CHAPTER 2

CHARACTERISTIC CLASSES

Introduction

In the second chapter, we develop the theory of characteristic classes. In section 2.1, we discuss the Chern classes of a complex vector bundle and in section 2.2 we discuss the Pontrjagin and Euler classes of a real vector bundle. We shall define the Todd class, the Hirzebruch L-polynomial, and the A-roof genus which will play a central role in our discussion of the index theorem. We also discuss the total Chern and Pontrjagin classes as well as the Chern character.

In section 2.3, we apply these ideas to the tangent space of a real manifold and to the holomorphic tangent space of a holomorphic manifold. We compute several examples defined by Clifford matrices and compute the Chern classes of complex projective space. We show that suitable products of projective spaces form a dual basis to the space of characteristic classes. Such products will be used in chapter three to find the normalizing constants which appear in the formula for the index theorem.

In section 2.4 and in the first part of section 2.5, we give a heat equation proof of the Gauss-Bonnet theorem. This is based on first giving an abstract characterization of the Euler form in terms of invariance theory. This permis us to identify the integrand of the heat equation with the Euler integrand. This gives a more direct proof of Theorem 2.4.8 which was first proved by Patodi using a complicated cancellation lemma. (The theorem for dimension $m = 2$ is due to McKean and Singer).

In the remainder of section 2.5, we develop a similar characterization of the Pontrjagin forms of the real tangent space. We shall wait until the third chapter to apply these results to obtain the Hirzebruch signature theorem. There are two different approaches to this result. We have presented both our original approach and also one modeled on an approach due to Atiyah, Patodi and Bott. This approach uses H. Weyl's theorem on the invariants of the orthogonal group and is not self-contained as it also uses facts about geodesic normal coordinates we shall not develop. The other approach is more combinatorial in flavor but is self-contained. It also generalizes to deal with the holomorphic case for a Kaehler metric.

The signature complex with coefficients in a bundle V gives rise to invariants which depend upon both the metric on the tangent space of M and on the connection 1-form of V. In section 2.6, we extend the results of section 2.5 to cover more general invariants of this type. We also construct dual bases for these more general invariants similar to those constructed in section 2.3 using products of projective spaces and suitable bundles over these spaces.

The material of sections 2.1 through 2.3 is standard and reviews the theory of characteristic classes in the context we shall need. Sections 2.4 through 2.6 deal with less standard material. The chapter is entirely self-contained with the exception of some material in section 2.5 as noted above. We have postponed a discussion of similar material for the holomorphic Kaehler case until sections 3.6 and 3.7 of chapter three since this material is not needed to discuss the signature and spin complexes.

2.1. Characteristic Classes Of a Complex Vector Bundle.

The characteristic classes are topological invariants of a vector bundle which are represented by differential forms. They are defined in terms of the curvature of a connection.

Let M be a smooth compact manifold and let V be a smooth complex vector bundle of dimension k over M. A connection ∇ on V is a first order partial differential operator $\nabla\colon C^\infty(V) \to C^\infty(T^*M \otimes V)$ such that:

$$\nabla(fs) = df \otimes s + f\nabla s \qquad \text{for } f \in C^\infty(M) \text{ and } s \in C^\infty(V).$$

Let $(s_1, \ldots, s_k)$ be a local frame for V. We can decompose any section $s \in C^\infty(V)$ locally in the form $s(x) = f_i(x)s_i(x)$ for $f_i \in C^\infty(M)$. We adopt the convention of summing over repeated indices unless otherwise specified in this subsection. We compute:

$$\nabla s = df_i \otimes s_i + f_i \nabla s_i = df_i \otimes s_i + f_i \omega_{ij} \otimes s_j \qquad \text{where } \nabla s_i = \omega_{ij} \otimes s_j.$$

The connection 1-form ω defined by

$$\omega = \omega_{ij}$$

is a matrix of 1-forms. The connection ∇ is uniquely determined by the derivation property and by the connection 1-form. If we specify ω arbitrarily locally, we can define ∇ locally. The convex combination of connections is again a connection, so using a partition of unity we can always construct connections on a bundle V.

If we choose another frame for V, we can express $s'_i = h_{ij}s_j$. If $h^{-1}_{ij}s'_j = s_i$ represents the inverse matrix, then we compute:

$$\begin{aligned}\nabla s'_i = \omega'_{ij} \otimes s'_j = \nabla(h_{ik}s_k) &= dh_{ik} \otimes s_k + h_{ik}\omega_{kl} \otimes s_l \\ &= (dh_{ik}h^{-1}_{kj} + h_{ik}\omega_{kl}h^{-1}_{lj}) \otimes s'_j.\end{aligned}$$

This shows ω' obeys the transformation law:

$$\omega'_{ij} = dh_{ik}h^{-1}_{kj} + h_{ik}\omega_{kl}h^{-1}_{lj} \qquad \text{i.e.,}\ \ \omega' = dh \cdot h^{-1} + h\omega h^{-1}$$

in matrix notation. This is, of course, the manner in which the 0^{th} order symbol of a first order operator transforms.

We extend ∇ to be a derivation mapping

$$C^\infty(\Lambda^p T^*M \otimes V) \to C^\infty(\Lambda^{p+1} T^*M \otimes V)$$

so that:

$$\nabla(\theta_p \otimes s) = d\theta_p \otimes s + (-1)^p \theta_p \wedge \nabla s.$$

We compute that:

$$\nabla^2(fs) = \nabla(df \otimes s + f\nabla s) = d^2 f \otimes s - df \wedge \nabla s + df \wedge \nabla s + f\nabla^2 s = f\nabla^2 s$$

so instead of being a second order operator, ∇^2 is a 0^{th} order operator. We may therefore express

$$\nabla^2(s)(x_0) = \Omega(x_0)s(x_0)$$

where the curvature Ω is a section to the bundle $\Lambda^2(T^*M) \otimes \mathrm{END}(V)$ is a 2-form valued endomorphism of V.

In local coordinates, we compute:

$$\Omega(s_i) = \Omega_{ij} \otimes s_j = \nabla(\omega_{ij} \otimes s_j) = d\omega_{ij} \otimes s_j - \omega_{ij} \wedge \omega_{jk} \otimes s_k$$

so that:

$$\Omega_{ij} = d\omega_{ij} - \omega_{ik} \wedge \omega_{kj} \qquad \text{i.e.,} \quad \Omega = d\omega - \omega \wedge \omega.$$

Since ∇^2 is a 0^{th} order operator, Ω must transform like a tensor:

$$\Omega'_{ij} = h_{ik}\Omega_{kl}h^{-1}_{lj} \qquad \text{i.e.,} \quad \Omega' = h\Omega h^{-1}.$$

This can also be verified directly from the transition law for ω'. The reader should note that in some references, the curvature is defined by $\Omega = d\omega + \omega \wedge \omega$. This sign convention results from writing $V \otimes T^*M$ instead of $T^*M \otimes V$ and corresponds to studying left invariant rather than right invariant vector fields on $\mathrm{GL}(k, C)$.

It is often convenient to normalize the choice of frame.

LEMMA 2.1.1. *Let ∇ be a connection on a vector bundle V. We can always choose a frame s so that at a given point x_0 we have*

$$\omega(x_0) = 0 \qquad \text{and} \qquad d\Omega(x_0) = 0.$$

PROOF: We find a matrix $h(x)$ defined near x_0 so that $h(x_0) = I$ and $dh(x_0) = -\omega(x_0)$. If $s'_i = h_{ij}s_j$, then it is immediate that $\omega'(x_0) = 0$. Similarly we compute $d\Omega'(x_0) = d(d\omega' - \omega' \wedge \omega')(x_0) = \omega'(x_0) \wedge d\omega'(x_0) - d\omega'(x_0) \wedge \omega'(x_0) = 0$.

We note that as the curvature is invariantly defined, we cannot in general find a parallel frame s so ω vanishes in a neighborhood of x_0 since this would imply $\nabla^2 = 0$ near x_0 which need not be true.

Let A_{ij} denote the components of a matrix in $\mathrm{END}(\mathbf{C}^k)$ and let $P(A) = P(A_{ij})$ be a polynomial mapping $\mathrm{END}(\mathbf{C}^k) \to \mathbf{C}$. We assume that P is invariant—i.e., $P(hAh^{-1}) = P(A)$ for any $h \in \mathrm{GL}(k, \mathbf{C})$. We define $P(\Omega)$ as an even differential form on M by substitution. Since P is invariant and since the curvature transforms tensorially, $P(\Omega) \overset{\text{def}}{=} P(\nabla)$ is defined independently of the particular local frame which is chosen.

There are two examples which are of particular interest and which will play an important role in the statement of the Atiyah-Singer index theorem. We define:

$$\begin{aligned} c(A) &= \det\left(I + \frac{i}{2\pi}A\right) \\ &= 1 + c_1(A) + \cdots + c_k(A) \qquad &\text{(the total Chern form)} \\ ch(A) &= \operatorname{Tr} e^{iA/2\pi} = \sum_j \operatorname{Tr}\left(\frac{iA}{2\pi}\right)^j \Big/ j! \qquad &\text{(the total Chern character)} \end{aligned}$$

The $c_j(A)$ represent the portion of $c(A)$ which is homogeneous of order j. $c_j(\nabla) \in \Lambda^{2j}(M)$. In a similar fashion, we decompose $ch(A) = \sum_j ch_j(A)$ for $ch_j(A) = \operatorname{Tr}(iA/2\pi)^j/j!$

Strictly speaking, $ch(A)$ is not a polynomial. However, when we substitute the components of the curvature tensor, this becomes a finite sum since $\operatorname{Tr}(\Omega^j) = 0$ if $2j > \dim M$. More generally, we can define $P(\Omega)$ if $P(A)$ is an invariant formal power series by truncating the power series appropriately.

As a differential form, $P(\nabla)$ depends on the connection chosen. We show $P(\nabla)$ is always closed. As an element of de Rham cohomology, $P(\nabla)$ is independent of the connection and defines a cohomology class we shall denote by $P(V)$.

Lemma 2.1.2. *Let P be an invariant polynomial.*
(a) $dP(\nabla) = 0$ so $P(\nabla)$ is a closed differential form.
(b) Given two connections ∇_0 and ∇_1, we can define a differential form $TP(\nabla_0, \nabla_1)$ so that $P(\nabla_1) - P(\nabla_0) = d\{TP(\nabla_1, \nabla_0)\}$.

Proof: By decomposing P as a sum of homogeneous polynomials, we may assume without loss of generality that P is homogeneous of order k. Let $P(A_1, \dots, A_k)$ denote the complete polarization of P. We expand $P(t_1A_1 + \cdots + t_kA_k)$ as a polynomial in the variables $\{t_j\}$. $1/k!$ times the coefficient of $t_1 \dots t_k$ is the polarization of P. P is a multi-linear symmetric function of its arguments. For example, if $P(A) = \operatorname{Tr}(A^3)$, then the polarization is given by $\frac{1}{2}\operatorname{Tr}(A_1A_2A_3 + A_2A_1A_3)$ and $P(A) = P(A, A, A)$.

Fix a point x_0 of M and choose a frame so $\omega(x_0) = d\Omega(x_0) = 0$. Then

$$dP(\Omega)(x_0) = dP(\Omega, \dots, \Omega)(x_0) = kP(d\Omega, \Omega, \dots, \Omega)(x_0) = 0.$$

Since x_0 is arbitrary and since $dP(\Omega)$ is independent of the frame chosen this proves $dP(\Omega) = 0$ which proves (a).

Let $\nabla_t = t\nabla_1 + (1-t)\nabla_0$ have connection 1-form $\omega_t = \omega_0 + t\theta$ where $\theta = \omega_1 - \omega_0$. The transformation law for ω implies relative to a new frame:

$$\begin{aligned}\theta' = \omega_1' - \omega_0' &= (dh \cdot h^{-1} + h\omega_1 h^{-1}) - (dh \cdot h^{-1} + h\omega_0 h^{-1})\\ &= h(\omega_1 - \omega_0)h^{-1} = h\theta h^{-1}\end{aligned}$$

so θ transforms like a tensor. This is of course nothing but the fact that the difference between two first order operators with the same leading symbol is a 0^{th} order operator.

Let Ω_t be the curvature of the connection ∇_t. This is a matrix valued 2-form. Since θ is a 1-form, it commutes with 2-forms and we can define

$$P(\theta, \Omega_t, \dots, \Omega_t) \in \Lambda^{2k-1}(T^*M)$$

by substitution. Since P is invariant, its complete polarization is also invariant so $P(h^{-1}\theta h, h^{-1}\Omega_t h, \dots, h^{-1}\Omega_t h) = P(\theta, \Omega_t, \dots, \Omega_t)$ is invariantly defined independent of the choice of frame.

We compute that:

$$P(\nabla_1) - P(\nabla_0) = \int_0^1 \frac{d}{dt} P(\Omega_t, \dots, \Omega_t)\, dt = k \int_0^1 P(\Omega_t', \Omega_t, \dots, \Omega_t)\, dt.$$

We define:

$$TP(\nabla_1, \nabla_0) = k \int_0^1 P(\theta, \Omega_t, \dots, \Omega_t)\, dt.$$

To complete the proof of the Lemma, it suffices to check

$$dP(\theta, \Omega_t, \dots, \Omega_t) = P(\Omega_t', \Omega_t, \dots, \Omega_t).$$

Since both sides of the equation are invariantly defined, we can choose a suitable local frame to simplify the computation. Let $x_0 \in M$ and fix t_0. We choose a frame so $\omega_t(x_0, t_0) = 0$ and $d\Omega_t(x_0, t_0) = 0$. We compute:

$$\begin{aligned}\Omega_t' &= \{d\omega_0 + t\,d\theta - \omega_t \wedge \omega_t\}'\\ &= d\theta - \omega_t' \wedge \omega_t - \omega_t \wedge \omega_t'\\ \Omega_t'(x_0, t_0) &= d\theta\end{aligned}$$

and

$$\begin{aligned}dP(\theta, \Omega_t, \dots, \Omega_t)(x_0, t_0) &= P(d\theta, \Omega_t, \dots, \Omega_t)(x_0, t_0)\\ &= P(\Omega_t', \Omega_t, \dots, \Omega_t)(x_0, t_0)\end{aligned}$$

which completes the proof.

TP is called the transgression of P and will play an important role in our discussion of secondary characteristic classes in Chapter 4 when we discuss the eta invariant with coefficients in a locally flat bundle.

We suppose that the matrix $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_k)$ is diagonal. Then modulo suitable normalizing constants $\left(\frac{i}{2\pi}\right)^j$, it is immediate that $c_j(A)$ is the j^{th} elementary symmetric function of the λ_i's since

$$\det(I + A) = \prod(1 + \lambda_j) = 1 + s_1(\lambda) + \cdots + s_k(\lambda).$$

If $P(\cdot)$ is any invariant polynomial, then $P(A)$ is a symmetric function of the λ_i's. The elementary symmetric functions form an algebra basis for the symmetric polynomials so there is a unique polynomial Q so that

$$P(A) = Q(c_1, \ldots, c_k)(A)$$

i.e., we can decompose P as a polynomial in the c_i's for diagonal matrices. Since P is invariant, this is true for diagonalizable matrices. Since the diagonalizable matrices are dense and P is continuous, this is true for all A. This proves:

Lemma 2.1.3. *Let $P(A)$ be a invariant polynomial. There exists a unique polynomial Q so that $P = Q(c_1(A), \ldots, c_k(A))$.*

It is clear that any polynomial in the c_k's is invariant, so this completely characterizes the ring of invariant polynomials; it is a free algebra in k variables $\{c_1, \ldots, c_k\}$.

If $P(A)$ is homogeneous of degree k and is defined on $k \times k$ matrices, we shall see that if $P(A) \neq 0$ as a polynomial, then there exists a holomorphic manifold M so that if $T_c(M)$ is the holomorphic tangent space, then $\int_M P(T_c(M)) \neq 0$ where $\dim M = 2k$. This fact can be used to show that in general if $P(A) \neq 0$ as a polynomial, then there exists a manifold M and a bundle V over M so $P(V) \neq 0$ in cohomology.

We can apply functorial constructions on connections. Define:

$$\begin{array}{ccc} \nabla_1 \oplus \nabla_2 & \text{on} & C^\infty(V_1 \oplus V_2) \\ \nabla_1 \otimes \nabla_2 & \text{on} & C^\infty(V_1 \otimes V_2) \\ \nabla_1^* & \text{on} & C^\infty(V_1^*) \end{array}$$

by:

$$(\nabla_1 \oplus \nabla_2)(s_1 \oplus s_2) = (\nabla_1 s_1) \oplus (\nabla_2 s_2)$$
$$\text{with } \omega = \omega_1 \oplus \omega_2 \text{ and } \Omega = \Omega_1 \oplus \Omega_2$$
$$(\nabla_1 \otimes \nabla_2)(s_1 \otimes s_2) = (\nabla_1 s_1) \otimes (\nabla_2 s_2)$$
$$\text{with } \omega = \omega_1 \otimes 1 + 1 \otimes \omega_2 \text{ and } \Omega = \Omega_1 \otimes 1 + 1 \otimes \Omega_2$$
$$(\nabla_1 s_1, s_1^*) + (s_1, \nabla_1^* s_1^*) = d(s_1, s_1^*)$$
$$\text{relative to the dual frame } \omega = -\omega_1^t \text{ and } \Omega = -\Omega_1^t.$$

In a similar fashion we can define an induced connection on $\Lambda^p(V)$ (the bundle of p-forms) and $S^p(V)$ (the bundle of symmetric p-tensors). If V has a given Hermitian fiber metric, the connection ∇ is said to be unitary or Riemannian if $(\nabla s_1, s_2) + (s_1, \nabla s_2) = d(s_1, s_2)$. If we identify V with V^* using the metric, this simply means $\nabla = \overline{\nabla}^*$. This is equivalent to the condition that ω is a skew adjoint matrix of 1-forms relative to a local orthonormal frame. We can always construct Riemannian connections locally by taking $\omega = 0$ relative to a local orthonormal frame and then using a partition of unity to construct a global Riemannian connection.

If we restrict to Riemannian connections, then it is natural to consider polynomials $P(A)$ which are defined for skew-Hermitian matrices $A + A^* = 0$ and which are invariant under the action of the unitary group. Exactly the same argument as that given in the proof of Lemma 2.1.3 shows that we can express such a P in the form $P(A) = Q(c_1(A), \dots, c_k(A))$ so that the $\mathrm{GL}(\cdot, \mathbf{C})$ and the $\mathrm{U}(\cdot)$ characteristic classes coincide. This is not true in the real category as we shall see; the Euler form is a characteristic form of the special orthogonal group which can not be defined as a characteristic form using the general linear group $\mathrm{GL}(\cdot, \mathbf{R})$.

The Chern form and the Chern character satisfy certain identities with respect to functorial constructions.

LEMMA 2.1.4.
(a)

$$c(V_1 \oplus V_2) = c(V_1)c(V_2)$$
$$c(V^*) = 1 - c_1(V) + c_2(V) - \cdots + (-1)^k c_k(V).$$

(b)

$$ch(V_1 \oplus V_2) = ch(V_1) + ch(V_2)$$
$$ch(V_1) \oplus V_2) = ch(V_1)ch(V_2).$$

PROOF: All the identities except the one involving $c(V^*)$ are immediate from the definition if we use the direct sum and product connections. If we fix a Hermitian structure on V, the identification of V with V^* is conjugate linear. The curvature of ∇^* is $-\Omega^t$ so we compute:

$$c(V^*) = \det\left(I - \frac{i}{2\pi}\Omega^t\right) = \det\left(I - \frac{i}{2\pi}\Omega\right)$$

which gives the desired identity.

If we choose a Riemannian connection on V, then $\Omega + \bar{\Omega}^t = 0$. This immediately yields the identities:

$$\overline{ch(V)} = ch(V) \qquad \text{and} \qquad \overline{c(V)} = c(V)$$

so these are real cohomology classes. In fact the normalizing constants were chosen so $c_k(V)$ is an integral cohomology class—i.e., if N_{2k} is any oriented submanifold of dimension $2k$, then $\int_{N_{2k}} c_k(V) \in \mathbf{Z}$. The $ch_k(V)$ are not integral for $k > 1$, but they are rational cohomology classes. As we shall not need this fact, we omit the proof.

The characteristic classes give cohomological invariants of a vector bundle. We illustrate this by constructing certain examples over even dimensional spheres; these examples will play an important role in our later development of the Atiyah-Singer index theorem.

Definition. *A set of matrices $\{e_0, \ldots, e_m\}$ are Clifford matrices if the e_i are self-adjoint and satisfy the commutation relations $e_i e_j + e_j e_i = 2\delta_{ij}$ where δ_{ij} is the Kronecker symbol.*

For example, if $m = 2$ we can take:

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

to be the Dirac matrices. More generally, we can take the e_i's to be given by the spin representations. If $x \in \mathbf{R}^{m+1}$, we define:

$$e(x) = \sum_i x_i e_i \qquad \text{so} \quad e(x)^2 = |x|^2 I.$$

Conversely let $e(x)$ be a linear map from $\mathbf{R}^m$ to the set of self-adjoint matrices with $e(x) = |x|^2 I$. If $\{v_0, \ldots, v_m\}$ is any orthonormal basis for $\mathbf{R}^m$, $\{e(v_0), \ldots, e(v_m)\}$ forms a set of Clifford matrices.

If $x \in S^m$, we let $\Pi_\pm(x)$ be the range of $\frac{1}{2}(1 + e(x)) = \pi_\pm(x)$. This is the span of the ± 1 eigenvectors of $e(x)$. If $e(x)$ is a $2k \times 2k$ matrix, then $\dim \Pi_\pm(x) = k$. We have a decomposition $S^m \times \mathbf{C}^{2k} = \Pi_+ \oplus \Pi_-$. We project the flat connection on $S^m \times \mathbf{C}^{2k}$ to the two sub-bundles to define connections $\nabla_\pm$ on $\Pi_\pm$. If $e^0_\pm$ is a local frame for $\Pi_\pm(x_0)$, we define $e_\pm(x) = \pi_\pm e^0_\pm$ as a frame in a neighborhood of x_0. We compute

$$\nabla_\pm e_\pm = \pi_\pm \, d\pi_\pm e^0_\pm, \qquad \Omega_\pm e_\pm = \pi_\pm \, d\pi_\pm \, d\pi_\pm e^0_\pm.$$

Since $e^0_\pm = e_\pm(x_0)$, this yields the identity:

$$\Omega_\pm(x_0) = \pi_\pm \, d\pi_\pm \, d\pi_\pm(x_0).$$

Since Ω is tensorial, this holds for all x.

