega_\Lambda = \Omega_{jk}\,\mathrm{ext}(e_k)\,\mathrm{int}(e_j)$$

$$\text{(b)} \qquad \omega_\Delta = \tfrac{1}{4}\omega_{jk} e_j * e_k \qquad \Omega_\Delta = \tfrac{1}{4}\Omega_{jk} e_j * e_k.$$

Proof: We sum over repeated indices in these expressions. We note (a) is true by definition on $\Lambda^1(V) = V$. Since both ω and Ω extend to act as derivations on the exterior algebra, this implies (a) is true on forms of all degree.

Let $\mathrm{so}(n) = \{A \in n \times n \text{ real matrices with } A + A^t = 0\}$ be the Lie algebra of $\mathrm{SO}(n)$. This is also the Lie algebra of $\mathrm{SPIN}(n)$ so we must identify this with an appropriate subset of the Clifford algebra in order to prove (b). We choose a representative element of so(n) and define

$$Ae_1 = e_2, \quad Ae_2 = -e_1, \quad Ae_j = 0 \qquad \text{for } j > 2;$$

i.e.,

$$A_{12} = 1, \quad A_{21} = -1, \quad A_{jk} = 0 \qquad \text{otherwise.}$$

If we let $g(t) \in \mathrm{SO}(n)$ be defined by

$$\begin{aligned} g(t)e_1 &= (\cos t)e_1 + (\sin t)e_2, \\ g(t)e_2 &= (\cos t)e_2 - (\sin t)e_1, \\ g(t)e_j &= e_j \qquad j > 2, \end{aligned}$$

then $g(0) = I$ and $g'(0) = A$. We must lift $g(t)$ from $\mathrm{SO}(n)$ to $\mathrm{SPIN}(n)$. Define:

$$\begin{aligned} h(t) &= \big(\cos(t/4)e_1 + \sin(t/4)e_2\big)\big(-\cos(t/4)e_1 + \sin(t/4)e_2\big) \\ &= \cos(t/2) + \sin(t/2)e_1e_2 \in \mathrm{SPIN}(n) \end{aligned}$$

Then $\rho(h) \in \mathrm{SO}(n)$ is defined by:

$$\rho(h)e_j = \big(\cos(t/2) + \sin(t/2)e_1e_2\big)e_j\big(\cos(t/2) - \sin(t/2)e_1e_2\big)$$

so that $\rho(h)e_j = e_j$ for $j > 2$. We compute:

$$\begin{aligned} \rho(h)e_1 &= \big(\cos(t/2)e_1 + \sin(t/2)e_2\big)\big(\cos(t/2) - \sin(t/2)e_1e_2\big) \\ &= \big(\cos^2(t/2) - \sin^2(t/2)\big)e_1 + 2\sin(t/2)\cos(t/2)e_2 \\ \rho(h)e_2 &= \big(\cos(t/2)e_2 - \sin(t/2)e_1\big)\big(\cos(t/2) - \sin(t/2)e_1e_2\big) \\ &= \big(\cos^2(t/2) - \sin^2(t/2)\big)e_2 - 2\sin(t/2)\cos(t/2)e_1 \end{aligned}$$

so that $\rho(h) = g$. This gives the desired lift from $\mathrm{SO}(n)$ to $\mathrm{SPIN}(n)$. We differentiate to get

$$h'(0) = \frac{1}{2}e_1e_2 = \frac{1}{4}A_{jk}e_j * e_k.$$

This gives the lift of a matrix in this particular form. Since the whole Lie algebra is generated by elements of this form, it proves that the lift of A_{jk} in general is given by $\frac{1}{4}A_{jk}e_j * e_k$. Since $\mathrm{SPIN}(n)$ acts on the Clifford algebra by Clifford multiplication on the left, this gives the action of the curvature and connection 1-form on the spin representations and completes the proof.

We can define $ch(\Delta^{\pm}(V))$ as SO characteristic forms. They can be expressed in terms of the Pontrjagin and Euler forms. We could use the explicit representation given by Lemma 3.3.5 to compute these forms; it is most easy, however, to compute in terms of generating functions. We introduce formal variables $\{x_j\}$ for $1 \le j \le (\dim V)/2$ so that

$$p(V) = \prod_j (1 + x_j^2) \qquad \text{and} \qquad e(V) = \prod_j x_j.$$

All these computations are really induced from the corresponding matrix identities and the $\{x_j\}$ arise from putting a skew-symmetric matrix A in block form.

Lemma 3.3.6. *Let V be an oriented real Riemannian vector bundle of dimension $n \equiv 0$ (2). Let $ch(\Delta^{\pm}(V))$ be a real characteristic form; these are well defined even if V does not admit a global spin structure. Then:*
(a) $ch(\Delta^{+}(V)) + ch(\Delta^{-}(V)) = ch(\Delta(V)) = \prod_j \{e^{x_j/2} + e^{-x_j/2}\}$.
(b)

$$\begin{aligned}(-1)^{n/2}\{ch(\Delta^{+}(V)) - ch(\Delta^{-}(V))\} &= \prod_j \{e^{x_j/2} - e^{-x_j/2}\} \\ &= e(V)(1 + \text{ higher order terms}).\end{aligned}$$

Proof: This is an identity among invariant polynomials in the Lie algebra so(n). We may therefore restrict to elements of the Lie algebra which split in block diagonal form. Using the multiplicative properties of the Chern character and Lemma 3.3.4, it suffices to prove this lemma for the special case that $n = 2$ so $\Delta^{\pm}(V)$ are complex line bundles. Let V_c be a complex line bundle and let V be the underlying real bundle. Then $x_1 = x = c_1(V_c) = e(V)$. Since:

$$V_c = \Delta^{-} \otimes \Delta^{-} \qquad \text{and} \qquad V_c^{*} = \Delta^{+} \otimes \Delta^{+}$$

we conclude:

$$x = 2c_1(\Delta^{-}) \qquad \text{and} \qquad -x = 2c_1(\Delta^{+})$$

which shows (a) and the first part of (b). We expand:

$$e^{x/2} - e^{-x/2} = x + \frac{1}{24}x^3 + \cdots$$

to see $ch(\Delta^{-}) - ch(\Delta^{+}) = e(V)(1 + \frac{1}{24}p_1(V) + \cdots)$ to complete the proof of (b).

There is one final calculation in characteristic classes which will prove helpful. We defined L and $\hat{A}$ by the generating functions:

$$\begin{aligned} L(x) &= \prod_j \frac{x_j}{\tanh x_j} \\ &= \prod_j x_j \frac{e^{x_j} + e^{-x_j}}{e^{x_j} - e^{-x_j}} \\ \hat{A}(x) &= \prod_j \frac{x_j}{\sinh(x_j/2)} = \prod_j \frac{x_j}{e^{x_j/2} - e^{-x_j/2}}. \end{aligned}$$

Lemma 3.3.7. *Let V be an oriented Riemannian real vector bundle of dimension $n = 2n_0$. If we compute the component of the differential form which is in $\Lambda^n(T^*M)$ then:*

$$\{L(V))\}_n = \{ch(\Delta(V)) \wedge \hat{A}(V)\}_n.$$

Proof: It suffices to compute the component of the corresponding symmetric functions which are homogeneous of order n_0. If we replace x_j by $x_j/2$ then:

$$\begin{aligned} \{L(x_j)\}n_0 &= \{2^{n_0} L(x_j/2)\}_{n_0} \\ &= \left\{ \prod_j x_j \tanh(x_j/2) \right\}_{n_0} \\ &= \left\{ \prod_j x_j \frac{e^{x/2} + e^{-x/2}}{e^{x_j/2} - e^{-x_j/2}} \right\}_{n_0} \\ &= \{ch(\Delta(V)) \wedge \hat{A}(V)\}_{n_0} \end{aligned}$$

which completes the proof.

3.4. The Spin Complex.

We shall use the spin complex chiefly as a formal construction to link the de Rham, signature, and Dolbeault complexes. Let M be a Riemannian manifold of even dimension m. Let $T(M)$ be the real tangent space. We assume that M is orientable and that $T(M)$ admits a spin structure. We let $\Delta^{\pm}(M)$ be the half-spin representations. There is a natural map given by Lemma 3.2.3 from the representations of

$$T^*(M) \to \mathrm{HOM}(\Delta^{\pm}(M), \Delta^{\mp}(M))$$

which we will call $c(\xi)$ (since it is essentially Clifford multiplication) such that $c(\xi)^2 = -|\xi|^2 I$. We extend the Levi-Civita connection to act naturally on these bundles and define the spin complex by the diagram:

$$A^{\pm}: C^\infty(\Delta^{\pm}(M)) \xrightarrow{\nabla} C^\infty(T^*M \otimes \Delta^{\pm}(M)) \xrightarrow{c} C^\infty(\Delta^{\mp}(M))$$

to be the operator with leading symbol c. $(A^+)^* = A^-$ and $A^{\pm}$ is elliptic since $c(\xi)^2 = -|\xi|^2 I$; this operator is called the Dirac operator.

Let V be a complex bundle with a Riemannian connection ∇. We define the spin complex with coefficients in ∇ by using the diagram:

$$\begin{aligned} A_V^{\pm}: C^\infty(\Delta^{\pm}(M) \otimes V) \to C^\infty(T^*M \otimes \Delta^{\pm}(M) \otimes V)& \\ \xrightarrow{c\otimes 1} C^\infty(\Delta^{\mp}(M) \otimes V)&. \end{aligned}$$

This is completely analogous to the signature complex with coefficients in V. We define:

$$\mathrm{index}(V, \mathrm{spin}) = \mathrm{index}(A_V^+);$$

a priori this depends on the particular spin structure chosen on V, but we shall show shortly that it does not depend on the particular spin structure, although it does depend on the orientation. We let

$$a_n^{\mathrm{spin}}(x, V)$$

denote the invariants of the heat equation. If we reverse the orientation of M, we interchange the roles of Δ^+ and of Δ^- so a_n^{spin} changes sign. This implies a_n^{spin} can be regarded as an invariantly defined m-form since the scalar invariant changes sign if we reverse the orientation.

Let X be an oriented coordinate system. We apply the Gramm-Schmidt process to the coordinate frame to construct a functorial orthonormal frame $\vec{s}(X)$ for $T(M) = T^*(M)$. We lift this to define two local sections $\vec{s}_i(X)$ to the principal bundle P_{SPIN} with $\vec{s}_1(X) = -\vec{s}_2(X)$. There is, of course, no cannonical way to prefer one over the other, but the pair is invariantly defined. Let $b^{\pm}$ be fixed bases for the representation space $\Delta^{\pm}$ and let

$b_i^\pm(X) = \vec{s}_i(X) \otimes b^\pm$ provide local frames for $\Delta^\pm(M)$. If we fix $i = 1$ or $i = 2$, the symbol of the spin complex with coefficients in V can be functorially expressed in terms of the 1-jets of the metric on M and in terms of the connection 1-form on V. The leading symbol is given by Clifford multiplication; the 0^{th} order term is linear in the 1-jets of the metric on M and in the connection 1-form on V with coefficients which are smooth functions of the metric on M. If we replace $b_1^\pm$ by $b_2^\pm = -b_1^\pm$ then the local representation of the symbol is unchanged since multiplication by -1 commutes with differential operators. Thus we may regard $a_n^{\text{spin}}(x, V) \in \mathcal{R}_{m,n,m,\dim V}$ as an invariantly defined polynomial which is homogeneous of order n in the jets of the metric and of the connection form on V which is m-form valued.

This interpretation defines $a_n^{\text{spin}}(x, V)$ even if the base manifold M is not spin. We can always define the spin complex locally as the $\mathbf{Z}_2$ indeterminacy in the choice of a spin structure will not affect the symbol of the operator. Of course, $\int_M a_m^{\text{spin}}(x, V)$ can only be given the interpretation of index(V, spin) if M admits a spin structure. In particular, if this integral is not an integer, M cannot admit a spin structure.

$a_n^{\text{spin}}(x, V)$ is a local invariant which is not affected by the particular global spin structure chosen. Thus index(V, spin) is independent of the particular spin structure chosen.

We use exactly the same arguments based on the theorem of the second chapter and the multiplicative nature of the twisted spin complex as were used to prove the Hirzebruch signature theorem to establish:

LEMMA 3.4.1.
(a) $a_n^{\text{spin}} = 0$ *if* $n < m$ *and* a_m^{spin} *is a characteristic form of* $T(M)$ *and of* V.
(b) $\int_M a_m^{\text{spin}}(x, V) = \text{index}(V, \text{spin})$.
(c) There exists a characteristic form $\hat{A}'$ *of* $T(M)$ *in the form:*

$$\hat{A}' = 1 + \hat{A}_1 + \cdots$$

which does not depend on the dimension of M *together with a universal constant* c *such that*

$$a_m^{\text{spin}} = \sum_{4s+2t=m} \hat{A}'_s \wedge c^t ch(V)_t.$$

We use the notation $\hat{A}'$ since we have not yet shown it is the A-roof genus defined earlier. In proving the formula splits into this form, we do not rely on the uniqueness property of the second chapter, but rather on the multiplicative properties of the invariants of the heat equation discussed in the first chapter.

The spin complex has an intimate relation with both the de Rham and signature complexes:

Lemma 3.4.2. *Let U be an open contractible subset of M. Over U, we define the signature, de Rham, and spin complexes. Then:*
(a) $(\Lambda^{\mathrm{e}} - \Lambda^{\mathrm{o}}) \otimes V \simeq (-1)^{m/2}(\Delta^+ - \Delta^-) \otimes (\Delta^+ - \Delta^-) \otimes V$.
(b) $(\Lambda^+ - \Lambda^-) \otimes V \simeq (\Delta^+ - \Delta^-) \otimes (\Delta^+ + \Delta^-) \otimes V$.
(c) These two isomorphisms preserve the unitary structures and the connections. They also commute with Clifford multiplication on the left.
(d) The natural operators on these complexes agree under this isomorphism.

Proof: (a)–(c) follow from previous results. The natural operators on these complexes have the same leading symbol and therefore must be the same since they are natural first order operators. This proves (d).

We apply this lemma in dimension $m = 2$ with $V =$ the trivial bundle and $M = S^2$:

$$\begin{aligned}\chi(S^2) = 2 = \operatorname{index}(d+\delta) &= \operatorname{index}(\Delta^- - \Delta^+, \mathrm{spin})\\ &= \int_{S^2} c\{c_1(\Delta^-) - c_1(\Delta^+)\}\\ &= c\int_{S^2} e(TM) = 2c\end{aligned}$$

by Lemmas 2.3.1, 3.3.5, and 3.3.4. This establishes that the normalizing constant of Lemma 3.4.1 must be 1 so that:

$$a_m^{\mathrm{spin}} = \sum_{4s+2t=m} \hat{A}' \wedge ch_t(V).$$

We apply this lemma to the twisted signature complex in dimension $m = 2$ with V a non-trivial line bundle over S^2 to conclude:

$$\begin{aligned}\operatorname{signature}(S^2, V) &= \operatorname{index}((\Delta^+ \oplus \Delta^-) \otimes V, \mathrm{spin})\\ &= \int_{S^2} ch((\Delta^+ \oplus \Delta^-) \otimes V) = \int_{S^2} 2 \cdot c_1(V).\end{aligned}$$

This shows that the normalizing constant of Lemma 3.1.4 for the twisted signature complex is 2 and completes the proof of Lemma 3.1.4. This therefore completes the proof of the Hirzebruch signature theorem in general.

The de Rham complex with coeffiecients in V is defined by the diagram:

$$C^\infty(\Lambda^{\mathrm{e,o}}(M) \otimes V) \xrightarrow{\nabla} C^\infty(T^*M \otimes \Lambda^{\mathrm{e,o}}(M) \otimes V) \xrightarrow{c\otimes 1} C^\infty(\Lambda^{\mathrm{o,e}}(M) \otimes V)$$

and we shall denote the operator by $(d+\delta)_V^{\mathrm{e}}$. The relations given by Lemma 3.4.2 give rise to relations among the local formulas:

$$\begin{aligned}a_n(x, (d+\delta)_V^{\mathrm{e}}) &= a_n(x, A^+_{(-1)^{m/2}(\Delta^+ - \Delta^-)\otimes V})\\ a_n(x, (d+\delta)_V^{+}) &= a_n(x, A^+_{(\Delta^+ - \Delta^-)\otimes V})\end{aligned}$$

where $(d+\delta)_V^+$ is the operator of the twisted signature complex. These relations are well defined regardless of whether or not M admits a spin structrue on $T(M)$.

We deal first with the de Rham complex. Using Lemma 3.4.1 and the fact that the normalizing constant c is 1, we conclude:

$$a_n(x,(d+\delta)_V^{\mathrm{e}}) = \hat{A}' \wedge ch((-1)^{m/2}(\Delta^+ - \Delta^-)) \wedge ch(V).$$

Since $ch((-1)^{m/2}(\Delta^+ - \Delta^-)) = e(M)$ is already a top dimensional form by Lemma 3.3.5(b), we conclude $a_n(x,(d+\delta)_V^{\mathrm{e}}) = (\dim V)e(M)$. This proves:

THEOREM 3.4.3. *Let $(d+\delta)_V^{\mathrm{e}}$ be the de Rham complex with coefficients in the bundle V and let $a_n(x,(d+\delta)_V^{\mathrm{e}})$ be the invariants of the heat equation. Then:*
(a) $a_n(x,(d+\delta)_V^{\mathrm{e}}) = 0$ *if* $n < m$.
(b) $a_m(x,(d+\delta)_V^{\mathrm{e}}) = (\dim V)e(M)$ *where* $e(M)$ *is the Euler form of* $T(M)$.
(c)
$$\operatorname{index}((d+\delta)_V^{\mathrm{e}}) = (\dim V)\chi(M) = \int_m (\dim V)e(M).$$

This shows that no information about V (except its dimension) is obtained by considering the de Rham complex with coefficients in V and it is for this reason we did not introduce this complex earlier. This gives a second proof of the Gauss-Bonnet theorem independent of the proof we gave earlier.

Next, we study the signature complex in order to compute $\hat{A}$. Using Lemma 3.4.1 with the bundle $V = 1$, we conclude for $m = 4k$,

$$a_m(x,(d+\delta)_V^+) = \{\hat{A}' \wedge ch(\Delta)\}_m.$$

Using Theorem 3.1.1 and Lemma 3.3.6, we compute therefore:

$$L_k = \{ch(\Delta) \wedge \hat{A}\}_m = \{ch(\Delta) \wedge \hat{A}'\}_m = a_m(x,(d+\delta)_V^+)$$

so as the Chern character is formally invertible; $\hat{A} = \hat{A}'$ is given by the generating function $z_j/\sinh(z_j/2)$ by Lemma 3.3.7. We can now improve Lemma 3.4.1 and determine all the relevant normalizing constants.

THEOREM 3.4.4. *Let $T(M)$ admit a spin structure, then:*
(a) $a_n^{\mathrm{spin}} = 0$ *if* $n < m$.
(b) $a_m^{\mathrm{spin}}(x,V) = \sum_{4s+2t=m} \hat{A}_s \wedge ch_t(V)$.
(c) $\operatorname{index}(V,\mathrm{spin}) = \int_M a_m^{\mathrm{spin}}(x,V)$.

3.5. The Riemann-Roch Theorem For Almost Complex Manifolds.

So far, we have discussed three of the four classical elliptic complexes, the de Rham, the signature, and the spin complexes. In this subsection, we define the Dolbeault complex for an almost complex manifold and relate it to the spin complex.

Let M be a Riemannian manifold of dimension $m = 2n$. An almost complex structure on M is a linear map $J: T(M) \to T(M)$ with $J^2 = -1$. The Riemannian metric G is unitary if $G(X,Y) = G(JX, JY)$ for all $X, Y \in T(M)$; we can always construct unitary metrics by averaging over the action of J. Henceforth we assume G is unitary. We extend G to be Hermitian on $T(M) \otimes \mathbf{C}$, $T^*(M) \otimes \mathbf{C}$, and $\Lambda(M) \otimes \mathbf{C}$.

Since $J^2 = -1$, we decompose $T(M) \otimes \mathbf{C} = T'(M) \oplus T''(M)$ into the $\pm i$ eigenspaces of J. This direct sum is orthogonal with respect to the metric G. Let $\Lambda^{1,0}(M)$ and $\Lambda^{0,1}(M)$ be the dual spaces in $T^*(M) \otimes \mathbf{C}$ to T' and T''. We choose a local frame $\{e_j\}$ for $T(M)$ so that $J(e_j) = e_{j+n}$ for $1 \le i \le n$. Let $\{e^j\}$ be the corresponding dual frame for $T^*(M)$. Then:

$$\begin{aligned} T'(M) &= \operatorname{span}_{\mathbf{C}}\{e_j - ie_{j+n}\}_{j=1}^n & T''(M) &= \operatorname{span}_{\mathbf{C}}\{e_j + ie_{j+n}\}_{j=1}^n \\ \Lambda^{1,0}(M) &= \operatorname{span}_{\mathbf{C}}\{e^j + ie^{j+n}\}_{j=1}^n & \Lambda^{0,1}(M) &= \operatorname{span}_{\mathbf{C}}\{e^j - ie^{j+n}\}_{j=1}^n. \end{aligned}$$

These four vector bundles are all complex vector bundles over M. The metric gives rise to natural isomorphisms $T'(M) \simeq \Lambda^{0,1}(M)$ and $T''(M) = \Lambda^{1,0}(M)$. We will use this isomorphism to identify e_j with e^j for much of what follows.

If we forget the complex structure on $T'(M)$, then the underlying real vector bundle is naturally isomorphic to $T(M)$. Complex multiplication by i on $T'(M)$ is equivalent to the endomorphism J under this identification. Thus we may regard J as giving a complex structure to $T(M)$.

The decomposition:

$$T^*(M) \otimes \mathbf{C} = \Lambda^{1,0}(M) \oplus \Lambda^{0,1}(M)$$

gives rise to a decomposition:

$$\Lambda(T^*M) \otimes \mathbf{C} = \bigoplus_{p,q} \Lambda^{p,q}(M)$$

for

$$\Lambda^{p,q}(M) = \Lambda^p(\Lambda^{1,0}(M)) \otimes \Lambda^q(\Lambda^{0,1}(M)).$$

Each of the bundles $\Lambda^{p,q}$ is a complex bundle over M and this decomposition of $\Lambda(T^*M) \otimes \mathbf{C}$ is orthogonal. Henceforth we will denote these bundles by T', T'', and $\Lambda^{p,q}$ when no confusion will arise.

If M is a holomorphic manifold, we let $z = (z_1, \dots, z_n)$ be a local holomorphic coordinate chart. We expand $z_j = x_j + iy_j$ and define:

$$\begin{aligned} \frac{\partial}{\partial z_j} &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), & \frac{\partial}{\partial \bar{z}_j} &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) \\ dz_j &= dx_j + idy_j, & d\bar{z}_j &= dx_j - idy_j. \end{aligned}$$

We define:

$$\partial(f) = \sum_j \frac{\partial f}{\partial z_j}\, dz_j \qquad \text{and} \qquad \bar{\partial}(f) = \sum_j \frac{\partial f}{\partial \bar{z}_j}\, d\bar{z}_j.$$

The Cauchy-Riemann equations imply that a function f is holomorphic if and only if $\bar{\partial}(f) = 0$. If $w = (w_1, \dots, w_n)$ is another holomorphic coordinate system, then:

$$\begin{aligned} \frac{\partial}{\partial w_j} &= \sum_k \frac{\partial z_k}{\partial w_j}\frac{\partial}{\partial z_k}, & \frac{\partial}{\partial \bar{w}_j} &= \sum_k \frac{\partial \bar{z}_k}{\partial \bar{w}_j}\frac{\partial}{\partial \bar{z}_k} \\ dw_j &= \sum_k \frac{\partial w_j}{\partial z_k}\, dz_k, & d\bar{w}_j &= \frac{\partial \bar{w}_j}{\partial \bar{z}_k}\, d\bar{z}_k. \end{aligned}$$

We define:

$$\begin{aligned} T'(M) &= \operatorname{span}\left\{\frac{\partial}{\partial z_j}\right\}_{j=1}^n, & T''(M) &= \operatorname{span}\left\{\frac{\partial}{\partial \bar{z}_j}\right\}_{j=1}^n \\ \Lambda^{1,0}(M) &= \operatorname{span}\{dz_j\}_{j=1}^n, & \Lambda^{0,1}(M) &= \operatorname{span}\{d\bar{z}_j\}_{j=1}^n \end{aligned}$$

then these complex bundles are invariantly defined independent of the choice of the coordinate system. We also note

$$\partial\colon C^\infty(M) \to C^\infty(\Lambda^{1,0}(M)) \qquad \text{and} \qquad \bar{\partial}\colon C^\infty(M) \to C^\infty(\Lambda^{0,1}(M))$$

are invariantly defined and decompose $d = \partial + \bar{\partial}$.

There is a natural isomorphism of $T'(M)$ with $T(M)$ as real bundles in this example and we let J be complex multiplication by i under this isomorphism. Equivalently:

$$J\left(\frac{\partial}{\partial x_j}\right) = \left(\frac{\partial}{\partial y_j}\right) \qquad \text{and} \qquad J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}.$$

$T'(M) = T_c(M)$ is the complex tangent bundle in this example; we shall reserve the notation $T_c(M)$ for the holomorphic case.

Not every almost complex structure arises from a complex structure; there is an integrability condition. If J is an almost complex structure, we decompose the action of exterior differentiation d on $C^\infty(\Lambda)$ with respect to the bigrading (p,q) to define:

$$\partial: C^\infty(\Lambda^{p,q}) \to C^\infty(\Lambda^{p+1,q}) \qquad \text{and} \qquad \bar\partial: C^\infty(\Lambda^{p,q}) \to C^\infty(\Lambda^{p,q+1}).$$

THEOREM 3.5.1 (NIRENBERG-NEULANDER). *The following are equivalent and define the notion of an integrable almost complex structure:*
(a) The almost complex structure J arises from a holomorphic structure on M.
(b) $d = \partial + \bar\partial$.
(c) $\bar\partial\bar\partial = 0$.
(d) $T'(M)$ is integrable—i.e., given $X, Y \in C^\infty(T'(M))$, then the Lie bracket $[X,Y] \in C^\infty(T'(M))$ where we extend $[\ \ ,\ \ ]$ to complex vector fields in the obvious fashion.

