CHAPTER 1

PSEUDO-DIFFERENTIAL OPERATORS

Introduction

In the first chapter, we develop the analysis needed to define the index of an elliptic operator and to compute the index using heat equation methods. Sections 1.1 and 1.2 are brief reviews of Sobolev spaces and pseudo-differential operators on Euclidean spaces. In section 1.3, we transfer these notions to compact Riemannian manifolds using partition of unity arguments. In section 1.4 we review the facts concerning Fredholm operators needed in section 1.5 to prove the Hodge decomposition theorem and to discuss the spectral theory of self-adjoint elliptic operators. In section 1.6 we introduce the heat equation and in section 1.7 we derive the local formula for the index of an elliptic operator using heat equation methods. Section 1.8 generalizes the results of section 1.7 to find a local formula for the Lefschetz number of an elliptic complex. In section 1.9, we discuss the index of an elliptic operator on a manifold with boundary and in section 1.10, we discuss the zeta and eta invariants.

Sections 1.1 and 1.4 review basic facts we need, whereas sections 1.8 through 1.10 treat advanced topics which may be omitted from a first reading. We have attempted to keep this chapter self-contained and to assume nothing beyond a first course in analysis. An exception is the de Rham theorem in section 1.5 which is used as an example.

A number of people have contributed to the mathematical ideas which are contained in the first chapter. We were introduced to the analysis of sections 1.1 through 1.7 by a course taught by L. Nirenberg. Much of the organization in these sections is modeled on his course. The idea of using the heat equation or the zeta function to compute the index of an elliptic operator seems to be due to R. Bott. The functional calculus used in the study of the heat equation contained in section 1.7 is due to R. Seeley as are the analytic facts on the zeta and eta functions of section 1.10.

The approach to Lefschetz fixed point theorems contained in section 1.8 is due to T. Kotake for the case of isolated fixed points and to S. C. Lee and the author in the general case. The analytic facts for boundary value problems discussed in section 1.9 are due to P. Greiner and R. Seeley.

1.1. Fourier Transform, Schwartz Class, And Sobolev Spaces.

The Sobolev spaces and Fourier transform provide the basic tools we shall need in our study of elliptic partial differential operators. Let $x = (x_1, \dots, x_m) \in \mathbf{R}^m$. If $x, y \in \mathbf{R}^m$, we define:

$$x \cdot y = x_1 y_1 + \cdots + x_m y_m \qquad \text{and} \qquad |x| = (x \cdot x)^{1/2}$$

as the Euclicean dot product and length. Let $\alpha = (\alpha_1, \dots, \alpha_m)$ be a multi-index. The α_j are non-negative integers. We define:

$$|\alpha| = \alpha_1 + \cdots + \alpha_m\,, \qquad \alpha! = \alpha_1! \dots \alpha_m!\,, \qquad x^\alpha = x_1^{\alpha_1} \dots x_m^{\alpha_m}.$$

Finally, we define:

$$d_x^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_m}\right)^{\alpha_m} \qquad \text{and} \qquad D_x^\alpha = (-i)^{|\alpha|} d_x^\alpha$$

as a convenient notation for multiple partial differentiation. The extra factors of $(-i)$ defining D_x^α are present to simplify later formulas. If $f(x)$ is a smooth complex valued function, then Taylor's theorem takes the form:

$$f(x) = \sum_{|\alpha| \le n} d_x^\alpha f(x_0) \frac{(x - x_0)^\alpha}{\alpha!} + O(|x - x_0|^{n+1}).$$

The Schwartz class $\mathcal{S}$ is the set of all smooth complex valued functions f on $\mathbf{R}^m$ such that for all α, β there are constants $C_{\alpha,\beta}$ such that

$$|x^\alpha D_x^\beta f| \le C_{\alpha,\beta}.$$

This is equivalent to assuming there exist estimates of the form:

$$|D_x^\beta f| \le C_{n,\beta}(1 + |x|)^{-n}$$

for all (n, β). The functions in $\mathcal{S}$ have all their derivatives decreasing faster at ∞ than the inverse of any polynomial.

For the remainder of Chapter 1, we let dx, dy, $d\xi$, etc., denote Lebesgue measure on $\mathbf{R}^m$ with an additional normalizing factor of $(2\pi)^{-m/2}$. With this normalization, the integral of the Gaussian distribution becomes:

$$\int e^{-\frac{1}{2}|x|^2}\, dx = 1.$$

We absorb the normalizing constant into the measure in order to simplify the formulas of the Fourier transform. If $C_0^\infty(\mathbf{R}^m)$ denotes the set of smooth functions of compact support on $\mathbf{R}^m$, then this is a subset of $\mathcal{S}$. Since $C_0(\mathbf{R}^m)$ is dense in $L^2(\mathbf{R}^m)$, $\mathcal{S}$ is dense in $L^2(\mathbf{R}^m)$.

We define the convolution product of two elements of $\mathcal{S}$ by:

$$(f * g)(x) = \int f(x - y) g(y)\, dy = \int f(y) g(x - y)\, dy.$$

This defines an associative and commutative multiplication. Although there is no identity, there do exist approximate identities:

Lemma 1.1.1. *Let $f \in \mathcal{S}$ with $\int f(x)\,dx = 1$. Define $f_u(x) = u^{-m} f(\frac{x}{u})$. Then for any $g \in \mathcal{S}$, $f_u * g$ converges uniformly to g as $u \to 0$.*

Proof: Choose C so $\int |f(x)|\,dx \le C$ and $|g(x)| \le C$. Because the first derivatives of g are uniformly bounded, g is uniformly continuous. Let $\varepsilon > 0$ and choose $\delta > 0$ so $|x-y| \le \delta$ implies $|g(x) - g(y)| \le \varepsilon$. Because $\int f_u(x)\,dx = 1$, we compute:

$$\begin{aligned} |f_u * g(x) - g(x)| &= \left| \int f_u(y)\{g(x-y) - g(x)\}\,dy \right| \\ &\le \int |f_u(y)\{g(x-y) - g(x)\}|\,dy. \end{aligned}$$

We decompose this integral into two pieces. If $|y| \le \delta$ we bound it by $C\varepsilon$. The integral for $|y| \ge \delta$ can be bounded by:

$$2C \int_{|y|\ge\delta} |f_u(y)|\,dy = 2C \int_{|y|\ge\delta/u} |f(y)|\,dy.$$

This converges to zero as $u \to 0$ so we can bound this by $C\varepsilon$ if $u < u(\varepsilon)$. This completes the proof.

A similar convolution smoothing can be applied to approximate any element of L^p arbitrarily well in the L^p norm by a smooth function of compact support.

We define the Fourier transform $\hat{f}(\xi)$ by:

$$\hat{f}(\xi) = \int e^{-ix\cdot\xi} f(x)\,dx \qquad \text{for } f \in \mathcal{S}.$$

For the moment $\xi \in \mathbf{R}^m$; when we consider operators on manifolds, it will be natural to regard ξ as an element of the fiber of the cotangent space. By integrating by parts and using Lebesgue dominated convergence, we compute:

$$D_\xi^\alpha\{\hat{f}(\xi)\} = (-1)^{|\alpha|}\{\widehat{x^\alpha f}\} \qquad \text{and} \qquad \xi^\alpha \hat{f}(\xi) = \{\widehat{D_x^\alpha f}\}.$$

This implies $\hat{f} \in \mathcal{S}$ so Fourier transform defines a map $\mathcal{S} \to \mathcal{S}$.