Let $m = 2j$ be even. We wish to compute ch_j. Suppose first $x_0 = (1, 0, \dots, 0)$ is the north pole of the sphere. Then:

$$\begin{aligned}
\pi_+(x_0) &= \frac{1}{2}(1 + e_0) \\
d\pi_+(x_0) &= \frac{1}{2}\sum_{i\geq 1} dx_i e_i \\
\Omega_+(x_0) &= \frac{1}{2}(1 + e_0)\left(\frac{1}{2}\sum_{i\geq 1} dx_i e_i\right)^2 \\
\Omega_+(x_0)^j &= \frac{1}{2}(1 + e_0)\left(\frac{1}{2}\sum_{i\geq 1} dx_i e_i\right)^{2j} \\
&= 2^{-m-1} m!(1 + e_0)(e_1 \dots e_m)(dx_1 \wedge \dots \wedge dx_m).
\end{aligned}$$

The volume form at x_0 is $dx_1 \wedge \dots \wedge dx_m$. Since e_1 anti-commutes with the matrix $e_1 \dots e_m$, this matrix has trace 0 so we compute:

$$ch_j(\Omega_+)(x_0) = \left(\frac{i}{2\pi}\right)^j 2^{-m-1} m! \operatorname{Tr}(e_0 \dots e_m) \operatorname{dvol}(x_0)/j!\,.$$

A similar computation shows this is true at any point x_0 of S^m so that:

$$\int_{S^m} ch_j(\Pi_+) = \left(\frac{i}{2\pi}\right)^j 2^{-m-1} m! \operatorname{Tr}(e_0 \dots e_m) \operatorname{vol}(s^m)/j!\,.$$

Since the volume of S^m is $j!2^{m+1}\pi^j/m!$ we conclude:

LEMMA 2.1.5. *Let $e(x)$ be a linear map from $\mathbf{R}^{m+1}$ to the set of self-adjoint matrices. We suppose $e(x)^2 = |x|^2 I$ and define bundles $\Pi_\pm(x)$ over S^m corresponding to the ± 1 eigenvalues of e. Let $m = 2j$ be even, then:*

$$\int_{S^m} ch_j(\Pi_+) = i^j 2^{-j} \operatorname{Tr}(e_0 \dots e_m).$$

In particular, this is always an integer as we shall see later when we consider the spin complex. If

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

then $\operatorname{Tr}(e_0 e_1 e_2) = -2i$ so $\int_{S^2} ch_1(\Pi_+) = 1$ which shows in particular Π_+ is a non-trivial line bundle over S^2.

There are other characteristic classes arising in the index theorem. These are most conveniently discussed using generating functions. Let

$$x_j = \frac{i}{2\pi}\lambda_j$$

be the normalized eigenvalues of the matrix A. We define:

$$c(A) = \prod_{j=1}^{k}(1 + x_j) \qquad \text{and} \qquad ch(A) = \sum_{j=1}^{k} e^{x_j}.$$

If $P(x)$ is a symmetric polynomial, we express $P(x) = Q(c_1(x), \ldots, c_k(x))$ to show $P(A)$ is a polynomial in the components of A. More generally if P is analytic, we first express P in a formal power series and then collect homogeneous terms to define $P(A)$. We define:

$$\textit{Todd class}:\ Td(A) = \prod_{j=1}^{k} \frac{x_j}{1 - e^{-x_j}}$$
$$= 1 + \frac{1}{2}c_1(A) + \frac{1}{12}(c_1^2 + c_2)(A) + \frac{1}{24}c_1(A)c_2(A) + \cdots$$

The Todd class appears in the Riemann-Roch theorem. It is clear that it is multiplicative with respect to direct sum—$Td(V_1 \oplus V_2) = Td(V_1)\, Td(V_2)$. The Hirzebruch L-polynomial and the $\hat{A}$ polynomial will be discussed in the next subsection. These are real characteristic classes which also are defined by generating functions.

If V is a bundle of dimension k, then $V \oplus 1$ will be a bundle of dimension $k+1$. It is clear $c(V \oplus 1) = c(V)$ and $Td(V \oplus 1) = Td(V)$ so these are stable characteristic classes. $ch(V)$ on the other hand is *not* a stable characteristic class since $ch_0(V) = \dim V$ depends explicitly on the dimension of V and changes if we alter the dimension of V by adding a trivial bundle.

2.2. Characteristic Classes of a Real Vector Bundle. Pontrjagin and Euler Classes.

Let V be a real vector bundle of dimension k and let $V_c = V \otimes \mathbf{C}$ denote the complexification of V. We place a real fiber metric on V to reduce the structure group from $\mathrm{GL}(k, \mathbf{R})$ to the orthogonal group $\mathrm{O}(k)$. We restrict henceforth to Riemannian connections on V, and to local orthonormal frames. Under these assumptions, the curvature is a skew-symmetric matrix of 2-forms:

$$\Omega + \Omega^t = 0.$$

Since V_c arises from a real vector bundle, the natural isomorphism of V with V^* defined by the metric induces a complex linear isomorphism of V_c with V_c^* so $c_j(V_c) = 0$ for j odd by Lemma 2.1.4. Expressed locally,

$$\det\left(I + \frac{i}{2\pi}A\right) = \det\left(I + \frac{i}{2\pi}A^t\right) = \det\left(I - \frac{i}{2\pi}A\right)$$

if $A + A^t = 0$ so $c(A)$ is an even polynomial in A. To avoid factors of i we define the Pontrjagin form:

$$p(A) = \det\left(I + \frac{1}{2\pi}A\right) = 1 + p_1(A) + p_2(A) + \cdots$$

where $p_j(A)$ is homogeneous of order $2j$ in the components of A; the corresponding characteristic class $p_j(V) \in H^{4j}(M; \mathbf{R})$. It is immediate that:

$$p_j(V) = (-1)^j c_{2j}(V_c)$$

where the factor of $(-1)^j$ comes form the missing factors of i.

The set of skew-symmetric matrices is the Lie algebra of $\mathrm{O}(k)$. Let $P(A)$ be an invariant polynomial under the action of $\mathrm{O}(k)$. We define $P(\Omega)$ for ∇ a Riemannian connection exactly as in the previous subsections. Then the analogue of Lemma 2.1.3 is:

Lemma 2.2.1. *Let $P(A)$ be a polynomial in the components of a matrix A. Suppose $P(A) = P(gAg^{-1})$ for every skew-symmetric A and for every $g \in \mathrm{O}(k)$. Then there exists a polynomial $Q(p_1, \ldots)$ so $P(A) = Q(p_1(A), \ldots)$ for every skew-symmetric A.*

Proof: It is important to note that we are not asserting that we have $P(A) = Q(p_1(A), \ldots)$ for every matrix A, but only for skew-symmetric A. For example, $P(A) = \mathrm{Tr}(A)$ vanishes for skew symmetric A but does not vanish in general.

It is not possible in general to diagonalize a skew-symmetric real matrix. We can, however, put it in block diagonal form:

$$A = \begin{pmatrix} 0 & -\lambda_1 & 0 & 0 & \dots\dots \\ \lambda_1 & 0 & 0 & 0 & \dots\dots \\ 0 & 0 & 0 & -\lambda_2 & \dots\dots \\ 0 & 0 & \lambda_2 & 0 & \dots\dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

If k is odd, then the last block will be a 1×1 block with a zero in it. We let $x_j = -\lambda_j/2\pi$; the sign convention is chosen to make the Euler form have the right sign. Then:

$$p(A) = \prod_j (1 + x_j^2)$$

where the product ranges from 1 through $\left[\frac{k}{2}\right]$.

If $P(A)$ is any invariant polynomial, then P is a symmetric function in the $\{x_j\}$. By conjugating A by an element of $\mathrm{O}(k)$, we can replace any x_j by $-x_j$ so P is a symmetric function of the $\{x_j^2\}$. The remainder of the proof of Lemma 2.2.1 is the same; we simply express $P(A)$ in terms of the elementary symmetric functions of the $\{x_j^2\}$.

Just as in the complex case, it is convenient to define additional characteristic classes using generating functions. The functions:

$$z/\tanh z \qquad \text{and} \qquad z\{2\sinh(z/2)\}$$

are both even functions of the parameter z. We define:

The Hirzebruch L-polynomial:

$$\begin{aligned} L(x) &= \prod_j \frac{x_j}{\tanh x_j} \\ &= 1 + \frac{1}{3}p_1 + \frac{1}{45}(7p_2 - p_1^2)\cdots \end{aligned}$$

The $\hat{A}$ (A-roof) genus:

$$\begin{aligned} A(x) &= \prod_j \frac{x_j}{2\sinh(x_j/2)} \\ &= 1 - \frac{1}{24}p_1 + \frac{1}{5760}(7p_1^2 - 4p_2) + \cdots . \end{aligned}$$

These characteristic classes appear in the formula for the signature and spin complexes.

For both the real and complex case, the characteristic ring is a pure polynomial algebra without relations. Increasing the dimension k just adds generators to the ring. In the complex case, the generators are the Chern classes $\{1, c_1, \ldots, c_k\}$. In the real case, the generators are the Pontrjagin classes $\{1, p_1, \ldots, p_{[k/2]}\}$ where $[k/2]$ is the greatest integer in $k/2$. There is one final structure group which will be of interest to us.

Let V be a vector bundle of real dimension k. V is orientable if we can choose a fiber orientation consistently. This is equivalent to assuming that the real line bundle $\Lambda^k(V)$ is trivial. We choose a fiber metric for V and an orientation for V and restrict attention to oriented orthonormal frames. This restricts the structure group from $\mathrm{O}(k)$ to the special orthogonal group $\mathrm{SO}(k)$.

If k is odd, no new characteristic classes result from the restriction of the fiber group. We use the final 1×1 block of 0 in the representation of A to replace any λ_i by $-\lambda_i$ by conjugation by an element of $\mathrm{SO}(k)$. If n is even, however, we cannot do this as we would be conjugating by a orientation reversing matrix.

Let $P(A)$ be a polymonial in the components of A. Suppose $P(A) = P(gAg^{-1})$ for every skew-symmetric A and every $g \in \mathrm{SO}(k)$. Let $k = 2\bar{k}$ be even and let $P(A) = P(x_1, \ldots, x_{\bar{k}})$. Fix $g_0 \in \mathrm{O}(k) - \mathrm{SO}(k)$ and define:

$$P_0(A) = \frac{1}{2}(P(A) + P(g_0 A g_0^{-1})) \quad \text{and} \quad P_1(A) = \frac{1}{2}(P(A) - P(g_0 A g_0^{-1})).$$

Both P_0 and P_1 are $\mathrm{SO}(k)$ invariant. P_0 is $\mathrm{O}(k)$ invariant while P_1 changes sign under the action of an element of $\mathrm{O}(k) - \mathrm{SO}(k)$.

We can replace x_1 by $-x_1$ by conjugating by a suitable orientation reversing map. This shows:

$$P_1(x_1, x_2, \ldots) = -P(-x_1, x_2, \ldots)$$

so that x_1 must divide every monomial of P_0. By symmetry, x_j divides every monomial of P_1 for $1 \le j \le \bar{k}$ so we can express:

$$P_1(A) = x_1 \ldots x_{\bar{k}} P_1'(A)$$

where $P_1'(A)$ is now invariant under the action of $\mathrm{O}(k)$. Since P_1' is polynomial in the $\{x_j^2\}$, we conclude that both P_0 and P_1' can be represented as polynomials in the Pontrjagin classes. We define:

$$e(A) = x_1 \ldots x_k \qquad \text{so} \quad e(A)^2 = \det(A) = p_{\bar{k}}(A) = \prod_j x_j^2$$

and decompose:

$$P = P_0 + e(A) P_1'.$$

$e(A)$ is a square root of the determinant of A.

It is not, of course, immediate that $e(A)$ is a polynomial in the components of A. Let $\{v_i\}$ be an oriented orthonormal basis for $\mathbf{R}^k$. We let $Av_i = \sum_j A_{ij}v_j$ and define

$$\omega(A) = \frac{1}{2\pi}\sum_{i<j} A_{ij}v_i \wedge v_j \in \Lambda^2(\mathbf{R}^k).$$

We let $v_1 \wedge \cdots \wedge v_k$ be the orientation class of $\mathbf{R}^k$ and define:

$$e(A) = (\omega(A)^{\bar{k}}, v_1 \wedge \cdots \wedge v_k)/k!$$

where $(\ ,\ )$ denotes the natural inner product on $\Lambda^k\mathbf{R}^k \simeq \mathbf{R}$. It is clear from the definition that $e(A)$ is invariant under the action of $\mathrm{SO}(k)$ since $\omega(A)$ is invariantly defined for skew-symmetric matrices A. It is clear that $e(A)$ is polynomial in the components of A. If we choose a block basis so that:

$$Av_1 = \lambda_1 v_2, \quad Av_2 = -\lambda_1 v_1, \quad Av_3 = \lambda_2 v_4, \quad Av_4 = -\lambda_2 v_3, \ldots$$

then we compute:

$$\begin{aligned}&\omega(A) = \{-\lambda_1 v_1 \wedge v_2 - \lambda_2 v_3 \wedge v_4 \ldots\}/2\pi = x_1 v_1 \wedge v_2 + x_2 v_3 \wedge v_4 \cdots \\ &e(A) = x_1 x_2 \ldots\end{aligned}$$

This new characteristic class is called the Euler class. While the Pontrjagin classes can be computed from the curvature of an arbitrary connection, the Euler class can only be computed from the curvature of a Riemannian connection. If Ω_{ij} are the matrix components of the curvature of V relative to some oriented orthomormal basis, then:

$$e(\Omega) = (-4\pi)^{-\bar{k}}/k! \sum_\rho \mathrm{sign}(\rho)\Omega_{\rho(1)\rho(2)} \ldots \Omega_{\rho(k-1)\rho(k)} \in \Lambda^k(M)$$

for $2\bar{k} = k = \dim(V)$. The sum ranges over all possible permutations ρ.

We define $e(V) = 0$ if $\dim(V)$ is odd. It is immediate that $e(V_1 \oplus V_2) = e(V_1)e(V_2)$ if we give the natural orientation and fiber metric to the direct sum and if we use the direct sum connection.

We illustrate this formula (and check that we have the correct normalizing constants) by studying the following simple example. Let $m = 2$ and let $M = S^2$ be the unit sphere. We calculate $e(TS^2)$. Parametrize M using spherical coordinates $f(u,v) = (\cos u \sin v, \sin u \sin v, \cos v)$ for $0 \le u \le 2\pi$ and $0 \le v \le \pi$. Define a local orthonormal frame for $T(\mathbf{R}^3)$ over S^2 by:

$$\begin{aligned}e_1 &= (\sin v)^{-1}\partial/\partial u = (-\sin u, \cos u, 0)\\ e_2 &= \partial/\partial v = (\cos u \cos v, \sin u \cos v, -\sin v)\\ e_3 &= N = (\cos u \sin v, \sin u \sin v, \cos v).\end{aligned}$$

The Euclidean connection $\tilde\nabla$ is easily computed to be:

$$\begin{aligned}
&\tilde\nabla_{e_1}e_1=(\sin v)^{-1}(-\cos u,-\sin u,0)=-(\cos v/\sin v)e_2-e_3\\
&\tilde\nabla_{e_1}e_2=(\cos v/\sin v)(-\sin u,\cos u,0)=(\cos v/\sin v)e_1\\
&\tilde\nabla_{e_2}e_1=0\\
&\tilde\nabla_{e_2}e_2=(-\cos u\sin v,-\sin u\sin v,-\cos v)=-e_3.
\end{aligned}$$

Covariant differentiation ∇ on the sphere is given by projecting back to $T(S^2)$ so that:

$$\begin{aligned}
&\nabla_{e_1}e_1=(-\cos v/\sin v)e_2 &&\nabla_{e_1}e_2=(\cos v/\sin v)e_1\\
&\nabla_{e_2}e_1=0 &&\nabla_{e_2}e_2=0
\end{aligned}$$

and the connection 1-form is given by:

$$\begin{aligned}
&\nabla e_1=(-\cos v/\sin v)e^1\otimes e_2 &&\text{so } \omega_{11}=0 \text{ and } \omega_{12}=-(\cos v/\sin v)e^1\\
&\nabla e_2=(\cos v/\sin v)e^1\otimes e_1 &&\text{so } \omega_{22}=0 \text{ and } \omega_{21}=(\cos v/\sin v)e^1.
\end{aligned}$$

As $du=e^1/\sin v$ we compute $\omega_{12}=-\cos v\,du$ so $\Omega_{12}=\sin v\,dv\wedge du=-e^1\wedge e^2$ and $\Omega_{21}=e^1\wedge e^2$. From this we calculate that:

$$e(\Omega)=-\frac{1}{4\pi}(\Omega_{12}-\Omega_{21})=\frac{e^1\wedge e^2}{2\pi}$$

and consequently $\int_{S^2}E_2=\mathrm{vol}(S^2)/2\pi=2=\chi(S^2)$.

There is a natural relation between the Euler form and $c_{\bar k}$. Let V be a complex vector space of dimension $\bar k$ and let V_r be the underlying real vector space of dimention $2\bar k=k$. If V has a Hermitian inner product, then V_r inherits a natural real inner product. V_r also inherits a natural orientation from the complex structure on V. If $\{e_j\}$ is a unitary basis for V, then $\{e_1,ie_1,e_2,ie_2,\dots\}$ is an oriented orthonormal basis for V_r. Let A be a skew-Hermitian matrix on V. The restriction of A to V_r defines a skew-symmetric matrix A_r on V_r. We choose a basis $\{e_j\}$ for V so $Ae_j=i\lambda_je_j$. Then:

$$x_j=-\lambda_j/2\pi \qquad\text{and}\qquad c_{\bar k}(A)=x_1\dots x_{\bar k}$$

defines the $\bar k^{\text{th}}$ Chern class. If $v_1=e_1$, $v_2=ie_1$, $v_3=e_2$, $v_4=ie_2,\dots$ then:

$$A_rv_1=\lambda_1v_2,\quad A_rv_2=-\lambda_1v_1,\quad A_rv_3=\lambda_2v_4,\quad A_rv_4=-\lambda_2v_3,\dots$$

so that

$$e(A_r)=c_{\bar k}(A).$$

We summarize these results as follows:

LEMMA 2.2.2. *Let $P(A)$ be a polynomial with $P(A) = P(gAg^{-1})$ for every skew symmetric A and every $g \in \mathrm{SO}(k)$.*
(a) If k is odd, then $P(A)$ is invariant under $\mathrm{O}(k)$ and is expressible in terms of Pontrjagin classes.
(b) If $k = 2\bar{k}$ is even, then we can decompose $P(A) = P_0(A) + e(A)P_1(A)$ where $P_i(A)$ are $\mathrm{O}(k)$ invariant and are expressible in terms of Pontrjagin classes.
(c) $e(A)$ is defined by:

$$e(A) = (-4\pi)^{-\bar{k}} \sum_{\rho} \operatorname{sign}(\rho) A_{\rho(1)\rho(2)} \ldots A_{\rho(k-1)\rho(k)} \, / k!$$

This satisfies the identity $e(A)^2 = p_{\bar{k}}(A)$. (We define $e(A) = 0$ if k is odd).

This completely describes the characteristic ring. We emphasize the conclusions are only applicable to skew-symmetric real matrices A. We have proved that $e(A)$ has the following functorial properties:

LEMMA 2.2.3.
(a) If we take the direct sum connection and metric, then $e(V_1 \oplus V_2) = e(V_1)e(V_2)$.
(b) If we take the metric and connection of V_r obtained by forgetting the complex structure on a complex bundle V then $e(V_r) = c_{\bar{k}}(V)$.

This lemma establishes that the top dimensional Chern class $c_{\bar{k}}$ does not really depend on having a complex structure but only depends on having an orientation on the underlying real vector bundle. The choice of the sign in computing $\det(A)^{1/2}$ is, of course, motivated by this normalization.

2.3. Characteristic Classes of Complex Projective Space.

So far, we have only discussed covariant differentiation from the point of view of the total covariant derivative. At this stage, it is convenient to introduce covariant differentiation along a direction. Let $X \in T(M)$ be a tangent vector and let $s \in C^\infty(V)$ be a smooth section. We define $\nabla_X s \in C^\infty(V)$ by:

$$\nabla_X s = X \cdot \nabla s$$

where "$\cdot$" denotes the natural pairing from $T(M) \otimes T^*(M) \otimes V \to V$. Let $[X, Y] = XY - YX$ denote the Lie-bracket of vector fields. We define:

$$\Omega(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

and compute that:

$$\Omega(fX, Y)s = \Omega(X, fY)s = \Omega(X, Y)fs = f\Omega(X, Y)s$$
$$\text{for} \quad f \in C^\infty(M),\ s \in C^\infty(V),\ \text{and } X, Y \in C^\infty(TM).$$

If $\{e_i\}$ is a local frame for $T(M)$, let $\{e^i\}$ be the dual frame for $T^*(M)$. It is immediate that:

$$\nabla s = \sum_i e^i \otimes \nabla_{e_i}(s)$$

so we can recover the total covariant derivative from the directional derivatives. We can also recover the curvature:

$$\Omega s = \sum_{i<j} e^i \wedge e^j \otimes \Omega(e_i, e_j)s.$$

If $V = T(M)$, there is a special connection called the Levi-Cività connection on $T(M)$. It is the unique Riemannian connection which is torsion free—i.e.,

$$(\nabla_X Y, Z) + (Y, \nabla_X Z) = X(Y, Z) \qquad \text{(Riemannian)}$$
$$\nabla_X Y - \nabla_Y X - [X, Y] = 0 \qquad \text{(Torsion free).}$$

Let $x = (x_1, \dots, x_m)$ be a system of local coordinates on M and let $\{\partial/\partial x_i\}$ be the coordinate frame for $T(M)$. The metric tensor is given by:

$$ds^2 = \sum_{i,j} g_{ij}\, dx^i \otimes dx^j \qquad \text{for} \quad g_{ij} = (\partial/\partial x_i, \partial/\partial x_j).$$

We introduce the Christoffel symbols Γ_{ij}^k and Γ_{ijk} by:

$$\nabla_{\partial/\partial x_i} \partial/\partial x_j = \sum_k \Gamma_{ij}^k \partial/\partial x_k$$
$$(\nabla_{\partial/\partial x_i} \partial/\partial x_j, \partial/\partial x_k) = \Gamma_{ijk}.$$

They are related by the formula:

$$\Gamma_{ijk} = \sum_l \Gamma_{ij}^l g_{lk}, \qquad \Gamma_{ij}^k = \sum_l \Gamma_{ijl} g^{lk}$$

where $g^{lk} = (dx_l, dx_k)$ is the inverse of the matrix g_{ij}. It is not difficult to compute that:

$$\Gamma_{ijk} = \tfrac{1}{2}\{g_{jk/i} + g_{ik/j} - g_{ij/k}\}$$

where we use the notation "/" to denote (multiple) partial differentiation.