PROOF: This is a fairly deep result and we shall not give complete details. It is worth, however, giving a partial proof of some of the implications to illustrate the concepts we will be working with. Suppose first M is holomorphic and let $\{z_j\}$ be local holomorphic coordinates on M. Define:

$$dz^I = dz_{i_1} \wedge \cdots \wedge dz_{i_p} \qquad \text{and} \qquad d\bar z^J = d\bar z_{j_1} \wedge \cdots \wedge d\bar z_{j_q}$$

then the collection $\{dz^I \wedge d\bar z^J\}$ gives a local frame for $\Lambda^{p,q}$ which is *closed*. If $\omega \in C^\infty(\Lambda^{p,q})$, we decompose $\omega = \sum f_{I,J}\, dz^I \wedge d\bar z^J$ and compute:

$$d\omega = d\left\{\sum f_{I,J}\, dz^I \wedge d\bar z^J\right\} = \sum df_{I,J} \wedge dz^I \wedge d\bar z^J.$$

On functions, we decompose $d = \partial + \bar\partial$. Thus $d\omega \in C^\infty(\Lambda^{p+1,q} \oplus \Lambda^{p,q+1})$ has no other components. Therefore $d\omega = \partial\omega + \bar\partial\omega$ so (a) implies (b).

We use the identity $d^2 = 0$ to compute $(\partial + \bar\partial)^2 = (\partial^2) + (\partial\bar\partial + \bar\partial\partial) + (\bar\partial^2) = 0$. Using the bigrading and decomposing this we conclude $(\partial^2) = (\partial\bar\partial + \bar\partial\partial) = (\bar\partial^2) = 0$ so (b) implies (c). Conversely, suppose that $\bar\partial\bar\partial = 0$ on $C^\infty(M)$. We must show $d: C^\infty(\Lambda^{p,q}) \to C^\infty(\Lambda^{p+1,q} \oplus \Lambda^{p,q+1})$ has no other components. Let $\{e_j\}$ be a local frame for $\Lambda^{1,0}$ and let $\{\bar e_j\}$ be the corresponding local frame for $\Lambda^{0,1}$. We decompose:

$$\begin{aligned} de_j &= \partial e_j + \bar\partial e_j + A_j \qquad && \text{for } A_j \in \Lambda^{0,2} \\ d\bar e_j &= \bar\partial \bar e_j + \partial \bar e_j + \bar A_j \qquad && \text{for } \bar A_j \in \Lambda^{2,0}. \end{aligned}$$

Then we compute:

$$\begin{aligned} d\Big(\sum f_j e_j\Big) &= \sum df_j \wedge e_j + f_j de_j \\ &= \sum\{\partial f_j \wedge e_j + \bar\partial f_j \wedge e_j + f_j \partial e_j + f_j \bar\partial e_j + f_j A_j\} \\ &= \partial\Big(\sum f_j e_j\Big) + \bar\partial\Big(\sum f_j e_j\Big) + \sum f_j A_j. \end{aligned}$$

Similarly

$$d\Big(\sum f_j\bar e_j\Big)=\partial\Big(\sum f_j\bar e_j\Big)+\bar\partial\Big(\sum f_j\bar e_j\Big)+\sum f_j\bar A_j.$$

If A and $\bar A$ denote the 0^{th} order operators mapping $\Lambda^{1,0}\to\Lambda^{0,2}$ and $\Lambda^{0,1}\to\Lambda^{2,0}$ then we compute:

$$\begin{aligned} d(\omega)&=\partial(\omega)+\bar\partial(\omega)+A\omega &&\text{for } \omega\in C^\infty(\Lambda^{1,0})\\ d(\bar\omega)&=\partial(\bar\omega)+\bar\partial(\bar\omega)+\bar A\bar\omega &&\text{for } \bar\omega\in C^\infty(\Lambda^{0,1}).\end{aligned}$$

Let $f\in C^\infty(M)$ and compute:

$$0=d^2f=d(\partial f+\bar\partial f)=(\partial^2+\bar A\bar\partial)f+(\partial\bar\partial+\bar\partial\partial)f+(\bar\partial^2+A\partial)f$$

where we have decomposed the sum using the bigrading. This implies that $(\bar\partial^2+A\partial)f=0$ so $A\partial f=0$. Since $\{\partial f\}$ spans $\Lambda^{1,0}$, this implies $A=\bar A=0$ since A is a 0^{th} order operator. Thus $de_j=\partial e_j+\bar\partial e_j$ and $d\bar e_j=\partial\bar e_j+\bar\partial\bar e_j$. We compute $d(e^I\wedge\bar e^J)$ has only $(p+1,q)$ and $(p,q+1)$ components so $d=\partial+\bar\partial$. Thus (b) and (c) are equivalent.

It is immediate from the definition that $X\in T'(M)$ if and only if $\omega(X)=0$ for all $\omega\in\Lambda^{0,1}$. If $X,Y\in C^\infty(T^*M)$, then Cartan's identity implies:

$$d\omega(X,Y)=X(\omega Y)-Y(\omega X)-\omega([X,Y])=-\omega([X,Y]).$$

If (b) is true then $d\omega$ has no component in $\Lambda^{2,0}$ so $d\omega(X,Y)=0$ which implies $\omega([X,Y])=0$ which implies $[X,Y]\in C^\infty(T'M)$ which implies (d). Conversely, if (d) is true, then $d\omega(X,Y)=0$ so $d\omega$ has no component in $\Lambda^{2,0}$ so $d=\partial+\bar\partial$ on $\Lambda^{0,1}$. By taking conjugates, $d=\partial+\bar\partial$ on $\Lambda^{1,0}$ as well which implies as noted above that $d=\partial+\bar\partial$ in general which implies (c).

We have proved that (b)–(d) are equivalent and that (a) implies (b). The hard part of the theorem is showing (b) implies (a). We shall not give this proof as it is quite lengthy and as we shall not need this implication of the theorem.

As part of the previous proof, we computed that $d-(\partial+\bar\partial)$ is a 0^{th} order operator (which vanishes if and only if M is holomorphic). We now compute the symbol of both ∂ and $\bar\partial$. We use the metric to identify $T(M)=T^*(M)$. Let $\{e_j\}$ be a local orthonormal frame for $T(M)$ such that $J(e_j)=e_{j+n}$ for $1\le j\le n$. We extend ext and int to be complex linear maps from $T^*(M)\otimes\mathbf{C}\to\mathrm{END}(\Lambda(T^*M)\otimes\mathbf{C})$.

Lemma 3.5.2. *Let ∂ and $\bar\partial$ be defined as before and let δ' and δ'' be the formal adjoints. Then:*

(a)

$$\partial\colon C^\infty(\Lambda^{p,q}) \to C^\infty(\Lambda^{p+1,q})$$
$$\text{and} \quad \sigma_L(\partial)(x,\xi) = \frac{i}{2}\sum_{j\le n}(\xi_j - i\xi_{j+n})\operatorname{ext}(e_j + ie_{j+n})$$
$$\bar\partial\colon C^\infty(\Lambda^{p,q}) \to C^\infty(\Lambda^{p,q+1})$$
$$\text{and} \quad \sigma_L(\bar\partial)(x,\xi) = \frac{i}{2}\sum_{j\le n}(\xi_j + i\xi_{j+n})\operatorname{ext}(e_j - ie_{j+n})$$
$$\delta'\colon C^\infty(\Lambda^{p,q}) \to C^\infty(\Lambda^{p-1,q})$$
$$\text{and} \quad \sigma_L(\delta')(x,\xi) = -\frac{i}{2}\sum_{j\le n}(\xi_j + i\xi_{j+n})\operatorname{int}(e_j - ie_{j+n})$$
$$\delta''\colon C^\infty(\Lambda^{p,q}) \to C^\infty(\Lambda^{p,q-1})$$
$$\text{and} \quad \sigma_L(\delta'')(x,\xi) = -\frac{i}{2}\sum_{j\le n}(\xi_j - i\xi_{j+n})\operatorname{int}(e_j + ie_{j+n}).$$

(b) If $\Delta'_c = (\partial + \delta')^2$ and $\Delta''_c = (\bar\partial + \delta'')^2$ then these are elliptic on Λ with $\sigma_L(\Delta'_c) = \sigma_l(\Delta''_c) = \frac{1}{2}|\xi|^2 I$.

PROOF: We know $\sigma_L(d)(x,\xi) = i\sum_j \xi_j \operatorname{ext}(e_j)$. We define

$$A(\xi) = \frac{1}{2}\sum_{j\le n}(\xi_{j+n})\operatorname{ext}(e_j + ie_{j+n})$$

then $A(\xi)\colon \Lambda^{p,q} \to \Lambda^{p+1,q}$ and $\bar A(\xi)\colon \Lambda^{p,q} \to \Lambda^{p,q+1}$. Since $iA(\xi) + i\bar A(\xi) = \sigma_L(d)$, we conclude that iA and $i\bar A$ represent the decomposition of $\sigma_L(d)$ under the bigrading and thus define the symbols of ∂ and $\bar\partial$. The symbol of the adjoint is the adjoint of the symbol and this proves (a). (b) is an immediate consequence of (a).

If M is holomorphic, the Dolbeault complex is the complex $\{\bar\partial, \Lambda^{0,q}\}$ and the index of this complex is called the arithmetic genus of M. If M is not holomorphic, but only has an almost complex structure, then $\bar\partial^2 \ne 0$ so we can not define the arithmetic genus in this way. Instead, we use a trick called "rolling up" the complex. We define:

$$\Lambda^{0,+} = \bigoplus_q \Lambda^{0,2q} \qquad \text{and} \qquad \Lambda^{0,-} = \bigoplus_q \Lambda^{0,2q+1}$$

to define a $\mathbf{Z}_2$ grading on the Dolbeault bundles. (These are also often denoted by $\Lambda^{0,\text{even}}$ and $\Lambda^{0,\text{odd}}$ in the literature). We consider the two term elliptic complex:

$$(\bar\partial + \delta'')_\pm\colon C^\infty(\Lambda^{0,\pm}) \to C^\infty(\Lambda^{0,\mp})$$

and define the arithmetic genus of M to be the index of $(\bar\partial+\delta'')_+$. The adjoint of $(\bar\partial+\delta'')_+$ is $(\bar\partial+\delta'')_-$ and the associated Laplacian is just Δ''_c restricted to $\Lambda^{0,\pm}$ so this is an elliptic complex. If M is holomorphic, then the index of this elliptic complex is equal to the index of the complex $(\bar\partial,\Lambda^{0,q})$ by the Hodge decomposition theorem.

We define:

$$c'(\xi)=(\sqrt{2})^{-1}\sum_{j\le n}\{(\xi_j+i\xi_{j+n})\operatorname{ext}(e_j-ie_{j+n}) \\ -(\xi_j-i\xi_{j+n})\operatorname{int}(e_j+ie_{j+n})\}$$

then it is immediate that:

$$c'(\xi)c'(\xi)=-|\xi|^2 \qquad \text{and} \qquad \sigma_L(\bar\partial+\delta'')=ic'(\xi)/\sqrt{2}.$$

Let ∇ be a connection on $\Lambda^{0,\pm}$, then we define the operator $A^\pm(\nabla)$ by the diagram:

$$A^\pm(\nabla)\colon C^\infty(\Lambda^{0,\pm})\xrightarrow{\nabla}C^\infty(T^*M\otimes\Lambda^{0,\pm})\xrightarrow{c'/\sqrt{2}}C^\infty(\Lambda^{0,\mp})$$

This will have the same leading symbol as $(\bar\partial+\delta'')_\pm$. There exists a unique connection so $A^\pm(\nabla)=(\bar\partial+\delta'')_\pm$ but as the index is constant under lower order perturbations, we shall not need this fact as the index of $A^+(\nabla)=\operatorname{index}(\bar\partial+\delta'')_+$ for any ∇. We shall return to this point in the next subsection.

We now let V be an arbitrary coefficient bundle with a connection ∇ and define the Dolbeault complex with coefficients in V using the diagram:

$$A^\pm\colon C^\infty(\Lambda^{0,\pm}\otimes V)\xrightarrow{\nabla}C^\infty(T^*M\otimes\Lambda^{0,\pm}\otimes V)\xrightarrow{c'/\sqrt{2}\otimes 1}C^\infty(\Lambda^{0,\mp}\otimes V)$$

and we define $\operatorname{index}(V,\text{Dolbeault})$ to be the index of this elliptic complex. The index is independent of the connections chosen, of the fiber metrics chosen, and is constant under perturbations of the almost complex structure.

If M is holomorphic and if V is a holomorphic vector bundle, then we can extend $\bar\partial\colon C^\infty(\Lambda^{0,q}\otimes V)\to C^\infty(\Lambda^{0,q+1}\otimes V)$ with $\bar\partial\bar\partial=0$. Exactly as was true for the arithmetic genus, the index of this elliptic complex is equal to the index of the rolled up elliptic complex so our definitions generalize the usual definitions from the holomorphic category to the almost complex category.

We will compute a formula for $\operatorname{index}(V,\text{Dolbeault})$ using the spin complex. There is a natural inclusion from $\mathrm{U}(\frac{m}{2})=\mathrm{U}(n)$ into $\mathrm{SO}(m)$, but this does not lift in general to $\mathrm{SPIN}(m)$. We saw earlier that $T(\mathbf{C}P_k)$

does not admit a spin structure if k is even, even though it does admit a unitary structure. We define $\mathrm{SPIN}_c(m) = \mathrm{SPIN}(m) \times S^1/\mathbf{Z}_2$ where we choose the $\mathbf{Z}_2$ identification $(g, \lambda) = (-g, -\lambda)$. There is a natural map $\rho_c\colon \mathrm{SPIN}_c(m) \to \mathrm{SO}(m) \times S^1$ induced by the map which sends $(g, \lambda) \mapsto (\rho(g), \lambda^2)$. This map is a $\mathbf{Z}_2$ double cover and is a group homomorphism.

There is a natural map of $\mathrm{U}(\frac{m}{2})$ into $\mathrm{SO}(m) \times S^1$ defined by sending $g \mapsto (g, \det(g))$. The interesting thing is that this inclusion does lift; we can define $f\colon \mathrm{U}(\frac{m}{2}) \to \mathrm{SPIN}_c(m)$ so the following diagram commutes:

$$\begin{tikzcd}
 & \mathrm{SPIN}_c(m) \arrow[d, "\rho_c = \rho \times \lambda^2"] \\
\mathrm{U}(\frac{m}{2}) \arrow[ur, "\mathrm{f}"] \arrow[r, "1 \times \det"] & \mathrm{SO}(m) \times S^1
\end{tikzcd}$$

We define the lifting as follows. For $U \in \mathrm{U}(\frac{m}{2})$, we choose a unitary basis $\{e_j\}_{j=1}^n$ so that $U(e_j) = \lambda_j e_j$. We define $e_{j+n} = ie_j$ so $\{e_j\}_{j=1}^{2n}$ is an orthogonal basis for $\mathbf{R}^m = \mathbf{C}^n$. Express $\lambda_j = e^{i\theta_j}$ and define:

$$f(U) = \prod_{j=1}^{n} \{\cos(\theta_j/2) + \sin(\theta_j/2) e_j e_{j+n}\} \times \prod_{j=1}^{n} e^{i\theta_j/2} \in \mathrm{SPIN}_c(m).$$

We note first that $e_j e_{j+n}$ is an invariant of the one-dimensional complex subspace spanned by e_j and does not change if we replace e_j by ze_j for $|z| = 1$. Since all the factors commute, the order in which the eigenvalues is taken does not affect the product. If there is a multiple eigenvalue, this product is independent of the particular basis which is chosen. Finally, if we replace θ_j by $\theta_j + 2\pi$ then both the first product and the second product change sign. Since $(g, \lambda) = (-g, -\lambda)$ in $\mathrm{SPIN}_c(m)$, this element is invariantly defined. It is clear that $f(I) = I$ and that f is continuous. It is easily verified that $\rho_c f(U) = i(U) \times \det(U)$ where $i(U)$ denotes the matrix U viewed as an element of $\mathrm{SO}(m)$ where we have forgotten the complex structure on $\mathbf{C}^n$. This proves that f is a group homomorphism near the identity and consequently f is a group homomorphism in general. ρ_c is a covering projection.

The cannonical bundle K is given by:

$$K = \Lambda^{n,0} = \Lambda^n(T'M)^*$$

so $K^* = \Lambda^n(T'M)$ is the bundle with transition functions $\det(U_{\alpha\beta})$ where the $U_{\alpha\beta}$ are the transitions for $T'M$. If we have a spin structure on M, we can split:

$$P_{\mathrm{SPIN}_c} = P_{\mathrm{SPIN}} \times P_{S^1}/\mathbf{Z}_2$$

where P_{S^1} represents a line bundle L_1 over M. From this description, it is clear $K^* = L_1 \otimes L_1$. Conversely, if we can take the square root of K (or equivalently of K^*), then M admits a spin structure so the obstruction to constructing a spin structure on an almost complex manifold is the obstruction to finding a square root of the cannonical bundle.

Let $V = \mathbf{C}^n = \mathrm{R}^m$ with the natural structures. We extend $\Delta^\pm$ to representations $\Delta^\pm$ of $\mathrm{SPIN}_c(m)$ in the natural way. We relate this representation to the Dolbeault representation as follows:

Lemma 3.5.3. *There is a natural isomorphism between $\Lambda^{0,\pm}$ and $\Delta_c^\pm$ which defines an equivalence of these two representations of $\mathrm{U}(\frac{m}{2})$. Under this isomorphism, the action of V by Clifford multiplication on the left is preserved.*

Proof: Let $\{e_j\}$ be an orthonormal basis for $\mathbf{R}^m$ with $Je_j = e_{j+n}$ for $1 \le j \le n$. Define:

$$\alpha_j = ie_je_{j+n}, \qquad \beta_j = e_j + ie_{j+n}, \qquad \bar\beta_j = e_j - ie_{j+n} \qquad \text{for } 1 \le j \le n.$$

We compute:

$$\beta_j\alpha_j = -\beta_j \quad \text{and} \quad \beta_j\alpha_k = \alpha_k\beta_j \qquad \text{for } k \ne j.$$

We define $\gamma = \beta_1 \dots \beta_n$ then γ spans $\Lambda^{n,0}$ and $\gamma\alpha_j = -\gamma,\ 1 \le j \le n$. We define:

$$\Lambda^{0,*} = \bigoplus_q \Lambda^{0,q} \qquad \text{and} \qquad \Lambda^{n,*} = \bigoplus_q \Lambda^{n,q} = \Lambda^{0,*}\gamma.$$

Since $\dim(\Lambda^{0,*}) = 2^n$, we conclude that $\Lambda^{0,*}\gamma$ is the simultaneous -1 eigenspace for all the α_j and that therefore:

$$\Delta_c = \Lambda^{0,*}\gamma$$

as a left representation space for $\mathrm{SPIN}(m)$. Again, we compute:

$$\alpha_j\beta_j = \beta_j, \quad \alpha_j\bar\beta_j = -\bar\beta_j, \quad \alpha_j\beta_k = \beta_k\alpha_j, \quad \alpha_j\bar\beta_k = \bar\beta_k\alpha_j \qquad \text{for } j \ne k$$

so that if $x \in \Lambda^{0,q}\gamma$ then since $\tau = \alpha_1 \dots \alpha_n$ we have $\tau x = (-1)^q x$ so that

$$\Delta_c^\pm = \Lambda^{0,\pm}\gamma.$$

We now study the induced representation of $\mathrm{U}(n)$. Let

$$g = \big(\cos(\theta/2) + \sin(\theta/2)e_1 J e_1\big) \cdot e^{i\theta/2} \in \mathrm{SPIN}_c(m).$$

We compute:

$$\begin{aligned} g\beta_1 &= \{\cos(\theta/2) - i\sin(\theta/2)\}e^{i\theta/2}\beta_1 = \beta_1 \\ g\bar\beta_1 &= \{\cos(\theta/2) + i\sin(\theta/2)\}e^{i\theta/2}\bar\beta_1 = e^{i\theta}\bar\beta_1 \\ g\beta_k &= \beta_k g \qquad \text{and} \qquad g\bar\beta_k = \bar\beta_k g \quad \text{for } k > 1. \end{aligned}$$

Consequently:

$$g\bar\beta_j\gamma = \begin{cases} \bar\beta_j\gamma & \text{if } j_1 > 1 \\ e^{i\theta}\bar\beta_j\gamma & \text{if } j_1 = 1. \end{cases}$$

A similar computation goes for all the other indices and thus we compute that if $Ue_j = e^{i\theta_j}e_j$ is unitary that:

$$f(U)\bar\beta_J\gamma = e^{i\theta_{j_1}} \dots e^{i\theta_{j_q}}\bar\beta_J\gamma$$

which is of course the natural represenation of $\mathrm{U}(n)$ on $\Lambda^{0,q}$.

Finally we compare the two actions of V by Clifford multiplication. We assume without loss of generality that $\xi = (1, 0, \dots, 0)$ so we must study Clifford multiplication by e_1 on Δ_c and

$$\{\mathrm{ext}(e_1 - ie_{1+n}) - \mathrm{int}(e_1 + ie_{1+n})\}/\sqrt{2} = c'(e_1)$$

on $\Lambda^{0,*}$. We compute:

$$\begin{aligned} e_1\beta_1 &= -1 + ie_1e_2 = (e_1 - ie_2)(e_1 + ie_2)/2 = \bar\beta_1\beta_1/2, \\ e_1\bar\beta_1\beta_1 &= -2\beta_1, \\ e_1\bar\beta_k &= -\bar\beta_k e_1 \qquad \text{for } k > 1. \end{aligned}$$

From this it follows immediately that:

$$e_1\bar\beta_J\gamma = (-1)^q \begin{cases} \bar\beta_1\bar\beta_J\gamma/2 & \text{if } j_1 > 1 \\ 2\bar\beta_{J'}\gamma & \text{where } J' = \{j_2, \dots, j_q\} \text{ if } j_1 = 1. \end{cases}$$

Similarly, we compute:

$$c'(e_1)\bar\beta_J = \begin{cases} \bar\beta_1\bar\beta_J/\sqrt{2} & \text{if } j_1 > 1 \\ -\sqrt{2}\bar\beta_{J'} & \text{for } J' = \{j_2, \dots, j_q\} \text{ if } j_1 = 1. \end{cases}$$

From these equations, it is immediate that if we define $T(\bar\beta_J) = \bar\beta_J\gamma$, then T will not preserve Clifford multiplication. We let $a(q)$ be a sequence of non-zero constants and define $T\colon \Lambda^{0,\pm} \to \Delta_c^\pm$ by:

$$T(\bar\beta_J) = a(|J|)\bar\beta_J\gamma.$$

Since the spaces $\Lambda^{0,q}$ are $\mathrm{U}(n)$ invariant, this defines an equivalence between these two representations of $\mathrm{U}(n)$. T will induce an equivalence between Clifford multiplication if and only if we have the relations:

$$(-1)^q a(q)/2 = a(q+1)/\sqrt{2} \qquad \text{and} \qquad \sqrt{2}a(q-1) = (-1)^q \cdot 2 \cdot a(q).$$

These give rise to the inductive relations:

$$a(q+1) = (-1)^q a(q)/\sqrt{2} \qquad \text{and} \qquad a(q) = (-1)^{q-1} a(q-1)/\sqrt{2}.$$

These relations are consistent and we set $a(q) = (\sqrt{2})^{-q}(-1)^{q(q-1)/2}$ to define the equivalence T and complete the proof of the Lemma.

From this lemma, we conclude:

Lemma 3.5.4. *There is a natural isomorphism of elliptic complexes $(\Delta_c^+ - \Delta_c^-) \otimes V \simeq (\Lambda^{0,+} - \Lambda^{0,-}) \otimes V$ which takes the operator of the* SPIN_c *complex to an operator which has the same leading symbol as the Dolbeault complex (and thus has the same index). Furthermore, we can represent the* SPIN_c *complex locally in terms of the* SPIN *complex in the form: $(\Delta_c^+ - \Delta_c^-) \otimes V = (\Delta^+ - \Delta^-) \otimes L_1 \otimes V$ where L_1 is a local square root of $\Lambda^n(T'M)$.*

We use this sequence of isomorphisms to define an operator on the Dolbeault complex with the same leading symbol as the operator $(\bar\partial + \partial'')$ which is locally isomorphic to the natural operator of the SPIN complex. The $\mathbf{Z}_2$ ambiguity in the definition of L_1 does not affect this construction. (This is equivalent to choosing an appropriate connection called the spin connection on $\Lambda^{0,\pm}$.) We can compute $\operatorname{index}(V, \text{Dolbeault})$ using this operator. The local invariants of the heat equation for this operator are the local invariants of the twisted spin complex and therefore arguing exactly as we did for the signature complex, we compute:

$$\operatorname{index}(V, \text{Dolbeault}) = \int_M \hat{A}(TM) \wedge ch(L_1) \wedge ch(V)$$

where $ch(L_1)$ is to be understood as a complex characteristic class of $T'(M)$.

Theorem 3.5.5 (Riemann-Roch). *Let $Td(T'M)$ be the Todd class defined earlier by the generating function $Td(A) = \prod_\nu x_\nu/(1-e^{-x_\nu})$. Then*

$$\operatorname{index}(V, \text{Dolbeault}) = \int_M Td(T'M) \wedge ch(V).$$

Proof: We must simply identify $\hat{A}(TM) \wedge ch(L_1)$ with $Td(T'M)$. We perform a computation in characteristic classes using the splitting principal.

We formally decompose $T'M = \tilde{L}_1 \oplus \cdots \oplus \tilde{L}_n$ as the direct sum of line bundles. Then $\Lambda^n(T'M) = \tilde{L}_1 \otimes \cdots \otimes \tilde{L}_n$ so $c_1(\Lambda^n T'M) = c_1(\tilde{L}_1) + \cdots + c_1(\tilde{L}_n) = x_1 + \cdots + x_n$. Since $L_1 \otimes L_1 = \Lambda^n T'M$, we conclude $c_1(L_1) = \frac{1}{2}(x_1 + \cdots + x_n)$ so that:

$$ch(L_1) = \prod_\nu e^{x_\nu/2}.$$

Therefore:

$$\begin{aligned}\hat{A}(M) \wedge ch(L_1) &= \prod_\nu x_\nu \frac{e^{x_\nu/2}}{e^{x_\nu/2} - e^{-x_\nu/2}} \\ &= \prod_\nu \frac{x_\nu}{1 - e^{-x_\nu}} = Td(T'M).\end{aligned}$$

Of course, this procedure is only valid if the given bundle does in fact split as the direct sum of line bundles. We can make this procedure a correct way of calculating characteristic classes by using flag manifolds or by actually calculating on the group representaton and using the fact that the diagonizable matrices are dense.