We compute the Fourier transform of the Gaussian distribution. Let $f_0(x) = \exp(-\frac{1}{2}|x|^2)$, then $f_0 \in \mathcal{S}$ and $\int f_0(x)\,dx = 1$. We compute:

$$\begin{aligned} \hat{f}_0(\xi) &= \int e^{-ix\cdot\xi} e^{-\frac{1}{2}|x|^2}\,dx \\ &= e^{-\frac{1}{2}|\xi|^2} \int e^{-(x+i\xi)\cdot(x+i\xi)/2}\,dx. \end{aligned}$$

We make a change of variables to replace $x + i\xi$ by x and to shift the contour in $\mathbf{C}^m$ back to the original contour $\mathbf{R}^m$. This shows the integral is 1 and $\hat{f}_0(\xi) = \exp\left(-\frac{1}{2}|\xi|^2\right)$ so the function f_0 is its own Fourier transform.

In fact, the Fourier transform is bijective and the Fourier inversion formula gives the inverse expressing f in terms of $\hat{f}$ by:

$$f(x) = \int e^{ix\cdot\xi}\,\hat{f}(\xi)\,d\xi = \hat{\hat{f}}(-x).$$

We define $T(f) = \hat{\hat{f}}(-x) = \int e^{ix\cdot\xi}\,\hat{f}(\xi)\,d\xi$ as a linear map from $\mathcal{S} \to \mathcal{S}$. We must show that $T(f) = f$ to prove the Fourier inversion formula.

Suppose first $f(0) = 0$. We expand:

$$f(x) = \int_0^1 \frac{d}{dt}\{f(tx)\}\,dt = \sum_j x_j \int_0^1 \frac{\partial f}{\partial x_j}(tx)\,dt = \sum_j x_j g_j$$

where the g_j are smooth. Let $\phi \in C_0^\infty(\mathbf{R}^m)$ be identically 1 near $x = 0$. Then we decompose:

$$f(x) = \phi f(x) + (1-\phi)f(x) = \sum_j x_j \phi g_j + \sum_j x_j \left\{\frac{x_j(1-\phi)f}{|x|^2}\right\}.$$

Since ϕg_j has compact support, it is in $\mathcal{S}$. Since ϕ is identically 1 near $x = 0$, $x_j(1-\phi)f/|x|^2 \in \mathcal{S}$. Thus we can decompose $f = \sum x_j h_j$ for $h_j \in \mathcal{S}$. We Fourier transform this identity to conclude:

$$\hat{f} = \sum_j \{\widehat{x_j h_j}\} = \sum_j i\frac{\partial \hat{h}_j}{\partial \xi_j}.$$

Since this is in divergence form, $T(f)(0) = \int \hat{f}(\xi)\,d\xi = 0 = f(0)$.

More generally, let $f \in \mathcal{S}$ be arbitrary. We decompose $f = f(0)f_0 + (f - f(0)f_0)$ for $f_0 = \exp(-\frac{1}{2}|x|^2)$. Since $\hat{f}_0 = f_0$ is an even function, $T(f_0) = f_0$ so that $T(f)(0) = f(0)f_0(0) + T(f - f(0)f_0) = f(0)f_0(0) = f(0)$ since $(f - f(0)f_0)(0) = 0$. This shows $T(f)(0) = f(0)$ in general.

We use the linear structure on $\mathbf{R}^m$ to complete the proof of the Fourier inversion formula. Let $x_0 \in \mathbf{R}^m$ be fixed. We let $g(x) = f(x + x_0)$ then:

$$\begin{aligned} f(x_0) = g(0) = T(g)(0) &= \int e^{-ix\cdot\xi}\,f(x + x_0)\,dx\,d\xi \\ &= \int e^{-ix\cdot\xi}\,e^{ix_0\cdot\xi}\,f(x)\,dx\,d\xi \\ &= T(f)(x_0). \end{aligned}$$

This shows the Fourier transform defines a bijective map $\mathcal{S} \to \mathcal{S}$. If we use the constants $C_{\alpha,\beta} = \sup_{x\in\mathbf{R}^m} |x^\alpha D_x^\beta f|$ to define a Frechet structure on $\mathcal{S}$, then the Fourier transform is a homeomorphism of topological vector spaces. It is not difficult to show $C_0^\infty(\mathbf{R}^m)$ is a dense subset of $\mathcal{S}$ in this topology. We can use either pointwise multiplication or convolution to define a multiplication on $\mathcal{S}$ and make $\mathcal{S}$ into a ring. The Fourier transform interchanges these two ring structures. We compute:

$$\begin{aligned}\hat{f}\cdot\hat{g} &= \int e^{-ix\cdot\xi} f(x) e^{-iy\cdot\xi} g(y)\,dx\,dy\\ &= \int e^{-i(x-y)\cdot\xi} f(x-y) e^{-iy\cdot\xi} g(y)\,dx\,dy\\ &= \int e^{-ix\cdot\xi} f(x-y) g(y)\,dx\,dy.\end{aligned}$$

The integral is absolutely convergent so we may interchange the order of integration to compute $\hat{f}\cdot\hat{g} = (\widehat{f*g})$. If we replace f by $\hat{f}$ and g by $\hat{g}$ we see $(f\cdot g)(-x) = (\widehat{\hat{f}*\hat{g}})$ using the Fourier inversion formula. We now take the Fourier transform and use the Fourier inversion formula to see $(\widehat{f\cdot g})(-\xi) = (\hat{f}*\hat{g})(-\xi)$ so that $(\widehat{f\cdot g}) = \hat{f}*\hat{g}$.

The final property we shall need of the Fourier transform is related to the L^2 inner product $(f,g) = \int f(x)\overline{g}(x)\,dx$. We compute:

$$\begin{aligned}(\hat{f},g) &= \int f(x)e^{-ix\cdot\xi}\overline{g}(\xi)\,dx\,d\xi = \int f(x)e^{-ix\cdot\xi}\overline{g}(\xi)\,d\xi\,dx\\ &= (f,\hat{g}(-x)).\end{aligned}$$

If we replace g by $\hat{g}$ then $(\hat{f},\hat{g}) = (f,\hat{\hat{g}}(-x)) = (f,g)$ so the Fourier transform is an isometry with respect to the L^2 inner product. Since $\mathcal{S}$ is dense in L^2, it extends to a unitary map $L^2(\mathbf{R}^m) \to L^2(\mathbf{R}^m)$. We summarize these properties of the Fourier transform as follows:

LEMMA 1.1.2. *The Fourier transform is a homeomorphism $\mathcal{S} \to \mathcal{S}$ such that:*
(a) $f(x) = \int e^{ix\cdot\xi}\hat{f}(\xi)\,d\xi = \int e^{i(x-y)\cdot\xi} f(y)\,dy\,d\xi$ (Fourier inversion formula);
(b) $D_x^\alpha f(x) = \int e^{ix\cdot\xi}\xi^\alpha\hat{f}(\xi)\,d\xi$ and $\xi^\alpha\hat{f}(\xi) = \int e^{-ix\cdot\xi} D_x^\alpha f(x)\,dx$;
(c) $\hat{f}\cdot\hat{g} = (\widehat{f*g})$ and $\hat{f}*\hat{g} = (\widehat{f\cdot g})$;
(d) The Fourier transform extends to a unitary map of $L^2(\mathbf{R}^m) \to L^2(\mathbf{R}^m)$ such that $(f,g) = (\hat{f},\hat{g})$. (Plancherel theorem).