The complete curvature tensor is defined by:

$$\Omega(\partial/\partial x_i, \partial/\partial x_j)\partial/\partial x_k = \sum_l R_{ijk}{}^l \partial/\partial x_l$$

or equivalently if we lower indices:

$$(\Omega(\partial/\partial x_i, \partial/\partial x_j)\partial/\partial x_k, \partial/\partial x_l) = R_{ijkl}.$$

The expression of the curvature tensor in terms of the derivatives of the metric is very complicated in general. By making a linear change of coordinates we can always normalize the metric so $g_{ij}(x_0) = \delta_{ij}$ is the Kronecker symbol. Similarly, by making a quadratic change of coordinates, we can further normalize the metric so $g_{ij/k}(x_0) = 0$. Relative to such a choice of coordinates:

$$\Gamma_{ij}^k(x_0) = \Gamma_{ijk}(x_0) = 0$$

and

$$R_{ijkl}(x_0) = R_{ijk}{}^l(x_0) = \tfrac{1}{2}\{g_{jl/ik} + g_{ik/lj} - g_{jk/il} - g_{il/jk}\}(x_0).$$

At any other point, of course, the curvature tensor is not as simply expressed. In general it is linear in the second derivatives of the metric, quadratic in the first derivatives of the metric with coefficients which depend smoothly on the g_{ij}'s.

We choose a local orthonormal frame $\{e_i\}$ for $T^*(M)$. Let $m = 2\bar{m}$ be even and let $\mathrm{orn} = * \cdot 1 = \mathrm{dvol}$ be the oriented volume. Let

$$e = E_m(g)\,\mathrm{orn}$$

be the Euler form. If we change the choice of the local orientation, then e and orn both change signs so $E_m(g)$ is scalar invariant of the metric. In terms of the curvature, if $2\bar{m} = m$, then:

$$c(m) = (-1)^{\bar{m}}(8\pi)^{-\bar{m}}\frac{1}{(\bar{m})!}$$

$$E_m = c(m)\sum_{\rho,\tau}\mathrm{sign}(\rho)\,\mathrm{sign}(\tau)R_{\rho(1)\rho(2)\tau(1)\tau(2)}\ldots R_{\rho(m-1)\rho(m)\tau(m-1)\tau(m)}$$

where the sum ranges over all permutations ρ, τ of the integers 1 through m. For example:

$$E_2 = -(2\pi)^{-1} R_{1212}$$
$$E_4 = (2\pi)^{-2} \sum_{i,j,k,l} \{R_{ijij} R_{klkl} + R_{ijkl} R_{ijkl} - 4R_{ijik} R_{ljlk}\}/8.$$

Let dvol be the Riemannian *measure* on M. If M is oriented, then

$$\int_M E_m \,\mathrm{dvol} = \int_M e$$

is independent of the orientation of M and of the metric. If M is not orientable, we pass to the double cover to see $\int_M E_m \,\mathrm{dvol}$ is a topological invariant of the manifold M. We shall prove later this integral is the Euler characteristic $\chi(M)$ but for the moment simply note it is not dependent upon a choice of orientation of M and is in fact defined even if M is not orientable.

It is worth computing an example. We let S^2 be the unit sphere in $\mathbf{R}^3$. Since this is homogeneous, E_2 is constant on S^2. We compute E_2 at the north pole $(0,0,1)$ and parametrize S^2 by $(u, v, (1-u^2-v^2)^{1/2})$. Then

$$\partial/\partial u = (1, 0, -u/(1-u^2-v^2)^{1/2}), \quad \partial/\partial v = (0, 1, -v/(1-u^2-v^2)^{1/2})$$
$$g_{11} = 1 + u^2/(1-u^2-v^2)$$
$$g_{22} = 1 + v^2/(1-u^2-v^2)$$
$$g_{12} = uv/(1-u^2-v^2).$$

It is clear $g_{ij}(0) = \delta_{ij}$ and $g_{ij/k}(0) = 0$. Therefore at $u = v = 0$,

$$\begin{aligned} E_2 &= -(2\pi)^{-1} R_{1212} = -(2\pi)^{-1}\{g_{11/22} + g_{22/11} - 2g_{12/12}\}/2 \\ &= -(2\pi)^{-1}\{0 + 0 - 2\}/2 = (2\pi)^{-1} \end{aligned}$$

so that

$$\int_{S^2} E_2 \,\mathrm{dvol} = (4\pi)(2\pi)^{-1} = 2 = \chi(S^2).$$

More generally, we can let $M = S^2 \times \cdots \times S^2$ where we have $\bar{m}$ factors and $2\bar{m} = m$. Since $e(V_1 \oplus V_2) = e(V_1)e(V_2)$, when we take the product metric we have $E_m = (E_2)^{\bar{m}} = (2\pi)^{-\bar{m}}$ for this example. Therefore:

$$\int_{S^2 \times \cdots \times S^2} E_m \,\mathrm{dvol} = 2^{\bar{m}} = \chi(S^2 \times \cdots \times S^2).$$

We will use this example later in this chapter to prove $\int_M E_m = \chi(M)$ in general. The natural examples for studying the Euler class are products of two dimensional spheres. Unfortunately, the Pontrjagin classes vanish identically on products of spheres so we must find other examples.

It is convenient at this point to discuss the holomorphic category. A manifold M of real dimension $m = 2\bar{m}$ is said to be holomorphic if we have local coordinate charts $z = (z_1, \dots, z_{\bar{m}})\colon M \to \mathbf{C}^{\bar{m}}$ such that two charts are related by holomorphic changes of coordinates. We expand $z_j = x_j + iy_j$ and define:

$$\begin{aligned} \partial/\partial z_j &= \tfrac{1}{2}(\partial/\partial x_j - i\partial/\partial y_j), & \partial/\partial \bar{z}_j &= \tfrac{1}{2}(\partial/\partial x_j + i\partial/\partial y_j) \\ dz_j &= dx_j + i\,dy_j, & d\bar{z}_j &= dx_j + i\,dy_j. \end{aligned}$$

We complexify $T(M)$ and $T^*(M)$ and define:

$$\begin{aligned} T_c(M) &= \operatorname{span}\{\partial/\partial z_j\}, \\ \Lambda^{1,0}(M) &= \operatorname{span}\{dz_j\}, \\ \Lambda^{0,1}(M) &= \operatorname{span}\{d\bar{z}_j\}. \end{aligned}$$

Then

$$\begin{aligned} \Lambda^1(M) &= \Lambda^{1,0}(M) \oplus \Lambda^{0,1}(M) \\ T_c(M)^* &= \Lambda^{1,0}(M). \end{aligned}$$

As complex bundles, $T_c(M) \simeq \Lambda^{0,1}(M)$. Define:

$$\begin{aligned} \partial(f) &= \sum_j \partial f/\partial z_j\, dz^j \colon C^\infty(M) \to C^\infty(\Lambda^{1,0}(M)) \\ \bar{\partial}(f) &= \sum_j \partial f/\partial \bar{z}_j\, d\bar{z}^j \colon C^\infty(M) \to C^\infty(\Lambda^{0,1}(M)) \end{aligned}$$

then the Cauchy-Riemann equations show f is holomorphic if and only if $\bar{\partial} f = 0$.

This decomposes $d = \partial + \bar{\partial}$ on functions. More generally, we define:

$$\begin{aligned} \Lambda^{p,q} &= \operatorname{span}\{dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}\} \\ \Lambda^n &= \bigoplus_{p+q=n} \Lambda^{p,q}. \end{aligned}$$

This spanning set of $\Lambda^{p,q}$ is closed. We decompose $d = \partial + \bar{\partial}$ where

$$\partial\colon C^\infty(\Lambda^{p,q}) \to C^\infty(\Lambda^{p+1,q}) \qquad \text{and} \qquad \bar{\partial}\colon C^\infty(\Lambda^{p,q}) \to C^\infty(\Lambda^{p,q+1})$$

so that $\partial\partial = \bar{\partial}\bar{\partial} = \bar{\partial}\partial + \partial\bar{\partial} = 0$. These operators and bundles are all invariantly defined independent of the particular holomorphic coordinate system chosen.

A bundle V over M is said to be holomorphic if we can cover M by holomorphic charts U_α and find frames s_α over each U_α so that on $U_\alpha \cap U_\beta$ the transition functions $s_\alpha = f_{\alpha\beta} s_\beta$ define holomorphic maps to $\mathrm{GL}(n, \mathbf{C})$. For such a bundle, we say a local section s is holomorphic if $s = \sum_\nu a_\alpha^\nu s_\alpha^\nu$ for holomorphic functions a_α^ν. For example, $T_c(M)$ and $\Lambda^{p,0}(M)$ are holomorphic bundles over M; $\Lambda^{0,1}$ is anti-holomorphic.

If V is holomorphic, we use a partition of unity to construct a Hermitian fiber metric h on V. Let $h_\alpha = (s_\alpha, s_\alpha)$ define the metric locally; this is a positive definite symmetric matrix which satisfies the transition rule $h_\alpha = f_{\alpha\beta} h_\beta \bar{f}_{\alpha\beta}$. We define a connection 1-form locally by:

$$\omega_\alpha = \partial h_\alpha h_\alpha^{-1}$$

and compute the transition rule:

$$\omega_\alpha = \partial\{f_{\alpha\beta} h_\beta \bar{f}_{\alpha\beta}\} \bar{f}_{\alpha\beta}^{-1} h_\beta^{-1} f_{\alpha\beta}^{-1}.$$

Since $f_{\alpha\beta}$ is holomorphic, $\bar\partial f_{\alpha\beta} = 0$. This implies $df_{\alpha\beta} = \partial f_{\alpha\beta}$ and $\partial \bar{f}_{\alpha\beta} = 0$ so that:

$$\omega_\alpha = \partial f_{\alpha\beta} f_{\alpha\beta}^{-1} + f_{\alpha\beta} \partial h_\beta h_\beta^{-1} f_{\alpha\beta}^{-1} = df_{\alpha\beta} f_{\alpha\beta}^{-1} + f_{\alpha\beta} \omega_\beta f_{\alpha\beta}^{-1}.$$

Since this is the transition rule for a connection, the $\{\omega_\alpha\}$ patch together to define a connection ∇_h.

It is immediate from the definition that:

$$(\nabla_h s_\alpha, s_\alpha) + (s_\alpha, \nabla_h s_\alpha) = \omega_\alpha h_\alpha + h_\alpha \omega_\alpha^* = \partial h_\alpha + \bar\partial h_\alpha = dh_\alpha$$

so ∇_h is a unitary connection on V. Since $\nabla_h s_\alpha \in C^\infty(\Lambda^{1,0} \otimes V)$ we conclude ∇_h vanishes on holomorphic sections when differentiated in anti-holomorphic directions (i.e., ∇_h is a holomorphic connection). It is easily verified that these two properties determine ∇_h.

We shall be particularly interested in holomorphic line bundles. If L is a line bundle, then h_α is a positive function on U_α with $h_\alpha = |f_{\alpha\beta}|^2 h_\beta$. The curvature in this case is a 2-form defined by:

$$\Omega_\alpha = d\omega_\alpha - \omega_\alpha \wedge \omega_\alpha = d(\partial h_\alpha h_\alpha^{-1}) = \bar\partial\partial \log(h_\alpha) = -\partial\bar\partial \log(h_\alpha).$$

Therefore:

$$c_1(L) = \frac{1}{2\pi i} \partial\bar\partial \log(h)$$

is independent of the holomorphic frame chosen for evaluation. $c_1(L) \in C^\infty(\Lambda^{1,1}(M))$ and $dc_1 = \partial c_1 = \bar\partial c_1 = 0$ so $c_1(L)$ is closed in all possible senses.

Let $\mathbf{C}P_n$ be the set of all lines through 0 in $\mathbf{C}^{n+1}$. Let $\mathbf{C}^* = \mathbf{C} - 0$ act on $\mathbf{C}^{n+1} - 0$ by complex multiplication, then $\mathbf{C}P_n = (\mathbf{C}^{n+1} - 0)/\mathbf{C}^*$ and we give $\mathbf{C}P_n$ the quotient topology. Let L be the tautological line bundle over $\mathbf{C}P_n$:

$$L = \{\, (x, z) \in \mathbf{C}P_n \times \mathbf{C}^{n+1} : z \in x \,\}.$$

Let L^* be the dual bundle; this is called the hyperplane bundle.

Let $(z_0, \dots, z_n)$ be the usual coordinates on $\mathbf{C}^{n+1}$. Let $U_j = \{\, z : z_j \neq 0 \,\}$. Since U_j is $\mathbf{C}^*$ invariant, it projects to define an open set on $\mathbf{C}P_n$ we again shall denote by U_j. We define

$$z_k^j = z_k / z_j$$

over U_j. Since these functions are invariant under the action of $\mathbf{C}^*$, they extend to continuous functions on U_j. We let $z^j = (z_0^j, \dots, \widehat{z_j^j}, \dots, z_n^j)$ where we have deleted $z_j^j = 1$. This gives local coordinates on U_j in $\mathbf{C}P_n$. The transition relations are:

$$z_\nu^j = (z_j^k)^{-1} z_\nu^k$$

which are holomorphic so $\mathbf{C}P_n$ is a holomorphic manifold.

We let $s^j = (z_0^j, \dots, z_n^j)$ be a section to L over U_j. Then $s^j = (z_j^k)^{-1} s^k$ transform holomorphically so L is a holomorphic line bundle over $\mathbf{C}P_n$. The coordinates z_j on $\mathbf{C}^{n+1}$ give linear functions on L and represent global holomorphic sections to the dual bundle L^*. There is a natural inner product on the trivial bundle $\mathbf{C}P_n \times \mathbf{C}^{n+1}$ which defines a fiber metric on L. We define:

$$x = -c_1(L) = \frac{i}{2\pi} \partial\bar\partial \log(1 + |z^j|^2)$$

then this is a closed 2-form over $\mathbf{C}P_n$.

$\mathrm{U}(n+1)$ acts on $\mathbf{C}^{n+1}$; this action induces a natural action on $\mathbf{C}P_n$ and on L. Since the metric on L arises from the invariant metric on $\mathbf{C}^{n+1}$, the 2-form x is invariant under the action of $\mathrm{U}(n+1)$.

LEMMA 2.3.1. *Let $x = -c_1(L)$ over $\mathbf{C}P_n$. Then:*
(a) $\int_{\mathbf{C}P_n} x^n = 1$.
(b) $H^*(\mathbf{C}P_n; \mathbf{C})$ is a polynomial ring with additive generators $\{1, x, \dots, x^n\}$.
(c) If $i\colon \mathbf{C}P_{n-1} \to \mathbf{C}P_n$ is the natural inclusion map, then $i^*(x) = x$.

PROOF: (c) is immediate from the naturality of the constructions involved. Standard methods of algebraic topology give the additive structure of $H^*(\mathbf{C}P_n; \mathbf{C}) = \mathbf{C} \oplus 0 \oplus \mathbf{C} \oplus 0 \cdots \oplus \mathbf{C}$. Since $x \in H^2(\mathbf{C}P_n; \mathbf{C})$ satisfies $x^n \neq 0$, (a) will complete the proof of (b). We fix a coordinate chart U_n. Since $\mathbf{C}P_n - U_n = \mathbf{C}P_{n-1}$, it has measure zero. It suffices to check that:

$$\int_{U_n} x^n = 1.$$

We identify $z \in U_n = \mathbf{C}^n$ with $(z, 1) \in \mathbf{C}^{n+1}$. We imbed $\mathrm{U}(n)$ in $\mathrm{U}(n+1)$ as the isotropy group of the vector $(0, \ldots, 0, 1)$. Then $\mathrm{U}(n)$ acts on $(z, 1)$ in $\mathbf{C}^{n+1}$ exactly the same way as $\mathrm{U}(n)$ acts on z in $\mathbf{C}^n$. Let $\mathrm{dvol} = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$ be ordinary Lebesgue measure on $\mathbf{C}^n$ without any normalizing constants of 2π. We parametrize $\mathbf{C}^n$ in the form (r, θ) for $0 \le r < \theta$ and $\theta \in S^{2n-1}$. We express $x^n = f(r, \theta)\,\mathrm{dvol}$. Since x is invariant under the action of $\mathrm{U}(n)$, $f(r, \theta) = f(r)$ is spherically symmetric and does not depend on the parameter θ.

We compute that:

$$\begin{aligned} x &= -\frac{1}{2\pi i}\partial\bar\partial \log\Big(1 + \sum z_j \bar z_j\Big) = -\frac{1}{2\pi i}\partial\Big(\sum_j z_j\, d\bar z_j/(1+|z|^2)\Big) \\ &= -\frac{1}{2\pi i}\Big\{\sum_j dz_j \wedge d\bar z_j/(1+r^2) - \sum_{j,k} z_j \bar z_k\, dz_k \wedge d\bar z_j/(1+r^2)^2\Big\}. \end{aligned}$$

We evaluate at the point $z = (r, 0, \ldots, 0)$ to compute:

$$\begin{aligned} x &= -\frac{1}{2\pi i}\Big\{\sum_j dz_j\, d\bar z_j/(1+r^2) - r^2\, dz_1 \wedge d\bar z_1/(1+r^2)^2\Big\} \\ &= -\frac{1}{2\pi i}\Big\{dz_1 \wedge d\bar z_1/(1+r^2)^2 + \sum_{j>1} dz_j \wedge d\bar z_j/(1+r^2)\Big\}. \end{aligned}$$

Consequently at this point,

$$\begin{aligned} x^n &= \Big(-\frac{1}{2\pi i}\Big)^n n!(1+r^2)^{-n-1}\, dz_1 \wedge d\bar z_1 \wedge \cdots \wedge dz_n \wedge d\bar z_n \\ &= \pi^{-n} n!(1+r^2)^{-n-1}\, dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \\ &= \pi^{-n} n!(1+r^2)^{-n-1}\,\mathrm{dvol}. \end{aligned}$$

We use the spherical symmetry to conclude this identitiy holds for all z.

We integrate over $U_n = \mathbf{C}^n$ and use spherical coordinates:

$$\begin{aligned} \int x^n &= n!\,\pi^{-n}\int (1+r^2)^{-n-1}\,\mathrm{dvol} = n!\,\pi^{-n}\int (1+r^2)^{-n-1} r^{2n-1}\, dr\, d\theta \\ &= n!\,\pi^{-n}\,\mathrm{vol}(S^{2n-1})\int_0^\infty (1+r^2)^{-n-1} r^{2n-1}\, dr \\ &= \frac{1}{2} n!\,\pi^{-n}\,\mathrm{vol}(S^{2n-1})\int_0^\infty (1+t)^{-n-1} t^{n-1}\, dt. \end{aligned}$$

We compute the volume of S^{2n-1} using the identity $\sqrt{\pi} = \int_{-\infty}^\infty e^{-t^2}\, dt$. Thus:

$$\begin{aligned} \pi^n &= \int e^{-r^2}\,\mathrm{dvol} = \mathrm{vol}(S^{2n-1})\int_0^\infty r^{2n-1} e^{-r^2}\, dr \\ &= \frac{1}{2}\,\mathrm{vol}(S^{2n-1})\int_0^\infty t^{n-1} e^{-t}\, dt = \frac{(n-1)!}{2}\,\mathrm{vol}(S^{2n-1}). \end{aligned}$$

We solve this for $\mathrm{vol}(S^{2n-1})$ and substitute to compute:

$$\begin{aligned}\int x^n &= n\int_0^\infty (1+t)^{-n-1}t^{n-1}\,dt = (n-1)\int_0^\infty (1+t)^{-n}t^{n-2}\,dt = \cdots\\ &= \int_0^\infty (1+t)^{-2}\,dt = 1\end{aligned}$$

which completes the proof.

x can also be used to define a $\mathrm{U}(n+1)$ invariant metric on $\mathbf{C}P_n$ called the Fubini-Study metric. We shall discuss this in more detail in Chapter 3 as this gives a Kaehler metric for $\mathbf{C}P_n$.