It is worth giving another proof of the Riemann-Roch formula to ensure that we have not made a mistake of sign somewhere in all our calculations. Using exactly the same multiplicative considerations as we used earlier and using the qualitative form of the formala for index$(V, \text{Dolbeault})$ given by the SPIN$_c$ complex, it is immediate that there is some formula of the form:

$$\text{index}(V, \text{Dolbeault}) = \int_N \sum_{s+t=n} Td'_s(T'M) \wedge c^t ch_t(V)$$

where c is some universal constant to be determined and where Td'_s is some characteristic form of $T'M$.

If $M = \mathbf{C}P_j$, then we shall show in Lemma 3.6.8 that the arithmetic genus of $\mathbf{C}P_j$ is 1. Using the multiplicative property of the Dolbeault complex, we conclude the arithmetic genus of $\mathbf{C}P_{j_1} \times \cdots \times \mathbf{C}P_{j_k}$ is 1 as well. If M^c_ρ are the manifolds of Lemma 2.3.4 then if we take $V = 1$ we conclude:

$$\begin{aligned}1 = \text{index}(1, \text{Dolbeault}) &= \text{ arithmetic genus of } \mathbf{C}P_{j_1} \times \cdots \times \mathbf{C}P_{j_k} \\ = \int_{M^c_\rho} Td'(T'M).\end{aligned}$$

We verified in Lemma 2.3.5 that $Td(T'M)$ also has this property so by the uniqueness assertion of Lemma 2.3.4 we conclude $Td = Td'$. We take $m = 2$ and decompose:

$$\Lambda^0 \oplus \Lambda^2 = \Lambda^{0,0} \oplus \Lambda^{1,1} \qquad \text{and} \qquad \Lambda^1 = \Lambda^{1,0} \oplus \Lambda^{0,1}$$

so that

$$\Lambda^{e} - \Lambda^{o} = (\Lambda^{0,0} - \Lambda^{0,1}) \otimes (\Lambda^{0,0} - \Lambda^{1,0}).$$

In dimension 2 (and more generally if M is Kaehler), it is an easy exercise to compute that $\Delta = 2(\bar\partial\delta'' + \delta''\bar\partial)$ so that the harmonic spaces are the same therefore:

$$\begin{aligned}\chi(M) &= \operatorname{index}(\Lambda^{0,0}, \text{Dolbeault}) - \operatorname{index}(\Lambda^{1,0}, \text{Dolbeault}) \\ &= -c\int_M ch(\Lambda^{1,0}) = c\int_M ch(T'M) = c\int_M e(M).\end{aligned}$$

The Gauss-Bonnet theorem (or the normalization of Lemma 2.3.3) implies that the normalizing constant $c = 1$ and gives another equivalent proof of the Riemann-Roch formula.

There are many applications of the Riemann-Roch theorem. We present a few in dimension 4 to illustrate some of the techniques involved. If $\dim M = 4$, then we showed earlier that:

$$c_2(T'M) = e_2(TM) \qquad \text{and} \qquad (2c_2 - c_1^2)(T'M) = p_1(T'M).$$

The Riemann-Roch formula expresses the arithmetic genus of M in terms of c_2 and c_1^2. Consequently, there is a formula:

$$\begin{aligned}\text{arithmetic genus } &= a_1\int_M e_2(TM) + a_2\int_M p_1(TM)/3 \\ &= a_1\chi(M) + a_2\,\text{signature}(M)\end{aligned}$$

where a_1 and a_2 are universal constants. If we consider the manifolds $S^2 \times S^2$ and $\mathbf{C}P_2$ we derive the equations:

$$1 = a_1 \cdot 4 + a_2 \cdot 0 \qquad \text{and} \qquad 1 = a_1 \cdot 3 + a_2 \cdot 1$$

so that $a_1 = a_2 = 1/4$ which proves:

Lemma 3.5.6. *If M is an almost complex manifold of real dimension 4, then:*

$$\text{arithmetic genus}(M) = \{\chi(M) + \text{signature}(M)\}/4.$$

Since the arithmetic genus is always an integer, we can use this result to obtain some non-integrability results:

Corollary 3.5.7. *The following manifolds do not admit almost complex structures:*
(a) S^4 (the four dimensional sphere).
(b) $\mathbf{C}P_2$ with the reversed orientation.

(c) $M_1 \# M_2$ where the M_i are 4-dimensional manifolds admitting almost complex structures (# denotes connected sum).

PROOF: We assume the contrary in each of these cases and attempt to compute the arithmetic genus:

$$\begin{aligned}
\text{a.g.}(S^4) &= \tfrac{1}{4}(2+0) = \tfrac{1}{2} \\
\text{a.g.}(-\mathbf{C}P_2) &= \tfrac{1}{4}(3-1) = \tfrac{1}{2} \\
\text{a.g.}(M_1 \# M_2) &= \tfrac{1}{4}(\chi(M_1 \# dM_2) + \operatorname{sign}(M_1 \# M_2) \\
&= \tfrac{1}{4}(\chi(M_1) + \chi(M_2) - 2 + \operatorname{sign}(M_1) + \operatorname{sign}(M_2)) \\
&= \text{a.g.}(M_1) + \text{a.g.}(M_2) - \tfrac{1}{2}.
\end{aligned}$$

In none of these examples is the arithmetic genus an integer which shows the impossibility of constructing the desired almost complex structure.

3.6. A Review of Kaehler Geometry.

In the previous subsection, we proved the Riemann-Roch theorem using the SPIN_c complex. This was an essential step in the proof even if the manifold was holomorphic.

Let M be holomorphic and let the coefficient bundle V be holomorphic (i.e., the transition functions are holomorphic). We can define the Dolbeault complex directly by defining:

$$\bar{\partial}_V : C^\infty(\Lambda^{0,q} \otimes V) \to C^\infty(\Lambda^{0,q+1} \otimes V) \qquad \text{by } \bar{\partial}_V(\omega \otimes s) = \bar{\partial}\omega \otimes s$$

where s is a local holomorphic section to V. It is immediate $\bar{\partial}_V \bar{\partial}_V = 0$ and that this is an elliptic complex; the index of this elliptic complex is just $\mathrm{index}(V, \mathrm{Dolbeault})$ as defined previously.

We let $a_n(x, V, \mathrm{Dolbeault})$ be the invariant of the heat equation for this elliptic complex; we use the notation $a_j(x, \mathrm{Dolbeault})$ when V is the trivial bundle. Then:

Remark 3.6.1. *Let $m = 2n > 2$. Then there exists a unitary Riemannian metric on the m-torus (with its usual complex structure) and a point x such that:*
(a) $a_j(x, \mathrm{Dolbeault}) \neq 0$ for j even and $j \geq n$,
(b) $a_m(x, \mathrm{Dolbeault}) \neq Td_n$ where both are viewed as scalar invariants.

The proof of this is quite long and combinatorial and is explained elsewhere; we simply present the result to demonstrate that it is not in general possible to prove the Riemann-Roch theorem directly by heat equation methods.

The difficulty is that the metric and the complex structure do not fit together properly. Choose local holomorphic coordinates $z = (z_1, \ldots, z_n)$ and extend the metric G to be Hermitian on $T(M) \otimes \mathbf{C}$ so that T' and T'' are orthogonal. We define:

$$g_{j\bar{k}} = G(\partial/\partial z_j, \partial/\partial z_k)$$

then the matrix $g_{j\bar{k}}$ is a positive definite Hermitian matrix which determines the original metric on $T(M)$:

$$ds^2 = 2\sum_{j,k} g_{j\bar{k}} dz^j \cdot d\bar{z}^k$$
$$G(\partial/\partial x_j, \partial/\partial x_k) = G(\partial/\partial y_j, \partial/\partial y_k) = (g_{j\bar{k}} + g_{k\bar{j}})$$
$$G(\partial/\partial x_j, \partial/\partial y_k) = -G(\partial/\partial y_j, \partial/\partial x_k) = \frac{1}{i}(g_{j\bar{k}} - g_{k\bar{j}})$$

We use this tensor to define the Kaehler 2-form:

$$\Omega = i\sum_{j,k} g_{j\bar{k}} dz^j \wedge d\bar{z}^k.$$

This is a real 2-form which is defined by the identity:

$$\Omega[X,Y] = -G(X,JY).$$

(This is a slightly different sign convention from that sometimes followed.) The manifold M is said to be Kaehler if Ω is closed and the heat equation gives a direct proof of the Riemann-Roch theorem for Kaehler manifolds.

We introduce variables:

$$g_{j\bar{k}/l} \qquad \text{and} \qquad g_{j\bar{k}/\bar{l}}$$

for the jets of the metric. On a Riemannian manifold, we can always find a coordinate system in which all the 1-jets of the metric vanish at a point; this concept generalizes as follows:

Lemma 3.6.2. *Let M be a holomorphic manifold and let G be a unitary metric on M. The following statements are equivalent:*
(a) The metric is Kaeher (i.e., $d\Omega = 0$).
(b) For every $z_0 \in M$ there is a holomorphic coordinate system Z centred at z_0 so that $g_{j\bar{k}}(Z,G)(z_0) = \delta_{jk}$ and $g_{j\bar{k}/l}(Z,G)(z_0) = 0$.
(c) For every $z_0 \in M$ there is a holomorphic coordinate system Z centred at z_0 so $g_{j\bar{k}}(Z,G)(z_0) = \delta_{jk}$ and so that all the 1-jets of the metric vanish at z_0.

Proof: We suppose first the metric is Kaehler and compute:

$$d\Omega = i\sum_{j,k}\{g_{j\bar{k}/l}dz^l \wedge dz^j \wedge d\bar{z}^k + g_{j\bar{k}/\bar{l}}d\bar{z}^l \wedge dz^j \wedge d\bar{z}^k\}.$$

Thus Kaehler is equivalent to the conditions:

$$g_{j\bar{k}/l} - g_{l\bar{k}/j} = g_{j\bar{k}/\bar{l}} - g_{j\bar{l}/\bar{k}} = 0.$$

By making a linear change of coordinates, we can assume that the holomorphic coordinate system is chosen to be orthogonal at the center z_0. We let

$$z'_j = z_j + \sum c^j_{kl} z_k z_l \qquad (\text{where } c^j_{kl} = c^j_{lk})$$

and compute therefore:

$$\begin{aligned}
dz'_j &= dz_j + 2\sum c^j_{kl} z_l dz_k \\
\partial/\partial z'_j &= \partial/\partial z_j - 2\sum c^k_{jl} z_l \partial/\partial z_k + O(z^2) \\
g'_{j\bar{k}} &= g_{j\bar{k}} - 2\sum c^k_{jl} z_l + \text{ terms in } \bar{z} + O(|z|^2) \\
g'_{j\bar{k}/l} &= g_{j\bar{k}/l} - 2c^k_{jl} \text{ at } z_0.
\end{aligned}$$

We define $c_{jl}^k = \frac{1}{2} g_{j\bar{k}/l}$ and observe that the Kaehler condition shows this is symmetric in the indices (j,l) so (a) implies (b). To show (b) implies (c) we simply note that $g_{j\bar{k}/\bar{l}} = \overline{g_{k\bar{j}/l}} = 0$ at z_0. To prove (c) implies (a) we observe $d\Omega(z_0) = 0$ so since z_0 was arbitrary, $d\Omega \equiv 0$.

If M is Kaehler, then $\delta\Omega$ is linear in the 1-jets of the metric when we compute with respect to a holomorphic coordinate system. This implies that $\delta\Omega = 0$ so Ω is harmonic. We compute that Ω^k is also harmonic for $1 \le k \le n$ and that $\Omega^n = c \cdot \mathrm{dvol}$ is a multiple of the volume form (and in particular is non-zero). This implies that if x is the element in $H^2(M;\mathbf{C})$ defined by Ω using the Hodge decomposition theorem, then $1, x, \ldots, x^n$ all represent non-zero elements in the cohomology ring of M. This can be used to show that there are topological obstructions to constructing Kaehler metrics:

REMARK 3.6.3. *If $M = S^1 \times S^{m-1}$ for m even and $m \ge 4$ then this admits a holomorphic structure. No holomorphic structure on M admits a Kaehler metric.*

PROOF: Since $H^2(M;\mathbf{C}) = 0$ it is clear M cannot admit a Kaehler metric. We construct holomorphic structures on M as follows: let $\lambda \in \mathbf{C}$ with $|\lambda| > 1$ and let $M_\lambda = \{\mathbf{C}^n - 0\}/\lambda$ (where we identify z and w if $z = \lambda^k w$ for some $k \in \mathbf{Z}$). The M_λ are all topologically $S^1 \times S^{m-1}$; for example if $\lambda \in \mathbf{R}$ and we introduce spherical coordinates (r,θ) on $\mathbf{C}^n = \mathbf{R}^m$ then we are identifying (r,θ) and $(\lambda^k r, \theta)$. If we let $t = \log r$, then we are identifying t with $t + k\log\lambda$ so the manifold is just $[1, \log\lambda] \times S^{m-1}$ where we identify the endpoints of the interval. The topological identification of M_λ for other λ is similar.

We now turn to the problem of constructing a Kaehler metric on $\mathbf{C}P_n$. Let L be the tautological line bundle over $\mathbf{C}P_n$ and let $x = -c_1(L) \in \Lambda^{1,1}(\mathbf{C}P_n)$ be the generator of $H^2(\mathbf{C}P_n;\mathbf{Z})$ discussed earlier. We expand x in local coordinates in the form:

$$x = \frac{i}{2\pi} \sum_{j,k} g_{j\bar{k}} dz^j \wedge d\bar{z}^k$$

and define $G(\partial/\partial z_k, \partial/\partial z_k) = g_{j\bar{k}}$. This gives an invariantly defined form called the Fubini-Study metric on $T_c(\mathbf{C}P_n)$. We will show G is positive definite and defines a unitary metric on the real tangent space. Since the Kaehler form of G, $\Omega = 2\pi x$, is a multiple of x, we conclude $d\Omega = 0$ so G will be a Kaehler metric.

The 2-form x is invariant under the action of $\mathrm{U}(n+1)$. Since $\mathrm{U}(n+1)$ acts transitively on $\mathbf{C}P_n$, it suffices to show G is positive definite and symmetric at a single point. Using the notation of section 2.3, let $\mathrm{U}_n = \{z \in \mathbf{C}P_n : z_{n+1}(z) \ne 0\}$. We identify U_n with $\mathbf{C}^n$ by identifying $z \in \mathbf{C}^n$ with the line in $\mathbf{C}^{n+1}$ through the point $(z,1)$. Then:

$$x = \frac{i}{2\pi}\partial\bar\partial\log(1+|z|^2)$$

so at the center $z=0$ we compute that:

$$x = \frac{i}{2\pi}\sum_j dz^j\wedge d\bar z^j \quad \text{so that} \quad g_{j\bar k} = \delta_{j\bar k}.$$

If G' is any other unitary metric on $\mathbf{C}P_n$ which is invariant under the action of $\mathrm{U}(n+1)$, then G' is determined by its value on $T_c(\mathbf{C}P_n)$ at a single point. Since $\mathrm{U}(n)$ preserves the origin of $\mathrm{U}_n = \mathbf{C}^n$, we conclude $G' = cG$ for some $c > 0$. This proves:

Lemma 3.6.4.
(a) There exists a unitary metric G on $\mathbf{C}P_n$ which is invariant under the action of $\mathrm{U}(n+1)$ such that the Kaehler form of G is given by $\Omega = 2\pi x$ where $x = -c_1(L)$ (L is the tautological bundle over $\mathbf{C}P_n$). G is a Kaehler metric.
(b) $\int_{\mathbf{C}P_n} x^n = 1$.
(c) If G' is any other unitary metric on $\mathbf{C}P_n$ which is invariant under the action of $\mathrm{U}(n+1)$ then $G' = cG$ for some constant $c > 0$.

A holomorphic manifold M is said to be Hodge if it admits a Kaehler metric G such that the Kaehler form Ω has the form $\Omega = c_1(L)$ for some holomorphic line bundle L over M. Lemma 3.6.4 shows $\mathbf{C}P_n$ is a Hodge manifold. Any submanifold of a Hodge manifold is again Hodge where the metric is just the restriction of the given Hodge metric. Thus every algebraic variety is Hodge. The somewhat amazing fact is that the converse is true:

Remark 3.6.5. *A holomorphic manifold M is an algebraic variety (i.e., is holomorphically equivalent to a manifold defined by algebraic equations in $\mathbf{C}P_n$ for some n) if and only if it admits a Hodge metric.*

We shall not prove this remark but simply include it for the sake of completeness. We also note in passing that there are many Kaehler manifolds which are not Hodge. The Riemann period relations give obstructions on complex tori to those tori being algebraic.

We now return to our study of Kaehler geometry. We wish to relate the $\bar\partial$ cohomology of M to the ordinary cohomology. Define:

$$H^{p,q}(M) = \ker\bar\partial / \operatorname{im}\bar\partial \quad \text{in bi-degree } (p,q).$$

Since the Dolbeault complex is elliptic, these groups are finite dimensional using the Hodge decomposition theorem.

Lemma 3.6.6. *Let M be Kaehler and let $\Omega = i\sum g_{j\bar{k}} dz^j \wedge d\bar{z}^k$ be the Kaehler form. Let $d = \partial + \bar{\partial}$ and let $\delta = \delta' + \delta''$. Then we have the identities:*
(a) $\bar{\partial}\operatorname{int}(\Omega) - \operatorname{int}(\Omega)\bar{\partial} = i\delta'$.
(b) $\bar{\partial}\delta' + \delta'\bar{\partial} = \partial\delta'' + \delta''\partial = 0$.
(c) $d\delta + \delta d = 2(\partial\delta' + \delta'\partial) = 2(\bar{\partial}\delta'' + \delta''\bar{\partial})$.
(d) We can decompose the de Rham cohomology in terms of the Dolbeault cohomology:

$$H^n(M;\mathbf{C}) = \bigoplus_{p+q=n} H^{p,q}(M;\mathbf{C}).$$

(e) There are isomorphisms $H^{p,q} \simeq H^{q,p} \simeq H^{n-p,n-q} \simeq H^{n-q,n-p}$.

Proof: We first prove (a). We define:

$$A = \bar{\partial}\operatorname{int}(\Omega) - \operatorname{int}(\Omega)\bar{\partial} - i\delta'.$$

If we can show that A is a 0^{th} order operator, then A will be a functorially defined endomorphism linear in the 1-jets of the metric. This implies $A = 0$ by Lemma 3.6.2. Thus we must check $\sigma_L(A) = 0$. We choose an orthonormal basis $\{e_j\}$ for $T(M) = T^*(M)$ so $Je_j = e_{j+n}$, $1 \le j \le n$. By Lemma 3.5.2, it suffices to show that:

$$\operatorname{ext}(e_j - ie_{j+n})\operatorname{int}(\Omega) - \operatorname{int}(\Omega)\operatorname{ext}(e_j - ie_{j+n}) = -i\cdot\operatorname{int}(e_j - ie_{j+n})$$

for $1 \le j \le n$. Since $g_{j\bar{k}} = \frac{1}{2}\delta_{j\bar{k}}$, Ω is given by

$$\Omega = \frac{i}{2}\sum_j (e_j + ie_{j+n}) \wedge (e_j - ie_{j+n}) = \sum_j e_j \wedge e_{j+n}.$$

The commutation relations:

$$\operatorname{int}(e_k)\operatorname{ext}(e_l) + \operatorname{ext}(e_l)\operatorname{int}(e_k) = \delta_{k,l}$$

imply that $\operatorname{int}(e_k \wedge e_{k+n}) = -\operatorname{int}(e_k)\operatorname{int}(e_{k+n})$ commutes with $\operatorname{ext}(e_j - ie_{j+n})$ for $k \ne j$ so these terms disappear from the commutator. We must show:

$$\begin{aligned}\operatorname{ext}(e_j - ie_{j+n})\operatorname{int}(e_j \wedge e_{j+n}) - \operatorname{int}(e_j \wedge e_{j+n})\operatorname{ext}(e_j - ie_{j+n})&\\ = -i\cdot\operatorname{int}(e_j - ie_{j+n}).&\end{aligned}$$

This is an immediate consequence of the previous commutation relations for int and ext together with the identity $\operatorname{int}(e_j)\operatorname{int}(e_k) + \operatorname{int}(e_k)\operatorname{int}(e_j) = 0$. This proves (a).

From (a) we compute:

$$\bar\partial\delta' + \delta'\bar\partial = i(-\bar\partial\,\mathrm{int}(\Omega)\bar\partial) + i(\bar\partial\,\mathrm{int}(\Omega)\bar\partial) = 0$$

and by taking complex conjugates $\partial\delta'' + \delta''\partial = 0$. This shows (b). Thus

$$\Delta = (d+\delta)^2 = d\delta+\delta d = \{\partial\delta'+\delta'\partial\}+\{\bar\partial\delta''+\delta''\bar\partial\} = \{\partial+\delta'\}^2+\{\bar\partial+\delta''\}^2$$

since all the cross terms cancel. We apply (a) again to compute:

$$\begin{aligned}\partial\bar\partial\,\mathrm{int}(\Omega) - \partial\,\mathrm{int}(\Omega)\bar\partial &= i\partial\delta'\\ \bar\partial\,\mathrm{int}(\Omega)\partial - \mathrm{int}(\Omega)\bar\partial\partial &= i\delta'\partial\end{aligned}$$

which yields the identity:

$$i(\partial\delta' + \delta'\partial) = \partial\bar\partial\,\mathrm{int}(\Omega) - \partial\,\mathrm{int}(\Omega)\bar\partial + \bar\partial\,\mathrm{int}(\Omega)\partial - \mathrm{int}(\Omega)\bar\partial\partial.$$

Since Ω is real, we take complex conjugate to conclude:

$$\begin{aligned}-i(\bar\partial\delta'' + \delta''\bar\partial) &= \bar\partial\partial\,\mathrm{int}(\Omega) - \bar\partial\,\mathrm{int}(\Omega)\partial + \partial\,\mathrm{int}(\Omega)\bar\partial - \mathrm{int}(\Omega)\partial\bar\partial\\ &= -i(\partial\delta' + \delta'\partial).\end{aligned}$$

This shows $(\bar\partial\delta'' + \delta''\bar\partial) = (\partial\delta' + \delta'\partial)$ so $\Delta = 2(\bar\partial\delta'' + \delta''\bar\partial)$ which proves (c).

$H^n(M;\mathbf{C})$ denotes the de Rham cohomology groups of M. The Hodge decomposition theorem identifies these groups with the null-space of $(d+\delta)$. Since $\mathrm{N}(d+\delta) = \mathrm{N}(\bar\partial + \delta'')$, this proves (d). Finally, taking complex conjugate and applying $*$ induces the isomorphisms:

$$H^{p,q} \simeq H^{q,p} \simeq H^{n-q,n-p} \simeq H^{n-p,n-q}.$$

In particular $\dim H^1(M;\mathbf{C})$ is even if M admits a Kaehler metric. This gives another proof that $S^1 \times S^{2n-1}$ does not admit a Kaehler metric for $n > 1$.

The duality operations of (e) can be extended to the Dolbeault complex with coefficients in a holomorphic bundle V. If V is holomorphic, the transition functions are holomorphic and hence commute with $\bar\partial$. We define:

$$\bar\partial\colon C^\infty(\Lambda^{p,q}\otimes V) \to (\Lambda^{p,q+1}\otimes V)$$

by defining $\bar\partial(\omega\otimes s) = \bar\partial\omega\otimes s$ relative to any local holomorphic frame for V. This complex is equivalent to the one defined in section 3.5 and defines cohomology classes:

$$H^{p,q}(V) = \ker\bar\partial/\operatorname{im}\bar\partial \qquad \text{on } C^\infty(\Lambda^{p,q}\otimes V).$$

It is immediate from the definition that:

$$H^{p,q}(V) = H^{0,q}(\Lambda^{p,0} \otimes V).$$

We define

$$\operatorname{index}(V,\bar\partial) = \operatorname{index}(V,\text{Dolbeault}) = \sum (-1)^q \dim H^{0,q}(V)$$

to be the index of this elliptic complex.

There is a natural duality (called Serre duality) which is conjugate linear $\bar{*}\colon H^{p,q}(V) \to H^{n-p,n-q}(V^*)$, where V^* is the dual bundle. We put a unitary fiber metric on V and define:

$$T(s_1)(s_2) = s_2 \cdot s_1$$

so $T\colon V \to V^*$ is conjugate linear. If $*\colon \Lambda^k \to \Lambda^{m-k}$ is the ordinary Hodge operator, then it extends to define $*\colon \Lambda^{p,q} \to \Lambda^{n-q,n-p}$. The complex conjugate operator defines $\bar{*}\colon \Lambda^{p,q} \to \Lambda^{n-p,n-q}$. We extend this to have coefficients in V by defining:

$$\bar{*}(\omega \otimes s) = \bar{*}\omega \otimes Ts$$

so $\bar{*}\colon \Lambda^{p,q} \otimes V \to \Lambda^{n-p,n-q} \otimes V^*$ is conjugate linear. Then:

Lemma3.6.7. *Let δ''_V be the adjoint of $\bar\partial_V$ on $C^\infty(\Lambda^{p,*} \otimes V)$ with respect to a unitary metric on M and a Hermitian metric on V. Then*
(a) $\delta'' = -\bar{*}\bar\partial_V \bar{*}$.
(b) $\bar{*}$ induces a conjugate linear isomorphism between the groups $H^{p,q}(V)$ and $H^{n-p,n-q}(V^*)$.