We note that without the normalizing constant of $(2\pi)^{-m/2}$ in the definition of the measures dx and $d\xi$ there would be various normalizing constants appearing in these identities. It is property (b) which will be of the most interest to us since it will enable us to interchange differentiation and multiplication.

We define the Sobolev space $H_s(\mathbf{R}^m)$ to measure L^2 derivatives. If s is a real number and $f \in \mathcal{S}$, we define:

$$|f|_s^2 = \int (1+|\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi.$$

The Sobolev space $H_s(\mathbf{R}^m)$ is the completion of $\mathcal{S}$ with respect to the norm $|_s$. The Plancherel theorem shows $H_0(\mathbf{R}^m)$ is isomorphic to $L^2(\mathbf{R}^m)$. More generally, $H_s(\mathbf{R}^m)$ is isomorphic to L^2 with the measure $(1+|\xi|^2)^{s/2} \, d\xi$. Replacing $(1+|\xi|^2)^s$ by $(1+|\xi|)^{2s}$ in the definition of $|_s$ gives rise to an equivalent norm since there exist positive constants c_i such that:

$$c_1(1+|\xi|^2)^s \le (1+|\xi|)^{2s} \le c_2(1+|\xi|^2)^s.$$

In some sense, the subscript "s" counts the number of L^2 derivatives. If $s = n$ is a positive integer, there exist positive constants c_1, c_2 so:

$$c_1(1+|\xi|^2)^n \le \sum_{|\alpha| \le n} |\xi^\alpha|^2 \le c_2(1+|\xi|^2)^n.$$

This implies that we could define

$$|f|_n^2 = \sum_{|\alpha|\le n} \int |\xi^\alpha \hat{f}|^2 \, d\xi = \sum_{|\alpha|\le n} \int |D_x^\alpha f|^2 \, dx$$

as an equivalent norm for $H_n(\mathbf{R}^m)$. With this interpretation in mind, it is not surprising that when we extend D_x^α to H_s, that $|\alpha|$ L^2 derivatives are lost.

LEMMA 1.1.3. *D_x^α extends to define a continuous map $D_x^\alpha\colon H_s \to H_{s-|\alpha|}$.*

PROOF: Henceforth we will use C to denote a generic constant. C can depend upon certain auxiliary parameters which will usually be supressed in the interests of notational clarity. In this proof, for example, C depends on (s, α) but not of course upon f. The estimate:

$$|\xi^\alpha|^2 (1+|\xi|^2)^{s-|\alpha|} \le C(1+|\xi|^2)^s$$

implies that:

$$|D_x^\alpha f|_{s-\alpha}^2 = \int |\xi^\alpha \hat{f}(\xi)|^2 (1+|\xi|^2)^{s-|\alpha|} \, d\xi \le C|f|_s^2$$

for $f \in \mathcal{S}$. Since H_s is the closure of $\mathcal{S}$ in the norm $|_s$, this completes the proof.

We can also use the sup norm to measure derivatives. If k is a non-negative integer, we define:

$$|f|_{\infty,k} = \sup_{x\in\mathbf{R}^m} \sum_{|\alpha|\le k} |D_x^\alpha f| \qquad \text{for} \qquad f \in \mathcal{S}.$$

The completion of $\mathcal{S}$ with respect to this norm is a subset of $C^k(\mathbf{R}^m)$ (the continuous functions on $\mathbf{R}^m$ with continuous partial derivatives up to order k). The next lemma relates the two norms $|_s$ and $|_{\infty,k}$. It will play an important role in showing the weak solutions we will construct to differential equations are in fact smooth.

Lemma 1.1.4. *Let k be a non-negative integer and let $s > k + \frac{m}{2}$. If $f \in H_s$, then f is C^k and there is an estimate $|f|_{\infty,k} \le C|f|_s$. (Sobolev Lemma).*

Proof: Suppose first $k = 0$ and $f \in \mathcal{S}$. We compute

$$\begin{aligned} f(x) &= \int e^{ix\cdot\xi}\,\hat f(\xi)\,d\xi \\ &= \int \{e^{ix\cdot\xi}\,\hat f(\xi)(1+|\xi|^2)^{s/2}\}\cdot\{(1+|\xi|^2)^{-s/2}\}\,d\xi. \end{aligned}$$

We apply the Cauchy-Schwarz inequality to estimate:

$$|f(x)|^2 \le |f|_s^2 \int (1+|\xi|^2)^{-s}\,d\xi.$$

Since $2s > m$, $(1+|\xi|^2)^{-s}$ is integrable so $|f(x)| \le C|f|_s$. We take the sup over $x \in \mathbf{R}^m$ to conclude $|f|_{\infty,0} \le C|f|_s$ for $f \in \mathcal{S}$. Elements of H_s are the limits in the $|_s$ norm of elements of $\mathcal{S}$. The uniform limit of continuous functions is continuous so the elements of H_s are continuous and the same norm estimate extends to H_s. If $k > 0$, we use the estimate:

$$|D_x^\alpha f|_{\infty,0} \le C|D_x^\alpha f|_{s-|\alpha|} \le C|f|_s \qquad \text{for } |\alpha| \le k \text{ and } s-k > \frac{m}{2}$$

to conclude $|f|_{\infty,k} \le C|f|_s$ for $f \in \mathcal{S}$. A similar argument shows that the elements of H_s must be C^k and that this estimate continues to hold.

If $s > t$, we can estimate $(1+|\xi|^2)^s \ge (1+|\xi|^2)^t$. This implies that $|f|_s \ge |f|_t$ so the identity map on $\mathcal{S}$ extends to define an injection of $H_s \to H_t$ which is norm non-increasing. The next lemma shows that this injection is compact if we restrict the supports involved.

LEMMA 1.1.5. *Let $\{f_n\} \in \mathcal{S}$ be a sequence of functions with support in a fixed compact set K. We suppose there is a constant C so $|f_n|_s \le C$ for all n. Let $s > t$. There exists a subsequence f_{n_k} which converges in H_t. (Rellich lemma).*

PROOF: Choose $g \in C_0(\mathbf{R}^m)$ which is identically 1 on a neighborhood of K. Then $gf_n = f_n$ so by Lemma 1.1.2(c) $\hat{f}_n = \hat{g} * \hat{f}_n$. We let $\partial_j = \dfrac{\partial}{\partial \xi_j}$ then $\partial_j(\hat{g} * \hat{f}_n) = \partial_j \hat{g} * \hat{f}_n$ so that:

$$|\partial_j \hat{f}_n(\xi)| \le \int |\{\partial_j \hat{g}(\xi - \zeta)\} \cdot \hat{f}_n(\zeta)| \, d\zeta.$$