There is a relation between L and $\Lambda^{1,0}(\mathbf{C}P_n)$ which it will be convenient to exploit:

Lemma 2.3.2. *Let $M = \mathbf{C}P_n$.*
(a) There is a short exact sequence of holomorphic bundles:

$$0 \to \Lambda^{1,0}(M) \to L \otimes 1^{n+1} \to 1 \to 0.$$

(b) There is a natural isomorphism of complex bundles:

$$T_c(M) \oplus 1 \simeq L^* \otimes 1^{n+1} = \underbrace{L^* \oplus \cdots \oplus L^*}_{n+1 \text{ times}}.$$

Proof: Rather than attempting to give a geometric proof of this fact, we give a combinatorial argument. Over U_k we have functions z_i^k which give coordinates $0 \le i \le n$ for $i \ne k$. Furthermore, we have a section s^k to L. We let $\{s_i^k\}_{i=0}^n$ give a frame for $L \otimes 1^{n+1} = L \oplus \cdots \oplus L$. We define $F\colon \Lambda^{1,0}(M) \to L \otimes 1^{n+1}$ on U_k by:

$$F(dz_i^k) = s_i^k - z_i^k s_k^k.$$

We note that $z_k^k = 1$ and $F(dz_k^k) = 0$ so this is well defined on U_k. On the overlap, we have the relations $z_i^k = (z_k^j)^{-1} z_i^j$ and:

$$s_i^k = (z_k^j)^{-1} s_i^j \qquad \text{and} \qquad dz_i^k = (z_k^j)^{-1}\,dz_i^j - (z_k^j)^{-2} z_i^j\,dz_k^j.$$

Thus if we compute in the coordinate system U_j we have:

$$\begin{aligned}F(dz_i^k) &= (z_k^j)^{-1}(s_i^j - z_i^j s_j^j) - (z_k^j)^{-2} z_i^j (s_k^j - z_k^j s_j^j)\\ &= s_i^k - z_i^j s_j^k - z_i^k s_k^k + z_i^j s_j^k = s_i^k - z_i^k s_k^k\end{aligned}$$

which agrees with the definition of F on the coordinate system U_k. Thus F is invariantly defined. It is clear F is holomorphic and injective. We let

$\nu = L \otimes 1^{n+1} / \operatorname{image}(F)$, and let π be the natural projection. It is clear that s_k^k is never in the image of F so $\pi s_k^k \neq 0$. Since

$$F(dz_j^k) = s_j^k - z_j^k s_k^k = z_j^k(s_j^j - s_k^k)$$

we conclude that $\pi s_j^j = \pi s_k^k$ so $s = \pi s_k^k$ is a globally defined non-zero section to ν. This completes the proof of (a). We dualize to get a short exact sequence:

$$0 \to 1 \to L^* \otimes 1^{n+1} \to T_c(M) \to 0.$$

These three bundles have natural fiber metrics. Any short exact sequence of vector bundles splits (although the splitting is not holomorphic) and this proves (b).

From assertion (b) it follows that the Chern class of $T_c(\mathbf{C}P_n)$ is given by:

$$c(T_c) = c(T_c \oplus 1) = c(L^* \oplus \cdots \oplus L^*) = c(L^*)^{n+1} = (1+x)^{n+1}.$$

For example, if $m = 1$, then $c(T_c) = (1+x)^2 = 1 + 2x$. In this case, $c_1(T_c) = e(T(S^2))$ and we computed $\int_{S^2} e(T(S)) = 2$ so $\int_{S^2} x = 1$ which checks with Lemma 2.3.1.

If we forget the complex structure on $T_c(M)$ when M is holomorphic, then we obtain the real tangent space $T(M)$. Consequently:

$$T(M) \otimes \mathbf{C} = T_c \oplus T_c^*$$

and

$$\begin{aligned} c(T(\mathbf{C}P_n) \otimes \mathbf{C}) &= c(T_c(\mathbf{C}P_n) \oplus T_c^*(\mathbf{C}P_n)) \\ &= (1+x)^{n+1}(1-x)^{n+1} = (1-x^2)^{n+1}. \end{aligned}$$

When we take into account the sign changes involved in defining the total Pontrjagin form, we conclude:

LEMMA 2.3.3.
(a) If $M = S^2 \times \cdots \times S^2$ has dimension $2n$, then $\int_M e(T(M)) = \chi(M) = 2^n$.
(b) If $M = \mathbf{C}P_n$ has dimension $2n$, then

$$c(T_c(M)) = (1+x)^{n+1} \qquad \text{and} \qquad p(T(M)) = (1+x^2)^{n+1}.$$

$x \in H^2(\mathbf{C}P_n; \mathbf{C})$ is the generator given by $x = c_1(L^*) = -c_1(L)$, L is the tautological line bundle over $\mathbf{C}P_n$, and L^* is the dual, the hyperplane bundle.

The projective spaces form a dual basis to both the real and complex characteristic classes. Let ρ be a partition of the positive integer k in the form $k = i_1 + \cdots + i_j$ where we choose the notation so $i_1 \geq i_2 \geq \cdots$. We let $\pi(k)$ denote the number of such partitions. For example, if $k = 4$ then $\pi(4) = 5$ and the possible partitions are:

$$4 = 4, \quad 4 = 3 + 1, \quad 4 = 2 + 2, \quad 4 = 2 + 1 + 1, \quad 4 = 1 + 1 + 1 + 1.$$

We define classifying manifolds:

$$M_\rho^c = \mathbf{C}P_{i_1} \times \cdots \times \mathbf{C}P_{i_j} \qquad \text{and} \qquad M_\rho^r = \mathbf{C}P_{2i_1} \times \cdots \times \mathbf{C}P_{2i_j}$$

to be real manifolds of dimension $2k$ and $4k$.

LEMMA 2.3.4. *Let k be a positive integer. Then:*
(a) Let constants $c(\rho)$ be given. There exists a unique polynomial $P(A)$ of degree k in the components of a $k \times k$ complex matrix which is $\mathrm{GL}(k, \mathbf{C})$ *invariant such that the characteristic class defined by P satisfies:*

$$\int_{M_\rho^c} P(T_c(M_\rho^c)) = c(\rho)$$

for every such partition ρ of k.
(b) Let constants $c(\rho)$ be given. There exists a polynomial $P(A)$ of degree $2k$ in the components of a $2k \times 2k$ real matrix which is $\mathrm{GL}(2k, \mathbf{R})$ *invariant such that the characteristic class defined by P satisfies:*

$$\int_{M_\rho^r} P(T(M_\rho^r)) = c(\rho)$$

for every such partition ρ of k. If P' is another such polynomial, then $P(A) = P'(A)$ for every skew-symmetric matrix A.

In other words, the real and complex characteristic classes are completely determined by their values on the appropriate classifying manifolds.

PROOF: We prove (a) first. Let $\mathcal{P}_k$ denote the set of all such polynomials $P(A)$. We define $c_\rho = c_{i_1} \ldots c_{i_j} \in \mathcal{P}_k$, then by Lemma 2.1.3 the $\{c_\rho\}$ form a basis for $\mathcal{P}_k$ so $\dim(\mathcal{P}_k) = \pi(k)$. The $\{c_\rho\}$ are not a very convenient basis to work with. We will define instead:

$$H_\rho = ch_{i_1} \ldots ch_{i_j} \in \mathcal{P}_k$$

and show that the matrix:

$$a(\rho, \tau) = \int_{M_\tau^c} H_\rho(T_c(M_\tau^c))$$

is a non-singular matrix. This will prove the H_ρ also form a basis for $\mathcal{P}_k$ and that the M_τ^c are a dual basis. This will complete the proof.

The advantage of working with the Chern character rather than with the Chern class is that:

$$ch_i(T_c(M_1 \times M_2)) = ch_i(T_cM_1 \oplus T_cM_2) = ch_i(T_cM_1) + ch_i(T_cM_2).$$

Furthermore, $ch_i(T_cM) = 0$ if $2i > \dim(M)$. We define the length $\ell(\rho) = j$ to be the number of elements in the partition ρ. Then the above remarks imply:

$$a(\rho, \tau) = 0 \qquad \text{if } \ell(\tau) > \ell(\rho).$$

Furthermore, if $\ell(\tau) = \ell(\rho)$, then $a(\rho, \tau) = 0$ unless $\tau = \rho$. We define the partial order $\tau > \rho$ if $\ell(\tau) > \ell(\rho)$ and extend this to a total order. Then $a(\rho, \tau)$ is a triangular matrix. To show it is invertible, it suffices to show the diagonal elements are non-zero.

We first consider the case in which $\rho = \tau = k$. Using the identity $T_c(\mathbf{C}P_k) \oplus 1 = (L^* \oplus \cdots \oplus L^*)$ $(k+1$ times), it is clear that $ch_k(T_c\mathbf{C}P_k) = (k+1)ch_k(L^*)$. For a line bundle, $ch_k(L^*) = c_1(L^*)^k/k!$. If $x = c_1(L^*)$ is the generator of $H^2(\mathbf{C}P_k; \mathbf{C})$, then $ch_k(T_c\mathbf{C}P_k) = (k+1)x^k/k!$ which does not integrate to zero. If $\rho = \tau = \{i_1, \ldots, i_j\}$ then:

$$a(\rho, \rho) = c \prod_{\nu=1}^{\ell(\rho)} a(i_\nu, i_\nu)$$

where c is a positive constant related to the multiplicity with which the i_ν appear. This completes the proof of (a).

To prove (b) we replace M_ρ^c by $M_{2\rho}^c = M_\rho^r$ and H_ρ by $H_{2\rho}$. We compute

$$\begin{aligned} ch_{2i}(T(M)) \stackrel{\text{def}}{=} ch_{2i}(T(M) \otimes \mathbf{C}) &= ch_{2i}(T_cM \oplus T_c^*M) \\ &= ch_{2i}(T_cM) + ch_{2i}(T_c^*M) \\ &= 2ch_{2i}(T_cM). \end{aligned}$$

Using this fact, the remainder of the proof of (b) is immediate from the calculations performed in (a) and this completes the proof.

The Todd class and the Hirzebruch L-polynomial were defined using generating functions. The generating functions were chosen so that they would be particularly simple on the classifying examples:

LEMMA 2.3.5.
(a) Let $x_j = -\lambda_j/2\pi i$ be the normalized eigenvalues of a complex matrix A. We define $Td(A) = Td(x) = \prod_j x_j/(1 - e^{-x_j})$ as the Todd class. Then:

$$\int_{M_\rho^c} Td(T_c(M_\rho^c)) = 1 \qquad \text{for all } \rho.$$

(b) Let $x_j = \lambda_j/2\pi$ where the eigenvalues of the skew-symmetric real matrix A are $\{\pm\lambda_1, \ldots\}$. We define $L(A) = L(x) = \prod_j x_j/\tanh x_j$ as the Hirzebruch L-polynomial. Then $\int_{M_\rho^r} L(T(M^r)) = 1$ for all ρ.

We will use this calculation to prove the integral of the Todd class gives the arithmetic genus of a complex manifold and that the integral of the Hirzebruch L-polynomial gives the signature of an oriented real manifold. In each case, we only integrate the part of the total class which is of the same degree as the dimension of the manifold.

Proof: Td is a multiplicative class:

$$Td(T_c(M_1 \times M_2)) = Td(T_c(M_1) \oplus T_c(M_2)) = Td(T_c(M_1))\, Td(T_c(M_2)).$$

Similarly the Hirzebruch polynomial is multiplicative. This shows it suffices to prove Lemma 2.3.5 in the case $\rho = k$ so $M = \mathbf{C}P_k$ or $\mathbf{C}P_{2k}$.

We use the decomposition $T_c(\mathbf{C}P_k) \oplus 1 = L^* \oplus \cdots \oplus L^*$ $(k+1$ times) to compute $Td(T_c(\mathbf{C}P_k)) = [\,Td(x)]^{k+1}$ where $x = c_1(L^*)$ is the generator of $H^2(\mathbf{C}P_k; \mathbf{R})$. Since x^k integrates to 1, it suffices to show the coefficient of x^k in $Td(x)^{k+1}$ is 1 or equivalently to show:

$$\operatorname{Res}_{x=0} x^{-k-1}[\,Td(x)^{k+1}] = \operatorname{Res}_{x=0}(1 - e^{-x})^{-k-1} = 1.$$

If $k = 0$, then:

$$(1 - e^{-x})^{-1} = \left(x - \frac{1}{2}x^2 + \cdots\right)^{-1} = x^{-1}\left(1 + \frac{1}{2}x + \cdots\right)$$

and the result follows. Similarly, if $k = 1$

$$(1 - e^{-x})^{-2} = x^{-2}(1 + x + \cdots)$$

and the result follows. For larger values of k, proving this directly would be a combinatorial nightmare so we use instead a standard trick from complex variables. If $g(x)$ is any meromorphic function, then $\operatorname{Res}_{x=0} g'(x) = 0$. We apply this to the function $g(x) = (1 - e^{-x})^{-k}$ for $k \geq 1$ to conclude:

$$\operatorname{Res}_{x=0}(1 - e^{-x})^{-k-1} e^{-x} = 0.$$

This implies immediately that:

$$\begin{aligned}\operatorname{Res}_{x=0}(1 - e^{-x})^{-k-1} &= \operatorname{Res}_{x=0}(1 - e^{-x})^{-k-1}(1 - e^{-x})\\ &= \operatorname{Res}_{x=0}(1 - e^{-x})^{-k} = 1\end{aligned}$$

by induction which completes the proof of assertion (a).

We now assume k is even and study $\mathbf{C}P_k$. Again, using the decomposition $T_c(\mathbf{C}P_k) \oplus 1 = L^* \oplus \cdots \oplus L^*$ it follows that

$$L(T(\mathbf{C}P_k)) = [L(x)]^{k+1} = \frac{x^{k+1}}{\tanh^{k+1} x}.$$

Since we are interested in the coefficient of x^k, we must show:

$$\mathrm{Res}_{x=0} \tanh^{-k-1} x = 1 \qquad \text{if } k \text{ is even}$$

or equivalently

$$\mathrm{Res}_{x=0} \tanh^{-k} x = 1 \qquad \text{if } k \text{ is odd.}$$

We recall that

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$
$$\tanh^{-1} x = \frac{2 + x^2 + \cdots}{2x + \cdots}$$

so the result is clear if $k = 1$. We now proceed by induction. We differentiate $\tanh^{-k} x$ to compute:

$$(\tanh^{-k} x)' = -k(\tanh^{-k-1} x)(1 - \tanh^2 x).$$

This implies

$$\mathrm{Res}_{x=0} \tanh^{-k+1} x = \mathrm{Res}_{x=0} \tanh^{-k-1} x$$

for any integer k. Consequently $\mathrm{Res}_{x=0} \tanh^{-k} x = 1$ for any odd integer k since these residues are periodic modulo 2. (The residue at k even is, of course, zero). This completes the proof.

2.4. The Gauss-Bonnet Theorem.

Let P be a self-adjoint elliptic partial differential operator of order $d > 0$. If the leading symbol of P is positive definite, we derived an asymptotic expansion for $\mathrm{Tr}\{e^{-tP}\}$ in section 1.7. This is too general a setting in which to work so we shall restrict attention henceforth to operators with leading symbol given by the metric tensor, as this is the natural category in which to work.

Let $P: C^\infty(V) \to C^\infty(V)$ be a second order operator. We choose a local frame for V. Let $x = (x_1, \dots, x_m)$ be a system of local coordinates. Let $ds^2 = g_{ij}\, dx^i\, dx^j$ be the metric tensor and let g^{ij} denote the inverse matrix; $dx^i \cdot dx^j = g^{ij}$ is the metric on the dual space $T^*(M)$. We assume P has the form:

$$P = -g^{ij} \frac{\partial^2}{\partial x_i \partial x_j} I + a_j \frac{\partial}{\partial x_j} + b$$

where we sum over repeated indices. The a_j and b are sections to $\mathrm{END}(V)$.

We introduce formal variables for the derivatives of the symbol of P. Let

$$g_{ij/\alpha} = d_x^\alpha g_{ij}, \qquad a_{j/\alpha} = d_x^\alpha a_j, \qquad b_{/\alpha} = d_x^\alpha b.$$

We will also use the notation $g_{ij/kl\dots}$, $a_{j/kl\dots}$, and $b_{/kl\dots}$ for multiple partial derivatives. We emphasize that these variables are not tensorial but depend upon the choice of a coordinate system (and local frame for V).

There is a natural grading on these variables. We define:

$$\mathrm{ord}(g_{ij/\alpha}) = |\alpha|, \qquad \mathrm{ord}(a_{j/\alpha}) = 1 + |\alpha|, \qquad \mathrm{ord}(b_{/\alpha}) = 2 + |\alpha|.$$

Let $\mathcal{P}$ be the non-commutative polynomial algebra in the variables of positive order. We always normalize the coordinate system so x_0 corresponds to the point $(0, \dots, 0)$ and so that:

$$g_{ij}(x_0) = \delta_{ij} \qquad \text{and} \qquad g_{ij/k}(x_0) = 0.$$

Let $K(t, x, x)$ be the kernel of e^{-tP}. Expand

$$K(t, x, x) \sim \sum_{n \geq 0} t^{\frac{n-m}{2}} e_n(x, P)$$

where $e_n(x, P)$ is given by Lemma 1.7.4. Lemma 1.7.5(c) implies:

LEMMA 2.4.1. *$e_n(x, P) \in \mathcal{P}$ is a non-commutative polynomial in the jets of the symbol of P which is homogeneous of order n. If $a_n(x, P) = \mathrm{Tr}\, e_n(x, P)$ is the fiber trace, then $a_n(x, P)$ is homogeneous of order n in* *the components (relative to some local frame) of the jets of the total symbol of P.*

We could also have proved Lemma 2.4.1 using dimensional analysis and the local nature of the invariants $e_n(x,P)$ instead of using the combinatorial argument given in Chapter 1.

We specialize to the case where $P = \Delta_p$ is the Laplacian on p-forms. If $p = 0$, then $\Delta_0 = -g^{-1}\partial/\partial x_i\{gg^{ij}\partial/\partial x_j\}$; $g = \det(g_{ij})^{1/2}$ defines the Riemannian volume $\mathrm{dvol} = g\,dx$. The leading symbol is given by the metric tensor; the first order symbol is linear in the 1-jets of the metric with coefficients which depend smoothly on the metric tensor. The 0^{th} order symbol is zero in this case. More generally:

LEMMA 2.4.2. *Let $x = (x_1, \dots, x_m)$ be a system of local coordinates on M. We use the dx^I to provide a local frame for $\Lambda(T^*M)$. Relative to this frame, we expand $\sigma(\Delta_p) = p_2 + p_1 + p_0$. $p_2 = |\xi|^2 I$. p_1 is linear in the 1-jets of the metric with coefficients which depend smoothly on the g_{ij}'s. p_0 is the sum of a term which is linear in the 2-jets of the metric and a term which is quadratic in the 1-jets of the metric with coefficients which depend smoothly on the g_{ij}'s.*

PROOF: We computed the leading symbol in the first chapter. The remainder of the lemma follows from the decomposition $\Delta = d\delta + \delta d = \pm d * d * \pm * d * d$. In flat space, $\Delta = -\sum_i \partial^2/\partial x_i^2$ as computed earlier. If the metric is curved, we must also differentiate the matrix representing the Hodge $*$ operator. Each derivative applied to "$*$" reduces the order of differentiation by one and increases the order in the jets of the metric by one.

Let $a_n(x,P) = \operatorname{Tr} e_n(x,P)$. Then:

$$\operatorname{Tr} e^{-tP} \sim \sum_{n \ge 0} t^{\frac{n-m}{2}} \int_M a_n(x,P)\,\mathrm{dvol}(x).$$

For purposes of illustration, we give without proof the first few terms in the asymptotic expansion of the Laplacian. We shall discuss such formulas in more detail in the fourth chapter.

LEMMA 2.4.3.
(a) $a_0(x,\Delta_p) = (4\pi)^{-1}\dim(\Lambda^p) = (4\pi)^{-1}\binom{m}{p}$.
(b) $a_2(x,\Delta_0) = (4\pi)^{-1}(-R_{ijij})/6$.
(c)

$$a_4(x,\Delta_0) = \frac{1}{4\pi}\cdot\frac{-12R_{ijij;kk} + 5R_{ijij}R_{klkl} - 2R_{ijik}R_{ljlk} + 2R_{ijkl}R_{ijkl}}{360}.$$

In this lemma, "$;$" denotes multiple covariant differentiation. a_0, a_2, a_4, and a_6 have been computed for Δ_p for all p, but as the formulas are extremely complicated, we shall not reproduce them here.

We consider the algebra generated by the variables $\{g_{ij/\alpha}\}_{|\alpha|\ge 2}$. If X is a coordinate system and G a metric and if P is a polynomial in these variables, we define $P(X,G)(x_0)$ by evaluation. We always normalize the choice of X so $g_{ij}(X,G)(x_0)=\delta_{ij}$ and $g_{ij/k}(X,G)(x_0)=0$, and we omit these variables from consideration. We say that P is invariant if $P(X,G)(x_0)=P(Y,G)(x_0)$ for any two normalized coordinate systems X and Y. We denote the common value by $P(G)(x_0)$. For example, the scalar curvature $K=-\frac{1}{2}R_{ijij}$ is invariant as are the $a_n(x,\Delta_p)$.

We let $\mathcal{P}_m$ denote the ring of all invariant polynomials in the derivatives of the metric for a manifold of dimension m. We defined $\operatorname{ord}(g_{ij/\alpha})=|\alpha|$; let $\mathcal{P}_{m,n}$ be the subspace of invariant polynomials which are homogeneous of order n. This is an algebraic characterization; it is also useful to have the following coordinate free characterization:

LEMMA 2.4.4. *Let $P\in\mathcal{P}_m$, then $P\in\mathcal{P}_{m,n}$ if and only if $P(c^2G)(x_0)=c^{-n}P(G)(x_0)$ for every $c\neq 0$.*

PROOF: Fix $c=0$ and let X be a normalized coordinate system for the metric G at the point x_0. We assume $x_0=(0,\dots,0)$ is the center of the coordinate system X. Let $Y=cX$ be a new coordinate system, then:

$$\begin{aligned} \partial/\partial y_i &= c^{-1}\partial/\partial x_i \qquad & c^2G(\partial/\partial y_i,\partial/\partial y_j) &= G(\partial/\partial x_i,\partial/\partial x_j)\\ d_y^\alpha &= c^{-|\alpha|}d_x^\alpha & g_{ij/\alpha}(Y,c^2G) &= c^{-|\alpha|}g_{ij/\alpha}(X,G). \end{aligned}$$

This implies that if A is any monomial of P that:

$$A(Y,c^2G)(x_0)=c^{-\operatorname{ord}(A)}A(X,G)(x_0).$$

Since Y is normalized coordinate system for the metric c^2G, $P(c^2G)(x_0)=P(Y,c^2G)(x_0)$ and $P(G)(x_0)=P(X,G)(x_0)$. This proves the Lemma.