Proof: It is important to note that this lemma, unlike the previous one, does not depend upon having a Kaehler metric. We suppose first V is holomorphically trivial and the metric on V is flat. Then $\Lambda \otimes V = \Lambda \otimes 1^k$ for some k and we may suppose $k = 1$ for the sake of simplicity. We noted

$$\delta = - * d* = -\bar{*}d\bar{*}$$

in section 1.5. This decomposes

$$\delta = \delta' + \delta'' = -\bar{*}\partial\bar{*} - \bar{*}\bar\partial\bar{*}.$$

Since $\bar{*}\colon \Lambda^{p,q} \to \Lambda^{n-p,n-q}$, we conclude $\delta' = -\bar{*}\partial\bar{*}$ and $\delta'' = -\bar{*}\bar\partial\bar{*}$. This completes the proof if V is trivial. More generally, we define

$$A = \delta''_V + \bar{*}\bar\partial\bar{*}.$$

This must be linear in the 1-jets of the metric on V. We can always normalize the choice of local holomorphic frame so the 1-jets vanish at a basepoint z_0 and thus $A = 0$ in general which proves (a). We identify $H^{p,q}(V)$ with $\mathrm{N}(\bar\partial_V)\cap\mathrm{N}(\delta''_V)$ using the Hodge decomposition theorem. Thus $\bar *\colon H^{p,q}(V) \to H^{n-p,n-q}(V^*)$. Since $\bar *^2 = \pm 1$, this completes the proof.

We defined $x = -c_1(L) \in \Lambda^{1,1}(\mathbf{C}P_n)$ for the standard metric on the tautological line bundle over $\mathbf{C}P_n$. In section 2.3 we showed that x was harmonic and generates the cohomology ring of $\mathbf{C}P_n$. Since x defines a Kaehler metric, called the Fubini-Study metric, Lemma 3.6.6 lets us decompose $H^n(\mathbf{C}P_n;\mathbf{C}) = \bigoplus_{p+q=n} H^{p,q}(\mathbf{C}P_n)$. Since $x^k \in H^{k,k}$ this shows:

Lemma 3.6.8. *Let $\mathbf{C}P_n$ be given the Fubini-Study metric. Let $x = -c_1(L)$. Then $H^{k,k}(M) \simeq \mathbf{C}$ is generated by x^k for $1 \le k \le n$. $H^{p,q}(M) = 0$ for $p \ne q$. Thus in particular,*

$$\operatorname{index}(\bar\partial) = \text{ arithmetic genus } = \sum (-1)^k \dim H^{0,k} = 1 \qquad \text{for } \mathbf{C}P_n.$$

Since the arithmetic genus is multiplicative with respect to products,

$$\operatorname{index}(\bar\partial) = 1 \qquad \text{for } \mathbf{C}P_{n_1} \times \cdots \times \mathbf{C}P_{n_k}.$$

We used this fact in the previous subsection to derive the normalizing constants in the Riemann-Roch formula.

We can now present another proof of the Riemann-Roch theorem for Kaehler manifolds which is based directly on the Dolbeault complex and not on the SPIN_c complex. Let Z be a holomorphic coordinate system and let

$$g_{u\bar v} = G(\partial/\partial z_u, \partial/\partial z_v)$$

represent the components of the metric tensor. We introduce additional variables:

$$g_{u\bar v/\alpha\bar\beta} = d_z^\alpha d_{\bar z}^\beta g_{u\bar v} \qquad \text{for } 1 \le u, v \le n,\ \alpha = (\alpha_1, \dots, \alpha_n), \text{ and } \beta = (\beta_1, \dots, \beta_n).$$

These variables represent the formal derivatives of the metric.

We must also consider the Dolbeault complex with coefficients in a holomorphic vector bundle V. We choose a local fiber metric H for V. If $s = (s_1, \dots, s_k)$ is a local holomorphic frame for V, we introduce variables:

$$h_{p,\bar q} = H(s_p, s_q), \qquad h_{p,\bar q/\alpha\bar\beta} = d_z^\alpha d_{\bar z}^\beta h_{p\bar q} \qquad \text{for } 1 \le p, q \le k = \dim V.$$

If Z is a holomorphic coordinate system centered at z_0 and if s is a local holomorphic frame, we normalize the choice so that:

$$g_{u\bar v}(Z, G)(z_0) = \delta_{u,v} \qquad \text{and} \qquad h_{p\bar q}(s, H)(z_0) = \delta_{p,q}.$$

The coordinate frame $\{\partial/\partial z_u\}$ and the holomorphic frame $\{s_p\}$ for V are orthogonal at z_0. Let $\mathcal{R}$ be the polynomial algebra in these variables. If $P \in \mathcal{R}$ we can evaluate $P(Z, G, s, H)(z_0)$ once a holomorphic frame Z is chosen and a holomorphic frame s is chosen. We say P is invariant if $P(Z, G, s, H)(z_0) = P(G, H)(z_0)$ is independent of the particular (Z, s) chosen; let $\mathcal{R}^c_{n,k}$ denote the sub-algebra of all invariant polynomials. As before, we define:

$$\operatorname{ord}(g_{u,\bar v/\alpha\bar\beta}) = \operatorname{ord}(s_{p\bar q/\alpha\bar\beta}) = |\alpha| + |\beta|.$$

We need only consider those variables of positive order as $g_{u,\bar v} = \delta_{j\bar v}$ and $s_{p,\bar q} = \delta_{p,q}$ at z_0. If $\mathcal{R}^c_{n,\nu,k}$ denotes the subspace of invariant polynomials homogenous of order ν in the jets of the metrics (G, H), then there is a direct sum decomposition:

$$\mathcal{R}^c_{n,k} = \bigoplus \mathcal{R}^c_{n,\nu,k}.$$

In the real case in studying the Euler form, we considered the restriction map from manifolds of dimension m to manifolds of dimension $m-1$. In the complex case, there is a natural restriction map

$$r\colon \mathcal{R}^c_{n,\nu,k} \to \mathcal{R}^c_{n-1,\nu,k}$$

which lowers the complex dimension by one (and the real dimension by two). Algebraically, r is defined as follows: let

$$\begin{aligned} &\deg_k(g_{u\bar v/\alpha\bar\beta}) = \delta_{k,u} + \alpha(k) &\qquad &\deg_{\bar k}(g_{u\bar v/\alpha\bar\beta}) = \delta_{k,v} + \beta(k)\\ &\deg_k(h_{p\bar q/\alpha\bar\beta}) = \alpha(k) & &\deg_{\bar k}(h_{p\bar q/\alpha\bar\beta}) = \beta(k). \end{aligned}$$

We define:

$$r(g_{u\bar v/\alpha\bar\beta}) = \begin{cases} g_{u\bar v/\alpha\bar\beta} & \text{if } \deg_n(g_{u\bar v/\alpha\bar\beta}) + \deg_{\bar n}(g_{u\bar v/\alpha\bar\beta}) = 0\\ 0 & \text{if } \deg_n(g_{u\bar v/\alpha\bar\beta}) + \deg_{\bar n}(g_{u\bar v/\alpha\bar\beta}) > 0 \end{cases}$$

$$r(h_{p\bar q/\alpha\bar\beta}) = \begin{cases} h_{p\bar q/\alpha\bar\beta} & \text{if } \deg_n(h_{p\bar q/\alpha\bar\beta}) + \deg_{\bar n}(h_{p\bar q/\alpha\bar\beta}) = 0\\ 0 & \text{if } \deg_n(h_{p\bar q/\alpha\bar\beta}) + \deg_{\bar n}(h_{p\bar q/\alpha\bar\beta}) > 0. \end{cases}$$

This defines a map $r\colon \mathcal{R}_n \to \mathcal{R}_{n-1}$ which is an algebra morphism; we simply set to zero those variables which do not belong to $\mathcal{R}_{n-1}$. It is immediate that r preserves both invariance and the order of a polynomial. Geometrically, we are considering manifolds of the form $M^n = M^{n-1} \times T_2$ where T_2 is the flat two torus, just as in the real case we considered manifolds $M^m = M^{m-1} \times S^1$.

Let $\mathcal{R}^{ch}_{n,p,k}$ be the space of p-forms generated by the Chern forms of $T_c(M)$ and by the Chern forms of V. We take the holomorphic connection defined by the metrics G and H on $T_c(M)$ and on V. If $P \in \mathcal{R}^{ch}_{n,m,k}$ then $*P$ is a scalar invariant. Since P vanishes on product metrics of the form $M^{n-1} \times T_2$ which are flat in one holomorphic direction, $r(*P) = 0$. This is the axiomatic characterization of the Chern forms which we shall need.

Theorem 3.6.9. *Let the complex dimension of M be n and let the fiber dimension of V be k. Let $P \in \mathcal{R}^c_{n,\nu,k}$ and suppose $r(P) = 0$. Then:*
(a) If $\nu < 2n$ then $P(G,H) = 0$ for all Kaehler metrics G.
(b) If $\nu = 2n$ then there exists a unique $Q \in \mathcal{R}^{ch}_{n,m,k}$ so $P(G,H) = *Q(G,H)$ for all Kaehler metrics G.

As the proof of this theorem is somewhat technical, we shall postpone the proof until section 3.7. This theorem is false if we don't restrict to Kaehler metrics. There is a suitable generalization to form valued invariants. We can apply Theorem 3.6.9 to give a second proof of the Riemann-Roch theorem:

Theorem 3.6.10. *Let V be a holomorphic bundle over M with fiber metric H and let G be a Kaehler metric on M. Let $\bar\partial_V : C^\infty(\Lambda^{0,q} \otimes V) \to C^\infty(\Lambda^{0,q+1} \otimes V)$ denote the Dolbeault complex with coefficients in V and let $a_\nu(x, \bar\partial_V)$ be the invariants of the heat equation. $a_\nu \in \mathcal{R}^c_{n,\nu,k}$ and*

$$\int_M a_\nu(x,\bar\partial_V)\,\mathrm{dvol}(x) = \begin{cases} 0 & \text{if } \nu \neq 2n \\ \mathrm{index}(\bar\partial_V) & \text{if } \nu = 2n. \end{cases}$$

Then
(a) $a_\nu(x,\bar\partial_V) = 0$ for $\nu < 2n$.
(b) $a_{2n}(x,\bar\partial_V) = *\{Td(T_cM) \wedge ch(V)\}_m$.

Proof: We note this implies the Riemann-Roch formula. By remark 3.6.1, we note this theorem is false in general if the metric is not assumed to be Kaehler. The first assertions of the theorem (including the homogeneity) follow from the results of Chapter 1, so it suffices to prove (a) and (b). The Dolbeault complex is multiplicative under products $M = M^{n-1} \times T_2$. If we take the product metric and assume V is the pull-back of a bundle over M_{n-1} then the natural decomposition:

$$\Lambda^{0,q} \otimes V \simeq \Lambda^{0,q}(M^{n-1}) \otimes V \;\oplus\; \Lambda^{0,q-1}(M^{n-1}) \otimes V$$

shows that $r(a_\nu) = 0$. By Theorem 3.6.9, this shows $a_\nu = 0$ for $\nu < 2n$ proving (a).

In the limiting case $\nu = 2n$, we conclude $*a_{2n}$ is a characteristic $2n$ form. We first suppose $m = 1$. Any one dimensional complex manifold is Kaehler. Then

$$\mathrm{index}(\bar\partial_V) = a_1 \int_M c_1(T_c(M)) + a_2 \int_M c_1(V)$$

where a_1 and a_2 are universal constants to be determined. Lemma 3.6.6 implies that $\dim H^{0,0} = \dim H^0 = 1 = \dim H^2 = \dim H^{1,1}$ while $\dim H^{1,0} = \dim H^{0,1} = g$ where g is the genus of the manifold. Consequently:

$$\chi(M) = 2 - 2g = 2\,\mathrm{index}(\bar\partial).$$

We specialize to the case $M = S^2$ and $V = 1$ is the trivial bundle. Then $g = 0$ and

$$1 = \operatorname{index}(\bar\partial_V) = a_1 \int_M c_1(T_c(M)) = 2a_1$$

so $a_1 = \frac{1}{2}$. We now take the line bundle $V = \Lambda^{1,0}$ so

$$\operatorname{index}(\bar\partial_V) = g - 1 = -1 = 1 + a_2 \int_M c_1(T_c^*) = 1 - 2a_2$$

so that $a_2 = 1$. Thus the Riemann-Roch formula in dimension 1 becomes:

$$\operatorname{index}(\bar\partial_V) = \frac{1}{2}\int_M c_1(T_c(M)) + \int_M c_1(V).$$

We now use the additive nature of the Dolbeault complex with respect to V and the multiplicative nature with respect to products $M_1 \times M_2$ to conclude that the characteristic m-form $*a_m$ must have the form:

$$\{T_d'(T_cM) \wedge ch(V)\}$$

where we use the fact the normalizing constant for $c_1(V)$ is 1 if $m = 1$. We now use Lemma 2.3.5(a) together with the fact that $\operatorname{index}(\bar\partial_V) = 1$ for products of complex projective spaces to show $Td' = Td$. This completes the proof.

3.7. An Axiomatic Characterization Of the Characteristic Forms for Holomorphic Manifolds with Kaehler Metrics.

In subsection 3.6 we gave a proof of the Riemann-Roch formula for Kaehler manifolds based on Theorem 3.6.9 which gave an axiomatic characterization of the Chern forms. This subsection will be devoted to giving the proof of Theorem 3.6.9. The proof is somewhat long and technical so we break it up into a number of steps to describe the various ideas which are involved.

We introduce the notation:

$$g_{u_0 v_0 / u_1 \dots u_j \bar v_1 \dots \bar v_k}$$

for the jets of the metric on M. Indices (u, v, w) will refer to $T_c(M)$ and will run from 1 thru $n = \frac{m}{2}$. We use $(\bar u, \bar v, \bar w)$ for anti-holomorphic indices. The symbol "$*$" will refer to indices which are not of interest in some particular argument. We let A_0 denote a generic monomial. The Kaehler condition is simply the identity:

$$g_{u\bar v/w} = g_{w\bar v/u} \qquad \text{and} \qquad g_{u\bar v/\bar w} = g_{u\bar w/\bar v}.$$

We introduce new variables:

$$g(u_0, \dots, u_j; \bar v_0, \dots, \bar v_k) = g_{u_0 v_0 / u_1 \dots u_j \bar v_1 \dots \bar v_k}.$$

If we differentiate the Kaehler identity, then we conclude $g(\vec u; \vec v)$ is symmetric in $\vec u = (u_0, \dots, u_j)$ and $\vec v = (v_0, \dots, v_j)$. Consequently, we may also use the multi-index notation $g(\alpha; \bar\beta)$ to denote these variables. We define:

$$\operatorname{ord}(g(\alpha;\bar\beta)) = |\alpha| + |\beta| - 2, \quad \deg_u g(\alpha;\bar\beta) = \alpha(u), \quad \deg_{\bar u} g(\alpha;\bar\beta) = \beta(u).$$

We already noted in Lemma 3.6.2 that it was possible to normalize the coordinates so $g_{u\bar v/w}(z_0) = g_{u\bar v/\bar w}(z_0) = 0$. The next lemma will permit us to normalize coordinates to arbitrarily high order modulo the action of the unitary group:

LEMMA 3.7.1. *Let G be a Kaehler metric and let $z_0 \in M$. Let $\nu \geq 2$ be given. Then there exists a holomorphic coordinate system Z centered at z_0 so that*
(a) $g_{u\bar v}(Z,G)(z_0) = \delta_{uv}$.
(b) $g_{u\bar v/\alpha}(Z,G)(z_0) = g_{u\bar v/\bar\alpha}(Z,G)(z_0) = 0$ for $1 \leq |\alpha| \leq \nu - 1$.
(c) Z is unique modulo the action of the unitary group $\mathrm{U}(n)$ and modulo coordinate transformations of order $\nu + 1$ in z.

PROOF: We proceed by induction. It is clear $\mathrm{U}(n)$ preserves such coordinate systems. The case $\nu = 2$ is just Lemma 3.6.2 and the uniqueness is clear. We now consider W given which satisfies (a) and (b) for

$1 \le |\alpha| \le \nu - 2$ and define a new coordinate system Z by setting:

$$w_u = z_u + \sum_{|\alpha|=\nu} C_{u,\alpha} z^\alpha$$

where the constants $C_{u,\alpha}$ remain to be determined. Since this is the identity transformation up to order ν, the $\nu - 2$ jets of the metric are unchanged so conditions (a) and (b) are preserved for $|\beta| < \nu - 1$. We compute:

$$\frac{\partial}{\partial z_u} = \frac{\partial}{\partial w_u} + \sum_{v,|\alpha|=\nu} C_{v,\alpha}\alpha(u) z^{\alpha_u} \frac{\partial}{\partial w_v}$$

where α_u is the multi-index defined by the identity $z_u z^{\alpha_u} = z^\alpha$ (and which is undefined if $\alpha(u) = 0$.) This implies immediately that:

$$g_{u\bar v}(Z, G) = g_{u\bar v}(W, G) + \sum_{|\alpha|=n} C_{v,\alpha}\alpha(u) z^{\alpha_u} + \text{ terms in } \bar z + O(z^{\nu+1}).$$

The $\partial/\partial z_u$ and $\partial/\partial w_u$ agree to first order. Consequently:

$$g_{u\bar v/\alpha_u}(Z, G) = g_{u\bar v/\alpha_u}(W, G) + \alpha!\, C_{v,\alpha} + O(z, \bar z).$$

Therefore the symmetric derivatives are given by:

$$g(\alpha; \bar v)(Z, G)(z_0) = g(\alpha; \bar v)(W, G)(z_0) + \alpha!\, C_{v,\alpha}$$

The identity $g(\alpha; \bar v)(Z, G)(z_0) = 0$ determines the $C_{v,\alpha}$ uniquely. We take complex conjugate to conclude $g(v, \bar\alpha)(Z, G)(z_0) = 0$ as well.

This permits us to choose the coordinate system so all the purely holomorphic and anti-holomorphic derivatives vanish at a single point. We restrict henceforth to variables $g(\alpha; \bar\beta)$ so that $|\alpha| \ge 2, |\beta| \ge 2$.

We introduced the notation $h_{p\bar q/\alpha\bar\beta}$ for the jets of the metric H on the auxilary coefficient bundle V.

LEMMA 3.7.2. *Let H be a fiber metric on a holomorphic bundle V. Then given $v \ge 1$ there exists a holomorphic frame s near z_0 for V so that:*
(a) $h_{p\bar q}(s, H)(z_0) = \delta_{pq}$.
(b) $h_{p\bar q/\alpha}(s, H)(z_0) = h_{p\bar q/\bar\alpha}(s, H)(z_0) = 0$ for $1 \le |\alpha| \le \nu$.
(c) The choice of s is unique modulo the action of the unitary group $\mathrm{U}(k)$ (where k is the fiber dimension of V) and modulo transformations of order $\nu + 1$.

PROOF: If $\nu = 0$, we make a linear change to assume (a). We proceed by induction assuming s' chosen for $\nu - 1$. We define:

$$s_p = s'_p + \sum_{q,\,|\alpha|=\nu} C_{q,\alpha} s'_q z^\alpha.$$

We adopt the notational conventions that indices (p, q) run from 1 through k and index a frame for V. We compute:

$$h_{p\bar{q}}(s, H) = h_{p\bar{q}}(s', H) + \sum_{|\alpha|=\nu} C_{q,\alpha} z^\alpha$$
$$+ \text{ terms in } \bar{z} + \text{ terms vanishing to order } \nu + 1$$
$$h_{p\bar{q}/\alpha}(x, H)(z_0) = h_{p\bar{q}/\alpha}(s', H)(z_0) + \alpha! C_{q,\alpha} \qquad \text{for } |\alpha| = \nu$$

and derivatives of lower order are not disturbed. This determines the $C_{q,\alpha}$ uniquely so $h_{p\bar{q}/\alpha}(s, H)(z_0) = 0$ and taking complex conjugate yields $h_{q\bar{p}/\bar{\alpha}}(s, H)(z_0) = 0$ which completes the proof of the lemma.

Using these two normalizing lemmas, we restrict henceforth to polynomials in the variables $\{g(\alpha; \bar{\beta}), h_{p\bar{q}/\alpha_1\bar{\beta}_1}\}$ where $|\alpha| \geq 2$, $|\beta| \geq 2$, $|\alpha_1| \geq 1$, $|\beta_1| \geq 1$. We also restrict to unitary transformations of the coordinate system and of the fiber of V.

If P is a polynomial in these variables and if A is a monomial, let $c(A, P)$ be the coefficient of A in P. A is a monomial of P if $c(A, P) \neq 0$. Lemma 2.5.1 exploited invariance under the group SO(2) and was central to our axiomatic characterization of real Pontrjagin forms. The natural groups to study here are U(1), SU(2) and the coordinate permutations. If we set $\partial/\partial w_1 = a\partial/\partial z_1$, $\partial/\partial \bar{w}_1 = \bar{a}\partial/\partial \bar{z}_1$ for $a\bar{a} = 1$ and if we leave the other indices unchanged, then we compute:

$$A(W, *) = a^{\deg_1(A)} \bar{a}^{\deg_{\bar{1}}(A)} A(Z, *).$$

If A is a monomial of an invariant polynomial P, then $\deg_1(A) = \deg_{\bar{1}}(A)$ follows from this identity and consequently $\deg_u(A) = \deg_{\bar{u}}(A)$ for all $1 \leq u \leq n$.

This is the only conclusion which follows from U(1) invariance so we now study the group SU(2). We consider the coordinate transformation:

$$\begin{aligned}
\partial/\partial w_1 &= a\partial/\partial z_1 + b\partial/\partial z_2, &\qquad \partial/\partial w_2 &= -\bar{b}\partial/\partial z_1 + \bar{a}\partial/\partial z_2,\\
\partial/\partial \bar{w}_1 &= \bar{a}\partial/\partial \bar{z}_1 + \bar{b}\partial/\partial \bar{z}_2, &\qquad \partial/\partial \bar{w}_2 &= -b\partial/\partial \bar{z}_1 + a\partial/\partial \bar{z}_2,
\end{aligned}$$
$$\partial/\partial w_u = \partial/\partial z_u \qquad \text{for } u > 2,$$
$$\partial/\partial \bar{w}_u = \partial/\partial \bar{z}_u \qquad \text{for } u > 2,$$
$$a\bar{a} + b\bar{b} = 1.$$

We let $j = \deg_1(A) + \deg_2(A) = \deg_{\bar{1}}(A) + \deg_{\bar{2}}(A)$ and expand

$$\begin{aligned}
A(W, *) = a^j \bar{a}^j A(Z, *) &+ a^{j-1}\bar{a}^j b A(1 \to 2 \text{ or } \bar{2} \to \bar{1})(Z, *)\\
&+ a^j \bar{a}^{j-1} \bar{b} A(2 \to 1 \text{ or } \bar{1} \to \bar{2})(Z, *)\\
&+ \text{other terms.}
\end{aligned}$$

The notation "$A(1 \to 2 \text{ or } \bar{2} \to \bar{1})$" indicates all the monomials (with multiplicity) of this polynomial constructed by either changing a single index $1 \to 2$ or a single index $\bar{2} \to \bar{1}$. This plays the same role as the polynomial $A^{(1)}$ of section 2.5 where we must now also consider holomorphic and anti-holomorphic indices.

Let P be U(2) invariant. Without loss of generality, we may assume $\deg_1(A) + \deg_2(A) = j$ is constant for all monomials A of P since this condition is U(2) invariant. (Of course, the use of the indices 1 and 2 is for notational convenience only as similar statements will hold true for any pairs of indices). We expand:

$$\begin{aligned} P(W,*) =& a^j \bar{a}^j P(Z,*) + a^{j-1}\bar{a}^j b P(1 \to 2 \text{ or } \bar{2} \to \bar{1}) \\ &+ a^j \bar{a}^{j-1}\bar{b} P(2 \to 1 \text{ or } \bar{1} \to \bar{2}) + O(b,\bar{b})^2. \end{aligned}$$

Since P is invariant, we conclude:

$$P(1 \to 2 \text{ or } \bar{2} \to \bar{1}) = P(2 \to 1 \text{ or } \bar{1} \to \bar{2}) = 0.$$

We can now study these relations. Let B be an arbitrary monomial and expand:

$$B(1 \to 2) = c_0 A_0 + \cdots + c_k A_k \qquad B(\bar{2} \to \bar{1}) = -(d_0 A_0' + \cdots + d_k A_k')$$

where the c's and d's are positive integers with $\sum c_\nu = \deg_1(B)$ and $\sum d_\nu = \deg_{\bar{2}}(B)$. Then it is immediate that the $\{A_j\}$ and the $\{A_j'\}$ denote disjoint collections of monomials since $\deg_1(A)_\nu = \deg_1(B) - 1$ while $\deg_1(A)_\nu' = \deg_1(B)$. We compute:

$$A_\nu(2 \to 1) = -c_\nu' B + \text{other terms} \qquad A_\nu'(\bar{1} \to \bar{2}) = d_\nu' B + \text{other terms}$$

where again the c's and d's are positive integers (related to certain multiplicities). Since $P(2 \to 1 \text{ or } \bar{1} \to \bar{2})$ is zero, if P is invariant, we conclude an identity:

$$c(B, P(2 \to 1 \text{ or } \bar{1} \to \bar{2})) = \sum_\nu -c_\nu' c(A_\nu, P) + \sum_\nu d_\nu' c(A_\nu', P) = 0.$$

By varying the creating monomial B (and also interchanging the indices 1 and 2), we can construct many linear equations among the coefficients. We note that in practice B will never be a monomial of P since $\deg_1(B) \neq \deg_{\bar{1}}(B)$. We use this principle to prove the following generalization of Lemmas 2.5.1 and 2.5.5:

LEMMA 3.7.3. *Let P be invariant under the action of* $\mathrm{U}(2)$ *and let A be a monomial of P.*
(a) If $A = g(\alpha;\bar\beta)A_0'$, then by changing only 1 and 2 indices and $\bar 1$ and $\bar 2$ indices we can construct a new monomial A_1 of P which has the form $A_1 = g(\alpha_1;\bar\beta_1)A_0''$ where $\alpha(1)+\alpha(2) = \alpha_1(1)$ and where $\alpha_1(2) = 0$.
(b) If $A = h_{p\bar q/\alpha\bar\beta}A_0'$ then by changing only 1 and 2 indices and $\bar 1$ and $\bar 2$ indices we can construct a new monomial A_1 of P which has the form $A_1 = h_{p\bar q/\alpha_1\bar\beta_1}A_0''$ where $\alpha(1)+\alpha(2) = \alpha_1(1)$ and where $\alpha_1(2) = 0$.