We apply the Cauchy-Schwarz inequality to estimate:

$$|\partial_j \hat{f}_n(\xi)| \le |f_n|_s \cdot \left\{ \int |\partial_j \hat{g}(\xi - \zeta)|^2 (1 + |\zeta|^2)^{-s} \, d\zeta \right\}^{1/2} \le C \cdot h(\xi)$$

where h is some continuous function of ξ. A similar estimate holds for $|\hat{f}_n(\xi)|$. This implies that the $\{\hat{f}_n\}$ form a uniformly bounded equi-continuous family on compact ξ subsets. We apply the Arzela-Ascoli theorem to extract a subsequence we again label by f_n so that $\hat{f}_n(\xi)$ converges uniformly on compact subsets. We complete the proof by verifying that f_n converges in H_t for $s > t$. We compute:

$$|f_j - f_k|_t^2 = \int |\hat{f}_j - \hat{f}_k|^2 (1 + |\xi|^2)^t \, d\xi.$$

We decompose this integral into two parts, $|\xi| \ge r$ and $|\xi| \le r$. On $|\xi| \ge r$ we estimate $(1 + |\xi|^2)^t \le (1 + r^2)^{t-s}(1 + |\xi|^2)^s$ so that:

$$\begin{aligned} \int_{|\xi| \ge r} |\hat{f}_j - \hat{f}_k|^2 (1 + |\xi|^2)^t \, d\xi &\le (1 + r^2)^{t-s} \int |\hat{f}_j - \hat{f}_k|^2 (1 + |\xi|^2)^s \, d\xi \\ &\le 2C(1 + r^2)^{t-s}. \end{aligned}$$

If $\varepsilon > 0$ is given, we choose r so that $2C(1 + r^2)^{t-s} < \varepsilon$. The remaining part of the integral is over $|\xi| \le r$. The $\hat{f}_j$ converge uniformly on compact subsets so this integral can be bounded above by ε if $j, k > j(\varepsilon)$. This completes the proof.

The hypothesis that the supports are uniformly bounded is essential. It is easy to construct a sequence $\{f_n\}$ with $|f_n|_s = 1$ for all n and such that the supports are pair-wise disjoint. In this case we can find $\varepsilon > 0$ so that $|f_j - f_k|_t > \varepsilon$ for all (j, k) so there is no convergent subsequence.

We fix $\phi \in \mathcal{S}$ and let $\phi_\varepsilon(x) = \phi(\varepsilon x)$. We suppose $\phi(0) = 1$ and fix $f \in \mathcal{S}$. We compute:

$$D_x^\alpha(f - \phi_\varepsilon f) = (1 - \phi_\varepsilon)D_x^\alpha f + \text{terms of the form } \varepsilon^j D_x^\beta \phi(\varepsilon x) D_x^\gamma f.$$

As $\varepsilon \to 0$, these other terms go to zero in L^2. Since $\phi_\varepsilon \to 1$ pointwise, $(1 - \phi_\varepsilon)D_x f$ goes to zero in L^2. This implies $\phi_\varepsilon f \to f$ in H_n for any $n \geq 0$ as $\varepsilon \to 0$ and therefore $\phi_\varepsilon f \to f$ in H_s for any s. If we take $\phi \in C_0^\infty(\mathbf{R}^m)$, this implies $C_0^\infty(\mathbf{R}^m)$ is dense in H_s for any s.

Each H_s space is a Hilbert space so it is isomorphic to its dual. Because there is no preferred norm for H_s, it is useful to obtain an invariant alternative characterization of the dual space H_s^*:

Lemma 1.1.6. *The L^2 pairing which maps $\mathcal{S} \times \mathcal{S} \to \mathbf{C}$ extends to a map of $H_s \times H_{-s} \to \mathbf{C}$ which is a perfect pairing and which identifies H_{-s} with H_s^*. That is:*

(a) $|(f, g)| \leq |f|_s |g|_{-s}$ *for* $f, g \in \mathcal{S}$,

(b) *given* $f \in \mathcal{S}$ *there exists* $g \in \mathcal{S}$ *so* $(f, g) = |f|_s |g|_{-s}$ *and we can define*

$$|f|_s = \sup_{g \in \mathcal{S},\, g \neq 0} \frac{|(f, g)|}{|g|_{-s}}.$$

Proof: This follows from the fact that H_s is L^2 with the weight function $(1+|\xi|^2)^s$ and H_{-s} is L^2 with the weight function $(1+|\xi|^2)^{-s}$. We compute:

$$(f, g) = (\hat{f}, \hat{g}) = \int \hat{f}(\xi)(1 + |\xi|^2)^{s/2}\overline{\hat{g}}(\xi)(1 + |\xi|^2)^{-s/2}\, d\xi$$

and apply the Cauchy-Schwartz inequality to prove (a).

To prove part (b), we note $|f|_s \geq \sup_{g \in \mathcal{S},\, g \neq 0} \dfrac{|(f, g)|}{|g|_{-s}}$. We take g to be defined by:

$$\hat{g} = \hat{f}(1 + |\xi|^2)^s \in \mathcal{S}$$

and note that $(f, g) = (\hat{f}, \hat{g}) = |f|_s^2$ and that $|g|_{-s}^2 = |f|_s^2$ to see that equality can occur in (a) which proves (b)

If $s > t > u$ then we can estimate:

$$(1 + |\xi|)^{2t} \leq \varepsilon(1 + |\xi|)^{2s} + C(\varepsilon)(1 + |\xi|)^{2u}$$

for any $\varepsilon > 0$. This leads immediately to the useful estimate:

LEMMA 1.1.7. *Let $s > t > u$ and let $\varepsilon > 0$ be given. Then*

$$|f|_t \leq \varepsilon|f|_s + C(\varepsilon)|f|_u.$$

If V is a finite dimensional vector space, let $C^\infty(V)$ be the space of smooth complex valued maps of $\mathbf{R}^m \to V$. We choose a fixed Hermitian inner product on V and define $\mathcal{S}(V)$ and $H_s(V)$ as in the scalar case. If $\dim(V) = k$ and if we choose a fixed orthonormal basis for V, then $\mathcal{S}(V)$ and $H_s(V)$ become isomorphic to the direct sum of k copies of $\mathcal{S}$ and of H_s. Lemmas 1.1.1 through 1.1.7 extend in the obvious fashion.

We conclude this subsection with an extremely useful if elementary estimate:

LEMMA 1.1.8. (PEETRE'S INEQUALITY). *Let s be real and $x, y \in \mathbf{R}^m$. Then $(1 + |x + y|)^s \leq (1 + |y|)^s(1 + |x|)^{|s|}$.*

PROOF: We suppose first $s > 0$. We raise the triangle inequality:

$$1 + |x + y| < 1 + |x| + |y| \leq (1 + |y|)(1 + |x|)$$

to the s^{th} power to deduce the desired inequality. We now suppose $s < 0$. A similar inequality:

$$(1 + |y|)^{-s} \leq (1 + |x + y|)^{-s}(1 + |x|)^{-s}$$

yields immediately:

$$(1 + |x + y|)^s \leq (1 + |y|)^s(1 + |x|)^{-s}$$

to complete the proof.

1.2. Pseudo-Differential Operators on $\mathbf{R}^m$.

A linear partial differential operator of order d is a polynomial expression $P = p(x, D) = \sum_{|\alpha|\le d} a_\alpha(x) D_x^\alpha$ where the $a_\alpha(x)$ are smooth. The symbol $\sigma P = p$ is defined by:

$$\sigma P = p(x, \xi) = \sum_{|\alpha|\le d} a_\alpha(x) \xi^\alpha$$

and is a polynomial of order d in the dual variable ξ. It is convenient to regard the pair (x, ξ) as defining a point of the cotangent space $T^*(\mathbf{R}^m)$; we will return to this point again when we discuss the effect of coordinate transformations. The leading symbol $\sigma_L P$ is the highest order part:

$$\sigma_L P(x, \xi) = \sum_{|\alpha|=d} a_\alpha(x) \xi^\alpha$$

and is a homogeneous polynomial of order d in ξ.