If $P\in\mathcal{P}_m$ we can always decompose $P=P_0+\cdots+P_n$ into homogeneous polynomials. Lemma 2.4.4 implies the P_j are all invariant separately. Therefore $\mathcal{P}_m$ has a direct sum decomposition $\mathcal{P}_m=\mathcal{P}_{m,0}\oplus\mathcal{P}_{m,1}\oplus\cdots\oplus\mathcal{P}_{m,n}\oplus\cdots$ and has the structure of a graded algebra. Using Taylor's theorem, we can always find a metric with the $g_{ij/\alpha}(X,G)(x_0)=c_{ij,\alpha}$ arbitrary constants for $|\alpha|\ge 2$ and so that $g_{ij}(X,G)(x_0)=\delta_{ij}$, $g_{ij/k}(X,G)(x_0)=0$. Consequently, if $P\in\mathcal{P}_m$ is non-zero as a polynomial, then we can always find G so $P(G)(x_0)\neq 0$ so P is non-zero as a formula. It is for this reason we work with the algebra of jets. This is a pure polynomial algebra. If we work instead with the algebra of covariant derivatives of the curvature tensor, we must introduce additional relations which correspond to the Bianchi identities as this algebra is not a pure polynomial algebra.

We note finally that $\mathcal{P}_{m,n}$ is zero if n is odd since we may take $c=-1$. Later in this chapter, we will let $\mathcal{P}_{m,n,p}$ be the space of p-form valued invariants which are homogeneous of order n. A similar argument will show $\mathcal{P}_{m,n,p}$ is zero if $n-p$ is odd.

Lemmas 2.4.1 and 2.4.2 imply:

LEMMA 2.4.5. *$a_n(x, \Delta_p)$ defines an element of $\mathcal{P}_{m,n}$.*

This is such an important fact that we give another proof based on Lemma 2.4.4 to illustrate the power of dimensional analysis embodied in this lemma. Fix $c > 0$ and let $\Delta_p(c^2G) = c^{-2}\Delta_p(G)$ be the Laplacian corresponding to the new metric. Since $\operatorname{dvol}(c^2G) = c^m \operatorname{dvol}(G)$, we conclude:

$$\begin{aligned}
e^{-t\Delta_p(c^2G)} &= e^{-tc^{-2}\Delta_p(G)} \\
K(t,x,x,\Delta_p(c^2G))\operatorname{dvol}(c^2G) &= K(c^{-2}t,x,x,\Delta_p(G))\operatorname{dvol}(G) \\
K(t,x,x,\Delta_p(c^2G)) &= c^{-m}K(c^{-2}t,x,x,\Delta_p(G)) \\
\sum_n t^{\frac{n-m}{2}} a_n(x,\Delta_p(c^2G)) &\sim \sum_n c^{-m}c^m c^{-n} t^{\frac{n-m}{2}} a_n(x,\Delta_p(G)) \\
a_n(x,\Delta_p(c^2G)) &= c^{-n}a_n(x,\Delta_p(G)).
\end{aligned}$$

We expand $a_n(x, \Delta_p(G)) = \sum_\nu a_{n,\nu,p} + r$ in a finite Taylor series about $g_{ij} = \delta_{ij}$ in the $g_{ij/\alpha}$ variables. Then if $a_{n,\nu,p}$ is the portion which is homogeneous of order ν, we use this identity to show $a_{n,\nu,p} = 0$ for $n \neq \nu$ and to show the remainder in the Tayor series is zero. This shows a_n is a homogeneous polynomial of order n and completes the proof.

Since $a_n(x, \Delta_p) = 0$ if n is odd, in many references the authors replace the asymptotic series by $t^{-\frac{m}{2}} \sum_n t^n a_n(x, \Delta_p)$, They renumber this sequence $a_0, a_1, \ldots$ rather than $a_0, 0, a_2, 0, a_4, \ldots$. We shall not adopt this notational convention as it makes dealing with boundary problems more cumbersome.

H. Weyl's theorem on the invariants of the orthogonal group gives a spanning set for the spaces $\mathcal{P}_{m,n}$:

LEMMA 2.4.6. *We introduce formal variables $R_{i_1i_2i_3i_4;i_5\ldots i_k}$ for the multiple covariant derivatives of the curvature tensor. The order of such a variable is $k + 2$. We consider the polynomial algebra in these variables and contract on pairs of indices. Then all possible such expressions generate $\mathcal{P}_m$. In particular:*

$$\begin{gathered}
\{1\} \text{ spans } \mathcal{P}_{m,0}, \qquad \{R_{ijij}\} \text{ spans } \mathcal{P}_{m,2} \\
\{R_{ijij;kk}, R_{ijij}R_{klkl}, R_{ijik}R_{ljlk}, R_{ijkl}R_{ijkl}\} \text{ spans } \mathcal{P}_{m,4}.
\end{gathered}$$

This particular spanning set for $\mathcal{P}_{m,4}$ is linearly independent and forms a basis if $m \geq 4$. If $m = 3$, $\dim(\mathcal{P}_{3,4}) = 3$ while if $m = 2$, $\dim(\mathcal{P}_{2,4}) = 2$ so there are relations imposed if the dimension is low. The study of these additional relations is closely related to the Gauss-Bonnet theorem.

There is a natural restriction map

$$r: \mathcal{P}_{m,n} \to \mathcal{P}_{m-1,n}$$

which is defined algebraically as follows. We let

$$\deg_k(g_{ij/\alpha}) = \delta_{i,k} + \delta_{j,k} + \alpha(k)$$

be the number of times an index k appears in the variable $g_{ij/\alpha}$. Let

$$r(g_{ij/\alpha}) = \begin{cases} g_{ij/\alpha} \in \mathcal{P}_{m-1} & \text{if } \deg_m(g_{ij/\alpha}) = 0 \\ 0 & \text{if } \deg_m(g_{ij/\alpha}) > 0. \end{cases}$$

We extend $r\colon \mathcal{P}_m \to \mathcal{P}_{m-1}$ to be an algebra morphism; $r(P)$ is a polynomial in the derivatives of a metric on a manifold of dimension $m-1$. It is clear r preserves the degree of homogeneity.

r is the dual of a natural extension map. Let G' be a metric on a manifold M' of dimension $m-1$. We define $i(G') = G + d\theta^2$ on the manifold $M = M' \times S^1$ where S^1 is the unit circle with natural parameter θ. If X' is a local coordinate system on M', then $i(X') = (x, \theta)$ is a local coordinate system on M. It is clear that:

$$(rP)(X', G')(x_0') = P(i(X'), i(G'))(x_0', \theta_0)$$

for any $\theta_0 \in S^1$; what we have done by restricting to product manifolds $M' \times S^1$ with product metrics $G' + d\theta^2$ is to introduce the relation which says the metric is flat in the last coordinate. Restiction is simply the dual of this natural extension; rP is invariant if P is invariant. Therefore $r : \mathcal{P}_{m,n} \to \mathcal{P}_{m,n-1}$.

There is one final description of the restriction map which will be useful. In discussing a H. Weyl spanning set, the indices range from 1 through m. We define the restriction by letting the indices range from 1 through $m-1$. Thus $R_{ijij} \in \mathcal{P}_{m,2}$ is its own restriction in a formal sense; of course $r(R_{ijij}) = 0$ if $m = 2$ since there are no non-trivial local invariants over a circle.

THEOREM 2.4.7.
(a) $r\colon \mathcal{P}_{m,n} \to \mathcal{P}_{m-1,n}$ *is always surjective.*
(b) $r\colon \mathcal{P}_{m,n} \to \mathcal{P}_{m-1,n}$ *is bijective if* $n < m$.
(c) $r\colon \mathcal{P}_{m,m} \to \mathcal{P}_{m-1,m}$ *has 1-dimensional kernel spanned by the Euler class* E_m *if* m *is even. If* m *is odd,* $\mathcal{P}_{m,m} = \mathcal{P}_{m-1,m} = 0$.
(c) is an axiomatic characterization of the Euler form. It is an expression of the fact that the Euler form is a unstable characteristic class as opposed to the Pontrjagin forms which are stable characteristic classes.

PROOF: (a) is consequence of H. Weyl's theorem. If we choose a H. Weyl spanning set, we let the indices range from 1 to m instead of from 1 to $m-1$ to construct an element in the inverse image of r. The proof of (b) and of (c) is more complicated and will be postponed until the next subsection. Theorem 2.4.7 is properly a theorem in invariance theory, but we have stated it at this time to illustrate how invariance theory can be used to prove index theorems using heat equation methods:

THEOREM 2.4.8. *Let $a_n(x, d+\delta) = \sum_p (-1)^p a_n(x, \Delta_p) \in \mathcal{P}_{n,m}$ be the invariant of the de Rham complex. We showed in Lemma 1.7.6 that:*

$$\int_M a_n(x, d+\delta)\,\mathrm{dvol}(x) = \begin{cases} \chi(M) & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$$

Then:
(a) $a_n(x, d+\delta) = 0$ if either m is odd or if $n < m$.
(b) $a_m(x, d+\delta) = E_m$ is m is even so $\chi(M) = \int_M E_m\,\mathrm{dvol}(x)$ (Gauss-Bonnet theorem).

PROOF: We suppose first that m is odd. Locally we can always choose an orientation for $T(M)$. We let $*$ be the Hodge operator then $*\Delta_p = \pm\Delta_{m-p}*$ locally. Since these operators are locally isomorphic, their local invariants are equal so $a_n(x, \Delta_p) = a_n(x, \Delta_{m-p})$. If m is odd, these terms cancel in the alternating sum to give $a_n(x, d+\delta) = 0$. Next we suppose m is even. Let $M = M' \times S^1$ with the product metric. We decompose any $\omega \in \Lambda(T^*M)$ uniquely in the form

$$\omega = \omega_1 + \omega_2 \wedge d\theta \qquad \text{for } \omega_i \in \Lambda(T^*M').$$

We define:

$$F(\omega) = \omega_1 \wedge d\theta + \omega_2$$

and compute easily that $F\Delta = \Delta F$ since the metric is flat in the S^1 direction. If we decompose $\Lambda(M) = \Lambda^{\mathrm{e}}(M) \oplus \Lambda^{\mathrm{o}}(M)$ into the forms of even and odd degree, then F interchanges these two factors. Therefore, $a_n(x, \Delta_e) = a_n(x, \Delta_0)$ so $a_n(x, d+\delta) = 0$ for such a product metric. This implies $r(a_n) = 0$. Therefore $a_n = 0$ for $n < m$ by Theorem 2.4.7. Furthermore:

$$a_m = c_m E_m$$

for some universal constant c_m. We show $c_m = 1$ by integrating over the classifying manifold $M = S^2 \times \cdots \times S^2$. Let $2\bar{m} = m$, then

$$2^{\bar{m}} = \chi(M) = \int_M a_m(x, d+\delta)\,\mathrm{dvol}(x) = \int_M E_m\,\mathrm{dvol}(x)$$

by Lemma 2.3.4. This completes the proof of the Gauss-Bonnet theorem.

2.5. Invariance Theory and the Pontrjagin Classes of the Tangent Bundle.

In the previous subsection, we gave in Theorem 2.4.7 an axiomatic characterization of the Euler class in terms of functorial properties. In this subsection we will complete the proof of Theorem 2.4.7. We will also give a similar axiomatic characterization of the Pontrjagin classes which we will use in our discussion of the signature complex in Chapter 3.

Let $T: \mathbf{R}^m \to \mathbf{R}^m$ be the germ of a diffeomorphism. We assume that:

$$T(0) = 0 \qquad \text{and} \qquad dT(x) = dT(0) + O(x^2) \qquad \text{for } dT(0) \in \mathrm{O}(k).$$

If X is any normalized coordinate system for a metric G, then TX is another normalized coordinate system for G. We define an action of the group of germs of diffeomorphisms on the polynomial algebra in the $\{g_{ij/\alpha}\},\ |\alpha| \geq 2$ variables by defining the evaluation:

$$(T^*P)(X,G)(x_0) = P(TX,G)(x_0).$$

Clearly P is invariant if and only if $T^*P = P$ for every such diffeomorphism T.

Let P be invariant and let A be a monomial. We let $c(A,P)$ be the coefficient of A in P; $c(A,P)$ defines a linear functional on $\mathcal{P}$ for any monomial A. We say A is a monomial of P if $c(A,P) \neq 0$. Let T_j be the linear transformation:

$$T_j(x_k) = \begin{cases} -x_j & \text{if } k = j \\ x_k & \text{if } k \neq j. \end{cases}$$

This is reflection in the hyperplane defined by $x_j = 0$. Then

$$T_j^*(A) = (-1)^{\deg_j(A)} A$$

for any monomial A. Since

$$T_j^* P = \sum (-1)^{\deg_j(A)} c(A,P)A = P = \sum c(A,P)A,$$

we conclude $\deg_j(A)$ must be even for any monomial A of P. If A has the form:

$$A = g_{i_1 j_1/\alpha_1} \dots g_{i_r j_r/\alpha_r}$$

we define the length of A to be:

$$\ell(A) = r.$$

It is clear $2\ell(A) + \operatorname{ord}(A) = \sum_j \deg_j(A)$ so $\operatorname{ord}(A)$ is necessarily even if A is a monomial of P. This provides another proof $\mathcal{P}_{m,n} = 0$ if n is odd.

In addition to the hyperplane reflections, it is convenient to consider coordinate permutations. If ρ is a permutation, then $T_\rho^*(A) = A^\rho$ is defined by replacing each index i by $\rho(i)$ in the variables $g_{ij/k\ldots}$. Since $T_\rho^*(P) = P$, we conclude the form of P is invariant under coordinate permutations or equivalently $c(A,P) = c(A^\rho, P)$ for every monomial A of P.

We can use these two remarks to begin the proof of Theorem 2.4.7. Fix $P \neq 0$ with $r(P) = 0$ and $P \in \mathcal{P}_{m,n}$. Let A be a monomial of P. Then $r(P) = 0$ implies $\deg_m(A) > 0$ is even. Since P is invariant under coordinate permutations, $\deg_k(A) \geq 2$ for $1 \leq k \leq m$. We construct the chain of inequalities:

$$\begin{aligned} 2m \leq \sum_k \deg_k(A) &= 2\ell(A) + \operatorname{ord}(A) \\ &= \sum_\nu (2 + |\alpha_\nu|) \leq \sum_\nu 2|\alpha_\nu| = 2\operatorname{ord}(A) = 2n. \end{aligned}$$

We have used the fact $|\alpha_\nu| \geq 2$ for $1 \leq \nu \leq \ell(A)$. This implies $m \leq n$ so in particular $P = 0$ if $n < m$ which proves assertion (b) of Theorem 2.4.7. If $n = m$, then all of the inequalities in this string must be equalities. This implies $2\ell(A) = m$, $\deg_k(A) = 2$ for all k, and $|\alpha_\nu| = 2$ for all ν. Thus the monomial A must have the form:

$$A = g_{i_1 j_1/k_1 l_1} \ldots g_{i_r j_r/k_r l_r} \qquad \text{for } 2r = m$$

and in particular, P only involves the second derivatives of the metric.

In order to complete the proof of Theorem 2.4.7, we must use some results involving invariance under circle actions:

LEMMA 2.5.1. *Parametrize the circle* $\mathrm{SO}(2)$ *by* $z = (a,b)$ *for* $a^2 + b^2 = 1$. *Let* T_z *be the coordinate transformation:*

$$y_1 = ax_1 + bx_2, \qquad y_2 = -bx_1 + ax_2, \qquad y_k = x_k \qquad \text{for } k > 2.$$

Let P *be a polynomial and assume* $T_z^* P = P$ *for all* $z \in S^1$. *Then:*
(a) If $g_{12/\alpha}$ *divides a monomial* A *of* P *for some* α, *then* $g_{11/\beta}$ *divides a monomial* B *of* P *for some* β.
(b) If $g_{ij/\alpha}$ *divides a monomial of* A *of* P *for some* (i,j), *then* $g_{kl/\beta}$ *divides a monomial* B *of* P *for some* β *and some* (k,l) *where* $\beta(1) = \alpha(1) + \alpha(2)$ *and* $\beta(2) = 0$.
Of course, the use of the indices 1 and 2 is for convenience only. This lemma holds true for any pair of indices under the appropriate invariance assumption.

We postpone the proof of this lemma for the moment and use it to complete the proof of Theorem 2.4.7. Let $P \neq 0 \in \mathcal{P}_{m,m}$ with $r(P) = 0$.

Let A be a monomial of P. We noted $\deg_k(A) = 2$ for $1 \le k \le m$ and A is a polynomial in the 2-jets of the metric. We decompose

$$A = g_{i_1 j_1/k_1 l_1} \dots g_{i_r j_r/k_r l_r} \qquad \text{for } 2r = m.$$

By making a coordinate permutation we can choose A so $i_1 = 1$. If $j_1 \ne 1$, we can make a coordinate permutation to assume $j_1 = 2$ and then apply Lemma 2.5.1(a) to assume $i_1 = j_1 = 1$. We let $P_1 = \sum c(A, P)A$ where the sum ranges over A of the form:

$$A = g_{11/k_1 l_1} \dots g_{i_r j_r/k_r l_r}$$

which are monomials of P. Then $P_1 \ne 0$ and P_1 is invariant under coordinate transformations which fix the first coordinate. Since $\deg_1(A) = 2$, the index 1 appears nowhere else in A. Thus k_1 is not 1 so we may make a coordinate permutation to choose A so $k_1 = 2$. If $l_2 \ne 2$, then $l_1 \ge 3$ so we may make a coordinate permutation to assume $l_1 = 3$. We then apply Lemma 2.5.1(b) to choose A a monomial of P_1 of the form:

$$A = g_{11/22} A'.$$

We have $\deg_1(A) = \deg_2(A) = 2$ so $\deg_1(A') = \deg_2(A') = 0$ so these indices do not appear in A'. We define $P_2 = \sum c(A, P)A$ where the sum ranges over those monomial A of P which are divisible by $g_{11/22}$, then $P_2 \ne 0$.

We proceed inductively in this fashion to show finally that

$$A_0 = g_{11/22} g_{33/44} \dots g_{m-1,m-1/mm}$$

is a monomial of P so $c(A_0, P) \ne 0$. The function $c(A_0, P)$ is a separating linear functional on the kernel of r in $\mathcal{P}_{m,m}$ and therefore

$$\dim(\{ P \in \mathcal{P}_{m,m} : r(P) = 0 \}) \le 1.$$

We complete the proof of Theorem 2.4.7(c) by showing that $r(E_m) = 0$. If E_m is the Euler form and $M = M' \times S^1$, then:

$$E_m(M) = E_m(M' \times S^1) = E_{m-1}(M')E_1(S^1) = 0$$

which completes the proof; $\dim \mathrm{N}(r) = 1$ and $E_m \in \mathrm{N}(r)$ spans.

Before we begin the proof of Lemma 2.5.1, it is helpful to consider a few examples. If we take $A = g_{11/11}$ then it is immediate:

$$\begin{aligned} &\partial/\partial y_1 = a\partial/\partial x_1 + b\partial/\partial x_2\,, \qquad \partial/\partial y_2 = -b\partial/\partial x_1 + a\partial/\partial x_2 \\ &\partial/\partial y_k = \partial/\partial x_k \qquad \text{for } k > 2. \end{aligned}$$

We compute $T_z^*(A)$ by formally replacing each index 1 by $a1+b2$ and each index 2 by $-b1+a2$ and expanding out the resulting expression. Thus, for example:

$$\begin{aligned}T_z^*(g_{11/11}) &= g_{a1+b2,a1+b2/a1+b2,a1+b2}\\ &= a^4 g_{11/11} + 2a^3 b g_{11/12} + 2a^3 b g_{12/11}\\ &\quad + a^2b^2 g_{11/22} + a^2b^2 g_{22/11} + 4a^2b^2 g_{12/12}\\ &\quad\quad + 2ab^3 g_{12/22} + 2ab^3 g_{22/12} + b^4 g_{22/22}.\end{aligned}$$

We note that those terms involving a^3b arose from changing exactly one index to another index ($1 \to 2$ or $2 \to 1$); the coefficient reflects the multiplicity. Thus in particular this polynomial is not invariant.

In computing invariance under the circle action, all other indices remain fixed so

$$\begin{aligned}T_z^*(g_{34/11} + g_{34/22}) &= (a^2 g_{34/11} + b^2 g_{34/22} + 2ab g_{34/12}\\ &\quad + a^2 g_{34/22} + b^2 g_{34/11} - 2ab g_{34/12})\\ &= (a^2+b^2)(g_{34/11} + g_{34/22})\end{aligned}$$

is invariant. Similarly, it is easy to compute:

$$T_z^*(g_{11/22} + g_{22/11} - 2g_{12/12}) = (a^2+b^2)^2(g_{11/22} + g_{22/11} - 2g_{12/12})$$

so this is invariant. We note this second example is homogeneous of degree 4 in the (a,b) variables since $\deg_1(A) + \deg_2(A) = 4$.

With these examples in mind, we begin the proof of Lemma 2.5.1. Let P be invariant under the action of the circle acting on the first two coordinates. We decompose $P = P_0 + P_1 + \cdots$ where each monomial A of P_j satisfies $\deg_1(A) + \deg_2(A) = j$. If A is such a monomial, then $T_z^*(A)$ is a sum of similar monomials. Therefore each of the P_j is invariant separately so we may assume $P = P_n$ for some n. By setting $a = b = -1$, we see n must be even. Decompose:

$$T_z^*(P) = a^n P^{(0)} + ba^{n-1}P^{(1)} + \cdots + b^n P^{(n)}$$

where $P = P^{(0)}$. We use the assumption $T_z^*(P) = P$ and replace b by $-b$ to see:

$$0 = T_{(a,b)}^*(P) - T_{(a,-b)}^*(P) = 2ba^{n-1}P^{(1)} + 2b^3a^{n-3}P^{(3)} + \cdots$$

We divide this equation by b and take the limit as $b \to 0$ to show $P^{(1)} = 0$. (In fact, it is easy to show $P^{(2j+1)} = 0$ and $P^{(2j)} = \binom{n/2}{j}P$ but as we shall not need this fact, we omit the proof).