PROOF: We prove (a) as the proof of (b) is the same. Choose A of this form so $\alpha(1)$ is maximal. If $\alpha(2)$ is zero, we are done. Suppose the contrary. Let $B = g(\alpha_1;\bar\beta)A_0'$ where $\alpha_1 = (\alpha(1)+1, \alpha(2)-1, \alpha(3), \dots, \alpha(n))$. We expand:

$$B(1\to 2) = \alpha_1(1)A + \text{ monomials divisible by } g(\alpha_1;\bar\beta)$$
$$B(\bar 2\to\bar 1) = \text{ monomials divisible by } g(\alpha_1;\bar\beta_1) \text{ for some } \beta_1.$$

Since A is a monomial of P, we use the principle described previously to conclude there is some monomial of P divisible by $g(\alpha_1;\bar\beta_1)$ for some β_1. This contradicts the maximality of α and shows $\alpha(2) = 0$ completing the proof.

We now begin the proof of Theorem 3.6.9. Let $0 \neq P \in \mathcal{R}^c_{n,\nu,k}$ be a scalar valued invariant homogeneous of order ν with $r(P) = 0$. Let A be a monomial of P. Decompose

$$A = g(\alpha_1;\bar\beta_1)\dots g(\alpha_r;\bar\beta_r)h_{p_1q_1/\alpha_{r+1}\bar\beta_{r+1}}\dots h_{p_s\bar q_s/\alpha_{r+s}\bar\beta_{r+s}}.$$

Let $\ell(A) = r+s$ be the length of A. We show $\ell(A) \geq n$ as follows. Without changing $(r, s, |\alpha_\nu|, |\beta_\nu|)$ we can choose A in the same form so that $\alpha_1(u) = 0$ for $u > 1$. We now fix the index 1 and apply Lemma 3.7.3 to the remaining indices to choose A so $\alpha_2(u) = 0$ for $\nu > 2$. We continue in this fashion to construct such an A so that $\deg_u(A) = 0$ for $u > r+s$. Since $\deg_n(A) = \deg_{\bar n}(A) > 0$, we conclude therefore that $r+s \geq n$. We estimate:

$$\begin{aligned}\nu = \operatorname{ord}(P) &= \sum_{\mu\leq r}\{|\alpha_\mu| + |\beta_\mu| - 2\} + \sum_{r<\mu\leq r+s}\{|\alpha_\mu| + |\beta_u|\}\\ &\geq 2r + 2s \geq 2n = m\end{aligned}$$

since $\operatorname{ord}(g(\alpha,\beta)) \geq 2$ and $\operatorname{ord}(h_{p\bar q/\alpha\bar\beta}) \geq 2$. This proves $\nu \geq m \geq 2$. Consequently, if $\nu < m$ we conclude $P = 0$ which proves the first assertion of Theorem 3.6.

We assume henceforth that we are in the limiting case $\nu = m$. In this case, all the inequalities must have been equalities. This implies $r+s = n$ and

$$|\alpha_\mu| + |\beta_\mu| = 4 \quad \text{for } \mu \leq r \quad \text{and} \quad |\alpha_\mu| + |\beta_\mu| = 2 \quad \text{for } r < \mu \leq s+r.$$

This holds true for every monomial of P since the construction of A_1 from A involved the use of Lemma 3.7.3 and does not change any of the orders involved. By Lemma 3.7.1 and 3.7.2 we assumed all the purely holomorphic and purely anti-holomorphic derivatives vanished at z_0 and consequently:

$$|\alpha_\mu| = |\beta_\mu| = 2 \quad \text{for } \mu \le r \quad \text{and} \quad |\alpha_\mu| = |\beta_\mu| = 1 \quad \text{for } r < \mu \le s + r.$$

Consequently, P is a polynomial in the $\{g(i_1 i_2; \bar{j}_1 \bar{j}_2) h_{p\bar{q}/i\bar{j}}\}$ variables; it only involves the mixed 2-jets involved.

We wish to choose a monomial of P in normal form to begin counting the number of possible such P. We begin this process with:

Lemma 3.7.4. *Let P satisfy the hypothesis of Theorem 3.6.8(b). Then there exists a monomial A of P which has the form:*

$$A = g(11; \bar{j}_1 \bar{j}_1') \dots g(tt; \bar{j}_t \bar{j}_t') h_{p_1 q_1/t+1, \bar{j}_{t+1}} \dots h_{p_s q_s/m, \bar{j}_n}$$

where $t + s = n$.

Proof: We let $*$ denote indices which are otherwise unspecified and let A_0 be a generic monomial. We have $\deg_n(A) > 0$ for every monomial A of P so $\deg_u(A) > 0$ for every index u as well. We apply Lemma 3.7.3 to choose A of the form:

$$A = g(11; \bar{j}_1 \bar{j}_1') A_0.$$

Suppose $r > 1$. If $A = g(11; *) g(11 : *) A_0$, then we could argue as before using Lemma 3.7.3 that we could choose a monomial A of P so $\deg_k(A) = 0$ for $k > 1 + \ell(A_0) = r + s - 1 = n - 1$ which would be false. Thus $A = g(11; *) g(jk; *) A_0$ where not both j and k are 1. We may apply a coordinate permutation to choose A in the form $A = g(11; *) g(2j; *) A_0$. If $j \ge 2$ we can apply Lemma 3.7.3 to choose $A = g(11; *) g(22; *) A_0$. Otherwise we suppose $A = g(11; *) g(12; *) A_0$. We let $B = g(11; *) g(22; *) A_0$ and compute:

$$\begin{aligned} B(2 \to 1) &= A + \text{ terms divisible by } g(11; *) g(22; *) \\ B(\bar{1} \to \bar{2}) &= \text{ terms divisible by } g(11; *) g(22; *) \end{aligned}$$

so that we conclude in any event we can choose $A = g(11; *) g(22; *) A_0$. We continue this argument with the remaining indices to construct:

$$A = g(11; *) g(22; *) \dots g(tt; *) h_{p_1 \bar{q}_1 / u_1 \bar{v}_1} \dots h_{p_s \bar{q}_s / u_s \bar{v}_s}.$$

We have $\deg_u(A) \ne 0$ for all u. Consequently, the indices $\{t+1, \dots, t+s\}$ must appear among the indices $\{u_1, \dots, u_s\}$. Since these two sets have s elements, they must coincide. Thus by rearranging the indices we can assume $u_\nu = \nu + t$ which completes the proof. We note $\deg_u(A) = 2$ for $u \le t$ and $\deg_u(A) = 1$ for $u > t$.

This lemma does not control the anti-holomorphic indices, we further normalize the choice of A in the following:

LEMMA 3.7.5. *Let P satisfy the hypothesis of Theorem 3.6.8(b). Then there exists a monomial A of P which has the form:*

$$A = g(11; \bar{j}_1 \bar{j}'_1) \dots g(tt; \bar{j}_t \bar{j}'_t) h_{p_1 \bar{q}_1 / t+1, \overline{t+1}} \dots h_{p_s \bar{q}_s / n\bar{n}}$$

for

$$1 \le j_\mu, j'_\mu \le t.$$

PROOF: If A has this form, then $\deg_j(A) = 1$ for $j > t$. Therefore $\deg_{\bar{j}}(A) = 1$ for $j > t$ which implies $1 \le j_\mu, j'_\mu \le t$ automatically. We say that $\bar{j}$ touches itself in A if A is divisible by $g(*; \bar{j}\bar{j})$ for some $*$. We say that j touches $\bar{j}$ in A if A is divisible by $h_{pq/j\bar{j}}$ for some (p, q). Choose A of the form given in Lemma 3.7.4 so the number of indices $j > t$ which touch $\bar{j}$ in A is maximal. Among all such A, choose A so the number of $\bar{j} \le \bar{t}$ which touch themselves in A is maximal. Suppose A does not satisfy the conditions of the Lemma. Thus A must be divisible by $h_{p\bar{q}/u\bar{v}}$ for $u \ne v$ and $u > t$. We suppose first $v > t$. Since $\deg_{\bar{v}}(A) = \deg_v(A) = 1$, v does not touch $\bar{v}$ in A. We let $A = h_{p\bar{q}/u\bar{v}} A_0$ and $B = h_{p\bar{q}/u\bar{u}} A_0$ then $\deg_{\bar{v}}(B) = 0$. We compute:

$$B(\bar{u} \to \bar{v}) = A + A_1 \qquad \text{and} \qquad B(v \to u) = A_2$$

where A_1 is defined by interchanging $\bar{u}$ and $\bar{v}$ and where A_2 is defined by replacing both v and $\bar{v}$ by u and $\bar{u}$ in A. Thus $\deg_v(A)_2 = 0$ so A_2 is not a monomial of A. Thus $A_1 = h_{p\bar{q}/u\bar{u}} A'_0$ must be a monomial of P. One more index (namely u) touches its holomorphic conjugate in A_1 than in A. This contradicts the maximality of A and consequently $A = h_{p\bar{q}/u\bar{v}} A'_0$ for $u > t$, $v \le t$. (This shows $h_{p_0 q_0 / u_0 \bar{v}_0}$ does not divide A for any $u_0 \ne v_0$ and $\bar{v}_0 > \bar{t}$.) We have $\deg_{\bar{u}}(A) = 1$ so the anti-holomorphic index $\bar{u}$ must appear somewhere in A. It cannot appear in a h variable and consequently A has the form $A = g(*; \bar{u}\bar{w}) h_{p\bar{q}/u\bar{v}} A''_0$. We define $B = g(*; \bar{w}\bar{w}) h_{p\bar{q}/u\bar{v}} A''_0$ and compute:

$$B(\bar{w} \to \bar{u}) = A + \text{ terms divisible by } g(*; \bar{w}\bar{w})$$
$$B(u \to w) = g(*; \bar{w}\bar{w}) h_{p\bar{q}/w\bar{v}} A''_0.$$

Since $g(*; \bar{w}\bar{w}) h_{p\bar{q}/w\bar{v}} A''_0$ does not have the index u, it cannot be a monomial of P. Thus terms divisible by $g(*; \bar{w}\bar{w})$ must appear in P. We construct these terms by interchanging a $\bar{w}$ and $\bar{u}$ index in P so the maximality of indices u_0 touching $\bar{u}_0$ for $u_0 > t$ is unchanged. Since $\bar{w}$ does not touch $\bar{w}$ in A, we are adding one additional index of this form which again contradicts the maximality of A. This final contradiction completes the proof.

This constructs a monomial A of P which has the form

$$A = A_0 A_1 \qquad \text{for} \begin{cases} \deg_u(A_0) = \deg_{\bar{u}}(A_0) = 0, & u > t, \\ \deg_u(A_1) = \deg_{\bar{u}}(A_1) = 0, & u \le t. \end{cases}$$

A_0 involves only the derivatives of the metric g and A_1 involves only the derivatives of h. We use this splitting in exactly the same way we used a similar splitting in the proof of Theorem 2.6.1 to reduce the proof of Theorem 3.6.9 to the following assertions:

Lemma 3.7.6. *Let P satisfy the hypothesis of Theorem 3.6.9(b).*
(a) If P is a polynomial in the $\{h_{p\bar q/u\bar v}\}$ variables, then $P = *Q$ for Q a Chern m form of V.
(b) If P is a polynomial in the $\{g(u_1u_2;\bar v_1\bar v_2)\}$ variables, then $P = *Q$ for Q a Chern m form of T_cM.

Proof: (a). If A is a monomial of P, then $\deg_u(A) = \deg_{\bar u}(A) = 1$ for all u so we can express

$$A = h_{p_1q_1/1\bar u_1} \dots h_{p_nq_n/n\bar u_n}$$

where the $\{u_\nu\}$ are a permutation of the indices i through n. We let $A = h_{*/1\bar u_1}h_{*/2\bar u_2}A_0'$ and $B = h_{*/1\bar u_1}h_{*/2\bar u_1}A_0'$. Then:

$$\begin{aligned} B(\bar u_1 \to \bar u_2) &= A + A_1 &&\text{for } A_1 = h_{*/1\bar u_2}h_{*/2\bar u_1}A_0' \\ B(u_2 \to u_1) &= A_2 &&\text{for } \deg_{u_2}(A_2) = 0. \end{aligned}$$

Therefore A_2 is not a monomial of P. Thus A_1 is a monomial of P and furthermore $c(A,P) + c(A_1,P) = 0$. This implies when we interchange $\bar u_1$ and $\bar u_2$ that we change the sign of the coefficient involved. This implies immediately we can express P is terms of expressions:

$$\begin{aligned} *(\Omega^V_{p_1\bar q_1} \wedge \cdots \wedge \Omega^V_{p_n\bar q_n}) &= \sum_{\vec u,\vec v} *(h_{p_1\bar q_1/u_1\bar v_1}\,du^1 \wedge d\bar v_1 \cdots \\ &\qquad\qquad \wedge h_{p_n\bar q_n/u_n\bar v_n}\,du^n \wedge dv^n) \\ &= n!\sum_\rho \operatorname{sign}(\rho) h_{p_1q_1/1\bar\rho(1)} \dots h_{p_nq_n/n\bar\rho(n)} + \cdots \end{aligned}$$

where ρ is a permutation. This implies $*P$ can be expressed as an invariant polynomial in terms of curvature which implies it must be a Chern form as previously computed.

The remainder of this section is devoted to the proof of (b).

Lemma 3.7.7. *Let P satisfy the hypothesis of Lemma 3.7.6(b). Then we can choose a monomial A of P which has the form:*

$$A = g(11;\bar u_1\bar u_1)\dots g(nn;\bar u_n\bar u_n).$$

This gives a normal form for a monomial. Before proving Lemma 3.7.7, we use this lemma to complete the proof of Lemma 3.7.6(b). By making a coordinate permutation if necessary we can assume A has either the form $g(11;\bar{1}\bar{1})A_0'$ or $g(11;\bar{2}\bar{2})A_0'$. In the latter case, we continue inductively to express $A = g(11;\bar{2}\bar{2})g(22;\bar{3}\bar{3})\dots g(u-1,u-1;\bar{u}\bar{u})g(uu;\bar{1}\bar{1})A_0'$ until the cycle closes. If we permit $u=1$ in this decomposition, we can also include the first case. Since the indices 1 through u appear exactly twice in A they do not appear in A_0'. Thus we can continue to play the same game to decompose A into cycles. Clearly A is determined by the length of the cycles involved (up to coordinate permutations); the number of such classifying monomials is $\pi(n)$, the number of partitions of n. This shows that the dimension of the space of polynomials P satisfying the hypothesis of Lemma 3.7.6(b) is $\le \pi(n)$. Since there are exactly $\pi(n)$ Chern forms the dimension must be exactly $\pi(n)$ and every such P must be a Chern m form as claimed.

We give an indirect proof to complete the proof of Lemma 3.7.7. Choose A of the form given by Lemma 3.7.5 so the number of anti-holomorphic indices which touch themselves is maximal. If every anti-holomorphic index touches itself, then A has the form of Lemma 3.7.7 and we are done. We suppose the contrary. Since every index appears exactly twice, every anti-holomorphic index which does not touch itself touches another index which also does not touch itself. Every holomorphic index touches only itself. We may choose the notation so $A = g(*;\bar{1}\bar{2})A_0'$. Suppose first $\bar{1}$ does not touch $\bar{2}$ in A_0'. Then we can assume A has the form:

$$A = g(*;\bar{1}\bar{2})g(*;\bar{1}\bar{3})g(*;\bar{2}\bar{k})A_0'$$

where possibly $k=3$ in this expression. The index 1 touches itself in A. The generic case will be:

$$A = g(11;*)g(*;\bar{1}\bar{2})g(*;\bar{1}\bar{3})g(*;\bar{2}\bar{k})A_0'.$$

The other cases in which perhaps $A = g(11;\bar{1}\bar{2})\dots$ or $g(11,\bar{1}\bar{3})\dots$ or $g(11;\bar{2}\bar{k})\dots$ are handled similarly. The holomorphic and anti-holomorphic indices do not interact. In exactly which variable they appear does not matter. This can also be expressed as a lemma in tensor algebras.

We suppose $k \ne 3$; we will never let the 2 and 3 variables interact so the case in which $k=3$ is exactly analogous. Thus A has the form:

$$A = g(11;*)g(*;\bar{1}\bar{2})g(*;\bar{1}\bar{3})g(*;\bar{2}\bar{k})g(*;\bar{3}\bar{\jmath})A_0'$$

where possibly $j=k$. Set $B = (11;*)g(*;\bar{2}\bar{2})g(*;\bar{1}\bar{3})g(*;\bar{2}\bar{k})g(*;\bar{3}\bar{\jmath})A_0'$ then:

$$B(\bar{2}\to\bar{1}) = 2A + A''$$
$$B(1\to 2) = 2A_1$$

for

$$\begin{aligned} A'' &= g(11;*)g(*;\bar2\bar2)g(*;\bar1\bar3)g(*;\bar1\bar k)g(*;\bar3\bar\jmath)A'_0 \\ A_1 &= g(12;*)g(*;\bar2\bar2)g(*;\bar1\bar3)g(*;\bar2\bar k)g(*;\bar3\bar\jmath)A'_0. \end{aligned}$$

A'' is also a monomial of the form given by Lemma 3.7.5. Since one more anti-holomorphic index touches itself in A'' then does in A, the maximality of A shows A'' is not a monomial of P. Consequently A_1 is a monomial of P. Set $B_1 = g(12;*)g(*;\bar2\bar2)g(*;\bar3\bar3)g(*;\bar2\bar k)g(*;\bar3\bar\jmath)A'_0$ then:

$$\begin{aligned} B_1(\bar3\to\bar1) &= 2A_1 + A_2 \\ B_1(1\to 3) &= A''' \end{aligned}$$

for

$$\begin{aligned} A_2 &= g(12;*)g(*;\bar2\bar2)g(*;\bar3\bar3)g(*;\bar2\bar k)g(*;\bar1\bar\jmath)A'_0 \\ A''' &= g(32;*)g(*;\bar2\bar2)g(*;\bar3\bar3)g(*;\bar2\bar k)g(*;\bar3\bar\jmath)A'_0. \end{aligned}$$

However, $\deg_1(A''')=0$. Since $r(P)=0$, A''' cannot be a monomial of P so A_2 is a monomial of P. Finally we set

$$B_2 = g(11;*)g(*;\bar2\bar2)g(*;\bar3\bar3)g(*;\bar2\bar k)g(*;\bar1\bar\jmath)A'_0$$

so

$$\begin{aligned} B_2(1\to 2) &= A_2 \\ B_2(\bar2\to\bar1) &= A_3 + 2A_4 \end{aligned}$$

for

$$\begin{aligned} A_3 &= g(11;*)g(*;\bar2\bar2)g(*;\bar3\bar3)g(*;\bar1\bar k)g(*;\bar1\bar\jmath)A'_0 \\ A_4 &= g(11;*)g(*;\bar1\bar2)g(*;\bar3\bar3)g(*;\bar2\bar k)g(*;\bar1\bar\jmath)A'_0. \end{aligned}$$

This implies either A_3 or A_4 is monomial of P. Both these have every holomorphic index touching itself. Furthermore, one more anti-holomorphic index (namely $\bar3$) touches itself. This contradicts the maximality of A.

In this argument it was very important that $\bar2\neq\bar3$ as we let the index $\bar1$ interact with each of these indices separately. Thus the final case we must consider is the case in which A has the form:

$$A = g(11;*)g(22;*)g(*;\bar1\bar2)g(*;\bar1\bar2)A_0.$$

So far we have not had to take into account multiplicities or signs in computing $A(1\to 2)$ etc; we have been content to conclude certain coefficients are non-zero. In studying this case, we must be more careful in our analysis as the signs involved are crucial. We clear the previous notation and define:

$$\begin{aligned} A_1 &= g(12;*)g(22;*)g(*;\bar1\bar2)g(*;\bar2\bar2)A_0 \\ A_2 &= g(12;*)g(22;*)g(*;\bar2\bar2)g(*;\bar1\bar2)A_0 \\ A_3 &= g(11;*)g(22;*)g(*;\bar1\bar1)g(*;\bar2\bar2)A_0 \\ A_4 &= g(22;*)g(22;*)g(*;\bar2\bar2)g(*;\bar2\bar2)A_0. \end{aligned}$$

We note that A_3 is not a monomial of P by the maximality of A. A_4 is not a monomial of P as $\deg_1(A_4) = 0$ and $r(P) = 0$. We let $B = g(11;*)g(22;*)g(*;\bar{1}\bar{2})g(*;\bar{2}\bar{2})A_0$. Then

$$B(\bar{2} \to \bar{1}) = 2A + A_3, \qquad B(1 \to 2) = 2A_1$$

so that A_1 must be a monomial of P since A is a monomial of P and A_4 is not. We now pay more careful attention to the multiplicities and signs:

$$A(\bar{1} \to \bar{2}) = B + \cdots \qquad \text{and} \qquad A_1(2 \to 1) = B + \cdots$$

However $\bar{1} \to \bar{2}$ introduces a $\bar{b}$ while $2 \to 1$ introduces a $-\bar{b}$ so using the argument discussed earlier we conclude not only $c(A_1, P) \neq 0$ but that $c(A, P) - c(A_1, P) = 0$ so $c(A_1, P) = c(A, P)$. A_2 behaves similarly so the analogous argument using $B_1 = g(11;*)g(22;*)g(*;\bar{2}\bar{2})g(*;\bar{1}\bar{2})$ shows $c(A_2, P) = c(A, P)$. We now study $B_2 = g(12;*)g(22;*)g(*,\bar{2}\bar{2})g(*;\bar{2}\bar{2})$ and compute:

$$B_2(1 \to 2) = A_4 \qquad B_2(\bar{2} \to \bar{1}) = 2A_1 + 2A_2$$

A_4 is not a monomial of P as noted above. We again pay careful attention to the signs:

$$A_1(\bar{1} \to \bar{2}) = B_2 \qquad \text{and} \qquad A_2(\bar{1} \to \bar{2}) = B_2$$

This implies $c(A_1, P) + c(A_2, P) = 0$. Since $c(A_1, P) = c(A_2, P) = c(A, P)$ this implies $2c(A, P) = 0$ so A was not a monomial of P. This final contradiction completes the proof.

This proof was long and technical. However, it is not a theorem based on unitary invariance alone as the restriction axiom plays an important role in the development. We know of no proof of Theorem 3.6.9 which is based only on H. Weyl's theorem; in the real case, the corresponding characterization of the Pontrjagin classes was based only on orthogonal invariance and we gave a proof based on H. Weyl's theorem in that case.

3.8. The Chern Isomorphism and Bott Periodicity.

In section 3.9 we shall discuss the Atiyah-Singer index theorem in general using the results of section 3.1. The index theorem gives a topological formula for the index of an arbitrary elliptic operator. Before begining the proof of that theorem, we must first review briefly the Bott periodicity theorem from the point of Clifford modules. We continue our consideration of the bundles over S^n constructed in the second chapter. Let

$$\begin{aligned}
\mathrm{GL}(k,\mathbf{C}) &= \{\, A : A \text{ is a } k\times k \text{ complex matrix with } \det(A)\neq 0 \,\}\\
\mathrm{GL}'(k,\mathbf{C}) &= \{\, A\in \mathrm{GL}(k,\mathbf{C}) : \det(A-it)\neq 0 \text{ for all } t\in\mathbf{R} \,\}\\
\mathrm{U}(k) &= \{\, A\in \mathrm{GL}(k,\mathbf{C}) : A\cdot A^* = I \,\}\\
\mathrm{S}(k) &= \{\, A\in \mathrm{GL}(k,\mathbf{C}) : A^2 = I \text{ and } A = A^* \,\}\\
\mathrm{S}_0(k) &= \{\, A\in \mathrm{S}(k) : \mathrm{Tr}(A) = 0 \,\}.
\end{aligned}$$

We note that $\mathrm{S}_0(k)$ is empty if k is odd. $\mathrm{U}(k)$ is compact and is a deformation retract of $\mathrm{GL}(k,\mathbf{C})$; $\mathrm{S}(k)$ is compact and is a deformation retract of $\mathrm{GL}'(k,\mathbf{C})$. $\mathrm{S}_0(k)$ is one of the components of $\mathrm{S}(k)$.

Let X be a finite simplicial complex. The suspension ΣX is defined by identifying $X\times\{\frac{\pi}{2}\}$ to a single point N and $X\times\{-\frac{\pi}{2}\}$ to single point S in the product $X\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. Let $D_\pm(X)$ denote the northern and southern "hemispheres" in the suspension; the intersection $D_+(X)\cap D_-(X) = X$.

[Reprinter's note: A figure depicting the suspension of X belongs here.]

We note both $D_+(X)$ and $D_-(X)$ are contractible. $\Sigma(X)$ is a finite simplicial complex. If S^n is the unit sphere in R^{n+1}, then $\Sigma(S^n) = S^{n+1}$. Finally, if W is a vector bundle over some base space Y, then we choose a fiber metric on W and let $S(W)$ be the unit sphere bundle. $\Sigma(W)$ is the fiberwise suspension of $S(W)$ over Y. This can be identified with $S(W\oplus 1)$.

It is beyond the scope of this book to develop in detail the theory of vector bundles so we shall simply state relevant facts as needed. We let $\mathrm{Vect}_k(X)$ denote the set of isomorphism classes of complex vector bundles over X of fiber dimension k. We let $\operatorname{Vect}(X) = \bigcup_k \operatorname{Vect}_k(X)$ be the set of isomorphism classes of complex vector bundles over X of all fiber dimensions. We assume X is connected so that the dimension of a vector bundle over X is constant.

There is a natural inclusion map $\operatorname{Vect}_k(X) \to \operatorname{Vect}_{k+1}(X)$ defined by sending $V \mapsto V \oplus 1$ where 1 denotes the trivial line bundle over X.

LEMMA 3.8.1. *If* $2k \geq \dim X$ *then the map* $\operatorname{Vect}_k(X) \to \operatorname{Vect}_{k+1}(X)$ *is bijective.*

We let $F(\operatorname{Vect}(X))$ be the free abelian group on these generators. We shall let (V) denote the element of $K(X)$ defined by $V \in \operatorname{Vect}(X)$. $K(X)$ is the quotient modulo the relation $(V \oplus W) = (V) + (W)$ for $V, W \in \operatorname{Vect}(X)$. $K(X)$ is an abelian group. The natural map $\dim\colon \operatorname{Vect}(X) \to \mathbf{Z}^+$ extends to $\dim\colon K(X) \to \mathbf{Z}$; $\widetilde{K}(X)$ is the kernel of this map. We shall say that an element of $K(X)$ is a virtual bundle; $\widetilde{K}(X)$ is the subgroup of virtual bundles of virtual dimension zero.