We can use the Fourier inversion formula to express:

$$Pf(x) = \int e^{ix\cdot\xi} p(x, \xi) \hat{f}(\xi)\, d\xi = \int e^{i(x-y)\cdot\xi}\, p(x, \xi) f(y)\, dy\, d\xi$$

for $f \in \mathcal{S}$. We note that since the second integral does not converge absolutely, we cannot interchange the dy and $d\xi$ orders of integration. We use this formalism to define the action of pseudo-differential operators (ΨDO's) for a wider class of symbols $p(x, \xi)$ than polynomials. We make the following

DEFINITION. *$p(x, \xi)$ is a symbol of order d and we write $p \in S^d$ if*
(a) *$p(x, \xi)$ is smooth in $(x, \xi) \in \mathbf{R}^m \times \mathbf{R}^m$ with compact x support,*
(b) *for all (α, β) there are constants $C_{\alpha,\beta}$ such that*

$$|D_x^\alpha D_\xi^\beta p(x, \xi)| \le C_{\alpha,\beta}(1 + |\xi|)^{d-|\beta|}.$$

For such a symbol p, we define the associated operator $P(x, D)$ by:

$$P(x, D)(f)(x) = \int e^{ix\cdot\xi} p(x, \xi) \hat{f}(\xi)\, d\xi = \int e^{(x-y)\cdot\xi}\, p(x, \xi) f(y)\, dy\, d\xi$$

as a linear operator mapping $\mathcal{S} \to \mathcal{S}$.

A differential operator has as its order a positive integer. The order of a pseudo-differential operator is not necessarily an integer. For example, if $f \in C_0^\infty(\mathbf{R}^m)$, define:

$$p(x, \xi) = f(x)(1 + |\xi|^2)^{d/2} \in S^d \qquad \text{for any } d \in \mathbf{R}.$$

This will be a symbol of order d. If $p \in S^d$ for all d, then we say that $p \in S^{-\infty}$ is infinitely smoothing. We adopt the notational convention of letting p, q, r denote symbols and P, Q, R denote the corresponding ΨDO's.

Because we shall be interested in problems on compact manifolds, we have assumed the symbols have compact x support to avoid a number of technical complications. The reader should note that there is a well defined theory which does not require compact x support.

When we discuss the heat equation, we shall have to consider a wider class of symbols which depend on a complex parameter. We postpone discussion of this class until later to avoid unnecessarily complicating the discussion at this stage. We shall phrase the theorems and proofs of this section in such a manner that they will generalize easily to the wider class of symbols.

Our first task is to extend the action of P from $\mathcal{S}$ to H_s.

Lemma 1.2.1. *Let $p \in S^d$ then $|Pf|_{s-d} \le C|f|_s$ for $f \in \mathcal{S}$. P extends to a continuous map $P\colon H_s \to H_{s-d}$ for all s.*

Proof: We compute $Pf(x) = \int e^{ix\cdot\xi} p(x,\xi)\hat{f}(\xi)\,d\xi$ so that the Fourier transform is given by:

$$\widehat{Pf}(\zeta) = \int e^{ix\cdot(\xi-\zeta)} p(x,\xi)\hat{f}(\xi)\,d\xi\,dx.$$

This integral is absolutely convergent since p has compact x support so we may interchange the order of integration. If we define

$$q(\zeta,\xi) = \int e^{-ix\cdot\zeta} p(x,\xi)\,dx$$

as the Fourier transform in the x direction, then

$$\widehat{Pf}(\zeta) = \int q(\zeta-\xi,\xi)\hat{f}(\xi)\,d\xi.$$

By Lemma 1.1.6, $|Pf|_{s-d} = \sup\limits_{g\in S} \dfrac{|(Pf,g)|}{|g|_{d-s}}$. We compute:

$$(Pf,g) = \int q(\zeta-\xi,\xi)\hat{f}(\xi)\bar{\hat{g}}(\zeta)\,d\xi\,d\zeta.$$

Define:

$$K(\zeta,\xi) = q(\zeta-\xi,\xi)(1+|\xi|)^{-s}(1+|\zeta|)^{s-d}$$

then:

$$(Pf,g) = \int K(\zeta,\xi)\hat{f}(\xi)(1+|\xi|)^s\bar{\hat{g}}(\zeta)(1+|\zeta|)^{d-s}\,d\xi\,d\zeta.$$

We apply the Cauchy-Schwarz inequality to estimate:

$$|(Pf,g)| \le \left\{ \int |K(\zeta,\xi)|\,|\hat f(\xi)|^2(1+|\xi|)^{2s}\,d\xi\,d\zeta \right\}^{1/2}$$
$$\times \left\{ \int |K(\zeta,\xi)|\,|\hat g(\zeta)|^2(1+|\zeta|)^{2d-2s}\,d\xi\,d\zeta \right\}^{1/2}.$$

We complete the proof by showing

$$\int |K(\zeta,\xi)|\,d\xi \le C \qquad \text{and} \qquad \int |K(\zeta,\xi)|\,d\zeta \le C$$

since then $|(Pf,g)| \le C|f|_s|g|_{d-s}$.

By hypothesis, p has suppport in a compact set $\mathcal{K}$ and we have estimates:

$$|D_x^\alpha p(x,\xi)| \le C_\alpha(1+|\xi|)^d.$$

Therefore:

$$|\zeta^\alpha q(\zeta,\xi)| = \left| \int e^{-ix\cdot\zeta} D_x^\alpha p(x,\xi)\,dx \right| \le C_\alpha(1+|\xi|)^d \operatorname{vol}(\mathcal{K}).$$

Therefore, for any integer k, $|q(\zeta,\xi)| \le C_k(1+|\xi|)^d(1+|\zeta|)^{-k}\operatorname{vol}(\mathcal{K})$ and:

$$|K(\zeta,\xi)| \le C_k(1+|\xi|)^{d-s}(1+|\zeta|)^{s-d}(1+|\zeta-\xi|)^{-k}\operatorname{vol}(\mathcal{K}).$$

We apply Lemma 1.1.8 with $x+y=\xi$ and $y=\zeta$ to estimate:

$$|K(\zeta,\xi)| \le C_k(1+|\zeta-\xi|)^{|d-s|-k}\operatorname{vol}(\mathcal{K}).$$

If we choose $k > \frac{m}{2} + |d-s|$, then this will be integrable and complete the proof.

Our next task is to show that the class of ΨDO's forms an algebra under the operations of composition and taking adjoint. Before doing that, we study the situation with respect to differential operators to motivate the formulas we shall derive. Let $P = \sum_\alpha p_\alpha(x)D_x^\alpha$ and let $Q = \sum_\alpha q_\alpha(x)D_x^\alpha$ be two differential operators. We assume p and q have compact x support. It is immediate that:

$$P^* = \sum_\alpha D_x^\alpha p_\alpha^* \qquad \text{and} \qquad PQ = \sum_{\alpha,\beta} p_\alpha(x)D_x^\alpha q_\beta(x)D_x^\beta$$

are again differential operators in our class. Furthermore, using Leibnitz's rule

$$D_x^\alpha(fg) = \sum_{\beta+\gamma=\alpha} D_x^\beta(f)\cdot D_x^\gamma(g)\cdot\frac{\alpha!}{\beta!\gamma!},$$
$$d_\xi^\beta(\xi^{\beta+\gamma}) = \xi^\gamma\cdot\frac{(\beta+\gamma)!}{\gamma!},$$

it is an easy combinatorial exercise to compute that:

$$\sigma(P^*) = \sum_\alpha d_\xi^\alpha D_x^\alpha p^*/\alpha! \qquad \text{and} \qquad \sigma(PQ) = \sum_\alpha d_\xi^\alpha p \cdot D_x^\alpha q/\alpha!\,.$$

The perhaps surprising fact is that these formulas remain true in some sense for ΨDO's, only the sums will become infinite rather than finite.