We let A_0 denote a variable monomial which will play the same role as the generic constant C of Chapter 1. We introduce additional notation we shall find useful. If A is a monomial, we decompose $T^*_{(a,b)}A = a^n A + a^{n-1}bA^{(1)} + \cdots$. If B is a monomial of $A^{(1)}$ then $\deg_1(B) = \deg_1(A) \pm 1$ and B can be constructed from the monomial A by changing exactly one index $1 \to 2$ or $2 \to 1$. If $\deg_1(B) = \deg_1(A) + 1$, then B is obtained from A by changing exactly one index $2 \to 1$; $c(B, A^{(1)})$ is a negative integer which reflects the multiplicity with which this change can be made. If $\deg_1(B) = \deg_1(A) - 1$, then B is obtained from A by changing exactly one index $1 \to 2$; $c(B, A^{(1)})$ is a positive integer. We define:

$$\begin{aligned} A(1 \to 2) &= \sum c(B, A^{(1)})B \qquad &&\text{summed over } \deg_1(B) = \deg_1(A) - 1 \\ A(2 \to 1) &= \sum -c(B, A^{(1)})B \qquad &&\text{summed over } \deg_1(B) = \deg_1(A) + 1 \\ A^{(1)} &= A(1 \to 2) - A(2 \to 1). \end{aligned}$$

For example, if $A = (g_{12/33})^2 g_{11/44}$ then $n = 6$ and:

$$\begin{aligned} A(1 \to 2) &= 2g_{12/33}g_{22/33}g_{11/44} + 2(g_{12/33})^2 g_{12/44} \\ A(2 \to 1) &= 2g_{12/33}g_{11/33}g_{11/44}. \end{aligned}$$

It is immediate from the definition that:

$$\sum_B c(B, A(1 \to 2)) = \deg_1(A) \qquad \text{and} \qquad \sum_B c(B, A(2 \to 1)) = \deg_2(A).$$

Finally, it is clear that $c(B, A^{(1)}) \neq 0$ if and only if $c(A, B^{(1)}) \neq 0$, and that these two coefficients will be opposite in sign, and not necessarily equal in magnitude.

LEMMA 2.5.2. *Let P be invariant under the action of* $\mathrm{SO}(2)$ *on the first two coordinates and let A be a monomial of P. Let B be a monomial of $A^{(1)}$. Then there exists a monomial of A_1 different from A so that*

$$c(B, A^{(1)})c(A, P)c(B, A_1^{(1)})c(A_1, P) < 0.$$

PROOF: We know $P^{(1)} = 0$. We decompose

$$P^{(1)} = \sum_A c(A, P)A^{(1)} = \sum_B c(A, P)c(B, A^{(1)})B.$$

Therefore $c(B, P^{(1)}) = 0$ implies

$$\sum_A c(A, P)c(B, A^{(1)}) = 0$$

for all monomials B. If we choose B so $c(B, A^{(1)}) \neq 0$ then there must be some other monomial A_1 of P which helps to cancel this contribution. The signs must be opposite which proves the lemma.

This lemma is somewhat technical and formidable looking, but it is exactly what we need about orthogonal invariance. We can now complete the proof of Lemma 2.5.1. Suppose first that $A = (g_{12/\alpha})^k A_0$ for $k > 0$ where $g_{12/\alpha}$ does not divide A_0. Assume $c(A, P) \neq 0$. Let $B = g_{11/\alpha}(g_{12/\alpha})^{k-1} A_0$ then $c(B, A^{(1)}) = -k \neq 0$. Choose $A_1 \neq A$ so $c(A_1, P) \neq 0$. Then $c(A_1, B^{(1)}) \neq 0$. If $c(A_1, B(2 \to 1)) \neq 0$ then A_1 is constructed from B by changing a $2 \to 1$ index so $A_1 = g_{11/\beta} \dots$ has the desired form. If $c(A_1, B(1 \to 2)) \neq 0$, we expand $B(1 \to 2) = A+$ terms divisible by $g_{11/\beta}$ for some β. Since $A_1 \neq A$, again A_1 has the desired form which proves (a).

To prove (b), we choose $g_{ij/\alpha}$ dividing some monomial of P. Let β be chosen with $\beta(1)+\beta(2) = \alpha(1)+\alpha(2)$ and $\beta(k) = \alpha(k)$ for $k > 2$ with $\alpha(1)$ maximal so $g_{uv/\beta}$ divides some monomial of P for some (u, v). Suppose $\beta(2) \neq 0$, we argue for a contradiction. Set $\gamma = (\beta(1)+1, \beta(2)-1, \beta(3), \dots, \beta(m))$. Expand $A = (g_{uv/\beta})^k A_0$ and define $B = g_{uv/\gamma}(g_{uv/\beta})^{k-1} A_0$ where $g_{uv/\beta}$ does not divide A_0. Then $c(B, A^{(1)}) = -\beta(2)k \neq 0$ so we may choose $A_1 \neq A$ so $c(A_1, B^{(1)}) \neq 0$. If $c(A_1, B(2 \to 1)) \neq 0$ then either A_1 is divisible by $g_{u'v'/\gamma}$ or by $g_{uv/\gamma'}$ where $\gamma'(1) = \gamma(1) + 1$, $\gamma'(2) = \gamma(2) - 1$, and $\gamma'(j) = \alpha(j)$ for $j > 2$. Either possibility contradicts the choice of β as maximal so $c(A_1, B(1 \to 2)) \neq 0$. However, $B(1 \to 2) = \beta(1)A+$ terms divisible by $g_{u'v'/\gamma}$ for some (u', v'). This again contradicts the maximality as $A \neq A_1$ and completes the proof of Lemma 2.5.1 and thereby of Theorems 2.4.7 and 2.4.8.

If $I = \{1 \leq i_1 \leq \dots \leq i_p \leq m\}$, let $|I| = p$ and $dx^I = dx_{i_1} \wedge \dots \wedge dx_{i_p}$. A p-form valued polynomial is a collection $\{P_I\} = P$ for $|I| = p$ of polynomials P_I in the derivatives of the metric. We will also sometimes write $P = \sum_{|I|=p} P_I \, dx^I$ as a formal sum to represent P. If all the $\{P_I\}$ are homogeneous of order n, we say P is homogeneous of order n. We define:

$$P(X, G)(x_0) = \sum_I P_I(X, G) \, dx^I \in \Lambda^p(T^*M)$$

to be the evaluation of such a polynomial. We say P is invariant if $P(X, G)(x_0) = P(Y, G)(x_0)$ for every normalized coordinate systems X and Y; as before we denote the common value by $P(G)(x_0)$. In analogy with Lemma 2.4.4 we have:

Lemma 2.5.3. *Let P be p-form valued and invariant. Then P is homogeneous of order n if and only if $P(c^2 G)(x_0) = c^{p-n} P(G)(x_0)$ for every $c \neq 0$.*

The proof is exactly the same as that given for Lemma 2.4.4. The only new feature is that $dy^I = c^p \, dx^I$ which contributes the extra feature of c^p in this equation.

We define $\mathcal{P}_{m,n,p}$ to be the vector space of all p-form valued invariants which are homogeneous of order n on a manifold of dimension m. If $\mathcal{P}_{m,*,p}$ denotes the vector space of all p-form valued invariant polynomials, then we have a direct sum decomposition $\mathcal{P}_{m,*,p} = \bigoplus_n \mathcal{P}_{m,n,p}$ exactly as in the scalar case $p = 0$.

We define $\deg_k(I)$ to be 1 if k appears in I and 0 if k does not appear in I.

LEMMA 2.5.4. *Let $P \in \mathcal{P}_{m,n,p}$ with $P \neq 0$. Then $n - p$ is even. If A is a monomial then A is a monomial of at most one P_I. If $c(A, P_I) \neq 0$ then:*

$$\deg_k(A) + \deg_k(I) \qquad \text{is always even.}$$

PROOF: Let T be the coordinate transformation defined by $T(x_k) = -x_k$ and $T(x_j) = x_j$ for $j \neq k$. Then

$$P = T^*(P) = \sum_{I,A} (-1)^{\deg_k(A)+\deg_k(I)} c(A, P_I)\, dx^I$$

which implies $\deg_k(A) + \deg_k(I)$ is even if $c(A, P_I) \neq 0$. Therefore

$$\sum_k \deg_k(A) + \deg_k(I) = 2\ell(A) + \operatorname{ord}(A) + p = 2\ell(A) + n + p$$

must be even. This shows $n+p$ is even if $P \neq 0$. Furthermore, if $c(A, P_I) \neq 0$ then I is simply the ordered collection of indices k so $\deg_k(A)$ is odd which shows A is a monomial of at most one P_I and completes the proof.

We extend the Riemannian metric to a fiber metric on $\Lambda(T^*M) \otimes \mathbf{C}$. It is clear that $P \cdot P = \sum_I P_I \bar{P}_I$ since the $\{dx^I\}$ form an orthonormal basis at x_0. $P \cdot P$ is a scalar invariant. We use Lemma 2.5.1 to prove:

LEMMA 2.5.5. *Let P be p-form valued and invariant under the action of* $\mathrm{O}(m)$. *Let A be a monomial of P. Then there is a monomial A_1 of P with $\deg_k(A_1) = 0$ for $k > 2\ell(A)$.*

PROOF: Let $r = \ell(A)$ and let $P_r' = \sum_{\ell(B)=r} c(B, P_I)\, dx^I \neq 0$. Since this is invariant under the action of $\mathrm{O}(m)$, we may assume without loss of generality $P = P_r$. We construct a scalar invariant by taking the inner product $Q = (P, P)$. By applying Lemma 2.5.1(a) and making a coordinate permutation if necessary, we can assume g_{11/α_1} divides some monomial of Q. We apply 2.5.1(b) to the indices > 1 to assume $\alpha_1(k) = 0$ for $k > 2$. g_{11/α_1} must divide some monomial of P. Let

$$P_1 = \sum_{A_0, I} c(g_{11/\alpha_1} A_0, P_I) g_{11/\alpha_1} A_0\, dx^I \neq 0.$$

This is invariant under the action of $\mathrm{O}(m-2)$ on the last $m-2$ coordinates. We let $Q_1 = (P_1, P_1)$ and let $g_{i_2j_2/\alpha_2}$ divide some monomial of Q_1. If both i_2 and j_2 are ≤ 2 we leave this alone. If $i_2 \le 2$ and $j_2 \ge 3$ we perform a coordinate permutation to assume $j_2 = 3$. In a similar fashion if $i_2 \ge 3$ and $j_2 \le 2$, we choose this variable so $i_2 = 3$. Finally, if both indices ≥ 3, we apply Lemma 2.5.1(a) to choose this variable so $i_2 = j_2 = 3$. We apply Lemma 2.5.1(b) to the variables $k \ge 4$ to choose this variable so $\alpha_2(k) = 0$ for $k > 4$. If $A_2 = g_{11/\alpha_1} g_{i_2j_2/\alpha_2}$ then:

$$\deg_k(A_2) = 0 \text{ for } k > 4 \text{ and } A_2 \text{ divides some monomial of } P.$$

We continue inductively to construct $A_r = g_{11/\alpha_1} \dots g_{i_rj_r/\alpha_r}$ so that

$$\deg_k(A_r) = 0 \text{ for } k > 2r \text{ and } A_r \text{ divides some monomial of } P.$$

Since every monomial of P has length r, this implies A_r itself is a monomial of P and completes the proof.

Let $P_j(G) = p_j(TM)$ be the j^{th} Pontrjagin form computed relative to the curvature tensor of the Levi-Civita connection. If we expand p_j in terms of the curvature tensor, then p_j is homogeneous of order $2j$ in the $\{R_{ijkl}\}$ tensor so p_j is homogeneous of order $4j$ in the jets of the metric. It is clear p_j is an invariantly defined $4j$-form so $p_j \in \mathcal{P}_{m,4j,4j}$. The algebra generated by the p_j is called the algebra of the Pontrjagin forms. By Lemma 2.2.2, this is also the algebra of real characteristic forms of $T(M)$. If ρ is a partition of $k = i_1 + \cdots + i_j$ we define $p_\rho = p_{i_1} \dots p_{i_j} \in \mathcal{P}_{m,4k,4k}$. The $\{p_\rho\}$ form a basis of the Pontrjagin $4k$ forms. By Lemma 2.3.4, these are linearly independent if $m = 4k$ since the matrix $\int_{M_\tau^r} p_\rho$ is non-singular. By considering products of these manifolds with flat tori T^{m-4k} we can easily show that the $\{p_\rho\}$ are linearly independent in $\mathcal{P}_{m,4k,4k}$ if $4k \le m$. We let $\pi(k)$ be the number of partitions of k; this is the dimension of the Pontrjagin forms.

The axiomatic characterization of the real characteristic forms of the tangent space which is the analogue of the axiomatic characterization of the Euler class given in Theorem 2.4.7 is the following:

LEMMA 2.5.6.
(a) $\mathcal{P}_{m,n,p} = 0$ *if* $n < p$.
(b) $\mathcal{P}_{m,n,n}$ *is spanned by the Pontrjagin forms—i.e.,*

$$\mathcal{P}_{m,n,n} = 0 \text{ if } n \text{ is not divisible by } 4k,$$
$$\mathcal{P}_{m,4k,4k} = \operatorname{span}\{p_\rho\} \text{ for } 4k \le m \text{ has dimension } \pi(k).$$

PROOF: By decomposing P into its real and imaginary parts, it suffices to prove this lemma for polynomials with real coefficients. Let $0 \ne P \in \mathcal{P}_{m,n,p}$

and let A be a monomial of P. Use Lemma 2.5.5 to find a monomial A_1 of some P_I where $I = \{1 \le i_1 < \cdots < i_p \le m\}$ so $\deg_k(A) = 0$ for $k > 2\ell(A)$. Since $\deg_{i_p}(A)$ is odd (and hence non-zero) and since $2\ell(A) \le \sum |\alpha_\nu| = n$ as A is a polynomial in the jets of order 2 and higher, we estimate:

$$p \le i_p \le 2\ell(A) \le n.$$

This proves $P = 0$ if $n < p$, which proves (a).

If $n = p$, then all of these inequalities must have been equalities. This shows that the higher order jets of the metric do not appear in P so P is a polynomial in the $\{g_{ij/kl}\}$ variables. Furthermore, $i_p = p$ and there is some monomial A so that

$$\deg_k(A) = 0 \text{ for } k > p \quad \text{and} \quad A\,dx^1 \wedge \cdots \wedge dx^p \text{ appears in } P.$$

There is a natural restriction map $r\colon \mathcal{P}_{m,n,p} \to \mathcal{P}_{m-1,n,p}$ defined in the same way as the restriction map $r\colon \mathcal{P}_{m,n,0} \to \mathcal{P}_{m-1,n,0}$ discussed earlier. This argument shows $r\colon \mathcal{P}_{m,n,n} \to \mathcal{P}_{m-1,n,n}$ is injective for $n < m$ since $r(A\,dx^1 \wedge \cdots \wedge dx^p) = A\,dx^1 \wedge \cdots \wedge dx^p$ appears in $r(P)$. The Pontrjagin forms have dimension $\pi(k)$ for $n = 4k$. By induction, $r^{m-n}\colon \mathcal{P}_{m,n,n} \to \mathcal{P}_{n,n,n}$ is injective so $\dim \mathcal{P}_{m,n,n} \le \dim \mathcal{P}_{n,n,n}$.

We shall prove Lemma 2.5.6(b) for the special case $n = m$. If n is not divisible by $4k$, then $\dim \mathcal{P}_{n,n,n} = 0$ which implies $\dim \mathcal{P}_{m,n,n} = 0$. If $n = 4k$, then $\dim \mathcal{P}_{n,n,n} = \pi(k)$ implies that $\pi(k) \le \dim \mathcal{P}_{m,n,n} \le \dim \mathcal{P}_{n,n,n} \le \pi(k)$ so $\dim \mathcal{P}_{m,n,n} = \pi(k)$. Since the Pontrjagin forms span a subspace of exactly dimension $\pi(k)$ in $\mathcal{P}_{m,n,n}$ this will complete the proof of (b).

This lemma is at the heart of our discussion of the index theorem. We shall give two proofs for the case $n = m = p$. The first is based on H. Weyl's theorem for the orthogonal group and follows the basic lines of the proof given in Atiyah-Bott-Patodi. The second proof is purely combinatorial and follows the basic lines of the original proof first given in our thesis. The H. Weyl based proof has the advantage of being somewhat shorter but relies upon a deep theorem we have not proved here while the second proof although longer is entirely self-contained and has some additional features which are useful in other applications.

We review H. Weyl's theorem briefly. Let V be a real vector space with a fixed inner product. Let $\mathrm{O}(V)$ denote the group of linear maps of $V \to V$ which preserve this inner product. Let $\bigotimes^k(V) = V \otimes \cdots \otimes V$ denote the k^{th} tensor prodect of V. If $g \in \mathrm{O}(V)$, we extend g to act orthogonally on $\bigotimes^k(V)$. We let $z \mapsto g(z)$ denote this action. Let $f\colon \bigotimes^k(V) \to \mathbf{R}$ be a multi-linear map, then we say f is $\mathrm{O}(V)$ invariant if $f(g(z)) = f(z)$ for every $g \in \mathrm{O}(V)$. By letting $g = -1$, it is easy to see there are no $\mathrm{O}(V)$ invariant maps if k is odd. We let $k = 2j$ and construct a map

$f_0 : \bigotimes^k(V) = (V \otimes V) \otimes (V \otimes V) \otimes \cdots \otimes (V \otimes V) \to \mathbf{R}$ using the metric to map $(V \otimes V) \to \mathbf{R}$. More generally, if ρ is any permutation of the integers 1 through k, we define $z \mapsto z_\rho$ as a map from $\bigotimes^k(V) \to \bigotimes^k(V)$ and let $f_\rho(z) = f_0(z_\rho)$. This will be $\mathrm{O}(V)$ invariant for any permutation ρ. H. Weyl's theorem states that the maps $\{f_\rho\}$ define a spanning set for the collection of $\mathrm{O}(V)$ invariant maps.

For example, let $k = 4$. Let $\{v_i\}$ be an orthonormal basis for V and express any $z \in \bigotimes^4(V)$ in the form $a_{ijkl} v_i \otimes v_j \otimes v_k \otimes v_l$ summed over repeated indices. Then after weeding out duplications, the spanning set is given by:

$$f_0(z) = a_{iijj}, \qquad f_1(z) = a_{ijij}, \qquad f_2(z) = a_{ijji}$$

where we sum over repeated indices. f_0 corresponds to the identity permutation; f_1 corresponds to the permutation which interchanges the second and third factors; f_2 corresponds to the permutation which interchanges the second and fourth factors. We note that these need not be linearly independent; if $\dim V = 1$ then $\dim(\bigotimes^4 V) = 1$ and $f_1 = f_2 = f_3$. However, once $\dim V$ is large enough these become linearly independent.

We are interested in p-form valued invariants. We take $\bigotimes^k(V)$ where $k - p$ is even. Again, there is a natural map we denote by

$$f^p(z) = f_0(z_1) \wedge \Lambda(z_2)$$

where we decompose $\bigotimes^k(V) = \bigotimes^{k-p}(V) \otimes \bigotimes^p(V)$. We let f_0 act on the first $k - p$ factors and then use the natural map $\bigotimes^p(V) \xrightarrow{\Lambda} \Lambda^p(V)$ on the last p factors. If ρ is a permutation, we set $f^p_\rho(z) = f^p(z_\rho)$. These maps are equivariant in the sense that $f^p_\rho(gz) = g f^p_\rho(z)$ where we extend g to act on $\Lambda^p(V)$ as well. Again, these are a spanning set for the space of equivariant multi-linear maps from $\bigotimes^k(V)$ to $\Lambda^p(V)$. If $k = 4$ and $p = 2$, then after eliminating duplications this spanning set becomes:

$$\begin{aligned} &f_1(z) = a_{iijk} v_j \wedge v_k\,, \quad f_2(z) = a_{ijik} v_j \wedge v_k\,, \quad f_3(z) = a_{ijki} v_j \wedge v_k \\ &f_4(z) = a_{jiki} v_j \wedge v_k\,, \quad f_5(z) = a_{jiik} v_i \wedge v_k\,, \quad f_6(z) = a_{jkii} v_j \wedge v_k\,. \end{aligned}$$

Again, these are linearly independent if $\dim V$ is large, but there are relations if $\dim V$ is small. Generally speaking, to construct a map from $\bigotimes^k(V) \to \Lambda^p(V)$ we must alternate p indices (the indices j, k in this example) and contract the remaining indices in pairs (there is only one pair i, i here).

Theorem 2.5.7 (H. Weyl's theorem on the invariants of the orthogonal group). *The space of maps $\{f^p_\rho\}$ constructed above span the space of equivariant multi-linear maps from $\bigotimes^k V \to \Lambda^p V$.*

The proof of this theorem is beyond the scope of the book and will be omitted. We shall use it to give a proof of Lemma 2.5.6 along the lines of the proof given by Atiyah, Bott, and Patodi. We will then give an independent proof of Lemma 2.5.6 by other methods which does not rely on H. Weyl's theorem.

We apply H. Weyl's theorem to our situation as follows. Let $P \in \mathcal{P}_{n,n,n}$ then P is a polynomial in the 2-jets of the metric. If we let X be a system of geodesic polar coordinates centered at x_0, then the 2-jets of the the metric are expressible in terms of the curvature tensor so we can express P as a polynomial in the $\{R_{ijkl}\}$ variables which is homogeneous of order $n/2$. The curvature defines an element $R \in \bigotimes^4(T(M))$ since it has 4 indices. There are, however, relations among the curvature variables:

$$R_{ijkl} = R_{klij}, \qquad R_{ijkl} = -R_{jikl}, \qquad \text{and} \qquad R_{ijkl} + R_{iklj} + R_{iljk} = 0.$$

We let V be the sub-bundle of $\bigotimes^4(T(M))$ consisting of tensors satisfying these 3 relations.