It is possible to define a kind of inverse in $\operatorname{Vect}(X)$:

LEMMA 3.8.2. *Given* $V \in \operatorname{Vect}(X)$, *there exists* $W \in \operatorname{Vect}(X)$ *so* $V \oplus W = 1^j$ *is isomorphic to a trivial bundle of dimension* $j = \dim V + \dim W$.

We combine these two lemmas to give a group structure to $\operatorname{Vect}_k(X)$ for $2k \geq \dim X$. Given $V, W \in \operatorname{Vect}_k(X)$, then $V \oplus W \in \operatorname{Vect}_{2k}(X)$. By Lemma 3.8.1, the map $\operatorname{Vect}_k(X) \to \operatorname{Vect}_{2k}(X)$ defined by sending $U \mapsto U \oplus 1^k$ is bijective. Thus there is a unique element we shall denote by $V + W \in \operatorname{Vect}_k(X)$ so that $(V + W) \oplus 1^k = V \oplus W$. It is immediate that this is an associative and commutative operation and that the trivial bundle 1^k functions as the unit. We use Lemma 3.8.2 to construct an inverse. Given $V \in \operatorname{Vect}_k(X)$ there exists j and $W \in \operatorname{Vect}_j(X)$ so $V \oplus W = 1^{j+k}$. We assume without loss of generality that $j \geq k$. By Lemma 3.8.1, we choose $\overline{W} \in \operatorname{Vect}_k(X)$ so that $V \oplus \overline{W} \oplus 1^{j-k} = V \oplus \overline{W} = 1^{j+k}$. Then the bijectivity implies $V \oplus \overline{W} = 1^{2k}$ so $\overline{W}$ is the inverse of V. This shows $\operatorname{Vect}_k(X)$ is a group under this operation.

There is a natural map $\operatorname{Vect}_k(X) \to \widetilde{K}(X)$ defined by sending $V \mapsto (V) - (1^k)$. It is immediate that:

$$\begin{aligned}
(V + W) - 1^k &= (V + W) + (1^k) - (1^{2k}) = ((V + W) \oplus 1^k) - (1^{2k}) \\
&= (V \oplus W) - (1^{2k}) \\
&= (V) + (W) - (1^{2k}) = (V) - (1^k) + (W) - (1^k)
\end{aligned}$$

so the map is a group homomorphism. $\widetilde{K}(X)$ is generated by elements of the form $(V) - (W)$ for $V, W \in \operatorname{Vect}_j X$ for some j. If we choose $\overline{W}$ so $W \oplus \overline{W} = 1^v$ then $(V) - (W) = (V \oplus \overline{W}) - (1^v)$ so $\widetilde{K}(X)$ is generated by elements of the form $(V) - (1^v)$ for $V \in \operatorname{Vect}_v(X)$. Again, by adding trivial factors, we may assume $v \geq k$ so by Lemma 3.8.1 $V = V_1 \oplus 1^{j-k}$ and $(V) - (1^j) = (V_1) - (1^k)$ for $V_1 \in \mathrm{Vect}_k(X)$. This implies the map $\mathrm{Vect}_k(X) \to \widetilde{K}(X)$ is subjective. Finally, we note that that in fact $K(X)$ is generated by $\mathrm{Vect}_k(X)$ subject to the relation $(V) + (W) = (V \oplus W) = (V + W) + (1^k)$ so that this map is injective and we may identify $\widetilde{K}(X) = \mathrm{Vect}_k(X)$ and $K(X) = \mathbf{Z} \oplus \widetilde{K}(X) = \mathbf{Z} \oplus \mathrm{Vect}_k(X)$ for any k such that $2k \geq \dim X$.

Tensor product defines a ring structure on $K(X)$. We define $(V) \otimes (W) = (V \otimes W)$ for $V, W \in \mathrm{Vect}(X)$. Since $(V_1 \oplus V_2) \otimes (W) = (V_1 \otimes W \oplus V_2 \otimes W)$, this extends from $F(\mathrm{Vect}(X))$ to define a ring structure on $K(X)$ in which the trivial line bundle 1 functions as a multiplicative identity. $\widetilde{K}(X)$ is an ideal of $K(X)$.

$K(X)$ is a $\mathbf{Z}$-module. It is convenient to change the coefficient group and define:

$$K(X; \mathbf{C}) = K(X) \otimes_{\mathbf{Z}} \mathbf{C}$$

to permit complex coefficients. $K(X; \mathbf{C})$ is the free $\mathbf{C}$-vector space generated by $\mathrm{Vect}(X)$ subject to the relations $V \oplus W = V + W$. By using complex coefficients, we eliminate torsion which makes calculations much simpler. The Chern character is a morphism:

$$ch\colon \mathrm{Vect}(X) \to H^{\mathrm{even}}(X; \mathbf{C}) = \bigoplus_q H^{2q}(X; \mathbf{C}).$$

We define ch using characteristic classes in the second section if X is a smooth manifold; it is possible to extend this definition using topological methods to more general topological settings. The identities:

$$ch(V \oplus W) = ch(V) + ch(W) \qquad \text{and} \qquad ch(V \otimes W) = ch(V)\, ch(W)$$

imply that we can extend:

$$ch\colon K(X) \to H^{\mathrm{even}}(X; \mathbf{C})$$

to be a ring homomorphism. We tensor this $\mathbf{Z}$-linear map with $\mathbf{C}$ to get

$$ch\colon K(X; \mathbf{C}) \to H^{\mathrm{even}}(X; \mathbf{C}).$$

Lemma 3.8.3 (Chern isomorphism). *$ch\colon K(X; \mathbf{C}) \to H^{\mathrm{even}}(X; \mathbf{C})$ is a ring isomorphism.*

If $\widetilde{H}^{\mathrm{even}}(X; \mathbf{C}) = \bigoplus_{q>o} H^{2q}(X; \mathbf{C})$ is the reduced even dimensional cohomology, then $ch\colon \widetilde{K}(X; \mathbf{C}) \to \widetilde{H}^{\mathrm{even}}(X; \mathbf{C})$ is a ring isomorphism. For this reason, $\widetilde{K}(X)$ is often refered to as reduced K-theory. We emphasize that in this isomorphism we are ignoring torsion and that torsion is crucial to understanding K-theory in general. Fortunately, the index is $\mathbf{Z}$-valued and we can ignore torsion in K-theory for an understanding of the index theorem.

We now return to studying the relation between $K(X)$ and $K(\Sigma X)$. We shall let $[X, Y]$ denote the set of homotopy classes of maps from X to Y. We shall always assume that X and Y are equipped with base points and that all maps are basepoint preserving. We fix $2k \ge \dim X$ and let $f: X \to S_0(2k)$. Since $f(x)$ is self-adjoint, and $f(x)^2 = I$, the eigenvalues of $f(x)$ are ± 1. Since $\operatorname{Tr} f(x) = 0$, each eigenvalue appears with multiplicity k. We let $\Pi_\pm(f)$ be the bundles over X which are sub-bundles of $X \times 1^k$ so that the fiber of $\Pi_\pm(f)$ at x is just the ± 1 eigenspace of $f(x)$. If we define $\pi_\pm(f)(x) = \frac{1}{2}(1 \pm f(x))$ then these are projections of constant rank k with range $\Pi_\pm(f)$. If f and f_1 are homotopic maps, then they determine isomorphic vector bundles. Thus the assignment $f \mapsto \Pi_+(f) \in \operatorname{Vect}_k(X)$ defines a map $\Pi_+: [X, S_0(2k)] \to \operatorname{Vect}_k(X)$.

Lemma 3.8.4. *The natural map $[X, S_0(2k)] \to \operatorname{Vect}_k(X)$ is bijective for $2k \ge \dim X$.*

Proof: Given $V \in \operatorname{Vect}_k(X)$ we choose $W \in \operatorname{Vect}_k(X)$ so $V \oplus W = 1^{2k}$. We choose fiber metrics on V and on W and make this sum orthogonal. By applying the Gram-Schmidt process to the given global frame, we can assume that there is an orthonormal global frame and consequently that $V \oplus W = 1^{2k}$ is an orthogonal direct sum. We let $\pi_+(x)$ be orthogonal projection on the fiber V_x of 1^{2k} and $f(x) = 2\pi_+(x) - I$. Then it is immediate that $f: X \to S_0(2k)$ and $\pi_+(f) = V$. This proves the map is subjective. The injectivity comes from the same sorts of considerations as were used to prove Lemma 3.8.1 and is therefore omitted.

If $f: X \to \mathrm{GL}'(k, \mathbf{C})$, we can extend the definition to let $\Pi_\pm(f)(x)$ be the span of the generalized eigenvectors of f corresponding to eigenvalues with positive/negative real part. Since there are no purely imaginary eigenvalues, $\Pi_\pm(f)$ has constant rank and gives a vector bundle over X.

In a similar fashion, we can classify $\operatorname{Vect}_k \Sigma X = [X, \mathrm{U}(k)]$. Since $\mathrm{U}(k)$ is a deformation retract of $\mathrm{GL}(k, \mathbf{C})$, we identify $[X, \mathrm{U}(k)] = [X, \mathrm{GL}(k, \mathbf{C})]$. If $g: X \to \mathrm{GL}(k, \mathbf{C})$, we use g as a clutching function to define a bundle over ΣX. Over $D_\pm(X)$, we take the bundles $D_\pm(X) \times \mathbf{C}^k$. $D_+(X) \cap D_-(X) = X$. On the overlap, we identify $(x, z)_+ = (x, z')_-$ if $z \cdot g(x) = z'$. If we let s^+ and s^- be the usual frames for C^k over $D_\pm$ then $\sum z_i^+ g_{ij} s_j^+ = \sum z_i^- s_i^-$ so that we identify the frames using the identity

$$s^- = g s^+.$$

We denote this bundle by V_g. Homotopic maps define isomorphic bundles so we have a map $[X, \mathrm{GL}(k, \mathbf{C})] \to \operatorname{Vect}_k \Sigma X$. Conversely, given a vector bundle V over ΣX we can always choose local trivializations for V over

$D_\pm(X)$ since these spaces are contractible. The transition function $s^- = gs^+$ gives a map $g: X \to \mathrm{GL}(k, \mathbf{C})$. It is convenient to assume X has a base point x_0 and to choose $s^- = s^+$ at x_0. Thus $g(x_0) = I$. This shows the map $[X, \mathrm{GL}(k, \mathbf{C})] \to \mathrm{Vect}_k \Sigma X$ is surjective. If we had chosen different trivializations $\tilde{s}^+ = h^+ s^+$ and $\tilde{s}^- = h^- s^-$ then we would have obtained a new clutching function $\tilde{g} = h^- g (h^+)^{-1}$. Since $h^\pm$ are defined on contractible sets, they are null homotopic so $\tilde{g}$ is homotopic to g. This proves:

LEMMA 3.8.5. *The map $[X, \mathrm{GL}(k, \mathbf{C})] = [X, \mathrm{U}(k)] \to \mathrm{Vect}_k(\Sigma X)$, given by associating to a map g the bundle defined by the clutching function g, is bijective.*

It is always somewhat confusing to try to work directly with this definition. It is always a temptation to confuse the roles of g and of its inverse as well as the transposes involved. There is another definition which avoids this difficulty and which will be very useful in computing specific examples. If $g: X \to \mathrm{GL}(k, \mathbf{C})$, we shall let $g(x)z$ denote matrix multiplication. We shall regard $\mathbf{C}^k$ as consisting of column vectors and let g act as a matrix from the left. This is, of course, the opposite convention from that used previously.

Let $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ be the suspension parameter. We define:

$$\Sigma: [X, \mathrm{GL}(k, \mathbf{C})] \to [\Sigma X, \mathrm{GL}'(2k, \mathbf{C})]$$
$$\Sigma: [X, \mathrm{GL}'(k, \mathbf{C})] \to [\Sigma X, \mathrm{GL}(k, \mathbf{C})]$$

by

$$\Sigma g(x, \theta) = \begin{pmatrix} (\sin\theta) I_k & (\cos\theta) g^*(x) \\ (\cos\theta) g(x) & -(\sin\theta) I_k \end{pmatrix}$$
$$\Sigma f(x, \theta) = (\cos\theta) f(x) - i(\sin\theta) I_k.$$

We check that Σ has the desired ranges as follows. If $g: X \to \mathrm{GL}(k, \mathbf{C})$, then it is immediate that Σg is self-adjoint. We compute:

$$(\Sigma g)^2 = \begin{pmatrix} (\sin^2\theta) I_k + (\cos^2\theta) g^* g & 0 \\ 0 & (\sin^2\theta) I_k + (\cos^2\theta) g g^* \end{pmatrix}.$$

This is non-singular since g is invertible. Therefore Σg is invertible. If $\theta = \pi/2$, then $\Sigma g = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ is independent of x. If g is unitary, $\Sigma g \in \mathrm{S}_0(2k)$. If $f \in \mathrm{GL}'(k, \mathbf{C})$, then f has no purely imaginary eigenvalues so Σf is non-singular.

Lemma 3.8.6. *Let $g\colon X \to \mathrm{GL}(k, \mathbf{C})$ and construct the bundle $\Pi_+(\Sigma g)$ over ΣX. Then this bundle is defined by the clutching function g.*

Proof: We may replace g by a homotopic map without changing the isomorphism class of the bundle $\Pi_+(\Sigma g)$. Consequently, we may assume without loss of generality that $g\colon X \to \mathrm{U}(k)$ so $g^*g = gg^* = I_k$. Consequently $(\Sigma g)^2 = I_{2k}$ and $\pi_+(\Sigma g) = \frac{1}{2}(I_{2k} + \Sigma g)$. If $z \in \mathbf{C}^k$, then:

$$\pi_+(\Sigma g)(x,\theta)\begin{pmatrix} z \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} z + (\sin\theta)z \\ (\cos\theta)g(x)z \end{pmatrix}.$$

This projection from $\mathbf{C}^k$ to $\Pi_+(\Sigma g)$ is non-singular away from the south pole S and can be used to give a trivialization of $\Pi_+(\Sigma g)$ on $\Sigma X - S$.

From this description, it is clear that $\Pi_+(\Sigma g)$ is spanned by vectors of the form $\frac{1}{2}\begin{pmatrix} (1+(\sin\theta))z \\ (\cos\theta)g(x)z \end{pmatrix}$ away from the south pole. At the south pole, $\Pi_+(\Sigma g)$ consists of all vectors of the form $\begin{pmatrix} 0 \\ w \end{pmatrix}$. Consequently, projection on the second factor $\pi_+(S)\colon \begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} 0 \\ b \end{pmatrix}$ is non-singular away from the north pole N and gives a trivialization of $\Pi_+(\Sigma g)$ on $\Sigma X - N$. We restrict to the equator X and compute the composite of these two maps to determine the clutching function:

$$\begin{pmatrix} z \\ 0 \end{pmatrix} \mapsto \frac{1}{2}\begin{pmatrix} z \\ g(x)z \end{pmatrix} \mapsto \frac{1}{2}\begin{pmatrix} 0 \\ g(x)z \end{pmatrix}.$$

The function $g(x)/2$ is homotopic to g which completes the proof.

This is a very concrete description of the bundle defined by the clutching function g. In the examples we shall be considering, it will come equipped with a natural connection which will make computing characteristic classes much easier.

We now compute the double suspension. Fix $f\colon X \to \mathrm{S}_0(2k)$ and $g\colon X \to \mathrm{GL}(k, \mathbf{C})$.

$$\begin{aligned}
&\Sigma^2 f(x,\theta,\phi) \\
&\quad = \begin{pmatrix} \sin\phi & \cos\phi\{(\cos\theta)f^* + i\sin\theta\} \\ \cos\phi\{(\cos\theta)f(x) - i\sin\theta\} & -\sin\phi \end{pmatrix} \\
&\Sigma^2 g(x,\theta,\phi) \\
&\quad = \begin{pmatrix} \cos\phi\sin\theta - i\sin\phi & \cos\phi\cos\theta g^*(x) \\ \cos\phi\cos\theta g(x) & -\cos\phi\cos\theta - i\sin\phi \end{pmatrix}
\end{aligned}$$

We let $\mathrm{U}(\infty)$ be the direct limit of the inclusions $\mathrm{U}(k) \to \mathrm{U}(k+1)\cdots$ and $\mathrm{S}_0(\infty)$ be the direct limit of the inclusions $\mathrm{S}_0(2k) \to \mathrm{S}_0(2k+2) \to \cdots$. Then we have identified:

$$\widetilde{K}(X) = [X, \mathrm{S}_0(\infty)] \qquad \text{and} \qquad \widetilde{K}(\Sigma X) = [\Sigma X, \mathrm{S}_0(\infty)] = [S, \mathrm{U}(\infty)]$$

We can now state:

THEOREM 3.8.7 (BOTT PERIODICITY). *The map*

$$\Sigma^2 \colon \widetilde{K}(X) = [X, \mathrm{S}_0(\infty)] \to \widetilde{K}(\Sigma^2 X) = [\Sigma^2 X, \mathrm{S}_0(\infty)]$$

induces a ring isomorphism. Similarly, the map

$$\Sigma^2 \colon \widetilde{K}(\Sigma X) = [X, \mathrm{U}(\infty)] \to \widetilde{K}(\Sigma^2 X) = [\Sigma^2 X, \mathrm{U}(\infty)]$$

is a ring isomorphism.

We note that $[X, \mathrm{U}(\infty)]$ inherits a natural additive structure from the group structure on $\mathrm{U}(\infty)$ by letting $g \oplus g'$ be the direct sum of these two maps. This group structure is compatible with the additive structure on $\widetilde{K}(\Sigma X)$ since the clutching function of the direct sum is the direct sum of the clutching functions. Similarly, we can put a ring structure on $[X, \mathrm{U}(\infty)]$ using tensor products to be compatible with the ring structure on $\widetilde{K}(\Sigma X)$.

The Chern character identifies $\widetilde{K}(X; \mathbf{C})$ with $\widetilde{H}^{\mathrm{even}}(X; \mathbf{C})$. We may identify $\Sigma^2 X$ with a certain quotient of $X \times S^2$. Bott periodicity in this context becomes the assertion $K(X \times S^2) = K(X) \otimes K(S^2)$ which is the Kunneth formula in cohomology.

We now consider the case of a sphere $X = S^n$. The unitary group $\mathrm{U}(2)$ decomposes as $\mathrm{U}(2) = \mathrm{U}(1) \times \mathrm{SU}(2) = S^1 \times S^3$ topologically so $\pi_1(\mathrm{U}(2)) = \mathbf{Z}$, $\pi_2(\mathrm{U}(2)) = 0$ and $\pi_3(\mathrm{U}(2)) = \mathbf{Z}$. This implies $\widetilde{K}(S^1) = \widetilde{K}(S^3) = 0$ and $\widetilde{K}(S^2) = \widetilde{K}(S^4) = \mathbf{Z}$. Using Bott periodicity, we know more generally that:

LEMMA 3.8.8 (BOTT PERIODICITY). *If n is odd, then*

$$\widetilde{K}(S^n) \simeq \pi_{n-1}(\mathrm{U}(\infty)) = 0.$$

If n is even, then

$$\widetilde{K}(S^n) \simeq \pi_{n-1}(\mathrm{U}(\infty)) = \mathbf{Z}.$$

It is useful to construct explicit generators of these groups. Let $x = (x_0, \ldots, x_n) \in S^n$, let $y = (x, x_{n+1}) \in S^{n+1}$, and let $z = (y, x_{n+2}) \in S^{n+2}$. If $f \colon S^n \to \mathrm{GL}'(k, \mathbf{C})$ and $g \colon X \to \mathrm{GL}(k, \mathbf{C})$ we extend these to be homogeneous of degree 1 with values in the $k \times k$ matrices. Then we compute:

$$\begin{aligned}
\Sigma f(y) &= f(x) - ix_{n+1} \\
\Sigma^2 f(z) &= \begin{pmatrix} x_{n+2} & f^*(x) + ix_{n+1} \\ f(x) - ix_{n+1} & -x_{n+2} \end{pmatrix} \\
\Sigma g(y) &= \begin{pmatrix} x_{n+1} & g^*(x) \\ g(x) & -x_{n+1} \end{pmatrix} \\
\Sigma^2 g(z) &= \begin{pmatrix} x_{n+1} - ix_{n+2} & g^*(x) \\ g(x) & -x_{n+1} - ix_{n+2} \end{pmatrix}.
\end{aligned}$$

If we suppose that f is self-adjoint, then we can express:

$$\Sigma^2 f(z) = x_{n+2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes I_k + x_{n+1} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \otimes I_k + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes f(x).$$

We can now construct generators for $\pi_{n-1}\,\mathrm{U}(\infty)$ and $\widetilde{K}(S^n)$ using Clifford algebras. Let $g(x_0, x_1) = x_0 - ix_1$ generate $\pi_1(S^1) = \mathbf{Z}$ then

$$\Sigma g(x_0, x_1, x_2) = x_0 e_0 + x_1 e_1 + x_2 e_2$$

for

$$e_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad e_1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The $\{e_j\}$ satisfy the relations $e_j e_k + e_k e_j = 2\delta_{jk}$ and $e_0 e_1 e_2 = -iI_2$. $\Pi_+(\Sigma g)$ is a line bundle over S^2 which generates $\widetilde{K}(S^2)$. We compute:

$$\Sigma^2 g(x_0, x_1, x_2, x_3) = x_0 e_0 + x_1 e_1 + x_2 e_2 - ix_3 I_2.$$

If we introduce $z_1 = x_0 + ix_1$ and $z_2 = x_2 + ix_3$ then

$$\Sigma^2 g(z_1, z_2) = \begin{pmatrix} \bar{z}_2 & z_1 \\ \bar{z}_1 & -z_2 \end{pmatrix}.$$

Consequently $\Sigma^2 g$ generates $\pi_3(\mathrm{U}(\infty))$. We suspend once to construct:

$$\begin{aligned}
&\Sigma^3 g(x_0, x_1, x_2, x_3, x_4) \\
&\qquad = x_0 e_0 \otimes e_0 + x_1 e_0 \otimes e_1 + x_2 e_0 \otimes e_2 + x_3 e_1 \otimes I + x_4 e_2 \otimes I.
\end{aligned}$$

The bundle $\Pi_+(\Sigma^3 g)$ is a 2-plane bundle over S^4 which generates $\widetilde{K}(S^4)$. We express $\Sigma^3 g = x_0 e_0^4 + \cdots + x_4 e_4^4$ then these matrices satisfy the commutation relations $e_j^4 e_j^4 + e_k^4 e_j^4 = 2\delta_{jk}$ and $e_0^4 e_1^4 e_2^4 e_3^4 e_4^4 = -I$.

We proceed inductively to define matrices e_j^{2k} for $0 \le j \le 2k$ so that $e_ie_j + e_je_i = 2\delta_{ij}$ and $e_0^{2k} \dots e_{2k}^{2k} = (-i)^k I$. These are matrics of shape $2^k \times 2^k$ such that $\Sigma^{2k-1}(g)(x) = \sum x_j e_j^{2k}$. In Lemma 2.1.5 we computed that:

$$\int_{S^{2k}} ch_k(\Pi_+\Sigma^{2k-1}g) = i^k 2^{-k}\,\mathrm{Tr}(e_0^{2k}\dots e_{2k}^{2k}) = 1.$$

These bundles generate $\widetilde{K}(S^{2k})$ and $\Sigma^{2k}(g)$ generates $\pi_{2k+1}(\mathrm{U}(\infty))$. We summarize these calculations as follows:

LEMMA 3.8.9. *Let $\{e_0, \dots, e_{2k}\}$ be a collection of self-adjoint matrices of shape $2^k \times 2^k$ such that $e_ie_j + e_je_i = 2\delta_{ij}$ and such that $e_0 \dots e_{2k} = (-i)^k I$. We define $e(x) = x_0e_0 + \cdots + x_{2k}e_{2k}$ for $x \in S^{2k}$. Let $\Pi_+(e)$ be the bundle of $+1$ eigenvectors of e over S^{2k}. Then $\Pi_+(e)$ generates $\widetilde{K}(S^{2k}) \simeq \mathbf{Z}$. $\Sigma e(y) = e(x) - ix_{2k+1}$ generates $\pi_{2k+1}(\mathrm{U}(\infty)) \simeq \mathbf{Z}$. $\int_{S^{2k}} ch_k(V) \colon \widetilde{K}(S^{2k}) \to \mathbf{Z}$ is an isomorphism and $\int_{S^{2k}} ch_k(\Pi_+e) = 1$.*

PROOF: We note that $\int_{S^{2k}} ch_k(V)$ is the index of the spin complex with coefficients in V since all the Pontrjagin forms of $T(S^{2k})$ vanish except for p_0. Thus this integral, called the topological charge, is always an integer. We have checked that the integral is 1 on a generator and hence the map is surjective. Since $\widetilde{K}(S^{2k}) = \mathbf{Z}$, it must be bijective.

It was extremely convenient to have the bundle with clutching function $\Sigma^{2k-2}g$ so concretely given so that we could apply Lemma 2.1.5 to compute the topological charge. This will also be important in the next chapter.