We introduce an equivalence relation on the class of symbols by defining $p \sim q$ if $p - q \in S^{-\infty}$. We note that if $p \in S^{-\infty}$ then $P\colon H_s \to H_t$ for all s and t by Lemma 1.2.1. Consequently by Lemma 1.1.4, $P\colon H_s \to C_0^\infty$ for all s so that P is infinitely smoothing in this case. Thus we mod out by infinitely smoothing operators.

Given symbols $p_j \in S^{d_j}$ where $d_j \to -\infty$, we write

$$p \sim \sum_{j=1}^\infty p_j$$

if for every d there is an integer $k(d)$ such that $k \geq k(d)$ implies that $p - \sum_{j=1}^k p_j \in S^d$. We emphasize that this sum does not in fact need to converge. The relation $p \sim \sum p_j$ simply means that the difference between P and the partial sums of the P_j is as smoothing as we like. It will turn out that this is the appropriate sense in which we will generalize the formulas for $\sigma(P^*)$ and $\sigma(PQ)$ from differential to pseudo-differential operators.

Ultimately, we will be interested in operators which are defined on compact manifolds. Consequently, it poses no difficulties to restrict the domain and the range of our operators. Let U be a open subset of $\mathbf{R}^m$ with compact closure. Let $p(x, \xi) \in S^d$ have x support in U. We restrict the domain of the operator P to $C_0^\infty(U)$ so $P\colon C_0^\infty(U) \to C_0^\infty(U)$. Let $\Psi_d(U)$ denote the space of all such operators. For $d \leq d'$, then $\Psi_d(U) \subseteq \Psi_{d'}(U)$. We define

$$\Psi(U) = \bigcup_d \Psi_d(U) \qquad \text{and} \qquad \Psi_{-\infty}(U) = \bigcap_d \Psi_d(U)$$

to be the set of all pseudo-differential operators on U and the set of infinitely smoothing pseudo-differential operators on U.

More generally, let $p(x, \xi)$ be a matrix valued symbol; we suppose the components of p all belong to S^d. The corresponding operator P is given by a matrix of pseudo-differential operators. P is a map from vector valued functions with compact support in U to vector valued functions with compact support in U. We shall not introduce separate notation for the shape of p and shall continue to denote the collection of all such operators by $\Psi_d(U)$. If p and q are matrix valued and of the proper shape, we define $p \cdot q$ and also the operator $P \cdot Q$ by matrix product and by composition. We also define p^* and P^* to be the matrix adjoint and the operator adjoint so that $(P^* f, g) = (f, P^* g)$ where f and g are vector valued and of compact support. Before studying the algebra structure on $\Psi(U)$, we must enlarge the class of symbols which we can admit:

LEMMA 1.2.2. *Let $r(x,\xi,y)$ be a matrix valued symbol which is smooth in (x,ξ,y). We suppose r has compact x support inside U and that there are estimates:*

$$|D_x^\alpha D_\xi^\beta D_y^\gamma r| \le C_{\alpha,\beta,\gamma}(1+|\xi|)^{d-|\beta|}$$

for all multi-indices (α,β,γ). If f is vector valued with compact support in U, we define:

$$Rf(x) = \int e^{i(x-y)\cdot\xi}\, r(x,\xi,y) f(y)\, dy\, d\xi.$$

Then this operator is in $\Psi_d(U)$ and the symbol is given by:

$$\sigma R(x,\xi) \sim \left\{ \sum_\alpha d_\xi^\alpha D_y^\alpha r(x,\xi,y)/\alpha! \right\}\Bigg|_{x=y}.$$

PROOF: We note that any symbol in S^d belongs to this class of operators if we define $r(x,\xi,y) = p(x,\xi)$. We restricted to vector valued functions with compact support in U. By multiplying r by a cut-off function in y with compact support which is 1 over U, we may assume without loss of generality the y support of r is compact as well. Define:

$$q(x,\xi,\zeta) = \int e^{-iy\cdot\zeta} r(x,\xi,y)\, dy$$

to be the Fourier transform of r in the y variable. Using Lemma 1.1.2 we see $\widehat{(rf)} = \hat{r} * \hat{f}$. This implies that:

$$\int e^{-iy\cdot\xi}\, r(x,\xi,y) f(y)\, dy = \int q(x,\xi,\xi-\zeta)\hat{f}(\zeta)\, d\zeta.$$

The argument given in the proof of Lemma 1.2.1 gives estimates of the form:

$$|q(x,\xi,\zeta)| \le C_k(1+|\xi|)^d(1+|\zeta|)^{-k} \qquad \text{and} \qquad |\hat{f}(\zeta)| \le C_k(1+|\zeta|)^{-k}$$

for any k. Consequently:

$$|q(x,\xi,\xi-\zeta)\hat{f}(\zeta)| \le C_k(1+|\xi|)^d(1+|\xi-\zeta|)^{-k}(1+|\zeta|)^{-k}.$$

We apply Lemma 1.1.8 to estimate:

$$|q(x,\xi,\xi-\zeta)\hat{f}(\zeta)| \le C_k(1+|\xi|)^{|d|-k}(1+|\zeta|)^{|d|-k}$$

so this is absolutely integrable. We change the order of integration and express:

$$Rf(x) = \int e^{ix\cdot\xi} q(x,\xi,\xi-\zeta)\hat{f}(\zeta)\,d\xi\,d\zeta.$$

We define:

$$p(x,\zeta) = \int e^{ix(\xi-\zeta)} q(x,\xi,\xi-\zeta)\,d\xi$$

and compute:

$$Rf(x) = \int e^{ix\cdot\zeta} p(x,\zeta)\hat{f}(\zeta)\,d\zeta$$

is a pseudo-differential operator once it is verified that $p(x,\zeta)$ is a symbol in the correct form.

We change variables to express:

$$p(x,\zeta) = \int e^{ix\cdot\xi} q(x,\xi+\zeta,\xi)\,d\xi$$

and estimate:

$$\begin{aligned} |q(x,\xi+\zeta,\xi)| &\le C_k(1+|\xi+\zeta|)^d(1+|\xi|)^{-k} \\ &\le C_k(1+|\zeta|)^d(1+|\xi|)^{|d|-k}. \end{aligned}$$

This is integrable so $|p(x,\zeta)| \le C'_k(1+|\zeta|)^d$. Similar estimates on $|D_x^\alpha D_\zeta^\beta$ $q(x,\xi+\zeta,\xi)|$ which arise from the given estimates for r show that $p \in S^d$ so that R is a pseudo-differential operator.