If $n = 2$, then $P: V \to \Lambda^2(T(M))$ is equivariant while more generally, $P: \bigotimes^{n/2}(V) \to \Lambda^n(T(M))$ is equivariant under the action of $\mathrm{O}(T(M))$. (We use the metric tensor to raise and lower indices and identify $T(M) = T^*(M)$). Since these relations define an $\mathrm{O}(T(M))$ invariant subspace of $\bigotimes^{2n}(T(M))$, we extend P to be zero on the orthogonal complement of $\bigotimes^{n/2}(V)$ in $\bigotimes^{2n}(T(M))$ to extend P to an equivariant action on the whole tensor algebra. Consequently, we can use H. Weyl's theorem to span $\mathcal{P}_{n,n,n}$ by expressions in which we alternate n indices and contract in pairs the remaining n indices.

For example, we compute:

$$p_1 = C \cdot R_{ijab} R_{ijcd} \, dx^a \wedge dx^b \wedge dx^c \wedge dx^d$$

for a suitable normalizing constant C represents the first Pontrjagin form. In general, we will use letters a, b, $c, \dots$ for indices to alternate on and indices i, j, $k, \dots$ for indices to contract on. We let P be such an element given by H. Weyl's theorem. There are some possibilities we can eliminate on a priori grounds. The Bianchi identity states:

$$R_{iabc} \, dx^a \wedge dx^b \wedge dx^c = \frac{1}{3}(R_{iabc} + R_{ibca} + R_{icab}) \, dx^a \wedge dx^b \wedge dx^c = 0$$

so that three indices of alternation never appear in any $R_{\dots}$ variable. Since there are $n/2$ R variables and n indices of alternation, this implies each R variable contains exactly two indices of alternation. We use the Bianchi identity again to express:

$$\begin{aligned} R_{iajb} \, dx^a \wedge dx^b &= \frac{1}{2}(R_{iajb} - R_{ibja}) \, dx^a \wedge dx^b \\ &= \frac{1}{2}(R_{iajb} + R_{ibaj}) \, dx^a \wedge dx^b \\ &= -\frac{1}{2} R_{ijba} \, dx^a \wedge dx^b = \frac{1}{2} R_{ijab} \, dx^a \wedge dx^b. \end{aligned}$$

This together with the other curvature identities means we can express P in terms of $R_{ijab}\,dx^a \wedge dx^b = \Omega_{ij}$ variables. Thus $P = P(\Omega_{ij})$ is a polynomial in the components of the curvature matrix where we regard $\Omega_{ij} \in \Lambda^2(T^*M)$. (This differs by various factors of 2 from our previous definitions, but this is irrelevant to the present argument). $P(\Omega)$ is an $\mathrm{O}(n)$ invariant polynomial and thus is a real characteristic form. This completes the proof.

The remainder of this subsection is devoted to giving a combinatorial proof of this lemma in the case $n = m = p$ independent of H. Weyl's theorem. Since the Pontrjagin forms span a subspace of dimension $\pi(k)$ if $n = 4k$ we must show $\mathcal{P}_{n,n,n} = 0$ if n is not divisible by 4 and that $\dim \mathcal{P}_{4k,4k,4k} \le \pi(k)$ since then equality must hold.

We showed $n = 2j$ is even and any polynomial depends on the 2-jets of the metric. We improve Lemma 2.5.1 as follows:

Lemma 2.5.8. *Let P satisfy the hypothesis of Lemma 2.5.1 and be a polynomial in the 2-jets of the metric. Then:*
(a) Let $A = g_{12/\alpha}A_0$ be a monomial of P. Either by interchanging 1 and 2 indices or by changing two 2 indices to 1 indices we can construct a monomial A_1 of the form $A_1 = g_{11/\beta}A_0'$ which is a monomial of P.
(b) Let $A = g_{ij/12}A_0$ be a monomial of P. Either by interchanging 1 and 2 indices or by changing two 2 indices to 1 indices we can construct a monomial of A_1 of the form $A_1 = g_{i'j'/11}A_0'$ which is a monomial of P.
(c) The monomial $A_1 \ne A$. If $\deg_1(A_1) = \deg_1(A)+2$ then $c(A,P)c(A_1,P) > 0$. Otherwise $\deg_1(A_1) \ne \deg_1(A)$ and $c(A,P)c(A_1,P) < 0$.

Proof: We shall prove (a) as (b) is the same; we will also verify (c). Let $B = g_{11/\alpha}A_0$ so $c(B, A(2 \to 1)) \ne 0$. Then $\deg_1(B) = \deg_1(A) + 1$. Apply Lemma 2.5.2 to find $A_1 \ne A$ so $c(B, A_1^{(1)}) \ne 0$. We noted earlier in the proof of 2.5.1 that A_1 must have the desired form. If $c(A_1, B(2 \to 1)) \ne 0$ then $\deg_1(A_1) = \deg_1(B)+1 = \deg_1(A)+2$ so A_1 is constructed from A by changing two 2 to 1 indices. Furthermore, $c(B, A_1^{(1)}) > 0$ and $c(B, A^{(1)}) < 0$ implies $c(A,P)c(A_1,P) > 0$. If, on the other hand, $c(A_1, B(1 \to 2)) \ne 0$ then $\deg_1(A_1) = \deg_1(B) - 1 = \deg_1(A)$ and A changes to A_1 by interchanging a 2 and a 1 index. Furthermore, $c(B, A_1^{(1)}) < 0$ and $c(B, A^{(1)}) < 0$ implies $c(A,P)c(A_1,P) < 0$ which completes the proof.

Let $P \in \mathcal{P}_{n,n,n}$ and express $P = P'\,dx^1 \wedge \cdots \wedge dx^n$. $P' = *P$ is a scalar invariant which changes sign if the orientation of the local coordinate system is reversed. We identify P with P' for notational convenience and henceforth regard P as a skew-invariant scalar polynomial. Thus $\deg_k(A)$ is odd for every k and every monomial A of P.

The indices with $\deg_k(A) = 1$ play a particularly important role in our discussion. We say that an index i touches an index j in the monomial A

if A is divisible by a variable $g_{ij/..}$ or $g_{../ij}$ where ".." indicate two indices which are not of interest.

LEMMA 2.5.9. *Let $P \in \mathcal{P}_{n,n,n}$ with $P \neq 0$. Then there exists a monomial A of P so*
(a) $\deg_k(A) \leq 3$ all k.
(b) If $\deg_k(A) = 1$ there exists an index $j(k)$ which touches k in A. The index $j(k)$ also touches itself in A.
(c) Let $\deg_j A + \deg_k A = 4$. Suppose the index j touches itself and the index k in A. There is a unique monomial A_1 different from A which can be formed from A by interchanging a j and k index. $\deg_j A_1 + \deg_k A_1 = 4$ and the index j touches itself and the index k in A_1. $c(A,P)+c(A_1,P) = 0$.

PROOF: Choose A so the number of indices with $\deg_k(A) = 1$ is minimal. Among all such choices, we choose A so the number of indices which touch themselves is maximal. Let $\deg_k(A) = 1$ then k touches some index $j = j(k) \neq k$ in A. Suppose A has the form $A = g_{jk/..}A_0$ as the other case is similar. Suppose first that $\deg_j(A) \geq 5$. Use lemma 2.5.8 to find $A_1 = g_{kk/..}A_0$. Then $\deg_k(A_1) \neq \deg_k(A)$ implies $\deg_k(A_1) = \deg_k(A) + 2 = 3$. Also $\deg_j(A_1) = \deg_j(A) - 2 \geq 3$ so A_1 has one less index with degree 1 which contradicts the minimality of the choice of A. We suppose next $\deg_j(A) = 1$. If T is the coordinate transformation interchanging x_j and x_k, then T reverses the orientation so $T^*P = -P$. However $\deg_j(A) = \deg_k(A) = 1$ implies $T^*A = A$ which contradicts the assumption that A is a monomial of P. Thus $\deg_j(A) = 3$ which proves (a).

Suppose j does not touch itself in A. We use Lemma 2.5.8 to construct $A_1 = g_{kk/..}A'_0$. Then $\deg_k(A_1) = \deg_k(A) + 2 = 3$ and $\deg_j(A_1) = \deg_j(A) - 2 = 1$. This is a monomial with the same number of indices of degree 1 but which has one more index (namely k) which touches itself. This contradicts the maximality of A and completes the proof of (b).

Finally, let $\mathcal{A}_\nu = \{\, k : \deg_k(A) = \nu \,\}$ for $\nu = 1,\ 3$. The map $k \mapsto j(k)$ defines an injective map from $\mathcal{A}_1 \to \mathcal{A}_3$ since no index of degree 3 can touch two indices of degree 1 as well as touching itself. The equalities:

$$n = \operatorname{card}(\mathcal{A}_1) + \operatorname{card}(\mathcal{A}_3) \quad \text{and} \quad 2n = \sum_k \deg_k(A) = \operatorname{card}(\mathcal{A}_1) + 3\operatorname{card}(\mathcal{A}_3)$$

imply $2\operatorname{card}(\mathcal{A}_3) = n$. Thus $\operatorname{card}(\mathcal{A}_1) = \operatorname{card}(\mathcal{A}_3) = n/2$ and the map $k \mapsto j(k)$ is bijective in this situation.

(c) follows from Lemma 2.5.8 where $j = 1$ and $k = 2$. Since $\deg_k(A) = 1$, A_1 cannot be formed by transforming two k indices to j indices so A_1 must be the unique monomial different from A obtained by interchanging these indices. For example, if $A = g_{jj/ab}g_{jk/cd}A_0$, then $A_1 = g_{jk/ab}g_{jj/cd}A_0$. The multiplicities involved are all 1 so we can conclude $c(A,P)+c(A_1,P) = 0$ and not just $c(A,P)c(A_1,P) < 0$.

Before further normalizing the choice of the monomial A, we must prove a lemma which relies on cubic changes of coordinates:

LEMMA 2.5.10. *Let P be a polynomial in the 2-jets of the metric invariant under changes of coordinates of the form $y_i = x_i + c_{jkl}x_jx_kx_l$. Then:*
(a) $g_{12/11}$ and $g_{11/12}$ divide no monomial A of P.
(b) Let $A = g_{23/11}A_0$ and $B = g_{11/23}A_0$ then A is a monomial of P if and only if B is a monomial of P. Furthermore, $c(A, P)$ and $c(B, P)$ have the same sign.
(c) Let $A = g_{11/22}A_0$ and $B = g_{22/11}A_0$, then A is a monomial of P if and only if B is a monomial of P. Furthermore, $c(A, P)$ and $c(B, P)$ have the same sign.

PROOF: We remark the use of the indices 1, 2, 3 is for notational convenience only and this lemma holds true for any triple of distinct indices. Since $g_{ij}(X, G) = \delta_{ij} + O(x^2)$, $g^{ij}{}_{/kl}(X, G)(x_0) = -g_{ij/kl}(X, G)(x_0)$. Furthermore, under changes of this sort, $d_y^\alpha(x_0) = d_x^\alpha(x_0)$ if $|\alpha| = 1$. We consider the change of coordinates:

$$\begin{aligned} y_2 &= x_2 + cx_1^3, & y_k &= k_x \quad \text{otherwise} \\ dy_2 &= dx_2 + 3cx_1^2\,dx_1, & dy_k &= dx_k \quad \text{otherwise} \end{aligned}$$

with

$$\begin{aligned} g^{12}(Y, G) &= g^{12}(X, G) + 3cx_1^2 + O(x^4), \\ g^{ij}(Y, G) &= g^{ij}(X, G) + O(x^4) \quad \text{otherwise} \\ g_{12/11}(Y, G)(x_0) &= g_{12/11}(X, G)(x_0) - 6c \\ g_{ij/kl}(Y, G)(x_0) &= g_{ij/kl}(X, G)(x_0) \quad \text{otherwise.} \end{aligned}$$

We decompose $A = (g_{12/11})^\nu A_0$. If $\nu > 0$, $T^*(A) = A - 6\nu c(g_{12/11})^{\nu-1}A_0 + O(c^2)$. Since $T^*(P) = P$, and since there is no way to cancel this additional contribution, A cannot be a monomial of P so $g_{12/11}$ divides no monomial of P.

Next we consider the change of coordinates:

$$\begin{aligned} y_1 &= x_1 + cx_1^2x_2, & y_k &= x_k \quad \text{otherwise} \\ dy_1 &= dx_1 + 2cx_1x_2\,dx_1 + cx_1^2\,dx_2, & dy_k &= dx_k \quad \text{otherwise} \end{aligned}$$

with

$$\begin{aligned} g_{11/12}(Y, G)(x_0) &= g_{11/12}(X, G)(x_0) - 4c \\ g_{12/11}(Y, G)(x_0) &= g_{12/11}(X, G)(x_0) - 2c \\ g_{ij/kl}(Y, G)(x_0) &= g_{ij/kl}(X, G)(x_0) \qquad \text{otherwise.} \end{aligned}$$

We noted $g_{12/11}$ divides no monomial of P. If $A = (g_{11/12})^\nu A_0$, then $\nu > 0$ implies $T^*(A) = A - 4cv(g_{11/12})^{\nu-1}A_0 + O(c^2)$. Since there would be no way to cancel such a contribution, A cannot be a monomial of P which completes the proof of (a).

Let A_0' be a monomial not divisible by any of the variables $g_{23/11}$, $g_{12/13}$, $g_{13/12}$, $g_{11/23}$ and let $A(p,q,r,s) = (g_{23/11})^p (g_{12/13})^q (g_{13/12})^r (g_{11/23})^s A_0'$. We set $c(p,q,r,s) = c(A(p,q,r,s), P)$ and prove (b) by establishing some relations among these coefficients. We first consider the change of coordinates,

$$\begin{aligned} y_2 &= x_2 + cx_1^2 x_3, & y_k &= x_k \quad \text{otherwise} \\ dy_2 &= dx_2 + 2cx_1x_3\,dx_1 + cx_1^2\,dx_3, & dy_k &= dx_k \quad \text{otherwise} \end{aligned}$$

with

$$\begin{aligned} g_{12/13}(Y,G)(x_0) &= g_{12/13}(X,G)(x_0) - 2c \\ g_{23/11}(Y,G)(x_0) &= g_{23/11}(X,G)(x_0) - 2c \\ g_{ij/kl}(Y,G)(x_0) &= g_{ij/kl}(X,G)(x_0) \qquad \text{otherwise.} \end{aligned}$$

We compute that:

$$\begin{aligned} &T^*(A(p,q,r,s)) \\ &\quad = A(p,q,r,s) + c\{-2qA(p,q-1,r,s) - 2pA(p-1,q,r,s)\} + O(c^2). \end{aligned}$$

Since $T^*(P) = P$ is invariant, we conclude

$$pc(p,q,r,s) + (q+1)c(p-1,q+1,r,s) = 0.$$

By interchanging the roles of 2 and 3 in the argument we also conclude:

$$pc(p,q,r,s) + (r+1)c(p-1,q,r+1,s) = 0.$$

(We set $c(p,q,r,s) = 0$ if any of these integers is negative.)

Next we consider the change of coordinates:

$$\begin{aligned} y_1 &= x_1 + cx_1x_2x_3, & y_k &= x_k \quad \text{otherwise} \\ dy_1 &= dx_1 + cx_1x_2\,dx_3 + cx_1x_3\,dx_2 + cx_2x_3\,dx_1, & dy_k &= dx_k \quad \text{otherwise} \end{aligned}$$

with

$$\begin{aligned} g_{11/23}(Y,G)(x_0) &= g_{11/23}(X,G)(x_0) - 2c \\ g_{12/13}(Y,G)(x_0) &= g_{12/13}(X,G)(x_0) - c \\ g_{13/12}(Y,G)(x_0) &= g_{13/12}(Y,G)(x_0) - c \end{aligned}$$

so that

$$\begin{aligned} T^*(A(p,q,r,s)) = A(p,q,r,s) &+ c\{-2sA(p,q,r,s-1) \\ &- rA(p,q,r-1,s) - qA(p,q-1,r,s)\} + O(c^2). \end{aligned}$$

This yields the identities

$$(q+1)c(p,q+1,r,s-1)+(r+1)c(p,q,r+1,s-1)+2sc(p,q,r,s)=0.$$

Let $p \neq 0$ and let $A = g_{23/11}A_0'$ be a monomial of P. Then $c(p-1,q+1,r,s)$ and $c(p-1,q,r+1,s)$ are non-zero and have the opposite sign as $c(p,q,r,s)$. Therefore $(q+1)c(p-1,q+1,r,s)+(r+1)c(p-1,q,r+1,s)$ is non-zero which implies $c(p-1,q,r,s+1)(s+1)$ is non-zero and has the same sign as $c(p,q,r,s)$. This shows $g_{11/23}A_0'$ is a monomial of P. Conversely, if A is not a monomial of P, the same argument shows $g_{11/23}A_0'$ is not a monomial of P. This completes the proof of (b).

The proof of (c) is essentially the same. Let A_0' be a monomial not divisible by the variables $\{g_{11/22}, g_{12/12}, g_{22/11}\}$ and let

$$A(p,q,r)=(g_{11/22})^p(g_{12/12})^q(g_{22/11})^r.$$

Let $c(p,q,r)=c(A(p,q,r),P)$. Consider the change of coordinates:

$$\begin{aligned} y_1 &= x_1 + cx_1x_2^2, & y_k &= x_k \quad \text{otherwise}\\ dy_1 &= dx_1 + cx_2^2\,dx_1 + 2cx_1x_2\,dx_2, & dy_k &= dx_k \quad \text{otherwise}\end{aligned}$$

with

$$\begin{aligned} g_{11/22}(Y,g)(x_0) &= g_{11/22}(X,G)(x_0) - 4c\\ g_{12/12}(Y,G)(x_0) &= g_{12/12}(X,G)(x_0) - 2c\\ g_{ij/kl}(Y,G)(x_0) &= g_{ij/kl}(X,G)(x_0) \qquad \text{otherwise.}\end{aligned}$$

This yields the relation $2pc(p,q,r)+(q+1)c(p-1,q+1,r)=0$. By interchanging the roles of 1 and 2 we obtain the relation $2rc(p,q,r)+(q+1)c(p,q+1,r-1)=0$ from which (c) follows.

This step in the argument is functionally equivalent to the use made of the $\{R_{ijkl}\}$ variables in the argument given previously which used H. Weyl's formula. It makes use in an essential way of the invariance of P under a wider group than just first and second order transformations. For the Euler form, by contrast, we only needed first and second order coordinate transformations.

We can now construct classifying monomials using these lemmas. Fix $n = 2n_1$ and let A be the monomial of P given by Lemma 2.5.9. By making a coordinate permutation, we may assume $\deg_k(A) = 3$ for $k \le n_1$ and $\deg_k(A) = 1$ for $k > n_1$. Let $x(i) = i + n_1$ for $1 \le i \le n_0$; we may assume the index I touches itself and $x(i)$ in A for $i \le n_0$.

We further normalize the choice of A as follows. Either $g_{11/ij}$ or $g_{ij/11}$ divides A. Since $\deg_1(A) = 3$ this term is not $g_{11/11}$ and by Lemma 2.5.10 it is not $g_{11/1x(1)}$ nor $g_{1x(1)/11}$. By Lemma 2.5.10(b) or 2.5.10(c), we may assume $A = g_{11/ij}\dots$ for i, $j \ge 2$. Since not both i and j can have degree

1 in A, by making a coordinate permutation if necessary we may assume that $i = 2$. If $j = 2$, we apply Lemma 2.5.9(c) to the indices 2 and $x(2)$ to perform an interchange and assume $A = g_{11/2x(2)} A_0$. The index 2 must touch itself elsewhere in A. We apply the same considerations to choose A in the form $A = g_{11/2x(2)} g_{22/ij} A_0$. If i or j is 1, the cycle closes and we express $A = g_{11/2x(2)} g_{11/1x(1)} A_0$ where $\deg_k(A_0) = 0$ for $k = 1, 2, 1 + n_1, 2 + n_1$. If $i, j > 2$ we continue this argument until the cycle closes. This permits us to choose A to be a monomial of P in the form:

$$A = g_{11/2x(2)} g_{22/3x(3)} \cdots g_{j-1,j-1/jx(j)} g_{jj/1x(1)} A_0$$

where $\deg_k(A_0) = 0$ for $1 \le k \le j$ and $n_1 + 1 \le k \le n_1 + j$.

We wish to show that the length of the cycle involved is even. We apply Lemma 2.5.10 to show

$$B = g_{2x(2)/11} g_{3x(3)/22} \cdots g_{1x(1)/jj} A_0$$

satisfies $c(A, P)c(B, P) > 0$. We apply Lemma 2.5.9 a total of j times to see

$$C = g_{22,1x(1)} g_{33/2x(2)} \cdots g_{11/jx(j)} A_0$$

satisfies $c(B, P)c(C, P)(-1)^j > 0$. We now consider the even permutation:

$$\begin{aligned} \rho(1) &= j \\ \rho(k) &= \begin{cases} k-1, & 2 \le k \le j \\ k, & j < k \le n_1 \end{cases} \\ \rho(k(j)) &= x(\rho(j)) \qquad \text{for } 1 \le j \le n_1 \end{aligned}$$

to see that $c(C_\rho, P)c(C, P) > 0$. However $A = C_\rho$ so $c(A, P)^2(-1)^j > 0$ which shows j is necessarily even. (This step is formally equivalent to using the skew symmetry of the curvature tensor to show that only polynomials of even degree can appear to give non-zero real characteristic forms).

We decompose A_0 into cycles to construct A inductively so that A has the form:

$$\begin{aligned} A = &\{g_{11/2x(2)} \cdots g_{i_1 i_1/1x(1)}\} \\ &\{g_{i_1+1,i_1+1/i_1+2,x(i_1+2)} \cdots g_{i_1+i_2,i_1+i_2/i_1,x(i_1)}\} \cdots \end{aligned}$$

where we decompose A into cycles of length $i_1, i_2, \ldots$ with $n_1 = i_1 + \cdots + i_j$. Since all the cycles must have even length, $\ell(A) = n/2$ is even so n is divisible by 4.