3.9. The Atiyah-Singer Index Theorem.

In this section, we shall discuss the Atiyah-Singer theorem for a general elliptic complex by interpreting the index as a map in K-theory. Let M be smooth, compact, and without boundary. For the moment we make no assumptions regarding the parity of the dimension m. We do not assume M is orientable. Let $P: C^\infty(V_1) \rightarrow \mathbf{C}^\infty(V_2)$ be an elliptic complex with leading symbol $p(x,\xi): S(T^*M) \rightarrow \mathrm{HOM}(V_1, V_2)$. We let $\Sigma(T^*M)$ be the fiberwise suspension of the unit sphere bundle $S(T^*M)$. We identify $\Sigma(T^*M) = S(T^*M \oplus 1)$. We generalize the construction of section 3.8 to define $\Sigma p: \Sigma(T^*M) \rightarrow \mathrm{END}(V_1 \oplus V_2)$ by:

$$\Sigma p(x,\xi,\theta) = \begin{pmatrix} (\sin\theta) I_{V_1} & (\cos\theta) p^*(x,\xi) \\ (\cos\theta) p(x,\xi) & -(\sin\theta) I_{V_2} \end{pmatrix}.$$

This is a self-adjoint invertible endomorphism. We let $\Pi_\pm(\Sigma p)$ be the subbundle of $V_1 \oplus V_2$ over $\Sigma(T^*M)$ corresponding to the span of the eigenvectors of Σp with positive/negative eigenvalues. If we have given connections on V_1 and V_2, we can project these connections to define natural connections on $\Pi_\pm(\Sigma p)$. The clutching function of $\Pi_+(\Sigma p)$ is p in a sense we explain as follows:

We form the disk bundles $D_\pm(M)$ over M corresponding to the northern and southern hemispheres of the fiber spheres of $\Sigma(T^*M)$. Lemma 3.8.6 generalizes immediately to let us identify $\Pi_+(\Sigma p)$ with the bundle $V_1^+ \cup V_2^-$ over the disjoint union $D_+(M) \cup D_-(M)$ attached using the clutching function p over their common boundary $S(T^*M)$. If $\dim V_1 = k$, then $\Pi_+(\Sigma p) \in \mathrm{Vect}_k(\Sigma(T^*M))$. Conversely, we suppose given a bundle $V \in \mathrm{Vect}_k(\Sigma(T^*M))$. Let $N: M \rightarrow \Sigma(T^*M)$ and $S: M \rightarrow \Sigma(T^*M)$ be the natural sections mapping M to the northern and southern poles of the fiber spheres; $N(x) = (x,0,1)$ and $S(x) = (x,0,-1)$ in $S(T^*M \oplus 1)$. N and S are the centers of the disk bundles $D_\pm(M)$. We let $N^*(V) = V_1$ and $S^*(V) = V_2$ be the induced vector bundles over M. $D_\pm(M)$ deformation retracts to $M \times \{N\}$ and $M \times \{S\}$. Thus V restricted to $D_\pm(M)$ is cannonically isomorphic to the pull back of V_1 and V_2. On the intersection $S(T^*M) = D_+(M) \cap D_-(M)$ we have a clutching or glueing function relating the two decompositions of V. This gives rise to a map $p: S(T^*M) \rightarrow \mathrm{HOM}(V_1, V_2)$ which is non-singular. The same argument as that given in the proof of Lemma 3.8.5 shows that V is completely determined by the isomorphism class of V_1 and of V_2 together with the homotopy class of the map $p: S(T^*M) \rightarrow \mathrm{HOM}(V_1, V_2)$.

Given an order ν we can recover the leading symbol p by extending p from $S(T^*M)$ to T^*M to be homogeneous of order ν. We use this to define an elliptic operator $P_\nu: C^\infty(V_1) \rightarrow C^\infty(V_2)$ with leading symbol p_ν. If Q_ν is another operator with the same leading symbol, then we define

$P_\nu(t) = tP_\nu + (1-t)Q_\nu$. This is a 1-parameter family of such operators with the same leading symbol. Consequently by Lemma 1.4.4 $\operatorname{index}(P_\nu) = \operatorname{index}(Q_\nu)$. Similarly, if we replace p by a homotopic symbol, then the index is unchanged. Finally, suppose we give two orders of homgeneity $\nu_1 > \nu_2$. We can choose a self-adjoint pseudo-differential operator R on $C^\infty(V_2)$ with leading symbol $|\xi|^{\nu_1-\nu_2} I_{V_2}$. Then we can let $P_{\nu_1} = RP_{\nu_2}$ so $\operatorname{index}(P_{\nu_1}) = \operatorname{index}(R)+\operatorname{index}(P_{\nu_2})$. Since R is self-adjoint, its index is zero so $\operatorname{index}(P_{\nu_1}) = \operatorname{index}(P_{\nu_2})$. This shows that the index only depends on the homotopy class of the clutching map over $S(T^*M)$ and is independent of the order of homogeneity and the extension and the particular operator chosen. Consequently, we can regard $\operatorname{index}\colon \operatorname{Vect}(\Sigma(T^*M)) \to \mathbf{Z}$ so that $\operatorname{index}(\Pi_+(\Sigma p)) = \operatorname{index}(P)$ if P is an elliptic operator. (We can always "roll up" an elliptic complex to give a 2-term elliptic complex in computing the index so it suffices to consider this case).

It is clear from our definition that $\Sigma(p \oplus q) = \Sigma(p) \oplus \Sigma(q)$ and therefore $\Pi_+(\Sigma(p\oplus q)) = \Pi_+(\Sigma(p)) \oplus \Pi_+(\Sigma(q))$. Since $\operatorname{index}(P \oplus Q) = \operatorname{index}(P) + \operatorname{index}(Q)$ we conclude:

$$\operatorname{index}(V \oplus W) = \operatorname{index}(V) + \operatorname{index}(W) \qquad \text{for } V, W \in \operatorname{Vect}(\Sigma(T^*M)).$$

This permits us to extend $\operatorname{index}\colon K(\Sigma(T^*M)) \to \mathbf{Z}$ to be $\mathbf{Z}$-linear. We tensor with the complex numbers to extend $\operatorname{index}\colon K(\Sigma(T^*M);\mathbf{C}) \to \mathbf{C}$, so that:

LEMMA 3.9.1. *There is a natural map* $\operatorname{index}\colon K(\Sigma(T^*M);\mathbf{C}) \to \mathbf{C}$ *which is linear so that* $\operatorname{index}(P) = \operatorname{index}(\Pi_+(\Sigma p))$ *if* $P\colon C^\infty V_1 \to C^\infty V_2$ *is an elliptic complex over* M *with symbol* p.

There is a natural projection map $\pi\colon \Sigma(T^*M) \to M$. This gives a natural map $\pi^*\colon K(M;\mathbf{C}) \to K(\Sigma(T^*M);\mathbf{C})$. If N denotes the north pole section, then $\pi N = 1_M$ so $N^*\pi^* = 1$ and consequently π^* is injective. This permits us to regard $K(M;\mathbf{C})$ as a subspace of $K(\Sigma(T^*M);\mathbf{C})$. If $V = \pi^* V_1$, then the clutching function defining V is just the identity map. Consequently, the corresponding elliptic operator P can be taken to be a self-adjoint operator on $C^\infty(V)$ which has index zero. This proves:

LEMMA 3.9.2. *If* $V \in K(\Sigma(T^*M);\mathbf{C})$ *can be written as* $\pi^* V_1$ *for* $V_1 \in K(M;\mathbf{C})$ *then* $\operatorname{index}(V) = 0$. *Thus* $\operatorname{index}\colon K(\Sigma(T^*M);\mathbf{C})/K(M;\mathbf{C}) \to \mathbf{C}$.

These two lemmas show that all the information contained in an elliptic complex from the point of view of computing its index is contained in the corresponding description in K-theory. The Chern character gives an isomorphism of $K(X;\mathbf{C})$ to the even dimensional cohomology. We will exploit this isomorphism to give a formula for the index in terms of cohomology.

In addition to the additive structure on $K(X;\mathbf{C})$, there is also a ring structure. This ring structure also has its analogue with respect to elliptic operators as we have discussed previously. The multiplicative nature of the four classical elliptic complexes played a fundamental role in determining the normalizing constants in the formula for their index.

We give $\Sigma(T^*M)$ the simplectic orientation. If $x = (x_1, \dots, x_m)$ is a system of local coordinates on M, let $\xi = (\xi_1, \dots, \xi_m)$ be the fiber coordinates on T^*M. Let u be the additional fiber coordinate on $T^*M \oplus 1$. We orient $T^*M \oplus 1$ using the form:

$$\omega_{2m+1} = dx_1 \wedge d\xi_1 \wedge \cdots \wedge dx_m \wedge d\xi_m \wedge du.$$

Let $\vec{N}$ be the outward normal on $S(T^*M \oplus 1)$ and define ω_{2m} on $S(T^*M \oplus 1)$ by:

$$\omega_{2m+1} = \vec{N} \wedge \omega_{2m} = \omega_{2m} \wedge \vec{N}.$$

This gives the orientation of Stokes theorem.

Lemma 3.9.3.
(a) Let $P: C^\infty(V_1) \to C^\infty(V_2)$ be an elliptic complex over M_1 and let $Q: C^\infty(W_1) \to C^\infty(W_2)$ be an elliptic complex over M_2. We assume P and Q are partial differential operators of the same order and we let $M = M_1 \times M_2$ and define over M

$$R = (P \otimes 1 + 1 \otimes Q) \oplus (P^* \otimes 1 - 1 \otimes Q^*): C^\infty(V_1 \otimes W_1 \oplus V_2 \otimes W_2) \\ \to C^\infty(V_2 \otimes W_1 \oplus V_1 \otimes W_2).$$

Then R is elliptic and $\operatorname{index}(R) = \operatorname{index}(P) \cdot \operatorname{index}(Q)$.
(b) The four classical elliptic complexes discussed earlier can be decomposed in this fashion over product manifolds.
(c) Let p and q be arbitrary elliptic symbols over M_1 and M_2 and define

$$r = \begin{pmatrix} p \otimes 1 & -1 \otimes q^* \\ 1 \otimes q & p^* \otimes 1 \end{pmatrix} \qquad \text{over } M = M_1 \times M_2.$$

Let $\theta_i \in H^{\text{even}}(M_i; \mathbf{C})$ for $i = 1, 2$, then:

$$\int_{\Sigma(T^*M)} \theta_1 \wedge \theta_2 \wedge ch(\Pi_+(\Sigma r)) \\ = \int_{\Sigma(T^*M_1)} \theta_1 \wedge ch(\Pi_+(\Sigma p)) \cdot \int_{\Sigma(T^*M_2)} \theta_2 \wedge ch(\Pi_+(\Sigma q)).$$

(d) Let q be an elliptic symbol over M_1 and let p be a self-adjoint elliptic symbol over M_2. Over $M = M_1 \times M_2$ define the self-adjoint elliptic symbol r by:

$$r = \begin{pmatrix} 1 \otimes p & q^* \otimes 1 \\ q \otimes 1 & -1 \otimes p \end{pmatrix}.$$

Let $\theta_1 \in \Lambda^{\mathrm{e}}(M_1)$ and $\theta_2 \in \Lambda^{\mathrm{o}}(M_2)$ be closed differential forms. Give $S(T^*M)$, $\Sigma(T^*M_1)$ and $S(T^*M_2)$ the orientations induced from the simplectic orientations. Then:

$$\int_{S(T^*M)} \theta_1 \wedge \theta_2 \wedge ch(\Pi_+ r)$$
$$= \int_{\Sigma(T^*M_1)} \theta_1 \wedge ch(\Pi_+(\Sigma q)) \cdot \int_{S(T^*M_2)} \theta_1 \wedge ch(\Pi_+(p)).$$

Remark: We will use (d) to discuss the eta invariant in Chapter 4; we include this integral at this point since the proof is similar to that of (c).

PROOF: We let (p, q, r) be the symbols of the operators involved. Then:

$$r = \begin{pmatrix} p \otimes 1 & -1 \otimes q^* \\ 1 \otimes q & p^* \otimes 1 \end{pmatrix} \qquad r^* = \begin{pmatrix} p^* \otimes 1 & 1 \otimes q^* \\ -1 \otimes q & p \otimes 1 \end{pmatrix}$$

so that:

$$r^*r = \begin{pmatrix} p^*p \otimes 1 + 1 \otimes q^*q & 0 \\ 0 & pp^* \otimes 1 + 1 \otimes qq^* \end{pmatrix}$$
$$rr^* = \begin{pmatrix} pp^* \otimes 1 + 1 \otimes q^*q & 0 \\ 0 & p^*p \otimes 1 + 1 \otimes qq^* \end{pmatrix}.$$

r^*r and rr^* are positive self-adjoint matrices if $(\xi^1, \xi^2) \neq (0, 0)$. This verifies the ellipticity. We note that if (P, Q) are pseudo-differential, R will still be formally elliptic, but the symbol will not in general be smooth at $\xi^1 = 0$ or $\xi^2 = 0$ and hence R will not be a pseudo-differential operator in that case. We compute:

$$R^*R = \begin{pmatrix} P^*P \otimes 1 + 1 \otimes Q^*Q & 0 \\ 0 & PP^* \otimes 1 + 1 \otimes QQ^* \end{pmatrix}$$
$$RR^* = \begin{pmatrix} PP^* \otimes 1 + 1 \otimes Q^*Q & 0 \\ 0 & P^*P \otimes 1 + 1 \otimes QQ^* \end{pmatrix}$$
$$\mathrm{N}(R^*R) = \mathrm{N}(P^*P) \otimes \mathrm{N}(Q^*Q) \ \oplus \ \mathrm{N}(PP^*) \otimes \mathrm{N}(QQ^*)$$
$$\mathrm{N}(RR^*) = \mathrm{N}(PP^*) \otimes \mathrm{N}(Q^*Q) \ \oplus \ \mathrm{N}(P^*P) \otimes \mathrm{N}(QQ^*)$$
$$\begin{aligned} \mathrm{index}(R) &= \{\dim \mathrm{N}(P^*P) - \dim \mathrm{N}(PP^*)\} \\ &\qquad\qquad \times \{\dim \mathrm{N}(Q^*Q) - \dim \mathrm{N}(QQ^*)\} \\ &= \mathrm{index}(P)\,\mathrm{index}(Q) \end{aligned}$$

which completes the proof of (a).

We verify (b) on the symbol level. First consider the de Rham complex and decompose:

$$\begin{aligned}\Lambda(T^*M_1) = \Lambda_1^{\mathrm{e}} \oplus \Lambda_1^{\mathrm{o}}, \qquad \Lambda(T^*M_2) = \Lambda_2^{\mathrm{e}} \oplus \Lambda_2^{\mathrm{o}},\\ \Lambda(T^*M) = (\Lambda_1^{\mathrm{e}} \otimes \Lambda_2^{\mathrm{e}} \ \oplus\ \Lambda_1^{\mathrm{o}} \otimes \Lambda_2^{\mathrm{o}}) \ \oplus\ (\Lambda_1^{\mathrm{o}} \otimes \Lambda_2^{\mathrm{e}} \ \oplus\ \Lambda_1^{\mathrm{e}} \otimes \Lambda_2^{\mathrm{o}}).\end{aligned}$$

Under this decomposition:

$$\sigma_L((d+\delta)_M)(\xi^1,\xi^2) = \begin{pmatrix} c(\xi^1)\otimes 1 & -1\otimes c(\xi^2) \\ 1\otimes c(\xi^2) & c(\xi^1)\otimes 1 \end{pmatrix}.$$

$c(\cdot)$ denotes Clifford multiplication. This verifies the de Rham complex decomposes properly.

The signature complex is more complicated. We decompose

$$\Lambda^{\pm}(T^*M_1) = \Lambda_1^{\pm,\mathrm{e}} \oplus \Lambda_1^{\pm,\mathrm{o}}$$

to decompose the signature complex into two complexes:

$$\begin{aligned}(d+\delta)\colon C^\infty(\Lambda_1^{+,\mathrm{e}}) \to C^\infty(\Lambda_1^{-,\mathrm{o}})\\ (d+\delta)\colon C^\infty(\Lambda_1^{+,\mathrm{o}}) \to C^\infty(\Lambda_1^{-,\mathrm{e}}).\end{aligned}$$

Under this decomposition, the signature complex of M decomposes into four complexes. If, for example, we consider the complex:

$$\begin{aligned}(d+\delta)\colon C^\infty(\Lambda_1^{+,\mathrm{e}} \otimes \Lambda_2^{+,\mathrm{e}} \ \oplus\ \Lambda_1^{-,\mathrm{o}} \otimes \Lambda_2^{-,\mathrm{o}}) \to \\ C^\infty(\Lambda_1^{-,\mathrm{o}} \otimes \Lambda_2^{+,\mathrm{e}} \ \oplus\ \Lambda_1^{+,\mathrm{e}} \otimes \Lambda_2^{-,\mathrm{o}})\end{aligned}$$

then the same argument as that given for the de Rham complex applies to show the symbol is:

$$\begin{pmatrix} c(\xi^1)\otimes 1 & -1\otimes c(\xi^2) \\ 1\otimes c(\xi^2) & c(\xi^1)\otimes 1 \end{pmatrix}.$$

If we consider the complex:

$$\begin{aligned}(d+\delta)\colon C^\infty(\Lambda_1^{+,\mathrm{o}} \otimes \Lambda_2^{+\mathrm{e}} \ \oplus\ \Lambda_1^{-,\mathrm{e}} \otimes \Lambda_2^{-,\mathrm{o}})\\ \to C^\infty(\Lambda_1^{-,\mathrm{e}} \otimes \Lambda_2^{+,\mathrm{e}} \ \oplus\ \Lambda_1^{+,\mathrm{o}} \otimes \Lambda_2^{-,\mathrm{o}})\end{aligned}$$

then we conclude the symbol is

$$\begin{pmatrix} c(\xi^1)\otimes 1 & 1\otimes c(\xi^2) \\ -1\otimes (\xi^2) & c(\xi^1)\otimes 1 \end{pmatrix}$$

which isn't right. We can adjust the sign problem for ξ^2 either by changing one of the identifications with $\Lambda^+(M)$ or by changing the sign of Q (which won't affect the index). The remaining two cases are similar. The spin and Dolbeault complexes are also similar. If we take coefficients in an auxilary bundle V, the symbols involved are unchanged and the same arguments hold. This proves (b).

The proof of (c) is more complicated. We assume without loss of generality that p and q are homogeneous of degree 1. Let (ξ^1, ξ^2, u) parametrize the fibers of $T^*M \oplus 1$. At $\xi^1 = (1/\sqrt{2}, 0, \dots, 0)$, $\xi^2 = (1/\sqrt{2}, 0, \dots, 0)$, $u = 0$ the orientation is given by:

$$\begin{aligned} &- dx_1^1 \wedge d\xi_1^1 \wedge dx_2^1 \wedge d\xi_2^1 \wedge \cdots \\ &\qquad \wedge dx_{m_1}^1 \wedge d\xi_{m_1}^1 \wedge dx_1^2 \wedge dx_2^2 \wedge d\xi_2^2 \wedge \cdots \wedge dx_{m_2}^2 \wedge d\xi_{m_2}^2 \wedge du. \end{aligned}$$

We have omitted $d\xi_2^1$ (and changed the sign) since it points outward at this point of the sphere to get the orientation on $\Sigma(T^*M)$.

In matrix form we have on $(V_1 \otimes W_1) \oplus (V_2 \otimes W_2) \oplus (V_2 \otimes W_1) \oplus (V_1 \otimes W_2)$ that:

$$\Sigma r = \begin{pmatrix} u\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} p^* \otimes 1 & 1 \otimes q^* \\ -1 \otimes q & p \otimes 1 \end{pmatrix} \\ \\ \begin{pmatrix} p \otimes 1 & -1 \otimes q^* \\ 1 \otimes q & p^* \otimes 1 \end{pmatrix} & -u\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}.$$

This is not a very convenient form to work with. We define

$$\begin{aligned} \gamma_1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \alpha = \begin{pmatrix} 0 & p^* \\ p & 0 \end{pmatrix} \qquad \text{on } V_1 \oplus V_2 = V \\ \gamma_2 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 0 & q^* \\ q & 0 \end{pmatrix} \qquad \text{on } W_1 \oplus W_2 = W \end{aligned}$$

and compute that $\Sigma r = u\gamma_1 \otimes \gamma_2 + \alpha \otimes I_W + \gamma_1 \otimes \beta$. We replace p and q by homotopic symbols so that:

$$p^*p = |\xi^1|^2 I_{V_1}, \quad pp^* = |\xi^1|^2 I_{V_2}, \quad q^*q = |\xi^2|^2 I_{W_1}, \quad qq^* = |\xi^2|^2 I_{W_2}.$$

Since $\{\gamma_1 \otimes \gamma_2, \alpha \otimes I_W, \gamma_1 \otimes \beta\}$ all anti-commute and are self-adjoint,

$$(\Sigma r)^2 = (u^2 + |\xi^1|^2 + |\xi^2|^2)I \qquad \text{on } V \otimes W.$$

We parametrize $\Sigma(T^*M)$ by $S(T^*M_1) \times \left[0, \frac{\pi}{2}\right] \times \Sigma(T^*M_2)$ in the form:

$$\big(\xi^1 \cos\theta, \xi^2 \sin\theta, u \sin\theta\big).$$

We compute the orientation by studying $\theta = \frac{\pi}{4}$, $\xi^1 = (1, 0, \dots, 0)$, $\xi^2 = (1, 0, \dots, 0)$, $u = 0$. Since $d\xi_1^1 = -\sin\theta\, d\theta$, the orientation is given by:

$$dx_1^1 \wedge d\theta \wedge dx_2^1 \wedge d\xi_2^1 \wedge \cdots \\ \wedge dx_{m_1}^1 \wedge d\xi_{m_1}^1 \wedge dx_1^2 \wedge dx_2^2 \wedge d\xi_2^2 \wedge \cdots \wedge dx_{m_2}^2 \wedge d\xi_{m_2}^2 \wedge du.$$

If we identify $S(T^*M_1) \times \left[0, \frac{\pi}{2}\right]$ with $D_+(T^*M_1)$ then this orientation is:

$$\omega_{2m_1}^1 \wedge \omega_{2m_2}^2$$

so the orientations are compatible.

In this parametrization, we compute $\Sigma r(\xi^1, \theta, \xi^2, u) = (\sin\theta)\gamma_1 \otimes (u\gamma_2 + \beta(\xi^2)) + (\cos\theta)\alpha(\xi^1) \otimes I_W$. Since $\Sigma q = u\gamma_2 + \beta(\xi^2)$ satisfies $(\Sigma q)^2 = |\xi^2|^2 + u_2^2$ on W, we may decompose $W = \Pi_+(\Sigma q) \oplus \Pi_-(\Sigma q)$. Then:

$$\begin{aligned} \Sigma r &= \{(\sin\theta)\gamma_1 + (\cos\theta)\alpha(\xi^1)\} \otimes I = \Sigma p(\xi^1, \theta) \otimes I && \text{on } V \otimes \Pi_+(\Sigma q) \\ \Sigma r &= \{\sin(-\theta)\gamma_1 + (\cos\theta)\alpha(\xi^1)\} \otimes I = \Sigma p(\xi^1, -\theta) \otimes I && \text{on } V \otimes \Pi_-(\Sigma q). \end{aligned}$$

Consequently:

$$\Pi_+(\Sigma r) = \left\{\Pi_+(\Sigma p)(\xi^1, \theta) \otimes \Pi_+(\Sigma q)\right\} \oplus \left\{\Pi_+(\Sigma p)(\xi^1, -\theta) \otimes \Pi_-(\Sigma q)\right\} \\ \text{over } (D_+M_1) \times \Sigma(T^*M_2).$$

If we replace $-\theta$ by θ in the second factor, we may replace $\Pi_+(\Sigma p)(\xi^1, -\theta) \otimes \Pi_-(\Sigma q)$ by $\Pi_+(\Sigma p)(\xi^1, \theta) \otimes \Pi_-(\Sigma q)$. Since we have changed the orientation, we must change the sign. Therefore:

$$\begin{aligned} &\int_{\Sigma(T^*M)} \theta_1 \wedge \theta_2 \wedge ch(\Pi_+(\Sigma r)) \\ &\qquad = \int_{D_+M_1} \theta_1 \wedge ch(\Pi_+(\Sigma p)) \cdot \int_{\Sigma(T^*M_2)} \theta_2 \wedge ch(\Pi_+(\Sigma q)) \\ &\qquad\qquad - \int_{D_-M_1} \theta_1 \wedge ch(\Pi_+(\Sigma p)) \cdot \int_{\Sigma(T^*M_2)} \theta_2 \wedge ch(\Pi_-(\Sigma q)). \end{aligned}$$

$ch(\Pi_+(\Sigma q)) + ch(\Pi_-(\Sigma q)) = ch(V_2)$ does not involve the fiber coordinates of $\Sigma(T^*M_2)$ and thus

$$\int_{\Sigma(T^*M_2)} \theta_2 \wedge ch(V_2) = 0.$$

We may therefore replace $-ch(\Pi_-(\Sigma q))$ by $ch(\Pi_+(\Sigma q))$ in evaluating the integral over $D_-(M_1) \times \Sigma(T^*M_2)$ to complete the proof of (c).

We prove (d) in a similar fashion. We suppose without loss of generality that p and q are homogeneous of degree 1. We parameterize $S(T^*M) = S(T^*M_1) \times [0, \frac{\pi}{2}] \times S(T^*M_2)$ in the form $(\xi^1 \cos\theta, \xi^2 \sin\theta)$. Then

$$r = \begin{pmatrix} \sin\theta & 0 \\ 0 & -\sin\theta \end{pmatrix} \otimes p(\xi^2) + \begin{pmatrix} 0 & (\cos\theta) q^*(\xi^1) \\ (\cos\theta) q(\xi^1) & 0 \end{pmatrix}.$$

Again we decompose $V_2 = \Pi_+(p) \oplus \Pi_-(p)$ so that

$$\begin{aligned} \Sigma r &= \big((\sin\theta)\gamma_1 + (\cos\theta)\alpha^1(\xi^1)\big) &\quad& \text{on } V \otimes \Pi_+(p) \\ \Sigma r &= \big(\sin(-\theta)\gamma_1 + (\cos\theta)\alpha^1(\xi^1)\big) && \text{on } V \otimes \Pi_-(p) \end{aligned}$$

where

$$\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \alpha^1 = \begin{pmatrix} 0 & q^* \\ q & 0 \end{pmatrix}.$$

The remainder of the argument is exactly as before (with the appropriate change in notation from (c)) and is therefore omitted.