We use Taylor's theorem on the middle variable of $q(x,\xi+\zeta,\xi)$ to expand:

$$q(x,\xi+\zeta,\xi) = \sum_{|\alpha|\le k} \frac{d_\zeta^\alpha q(x,\zeta,\xi)\xi^\alpha}{\alpha!} + q_k(x,\zeta,\xi).$$

The remainder q_k decays to arbitrarily high order in (ξ,ζ) and after integration gives rise to a symbol in S^{d-k} which may therefore be ignored. We integrate to conclude

$$\begin{aligned} p(x,\zeta) &= \sum_{|\alpha|\le k} \int e^{ix\cdot\xi} \frac{d_\zeta^\alpha q(x,\zeta,\xi)\xi^\alpha}{\alpha!}\,d\xi + \text{remainder} \\ &= \sum_{|\alpha|\le k} \left.\frac{d_\zeta^\alpha D_y^\alpha r(x,\zeta,y)}{\alpha!}\right|_{x=y} + \text{a remainder} \end{aligned}$$

using Lemma 1.1.2. This completes the proof of the lemma.

We use this technical lemma to show that the pseudo-differential operators form an algebra:

LEMMA 1.2.3. *Let $P \in \Psi_d(U)$ and let $Q \in \Psi_e(U)$. Then:*
(a) If U' is any open set with compact closure containing $\overline{U}$, then $P^* \in \Psi_d(U')$ and $\sigma(P^*) \sim \sum_\alpha d_\xi^\alpha D_x^\alpha p^*/\alpha!$.
(b) Assume that P and Q have the proper shapes so PQ and pq are defined. Then $PQ \in \Psi_{d+e}(U)$ and $\sigma(PQ) \sim \sum_\alpha d_\xi^\alpha p \cdot D_x^\alpha q/\alpha!$.

PROOF: The fact that P^* lies in a larger space is only a slight bit of technical bother; this fact plays an important role in considering boundary value problems of course. Let $(f, g) = f \cdot g$ be the pointwise Hermitian inner product. Fix $\phi \in C_0^\infty(U')$ to be identically 1 on U and compute:

$$\begin{aligned}(Pf, g) &= \int e^{i(x-y)\cdot\xi}\, p(x,\xi)\phi(y)f(y) \cdot g(x)\, dy\, d\xi\, dx \\ &= \int f(y) \cdot e^{i(y-x)\cdot\xi}\, p^*(x,\xi)\phi(y)g(x)\, dy\, d\xi\, dx\end{aligned}$$

since the inner product is Hermitian. By approximating $p^*(x,\xi)$ by functions with compact ξ support, we can justify the use of Fubini's theorem to replace $dy\, d\xi\, dx$ by $dx\, d\xi\, dy$ and express:

$$\begin{aligned}(Pf, g) &= \int f(y) \cdot e^{i(y-x)\cdot\xi}\, p^*(x,\xi)\phi(y)g(x)\, dx\, d\xi\, dy \\ &= (f, P^*g)\end{aligned}$$

where we define:

$$P^*g(y) = \int e^{i(y-x)\cdot\xi}\, p^*(x,\xi)\phi(y)g(x)\, dx\, d\xi.$$

This is an operator of the form discussed in Lemma 1.2.2 so $P^* \in \Psi_d(U')$ and we compute:

$$\sigma(P^*) \sim \sum_\alpha d_\xi^\alpha D_x^\alpha p^*/\alpha!$$

since $\phi = 1$ on the support of p. This completes the proof of (a). We note that we can delete the factor of ϕ from the expression for P^*g since it was only needed to prove P^* was a ΨDO.

We use (a) to prove (b). Since:

$$Q^*g(y) = \int e^{i(y-x)\cdot\xi}\, q^*(x,\xi)g(x)\, dx\, d\xi$$

the Fourier inversion formula implies:

$$(\widehat{Q^*g}) = \int e^{-ix\cdot\xi}\, q^*(x,\xi)g(x)\, dx.$$

If $\tilde{q}$ is the symbol of Q^*, then we interchange the roles of Q and Q^* to see:

$$\widehat{(Qg)} = \int e^{-ix\cdot\xi}\,\tilde{q}^*(x,\xi)g(x)\,dx.$$

Therefore:

$$\begin{aligned} PQg(x) &= \int e^{ix\cdot\xi}p(x,\xi)\widehat{(Qg)}(\xi)\,d\xi \\ &= \int e^{i(x-y)\cdot\xi}\,p(x,\xi)\tilde{q}^*(y,\xi)g(y)\,dy\,d\xi \end{aligned}$$

which is an operator of the form discussed in Lemma 1.2.2 if $r(x,\xi,y) = p(x,\xi)\tilde{q}^*(y,\xi)$. This proves PQ is a pseudo-differential operator of the correct order. We compute the symbol of PQ to be:

$$\sim \sum_\alpha d_\xi^\alpha D_y^\alpha(p(x,\xi)\tilde{q}^*(y,\xi))/\alpha! \quad \text{evaluated at } x = y.$$

We use Leibnitz's formula and expand this in the form:

$$\sim \sum_{\beta,\gamma} d_\xi^\beta p(x,\xi) D_y^\beta d_\xi^\gamma D_y^\gamma \tilde{q}^*/\beta!\gamma!\,.$$

The sum over γ yields the symbol of $Q^{**} = Q$ so we conclude finally

$$\sigma(PQ) \sim \sum_\beta d_\xi^\beta p(x,\xi) D_x^\beta q(x,\xi)/\beta!$$

which completes the proof.

Let $K(x,y)$ be a smooth matrix valued function with compact x support in U. If f is vector valued with compact support in U, we define:

$$P(K)(f)(x) = \int K(x,y)f(y)\,dy.$$

LEMMA 1.2.4. *Let $K(x,y)$ be smooth with compact x support in U, then $P(K) \in \Psi_{-\infty}(U)$.*

PROOF: We let $\phi(\xi) \in C_0^\infty(\mathbf{R}^m)$ with $\int \phi(\xi)\,d\xi = 1$. Define:

$$r(x,\xi,y) = e^{i(y-x)\cdot\xi}\,\phi(\xi)K(x,y)$$

then this is a symbol in $S^{-\infty}$ of the sort discussed in Lemma 1.2.2. It defines an infinitely smoothing operator. It is immediate that:

$$P(K)(f)(x) = \int e^{i(x-y)\cdot\xi}\,r(x,\xi,y)f(y)\,dy\,d\xi.$$

Conversely, it can be shown that any infinitely smoothing map has a smooth kernel. In general, of course, it is not possible to represent an arbitrary pseudo-differential operator by a kernel. If P is smoothing enough, however, we can prove:

LEMMA 1.2.5. *Let r satisfy the hypothesis of Lemma 1.2.2 where $d < -m-k$. We define $K(x,y) = \int e^{i(x-y)\cdot\xi}\, r(x,\xi,y)\, d\xi$. Then K is C^k in (x,y) and $Rf(x) = \int K(x,y)f(y)\,dy$.*

PROOF: If we can show K is well defined, then the representation of R in terms of the kernel K will follow from Fubini's theorem. We estimate:

$$D_x^\alpha D_y^\beta K(x,y) = \sum_{\substack{\alpha=\alpha_1+\alpha_2\\ \beta=\beta_1+\beta_2}} \frac{\alpha!\,\beta!}{\alpha_1!\,\alpha_2!\,\beta_1!\,\beta_2!}(-1)^{|\beta_1|} \times \left\{ \int e^{i(x-y)\cdot\xi}\, \xi^{\alpha_1+\beta_1}\, D_x^{\alpha_2} D_y^{\beta_2} r(x,\xi,y)\, d\xi \right\}.$$

Since we can estimate:

$$|\xi^{\alpha_1+\beta_1} D_x^{\alpha_2} D_y^{\beta_2} r(x,\xi,y)| \le C(1+|\xi|)^{d+|\alpha|+|\beta|}$$

this will be integrable for $|\alpha|+|\beta| \le k$. Thus K is C^k and the representation of R follows immediately.