We let $n = 4k$ and let ρ be a partition of $k = k_1 + \cdots + k_j$. We let A_ρ be defined using the above equation where $i_1 = 2k_1$, $i_2 = 2k_2, \ldots$. By making a coordinate permutation we can assume $i_1 \ge i_2 \ge \cdots$. We have shown that if $P = 0$, then $c(A_\rho, P) = 0$ for some ρ. Since there are exactly $\pi(k)$ such partitions, we have constructed a family of $\pi(k)$ linear functionals on $\mathcal{P}_{n,n,n}$ which form a seperating family. This implies $\dim \mathcal{P}_{n,n,n} < \pi(k)$ which completes the proof.

We conclude this subsection with a few remarks on the proofs we have given of Theorem 2.4.7 and Lemma 2.5.6. We know of no other proof of Theorem 2.4.7 other than the one we have given. H. Weyl's theorem is only used to prove the surjectivity of the restriction map r and is inessential in the axiomatic characterization of the Euler form. This theorem gives an immediate proof of the Gauss-Bonnet theorem using heat equation methods. It is also an essential step in settling Singer's conjecture for the Euler form as we shall discuss in the fourth chapter. The fact that $r(E_m) = 0$ is, of course, just an invariant statement of the fact E_m is an unstable characteristic class; this makes it difficult to get hold of axiomatically in contrast to the Pontrjagin forms which are stable characteristic classes.

We have discussed both of the proofs of Lemma 2.5.6 which exist in the literature. The proof based on H. Weyl's theorem and on geodesic normal coordinates is more elegant, but relies heavily on fairly sophisticated theorems. The second is the original proof and is more combinatorial. It is entirely self-contained and this is the proof which generalizes to Kaehler geometry to yield an axiomatic characterization of the Chern forms of $T_c(M)$ for a holomorphic Kaehler manifold. We shall discuss this case in more detail in section 3.7.

2.6. Invariance Theory and Mixed Characteristic Classes of the Tangent Space and of a Coefficient Bundle.

In the previous subsection, we gave in Lemma 2.5.6 an axiomatic characterization of the Pontrjagin forms in terms of functorial properties. In discussing the Hirzebruch signature formula in the next chapter, it will be convenient to have a generalization of this result to include invariants which also depend on the derivatives of the connection form on an auxilary bundle.

Let V be a complex vector bundle. We assume V is equipped with a Hermitian fiber metric and let ∇ be a Riemannian or unitary connection on V. We let $\vec{s} = (s_1, \dots, s_a, \dots, s_v)$ be a local orthonormal frame for V and introduce variables ω_{abi} for the connection 1-form;

$$\nabla(s_a) = \omega_{abi}\, dx^i \otimes s_b, \qquad \text{i.e., } \nabla\vec{s} = \omega \otimes \vec{s}.$$

We introduce variables $\omega_{abi/\alpha} = d_x^\alpha(\omega_{abi})$ for the partial derivatives of the connection 1-form. We shall also use the notation $\omega_{abi/jk\dots}$. We use indices $1 \le a, b, \dots \le v$ to index the frame for V and indices $1 \le i, j, k \le m$ for the tangent space variables. We define:

$$\operatorname{ord}(\omega_{abi/\alpha}) = 1 + |\alpha| \qquad \text{and} \qquad \deg_k(\omega_{abi/\alpha}) = \delta_{i,k} + \alpha(k).$$

We let $\mathcal{Q}$ be the polynomial algebra in the $\{\omega_{abi/\alpha}\}$ variables for $|\alpha| \ge 1$; if $Q \in \mathcal{Q}$ we define the evaluation $Q(X, \vec{s}, \nabla)(x_0)$. We normalize the choice of frame $\vec{s}$ by requiring $\nabla(\vec{s})(x_0) = 0$. We also normalize the coordinate system X as before so $X(x_0) = 0$, $g_{ij}(X, G)(x_0) = \delta_{ij}$, and $g_{ij/k}(X, G)(x_0) = 0$. We say Q is invariant if $Q(X, \vec{s}, \nabla)(x_0) = Q(Y, \vec{s}\,', \nabla)(x_0)$ for any normalized frames $\vec{s}$, $\vec{s}\,'$ and normalized coordinate systems X, Y; we denote this common value by $Q(\nabla)$ (although it also depends in principle on the metric tensor and the 1-jets of the metric tensor through our normalization of the coordinate system X). We let $\mathcal{Q}_{m,p,v}$ denote the space of all invariant p-form valued polynomials in the $\{\omega_{abi/\alpha}\}$ variables for $|\alpha| \ge 1$ defined on a manifold of dimension m and for a vector bundle of complex fiber dimension v. We let $\mathcal{Q}_{m,n,p,v}$ denote the subspace of invariant polynomials homogeneous of order n in the jets of the connection form. Exactly as was done for the $\mathcal{P}_*$ algebra in the jets of the metric, we can show there is a direct sum decomposition

$$\mathcal{Q}_{m,p,v} = \bigoplus_n \mathcal{Q}_{m,n,p,v} \qquad \text{and} \qquad \mathcal{Q}_{m,n,p,v} = 0 \qquad \text{for } n - p \text{ odd}.$$

Let $Q(gAg^{-1}) = Q(A)$ be an invariant polynomial of order q in the components of a $v \times v$ matrix. Then $Q(\Omega)$ defines an element of $\mathcal{Q}_{m,2q,q,v}$ for any $m \ge 2q$. By taking $Q \cdot \bar{Q}$ we can define scalar valued invariants and by taking $\delta(Q)$ we can define other form valued invariants in $\mathcal{Q}_{m,2q+1,2q-1,v}$. Thus there are a great many such form valued invariants.

In addition to this algebra, we let $\mathcal{R}_{m,n,p,v}$ denote the space of p-form valued invariants which are homogeneous of order n in the $\{g_{ij/\alpha}, \omega_{abk/\beta}\}$ variables for $|\alpha| \ge 2$ and $|\beta| \ge 1$. The spaces $\mathcal{P}_{m,n,p}$ and $\mathcal{Q}_{m,n,p,v}$ are both subspaces of $\mathcal{R}_{m,n,p,v}$. Furthermore wedge product gives a natural map $\mathcal{P}_{m,n,p} \otimes \mathcal{Q}_{m,n',p',v} \to \mathcal{R}_{m,n+n',p+p',v}$. We say that $R \in \mathcal{R}_{m,p,p,v}$ is a characteristic form if it is in the linear span of wedge products of Pontrjagin forms of $T(M)$ and Chern forms of V. The characteristic forms are characterized abstractly by the following Theorem. This is the generalization of Lemma 2.5.6 which we shall need in discussing the signature and spin complexes.

THEOREM 2.6.1.
(a) $\mathcal{R}_{m,n,p,v} = 0$ *if* $n < p$ *or if* $n = p$ *and* n *is odd.*
(b) If $R \in \mathcal{R}_{m,n,n,v}$ *then* R *is a characteristic form.*

PROOF: The proof of this fact relies heavily on Lemma 2.5.6 but is much easier. We first need the following generalization of Lemma 2.5.1:

LEMMA 2.6.2. *Using the notation of Lemma 2.5.1 we define* $T_z^*(R)$ *if* R *is a scalar invariant in the* $\{g_{ij/\alpha}, \omega_{abj/\beta}\}$ *variables.*
(a) Let $g_{12/\alpha}$ *divide some monomial of* R*, then* $g_{../\beta}$ *divides some other monomial of* R.
(b) Let $g_{../\alpha}$ *divide some monomial of* R*, then* $g_{../\beta}$ *divides some other monomial of* R *where* $\beta(1) = \alpha(1) + \alpha(2)$ *and* $\beta(2) = 0$.
(c) Let $\omega_{abi/\alpha}$ *divide some monomial of* R *for* $i > 2$*, then* $\omega_{abi/\beta}$ *divides some other monomial of* R *where* $\beta(1) = \alpha(1) + \alpha(2)$ *and* $\beta(2) = 0$.

The proof of this is exactly the same as that given for Lemma 2.5.1 and is therefore omitted.

The proof of Theorem 2.6.1 parallels the proof of Lemma 2.5.6 for a while so we summarize the argument briefly. Let $0 \ne R \in \mathcal{R}_{m,n,p,v}$. The same argument given in Lemma 2.5.3 shows an invariant polynomial is homogeneous of order n if $R(c^2 G, \nabla) = c^{p-n} R(G, \nabla)$ which gives a invariant definition of the order of a polynomial. The same argument as given in Lemma 2.5.4 shows $n - p$ must be even and that if A is a monomial of R, A is a monomial of exactly one of the R_I. If $c(A, R_I) = 0$ then $\deg_k(A) + \deg_k(I)$ is always even. We decompose A in the form:

$$A = g_{i_1 j_1/\alpha_1} \dots g_{i_q j_q/\alpha_q} \omega_{a_1 b_1 k_1/\beta_1} \dots \omega_{a_r b_r k_r/\beta_r} = A^g A^\omega$$

and define $\ell(A) = q + r$ to the length of A. We argue using Lemma 2.6.2 to choose A so $\deg_k(A^g) = 0$ for $k > 2q$. By making a coordinate permutation we can assume that the $k_\nu \le 2q + r$ for $1 \le \nu \le r$. We apply Lemma 2.6.2(c) a total of r times to choose the β_i so $\beta_1(k) = 0$ for $k > 2q+r+1$, $\beta_2(k) = 0$ for $k > 2q+r+2, \ldots$, $\beta_r(k) = 0$ for $k > 2q+2r$. This chooses A so $\deg_k(A) = 0$ for $k > 2\ell(A)$. If A is a monomial of R_I for $I = \{1 \le i_1 < \cdots < i_p \le m\}$ then $\deg_{i_p}(A)$ is odd. We estimate $p \le i_p \le 2\ell(A) \le \sum |\alpha_\nu| + \sum(|\beta_\mu|+1) = n$ so that $\mathcal{P}_{m,n,p,v} = \{0\}$ if $n < p$ or if $n - p$ is odd which proves (a) of Theorem 2.6.1.

In the limiting case, we must have equalities so $|\alpha_\nu| = 2$ and $|\beta_\mu| = 1$. Furthermore, $i_p = p$ so there is some monomial A so $\deg_k(A) = 0$ for $k > p = n$ and $A\,dx^1 \wedge \cdots \wedge dx^p$ appears in R. There is a natural restriction map

$$r\colon \mathcal{R}_{m,n,p,v} \to \mathcal{R}_{m-1,n,p,v}$$

and our argument shows $r\colon \mathcal{R}_{m,n,n,v} \to \mathcal{R}_{m-1,n,n,v}$ is injective for $n < m$. Since the restriction of a characteristic form is a characteristic form, it suffices to prove (b) of Theorem 2.6.1 for the case $m = n = p$.

Let $0 \ne R \in \mathcal{R}_{n,n,n,v}$ then R is a polynomial in the $\{g_{ij/kl}, \omega_{abi/j}\}$ variables. The restriction map r was defined by considering products $M_1 \times S^1$ but there are other functorial constructions which give rise to useful projections. Fix non-negative integers (s,t) so that $n = s + t$. Let M_1 be a Riemannian manifold of dimension s. Let M_2 be the flat torus of dimension t and let V_2 be a vector bundle with connection ∇_2 over M_2. Let $M = M_1 \times M_2$ with the product metric and let V be the natural extension of V_2 to M which is flat in the M_1 variables. More exactly, if $\pi_2\colon M \to M_2$ is a projection on the second factor, then $(V,\nabla) = \pi_2^*(V_2,\nabla_2)$ is the pull back bundle with the pull back connection. We define

$$\pi_{(s,t)}(R)(G_1,\nabla_2) = R(G_1 \times 1, \nabla).$$

Using the fact that $\mathcal{P}_{s,n_1,p_1} = 0$ for $s < p_1$ or $n_1 < p_1$ and the fact $\mathcal{Q}_{t,n_2,p_2,k} = 0$ for $t < p_2$ or $n_2 < p_2$ it follows that $\pi_{(s,t)}$ defines a map

$$\pi_{(s,t)}\colon \mathcal{R}_{n,n,v} \to \mathcal{P}_{s,s,s} \otimes \mathcal{Q}_{t,t,t,v}.$$

More algebraically, let $A = A^g A^\omega$ be a monomial, then we define:

$$\pi_{(s,t)}(A) = \begin{cases} 0 & \text{if } \deg_k(A^g) > 0 \text{ for } k > s \text{ or } \deg_k(A^\omega) > 0 \text{ for } k \le s \\ A & \text{otherwise.} \end{cases}$$

The only additional relations imposed are to set $g_{ij/kl} = 0$ if any of these indices exceeds s and to set $\omega_{abi/j} = 0$ if either i or j is less than or equal to s.

We use these projections to reduce the proof of Theorem 2.6.1 to the case in which $R \in \mathcal{Q}_{t,t,t,v}$. Let $0 \ne R \in \mathcal{R}_{n,n,n,v}$ and let $A = A^g A^\omega$ be a monomial of R. Let $s = 2\ell(A^g) = \operatorname{ord}(A^g)$ and let $t = n - s = 2\ell(A^\omega) =$

$\operatorname{ord}(A^\omega)$. We choose A so $\deg_k(A^g) = 0$ for $k > s$. Since $\deg_k(A) \geq 1$ must be odd for each index k, we can estimate:

$$t \leq \sum_{k>s} \deg_k(A) = \sum_{k>s} \deg_k(A^\omega) \leq \sum_k \deg_k(A^\omega) = \operatorname{ord}(A^\omega) = t.$$

As all these inequalities must be equalities, we conclude $\deg_k(A^\omega) = 0$ for $k \leq s$ and $\deg_k(A^\omega) = 1$ for $k > s$. This shows in particular that $\pi_{(s,t)}(R) \neq 0$ for some (s,t) so that

$$\bigoplus_{s+t=n} \pi_{(s,t)} \colon \mathcal{R}_{n,n,n,v} \to \bigoplus_{s+t=n} \mathcal{P}_{s,s,s} \otimes \mathcal{Q}_{t,t,t,v}$$

is injective.

We shall prove that $\mathcal{Q}_{t,t,t,v}$ consists of characteristic forms of V. We showed earlier that $\mathcal{P}_{s,s,s}$ consists of Pontrjagin forms of $T(M)$. The characteristic forms generated by the Pontrjagin forms of $T(M)$ and of V are elements of $\mathcal{R}_{n,n,n,v}$ and $\pi_{(s,t)}$ just decomposes such products. Thererfore π is surjective when restricted to the subspace of characteristic forms. This proves π is bijective and also that $\mathcal{R}_{n,n,n,v}$ is the space of characteristic forms. This will complete the proof of Theorem 2.6.1.

We have reduced the proof of Theorem 2.6.1 to showing $\mathcal{Q}_{t,t,t,v}$ consists of the characteristic forms of V. We noted that $0 \neq Q \in \mathcal{Q}_{t,t,t,v}$ is a polynomial in the $\{\omega_{abi/j}\}$ variables and that if A is a monomial of Q, then $\deg_k(A) = 1$ for $1 \leq k \leq t$. Since $\operatorname{ord}(A) = t$ is even, we conclude $\mathcal{Q}_{t,t,t,v} = 0$ if t is odd.

The components of the curvature tensor are given by:

$$\Omega_{abij} = \omega_{abi/j} - \omega_{abi/j} \qquad \text{and} \qquad \Omega_{ab} = \sum \Omega_{abij}\, dx_i \wedge dx_j$$

up to a possible sign convention and factor of $\frac{1}{2}$ which play no role in this discussion. If A is a monomial of P, we decompose:

$$A = \omega_{a_1 b_1 i_1/i_2} \dots \omega_{a_u b_u i_{t-1}/i_t} \qquad \text{where } 2u = t.$$

All the indices i_ν are distinct. If ρ is a permutation of these indices, then $c(A,P) = \operatorname{sign}(\rho) c(A^\rho, P)$. This implies we can express P in terms of the expressions:

$$\begin{aligned} \bar{A} &= (\omega_{a_1 b_1 i_1/i_2} - \omega_{a_1 b_1 i_2/i_1}) \dots (\omega_{a_u b_u i_{t-1}/i_t} - \omega_{a_u b_u i_t/i_{t-1}}) \\ &\qquad\qquad\qquad\qquad\qquad dx_{i_1} \wedge \dots \wedge dx_{i_t} \\ &= \Omega_{a_1 b_1 i_1 i_2} \dots \Omega_{a_u b_u i_{t-1} i_t}\, dx_{i_1} \wedge \dots \wedge dx_{i_t} \end{aligned}$$

Again, using the alternating nature of these expression, we can express P in terms of expressions of the form:

$$\Omega_{a_1 b_1} \wedge \dots \wedge \Omega_{a_u b_u}$$

so that $Q = Q(\Omega)$ is a polynomial in the components Ω_{ab} of the curvature. Since the value of Q is independent of the frame chosen, Q is the invariant under the action of $\mathrm{U}(v)$. Using the same argument as that given in the proof of Lemma 2.1.3 we see that in fact Q is a characteristic form which completes the proof.

We conclude this subsection with some uniqueness theorems regarding the local formulas we have been considering. If M is a holomorphic manifold of real dimension m and if V is a complex vector bundle of fiber dimension v, we let $\mathcal{R}^{ch}_{m,m,p,v}$ denote the space of p-form valued invariants generated by the Chern forms of V and of $T_c(M)$. There is a suitable axiomatic characterization of these spaces using invariance theory for Kaehler manifolds which we shall discuss in section 3.7. The uniqueness result we shall need in proving the Hirzebruch signature theorem and the Riemann-Roch theorem is the following:

LEMMA 2.6.3.
(a) Let $0 \neq R \in \mathcal{R}_{m,m,m,v}$ then there exists (M, V) so M is oriented and $\int_M R(G, \nabla) \neq 0$.
(b) Let $0 \neq R \in \mathcal{R}^{ch}_{m,m,m,v}$ then there exists (M, V) so M is a holomorphic manifold and $\int_M R(G, \nabla) \neq 0$.

PROOF: We prove (a) first. Let $\rho = \{1 \leq i_1 \leq \cdots \leq i_\nu\}$ be a partition $k(\rho) = i_1 + \cdots + i_\nu$ for $4k = s \leq m$. Let $\{M^r_\rho\}$ be the collection of manifolds of dimension s discussed in Lemma 2.3.4. Let P_ρ be the corresponding real characteristic form so

$$\int_{M^r_\rho} P_\tau(G) = \delta_{\rho,\tau}.$$

The $\{P_\rho\}$ forms a basis for $\mathcal{P}_{m,s,s}$ for any $s \leq m$. We decompose

$$R = \sum_\rho P_\rho Q_\rho \qquad \text{for } Q_\rho \in \mathcal{Q}_{m,t,t,k}, \text{ where } t + s = m.$$

This decomposition is, of course, nothing but the decomposition defined by the projections $\pi_{(s,t)}$ discussed in the proof of the previous lemma.

Since $R \neq 0$, at least one of the $Q_\rho \neq 0$. We choose ρ so $k(\rho)$ is maximal with $Q_\rho \neq 0$. We consider $M = M^r_\rho \times M_2$ and $(V, \nabla) = \pi_2^*(V_2, \nabla_2)$ where (V_2, ∇_2) is a bundle over M_2 which will be specified later. Then we compute:

$$\int_M P_\tau(G) Q_\tau(\nabla) = 0$$

unless $\operatorname{ord}(Q_\tau) \leq \dim(M_2)$ since ∇ is flat along M^r_ρ. This implies $k(\tau) \geq k(\rho)$ so if this integral is non-zero $k(\tau) = k(\rho)$ by the maximality of ρ. This implies

$$\int_M P_\tau(G) Q_\tau(\nabla) = \int_{M^r_\rho} P_\tau(G) \cdot \int_{M_2} Q_\tau(\nabla_2) = \delta_{\rho,\tau} \int_{M_2} Q_\tau(\nabla_2).$$

This shows that

$$\int_M R(G,\nabla) = \int_{M_2} Q_\rho(\nabla_2)$$

and reduces the proof of this lemma to the special case $Q \in \mathcal{Q}_{m,m,m,v}$.

Let A be a $v \times v$ complex matrix and let $\{x_1, \dots, x_v\}$ be the normalized eigenvalues of A. If $2k = m$ and if ρ is a partition of k we define

$$x_\rho = x_1^{i_1} \dots x_\nu^{i_\nu},$$

then x_ρ is a monomial of $Q(A)$ for some ρ. We let $M = M_1 \times \cdots \times M_\nu$ with $\dim(M_j) = 2i_j$ and let $V = L_1 \oplus \cdots \oplus L_\nu \oplus 1^{v-\nu}$ where the L_j are line bundles over M_j. If $c(x_\rho, Q)$ is the coefficient of x in Q, then:

$$\int_M Q(\nabla) = c(x_\rho, Q) \prod_{j=1}^{\nu} \int_{M_j} c_1(L_j)^{i_j}.$$

This is, of course, nothing but an application of the splitting principle. This reduces the proof of this lemma to the special case $Q \in \mathcal{Q}_{m,m,m,1}$.

If $Q = c_1^k$ we take $M = S^2 \times \cdots \times S^2$ to be the k-fold product of two dimensional spheres. We let L_j be a line bundle over the j^{th} factor of S^2 and let $L = L_1 \otimes \cdots \otimes L_k$. Then $c_1(L) = c_1(L_j)$ so

$$\int_M c_1(L)^k = k! \prod_{j=1}^{k} \int_{S^2} c_1(L_j)$$

which reduces the proof to the case $m = 2$ and $k = 1$. We gave an example in Lemma 2.1.5 of a line bundle over S^2 so $\int_{S^2} c_1(L) = 1$. Alternatively, if we use the Gauss-Bonnet theorem with $L = T_c(S^2)$, then $\int_{S^2} c_1(T_c(S^2)) = \int_{S^2} e_2(T(S^2)) = \chi(S^2) = 2 \neq 0$ completes the proof of (a). The proof of (b) is the same where we replace the real manifolds M_ρ^r by the corresponding manifolds M_ρ^c and the real basis P_ρ by the corresponding basis P_ρ^c of characteristic forms of $T_c(M)$. The remainder of the proof is the same and relies on Lemma 2.3.4 exactly as for (a) and is therefore omitted in the interests of brevity.