We now check a specific example to verify some normalizing constants:

LEMMA 3.9.4. *Let $M = S^1$ be the unit circle. Define $P\colon C^\infty(M) \to C^\infty(M)$ by:*

$$P(e^{in\theta}) = \begin{cases} ne^{i(n-1)\theta} & \text{for } n \ge 0 \\ ne^{in\theta} & \text{for } n \le 0 \end{cases}$$

then P is an elliptic pseudo-differential operator with $\operatorname{index}(P) = 1$. *Furthermore:*

$$\int_{\Sigma(T^*S^1)} ch(\Pi_+(\Sigma p)) = -1.$$

PROOF: Let $P_0 = -i\partial/\partial\theta$ and let $P_1 = \{-\partial^2/\partial\theta^2\}^{1/2}$. P_0 is a differential operator while P_1 is a pseudo-differential operator by the results of section 1.10. It is immediate that:

$$\begin{aligned} \sigma_L(P_0) &= \xi, &\qquad P_0(e^{in\theta}) &= ne^{in\theta} \\ \sigma_L(P_1) &= |\xi|, & P_1(e^{in\theta}) &= |n|e^{in\theta}. \end{aligned}$$

We define:

$$Q_0 = \frac{1}{2}e^{-i\theta}(P_0 + P_1)$$

$$\sigma_L Q_0 = \begin{cases} \xi e^{-i\theta} & \xi \ge 0 \\ 0 & \xi \le 0 \end{cases}$$

$$Q_0(e^{in\theta}) = \begin{cases} ne^{i(n-1)\theta} & n \ge 0 \\ 0 & n \le 0 \end{cases}$$

and

$$
\begin{aligned}
Q_1 &= \frac{1}{2}(P_0 - P_1) \\
\sigma_L Q_1 &= \begin{cases} 0 & \xi \geq 0 \\ \xi & \xi \leq 0 \end{cases} \\
Q_1(e^{in\theta}) &= \begin{cases} 0 & n \geq 0 \\ ne^{in\theta} & n \leq 0. \end{cases}
\end{aligned}
$$

Consequently, $P = Q_0 + Q_1$ is a pseudo-differential operator and

$$
\sigma_L P = p = \begin{cases} \xi e^{-i\theta} & \xi > 0 \\ \xi & \xi < 0. \end{cases}
$$

It is clear P is surjective so $\mathrm{N}(P^*) = 0$. Since $\mathrm{N}(P)$ is the space of constant functions, $\mathrm{N}(P)$ is one dimensional so $\mathrm{index}(P) = 1$. We compute:

$$
\Sigma p(\theta, \xi, t) = \begin{cases} \begin{pmatrix} t & \xi e^{+i\theta} \\ \xi e^{-i\theta} & -t \end{pmatrix} & \text{if } \xi \geq 0 \\ \begin{pmatrix} t & \xi \\ \xi & -t \end{pmatrix} & \text{if } \xi \leq 0. \end{cases}
$$

Since Σp does not depend on θ for $\xi \leq 0$, we may restrict to the region $\xi \geq 0$ in computing the integral. (We must smooth out the symbol to be smooth where $\xi = 0$ but suppress these details to avoid undue technical complications).

It is convenient to introduce the parameters:

$$
u = t, \qquad v = \xi \cos\theta, \qquad w = \xi \sin\theta \qquad \text{for } u^2 + v^2 + w^2 = 1
$$

then this parametrizes the region of $\Sigma(T^*S^1)$ where $\xi \geq 0$ in a 1-1 fashion except where $\xi = 0$. Since $du \wedge dv \wedge dw = -\xi\, d\theta \wedge d\xi \wedge dt$, S^2 inherits the reversed orientation from its natural one. Let

$$
e_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}
$$

so $\Sigma p(u, v, w) = ue_0 + ve_1 + we_2$. Then by Lemma 2.1.5 we have

$$
-\int_{S^2} ch(\Pi_+(\Sigma p)) = (-1) \cdot \left(\frac{i}{2}\right) \cdot \mathrm{Tr}(e_0 e_1 e_2) = -1
$$

which completes the proof.

We can now state the Atiyah-Singer index theorem:

THEOREM 3.9.5 (THE INDEX THEOREM). *Let $P: C^\infty(V_1) \to C^\infty(V_2)$ be an elliptic pseudo-differential operator. Let $Td_r(M) = Td(TM \otimes \mathbf{C})$ be the Todd class of the complexification of the real tangent bundle. Then:*

$$\operatorname{index}(P) = (-1)^{\dim M} \int_{\Sigma(T^*M)} Td_r(M) \wedge ch(\Pi_+(\Sigma p)).$$

We remark that the additional factor of $(-1)^{\dim M}$ could have been avoided if we changed the orientation of $\Sigma(T^*M)$.

We begin the proof by reducing to the case $\dim M = m$ even and M orientable. Suppose first that m is odd. We take $Q: C^\infty(S^1) \to C^\infty(S^1)$ to be the operator defined in Lemma 3.9.4 with index $+1$. We then form the operator R with $\operatorname{index}(R) = \operatorname{index}(P)\operatorname{index}(Q) = \operatorname{index}(P)$ defined in Lemma 3.9.3. Although R is not a pseudo-differential operator, it can be uniformly approximated by pseudo-differential operators in the natural Fredholm topology (once the order of Q is adjusted suitably). (This process does not work when discussing the twisted eta invariant and will involve us in additional technical complications in the next chapter). Therefore:

$$\begin{aligned}
&(-1)^{m+1} \int_{\Sigma(T^*(M\times S^1))} Td_r(M \times S^1) \wedge ch(\Pi_+(\Sigma r)) \\
&= (-1)^m \int_{\Sigma(T^*M)} Td_r(M) \wedge ch(\Pi_+(\Sigma p)) \cdot (-1) \int_{\Sigma(T^*S^1)} ch(\Pi_+(\Sigma q)) \\
&= (-1)^m \int_{\Sigma(T^*M)} Td_r(M) \wedge ch(\Pi_+(\Sigma p)).
\end{aligned}$$

To show the last integral gives $\operatorname{index}(P)$ it suffices to show the top integral gives $\operatorname{index}(R)$ and therefore reduces the proof of Theorem 3.9.3 to the case $\dim M = m$ even.

If M is not orientable, we let M' be the orientable double cover of M. It is clear the formula on the right hand side of the equation multiplies by two. More careful attention to the methods of the heat equation for pseudo-differential operators gives a local formula for the index even in this case as the left hand side is also multiplied by two under this double cover.

This reduces the proof of Theorem 3.9.3 to the case $\dim M = m$ even and M orientable. We fix an orientation on M henceforth.

LEMMA 3.9.6. *Let $P: C^\infty(\Lambda^+) \to C^\infty(\Lambda^-)$ be the operator of the signature complex. Let $\omega = ch_{m/2}(\Pi_+(\Sigma p)) \in H^m(\Sigma(T^*M); \mathbf{C})$. Then if ω_M is the orientation class of M*
(a) $\omega_M \wedge \omega$ gives the orientation of $\Sigma(T^*M)$.

(b) If S^m is a fiber sphere of $\Sigma(T^*M)$, then $\int_{S^m}\omega = 2^{m/2}$.

PROOF: Let $(x_1,\dots,x_m)$ be an oriented local coordinate system on M so that the $\{dx_j\}$ are orthonormal at $x_0 \in M$. If $\xi = (\xi,\dots,\xi_m)$ are the dual fiber coordinates for T^*M), then:

$$p(\xi) = \sum_j i\xi_j(c(dx_j)) = \sum_j i\xi_j(\operatorname{ext}(dx_j) - \operatorname{int}(\xi_j))$$

gives the symbol of $(d+\delta)$; $c(\cdot)$ denotes Clifford multiplication as defined previously. We let $e_j = ic(dx_j)$; these are self-adjoint matrices such that $e_je_k + e_ke_j = 2\delta_{jk}$. The orientation class is defined by:

$$e_0 = i^{m/2}c(dx_1)\dots c(dx_m) = (-i)^{m/2}e_1\dots e_m.$$

The bundles $\Lambda^\pm$ are defined as the ± 1 eigenspaces of e_0. Consequently,

$$\Sigma p(\xi,t) = te_0 + \sum \xi_j e_j.$$

Therefore by Lemma 2.1.5 when S^m is given its natural orientation,

$$\begin{aligned}\int_{S^m} ch_{m/2}\,\Pi_+(\Sigma p) &= i^{m/2}2^{-m/2}\operatorname{Tr}(e_0e_1\dots e_m)\\ &= i^{m/2}2^{-m/2}\operatorname{Tr}(e_0 i^{m/2}e_0)\\ &= (-1)^{m/2}2^{-m/2}\operatorname{Tr}(I) = (-1)^{m/2}2^{-m/2}2^m\\ &= (-1)^{m/2}2^{m/2}.\end{aligned}$$

However, S^m is in fact given the orientation induced from the orientation on $\Sigma(T^*M)$ and on M. At the point $(x,0,\dots,0,1)$ in $T^*M \oplus \mathbf{R}$ the natural orientations are:

$$\begin{aligned}\text{of } X\text{:}\quad & dx_1\wedge\dots\wedge dx_m,\\ \text{of } \Sigma(T^*M)\text{:}\quad & dx_1\wedge d\xi_1\wedge\dots\wedge dx_m\wedge d\xi_m\\ & \quad = (-1)^{m/2}dx_1\wedge\dots\wedge dx_m\wedge d\xi_1\wedge\dots\wedge d\xi_m\\ \text{of } S^m\text{:}\quad & (-1)^{m/2}d\xi_1\wedge\dots\wedge d\xi_m.\end{aligned}$$

Thus with the induced orientation, the integral becomes $2^{m/2}$ and the lemma is proved.

Consequently, ω provides a cohomology extension and:

LEMMA 3.9.7. *Let $\rho\colon \Sigma(T^*M) \to M$ where M is orientable and even dimensional. Then*
(a) $\rho^*\colon H^*(M;\mathbf{C}) \to H^*(\Sigma(T^*M);\mathbf{C})$ is injective.

(b) If ω is as defined in Lemma 3.9.6, then we can express any $\alpha \in H^*(\Sigma(T^*M);\mathbf{C})$ uniquely as $\alpha = \rho^*\alpha_1 + \rho^*\alpha_2 \wedge \omega$ for $\alpha_i \in H^*(M;\mathbf{C})$.

Since ρ^* is injective, we shall drop it henceforth and regard $H^*(M;\mathbf{C})$ as being a subspace of $H^*(\Sigma(T^*M);\mathbf{C})$.

The Chern character gives an isomorphism $K(X;\mathbf{C}) \simeq H^{\mathrm{e}}(X;\mathbf{C})$. When we interpret Lemma 3.9.7 in K-theory, we conclude that we can decompose $K(\Sigma(T^*M);\mathbf{C}) = K(M;\mathbf{C}) \oplus K(M;\mathbf{C}) \otimes \Pi_+(\Sigma p)$; $ch(V)$ generates $H^{\mathrm{e}}(M;\mathbf{C})$ as V ranges over $K(M;\mathbf{C})$. Therefore $K(\Sigma(T^*M);\mathbf{C})/K(M;\mathbf{C})$ is generated as an additive module by the twisted signature complex with coefficients in bundles over M. $\Pi_+(\Sigma p_V) = V \otimes \Pi_+(\Sigma p)$ if p_V is the symbol of the signature complex with coefficients in V.

In Lemma 3.9.2, we interpreted the index as a map in K-theory. Since it is linear, it suffices to compute on the generators given by the signature complex with coefficients in V. This proves:

LEMMA 3.9.8. *Assume M is orientable and of even dimension m. Let P_V be the operator of the signature complex with coefficients in V. The bundles $\{\Pi_+(\Sigma p)_V\}_{V \in \mathrm{Vect}(M)}$ generate $K(\Sigma(T^*M);\mathbf{C})/K(M;\mathbf{C})$ additively. It suffices to prove Theorem 3.9.5 in general by checking it on the special case of the operators P_V.*

We will integrate out the fiber coordinates to reduce the integral of Theorem 3.9.5 from $\Sigma(T^*M)$ to M. We proceed as follows. Let W be an oriented real vector bundle of fiber dimension $k+1$ over M equipped with a Riemannian inner product. Let $S(W)$ be the unit sphere bundle of W. Let $\rho: S(W) \to M$ be the natural projection map. We define a map $\mathcal{I}: C^\infty(\Lambda(S(W))) \to C^\infty(\Lambda(M))$ which is a $C^\infty(\Lambda(M))$ module homomorphism and which commutes with integration—i.e., if $\alpha \in C^\infty(\Lambda(S(W)))$ and $\beta \in C^\infty(\Lambda(M))$, we require the map $\alpha \mapsto \mathcal{I}(\alpha)$ to be linear so that $\mathcal{I}(\rho^*\beta \wedge \alpha) = \beta \wedge \mathcal{I}(\alpha)$ and $\int_{S(W)} \alpha = \int_M \mathcal{I}(\alpha)$.

We construct $\mathcal{I}$ as follows. Choose a local orthonormal frame for W to define fiber coordinates $u = (u_0, \dots, u_k)$ on W. This gives a local representation of $S(W) = V \times S^k$ over the coordinate patch $\mathcal{U}$ on M. If $\alpha \in C^\infty(\Lambda(S(W)))$ has support over V, we can decompose $\alpha = \sum_\nu \beta_\nu \wedge \alpha_\nu$ for $\beta_\nu \in C^\infty(\Lambda(\mathcal{U}))$ and $\alpha_\nu \in C^\infty(\Lambda(S^k))$. We permit the α_ν to have coefficients which depend upon $x \in \mathcal{U}$. This expression is, of course, not unique. Then $\mathcal{I}(\alpha)$ is necessarily defined by:

$$\mathcal{I}(\alpha)(x) = \sum \beta_\nu \int_{S^k} \alpha_\nu(x).$$

It is clear this is independent of the particular way we have decomposed α. If we can show $\mathcal{I}$ is independent of the frame chosen, then this will define $\mathcal{I}$ in general using a partition of unity argument.

Let $u_i' = a_{ij}(x)u_j$ be a change of fiber coordinates. Then we compute:

$$du_i' = a_{ij}(x)\,du_j + da_{ij}(x)u_j.$$

Clearly if a is a constant matrix, we are just reparamatrizing S^k so the integral is unchanged. We fix x_0 and suppose $a(x_0) = I$. Then over x_0,

$$du_I = dv_I + \sum_{|I|<|J|} c_{I,J} \wedge dv_J \qquad \text{where } c_{I,J} \in \Lambda^{|I|-|J|}(M).$$

To integrate and get an answer different from 0 over S^k, we must have $|I| = k$ so these error terms integrate to zero and $\mathcal{I}$ is invariantly defined.

We specialize to the case $W = T^*M \oplus 1$. The orientation of M induces a natural orientation of $T^*M \oplus 1$ as a bundle in such a way as to agree with the orientation of $T^*M \oplus 1$ as a topological space. Let $\alpha = ch(\Pi_+(\Sigma p))$, so $\mathcal{I}(\alpha) \in C^\infty(\Lambda(M))$. If we reverse the orientation of M, then we interchange the roles of Λ^+ and of Λ^-. This has the effect of replacing the parameter u by $-u$ which is equivalent to reversing the orientation of $T^*M \oplus 1$ as a topological space. Since both orientations have been reversed, the orientation of the fiber is unchanged so $\mathcal{I}(\alpha)$ is invariantly defined independent of any local orientation of M. It is clear from the definition that $\mathcal{I}(\alpha)$ is a polynomial in the jets of the metric and is invariant under changes of the metric by a constant factor. Therefore Theorems 2.5.6 and Lemma 2.5.3 imply $\mathcal{I}(\alpha)$ is a real characteristic form. By Lemma 3.9.6, we can expand $\mathcal{I}(\alpha) = 2^{m/2} + \cdots$.

We solve the equation:

$$\{\mathcal{I}(ch(\Pi_+(\Sigma p))) \wedge \mathit{Todd}(m)\}_{m-4s} = 2^{(m-4s)/2} L_s$$

recursively to define a real characteristic class we shall call $\mathit{Todd}(m)$ for the moment. It is clear $\mathit{Todd}(m) = 1 + \cdots$. In this equation, L_s is the Hirzebruch genus.

Lemma 3.9.9. *Let $\mathit{Todd}(m)$ be the real characteristic class defined above. Then if P is any elliptic pseudo-differential operator,*

$$\operatorname{index}(P) = \int_{\Sigma(T^*M)} \mathit{Todd}(m) \wedge ch(\Pi_+(\Sigma p)).$$

Proof: By Lemma 3.9.8 it suffices to prove this identity if P is the operator of the twisted signature complex. By Theorem 3.1.5,

$$\begin{aligned}
\operatorname{index}(P_V^{\text{signature}}) &= \int_M \sum_{2t+4s=m} ch_t(V) 2^t \wedge L_s \\
&= \int_M ch(V) \wedge \mathcal{I}(ch(\Pi_+(\Sigma p))) \wedge \mathit{Todd}(m) \\
&= \int_{\Sigma(T^*M)} ch(V) \wedge ch(\Pi_+(\Sigma p)) \wedge \mathit{Todd}(m) \\
&= \int_{\Sigma(T^*M)} \mathit{Todd}(m) \wedge ch(\Pi_+(\Sigma p_V)).
\end{aligned}$$

It is clear *Todd*(m) is uniquely determined by Lemma 3.9.9. Both the index of an elliptic operator and the formula of Lemma 3.9.9 are multiplicative with respect to products by Lemma 3.9.3 so *Todd*(m) is a multiplicative characteristic form. We may therefore drop the dependence upon the dimension m and simply refer to *Todd*. We complete the proof of the Atiyah-Singer index theorem by identifying this characteristic form with the real Todd polynomial of $T(M)$.

We work with the Dolbeault complex instead of with the signature complex since the representations involved are simpler. Let m be even, then $(-1)^m = 1$. Let M be a holomorphic manifold with the natural orientation. We orient the fibers of T^*M using the natural orientation which arises from the complex structure on the fibers. If ξ are the fiber coordinates, this gives the orientation:

$$dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \wedge d\xi_1 \wedge \cdots \wedge d\xi_m \qquad \text{where } m = 2n.$$

This gives the total space T^*M an orientation which is $(-1)^n$ times the simplectic orientation. Let S^m denote a fiber sphere of $\Sigma(T^*M)$ with *this* orientation and let q be the symbol of the Dolbeault complex. Then:

$$\operatorname{index}(\bar\partial) = \int_{\Sigma(T^*M)} Todd \wedge ch(\Pi_+\Sigma q) = (-1)^n \int_M Todd \wedge \mathcal{I}\,(ch(\Pi_+\Sigma q)).$$

We define the complex characteristic form

$$\mathcal{S} = (-1)^{m/2}\mathcal{I}\,(ch(\Pi_+(\Sigma q)))$$

then the Riemann-Roch formula implies that:

$$\mathcal{S} \wedge Todd(m) = Todd(T_cM).$$

It is convenient to extend the definition of the characteristic form $\mathcal{S}$ to arbitrary complex vector bundles V. Let $W = \Lambda^*(V)$ and let $q(v)\colon V \to \mathrm{END}(W)$ be defined by Lemma 3.5.2(a) to be the symbol of $\bar\partial + \delta''$ if $V = T_c(M)$. We let $\mathrm{ext}\colon V \to \mathrm{HOM}(W, W)$ be exterior multiplication. This is complex linear and we let int be the dual of ext; $\mathrm{int}(\lambda v) = \bar\lambda\,\mathrm{int}(v)$ for $\lambda \in \mathbf{C}$. ext is invariantly defined while int requires the choice of a fiber matric. We let $q = \mathrm{ext}(v) - \mathrm{int}(v)$ (where we have deleted the factor of $i/2$ which appears in Lemma 3.5.2 for the sake of simplicity).

We regard q as a section to the bundle $\mathrm{HOM}(V, \mathrm{HOM}(W, W))$. Fix a Riemannian connection on V and covariantly differentiate q to compute $\nabla q \in C^\infty(T^*M \otimes \mathrm{HOM}(V, \mathrm{HOM}(W, W)))$. Since the connection is Riemannian, $\nabla q = 0$; this is not true in general for non-Riemannian connections.

If V is trivial with flat connection, the bundles $\Pi_\pm(\Sigma q)$ have curvature $\pi_\pm d\pi_\pm d\pi_\pm$ as computed in Lemma 2.1.5. If V is not flat, the curvature of

V enters into this expression. The connection and fiber metric on V define a natural metric on T^*V. We use the splitting defined by the connection to decompose T^*V into horizontal and vertical components. These components are orthogonal with respect to the natural metric on V. Over V, q becomes a section to the bundle $\mathrm{HOM}(V, V)$. We let ∇^V denote covariant differentiation over V, then $\nabla^V q$ has only vertical components in T^*V and has no horizontal components.

The calculation performed in Lemma 2.1.5 shows that in this more general setting that the curvatures of $\nabla_\pm$ are given by:

$$\Omega_\pm = \pi_+(\nabla^V \pi_+ \wedge \nabla^V \pi_+ + \rho^* \Omega_W).$$

If we choose a frame for V and W which is covariant constant at a point x_0, then $\nabla^V \pi_+ = d\pi_+$ has only vertical components while $\rho^*\Omega_W$ has only horizontal components. If Ω_V is the curvature of V, then $\Omega_W = \Lambda(\Omega_V)$.

Instead of computing $\mathcal{S}$ on the form level, we work with the corresponding invariant polynomial.

Lemma 3.9.10. *Let A be an $n \times n$ complex skew-adjoint matrix. Let $B = \Lambda(A)$ acting on $\Lambda(\mathbf{C}^n) = \mathbf{C}^{2^n}$. Define:*

$$\mathcal{S}(A) = \sum_\nu (-1)^n \left(\frac{i}{2\pi}\right)^\nu \frac{1}{\nu!} \int_{S^{2n}} \mathrm{Tr}\{(\pi_+ d\pi_+ d\pi_+ + B)^\nu\}.$$

If $x_j = i\lambda_j/2\pi$ are the normalized eigenvectors of A, then

$$\mathcal{S}(A) = \prod_j \frac{e^{x_j} - 1}{x_j}.$$

Proof: If $V = V_1 \oplus V_2$ and if $A = A_1 \oplus A_2$, then the Dolbeault complex decomposes as a tensor product by Lemma 3.9.3. The calculations of Lemma 3.9.3 using the decomposition of the bundles $\Pi_\pm$ shows $\mathcal{S}(A)$ is a multiplicative characteristic class. To compute the generating function, it suffices to consider the case $n = 1$.

If $n = 1$, $A = \lambda$ so that $B = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}$, if we decompose $W = \Lambda^{0,0} \oplus \Lambda^{0,1} = 1 \oplus V$. If $x + iy$ give the usual coordinates on $V = \mathbf{C}$, then:

$$q(x, y) = x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

by Lemma 3.5.2 which gives the symbol of the Dolbeault complex. Therefore:

$$q(x, y, u) = x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + u \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = xe_0 + ye_1 + ue_2.$$

We compute:

$$e_0 e_1 e_2 = i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \pi_+ \, d\pi_+ \, d\pi_+ = \frac{i}{2} \pi_+ \operatorname{dvol}.$$

Since $n = 1$, $(-1)^n = -1$ and:

$$\begin{aligned} \mathcal{S}(A) &= \sum_{j>0} -\frac{i}{2} \left(\frac{i}{2\pi} \right)^j \frac{1}{j!} \int \operatorname{Tr}((\pi_+ \operatorname{dvol} + \pi_+ B)^j) \\ &= \sum_{j>0} -\frac{i}{2} \left(\frac{i}{2\pi} \right)^j \frac{1}{(j-1)!} \int \operatorname{Tr}(\pi_+ B)^{j-1} \operatorname{dvol}. \end{aligned}$$

We calculate that:

$$\pi_+ B = \frac{1}{2} \begin{pmatrix} 1+u & x-iy \\ x+iy & 1-u \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & * \\ 0 & (1-u)\lambda \end{pmatrix}$$

$$(\pi_+ B)^{j-1} = 2^{1-j} \lambda^{j-1} (1-u)^{j-1} \begin{pmatrix} 0 & * \\ 0 & 1 \end{pmatrix}$$

where "$*$" indicates a term we are not interested in. We use this identity and re-index the sum to express:

$$\mathcal{S}(A) = \sum_{j \geq 0} \frac{1}{4\pi} \left(\frac{i\lambda}{2\pi} \right)^j \frac{1}{j!} 2^{-j} \int_{S^2} (1-u)^j \operatorname{dvol}.$$

We introduce the integrating factor of e^{-r^2} to compute:

$$\begin{aligned} \int_{R^3} u^{2k} e^{-r^2} \, dx \, dy \, du = \pi \int_R u^{2k} e^{-u^2} \, du & \\ &= \int_0^\infty r^{2k+2} e^{-r^2} \, dr \cdot \int_{S^2} u^{2k} \operatorname{dvol} \\ &= (2k+1)/2 \int_0^\infty r^{2k} e^{-r^2} \, dr \cdot \int_{S^2} u^{2k} \operatorname{dvol} \end{aligned}$$

so that:

$$\int_{S^2} u^{2k} \operatorname{dvol} = 4\pi/(2k+1).$$

The terms of odd order integrate to zero so:

$$\begin{aligned} \int_{S^2} (1-u)^j \operatorname{dvol} = 4\pi \sum \binom{j}{2k} \cdot \frac{1}{2k+1} &= 4\pi \int_0^1 \sum \binom{j}{2k} t^{2k} \, dt \\ &= 2\pi \int_0^1 (1+t)^j + (1-t)^j \, dt \\ &= 2\pi \frac{(1+t)^{j+1} - (1-t)^{j+1}}{j+1} \Bigg|_0^1 \\ &= 4\pi \cdot \frac{2^j}{j+1}. \end{aligned}$$

We substitute this to conclude:

$$\mathcal{S}(A) = \sum_{j \geq 0} \left(\frac{i\lambda}{2\pi}\right)^j \frac{1}{(j+1)!}.$$

If we introduce $x = i\lambda/2\pi$ then

$$\mathcal{S}(x) = \sum_{j \geq 0} \frac{x^j}{(j+1)!} = \frac{e^x - 1}{x}$$

which gives the generating function for $\mathcal{S}$. This completes the proof of the lemma.

We can now compute *Todd*. The generating function of $\mathit{Todd}(T_c)$ is $x/(1 - e^{-x})$ so that $\mathit{Todd} = \mathcal{S}^{-1} \cdot \mathit{Todd}(T_c)$ will have generating function:

$$\frac{x}{1 - e^{-x}} \cdot \frac{x}{e^x - 1} = \frac{x}{1 - e^{-x}} \cdot \frac{-x}{1 - e^x}$$

which is, of course, the generating function for the real *Todd* class. This completes the proof. We have gone into some detail to illustrate that it is not particularly difficult to evaluate the integrals which arise in applying the index theorem. If we had dealt with the signature complex instead of the Dolbeault complex, the integrals to be evaluated would have been over S^4 instead of S^2 but the computation would have been similar.