In Lemma 1.2.2 we computed the symbol of the pseudo-differential operator defined by $r(x,\xi,y)$ in terms of $d_\xi^\alpha D_y^\alpha r$ when $x = y$. This implies the singular (i.e., the non-smoothing part) of R is concentrated near the diagonal $x = y$. We make this more precise:

LEMMA 1.2.6. *Let $r(x,\xi,y)$ satisfy the hypothesis of Lemma 1.2.2. Suppose the x support of r is disjoint from the y support of r, then R is infinitely smoothing and is represented by a smooth kernel function $K(x,y)$.*

PROOF: We would like to define $K(x,y) = \int e^{(x-y)\cdot\xi}\, r(x,\xi,y)\, d\xi$. Unfortunately, this integral need not converge in general. By hypothesis, $|x-y| \ge \varepsilon > 0$ on the support of r. We define the Laplacian $\Delta_\xi = \sum_\nu D_{\xi_\nu}^2$. Since $\Delta_\xi e^{i(x-y)\cdot\xi} = |x-y|^2 e^{i(x-y)\cdot\xi}$ we integrate by parts in a formal sense k times to express:

$$Rf(x) = \int e^{i(x-y)\cdot\xi}\, |x-y|^{-2k} \Delta_\xi^k r(x,\xi,y) f(y)\, dy\, d\xi.$$

This formal process may be justified by first approximating r by a function with compact ξ support. We now define

$$K(x,y) = \int e^{i(x-y)\cdot\xi}\, |x-y|^{-2k} \Delta_\xi^k r(x,\xi,y)\, d\xi$$

for any k sufficiently large. Since $\Delta_\xi^k r$ decays to arbitrarily high order in ξ, we use the same argument as that given in Lemma 1.2.5 to show that

$K(x, y)$ is arbitrarily smooth in (x, y) and hence is C^∞. This completes the proof.

We note that in general $K(x, y)$ will become singular at $x = y$ owing to the presence of the terms $|x-y|^{-2k}$ if we do not assume the support of x is disjoint from the support of y.

A differential operator P is local in the sense that if $f = 0$ on some open subset of U, then $Pf = 0$ on that same subset since differentiation is a purely local process. ΨDO's are not local in general since they are defined by the Fourier transform which smears out the support. Nevertheless, they do have a somewhat weaker property, they do not smear out the singular support of a distribution f. More precisely, let $f \in H_s$. If $\phi \in C_0^\infty(U)$, we define the map $f \mapsto \phi f$. If we take $r(x, \xi, y) = \phi(x)$ and apply Lemma 1.2.2, then we see that this is a pseudo-differential operator of order 0. Therefore $\phi f \in H_s$ as well. This gives a suitable notion of restriction. We say that f is smooth on an open subset U' of U if and only if $\phi f \in C^\infty$ for every such ϕ. An operator P is said to be pseudo-local if f is smooth on U' implies Pf is smooth on U'.

LEMMA 1.2.7. *Pseudo-differential operators are pseudo-local.*

PROOF: Let $P \in \Psi_d(U)$ and let $f \in H_s$. Fix $x \in U'$ and choose $\phi \in C_0^\infty(U')$ to be identically 1 near x. Choose $\psi \in C_0^\infty(U')$ with support contained in the set where ϕ is identically 1. We must verify that $\psi P f$ is smooth. We compute:

$$\psi P f = \psi P \phi f + \psi P(1-\phi) f.$$

By hypothesis, ϕf is smooth so $\psi P \phi f$ is smooth. The operator $\psi P(1-\phi)$ is represented by a kernel of the form $\psi(x) p(x, \xi)(1-\phi(y))$ which has disjoint x and y support. Lemma 1.2.6 implies $\psi P(1-\phi) f$ is smooth which completes the proof.

In Lemmas 1.2.2 and 1.2.3 we expressed the symbol of an operator as a infinite asymptotic series. We show that the algebra of symbols is complete in a certain sense:

LEMMA 1.2.8. *Let $p_j \in S^{d_j}(U)$ where $d_j \to -\infty$. Then there exists $p \sim \sum_j p_j$ which is a symbol in our class. p is a unique modulo $S^{-\infty}$.*

PROOF: We may assume without loss of generality that $d_1 > d_2 > \cdots \to -\infty$. We will construct $p \in S^{d_1}$. The uniqueness is clear so we must prove existence. The p_j all have support inside U; we will construct p with support inside U' where U' is any open set containing the closure of U.

Fix a smooth function ϕ such that:

$$0 \le \phi \le 1, \qquad \phi(\xi) = 0 \text{ for } |\xi| \le 1, \qquad \phi(\xi) = 1 \text{ for } |\xi| \ge 2.$$

We use ϕ to cut away the support near $\xi = 0$. Let $t_j \to 0$ and define:

$$p(x,\xi) = \sum_j \phi(t_j\xi)p_j(x,\xi).$$

For any fixed ξ, $\phi(t_j\xi) = 0$ for all but a finite number of j so this sum is well defined and smooth in (x,ξ). For $j > 1$ we have

$$|p_j(x,\xi)| \le C_j(1+|\xi|)^{d_j} = C_j(1+|\xi|)^{d_1}(1+|\xi|)^{d_j-d_1}.$$

If $|\xi|$ is large enough, $(1+|\xi|)^{d_j-d_1}$ is as small as we like and therefore by passing to a subsequence of the t_j we can assume

$$|\phi(t_j\xi)p_j(x,\xi)| \le 2^{-j}(1-|\xi|)^{d_1} \qquad \text{for } j > 1.$$

This implies that $|p(x,\xi)| \le (C_1+1)(1+|\xi|)^{d_1}$. We use a similar argument with the derivatives and use a diagonalization argument on the resulting subsequences to conclude $p \in S^d$. The supports of all the p_j are contained compactly in U so the support of p is contained in $\overline{U}$ which is contained in U'.

We now apply exactly the same argument to $p_{d_2} + \cdots$ to assume that $p_{d_2} + \cdots \in S^{d_2}$. We continue in this fashion and use a diagonalization argument on the resulting subsequences to conclude in the end that

$$\sum_{j=j_0}^{\infty} \phi(t_j\xi)p_j(x,\xi) \in S^k \qquad \text{for } k = d_{j_0}.$$

Since $p_j - \phi(t_j\xi)p_j \in S^{-\infty}$, this implies $p - \sum_{j=1}^{j_0} p_j \in S^k$ and completes the proof.

If $K(x,y)$ is smooth with compact x,y support in U, then $P(K) \in \Psi_{-\infty}(U)$ defines a continuous operator from $H_s \to H_t$ for any s,t. Let $|P|_{s,t}$ denote the operator norm so $|Pf|_t \le |P|_{s,t}|f|_s$ for any $f \in \mathcal{S}$. It will be convenient to be able to estimate $|K|_{\infty,k}$ in terms of these norms:

LEMMA 1.2.9. *Let $K(x,y)$ be a smooth kernel with compact x,y support in U. Let $P = P(K)$ be the operator defined by K. If k is a non-negative integer, then $|K|_{\infty,k} \le C(k)|P|_{-k,k}$*

PROOF: By arguing separately on each entry in the matrix K, we may reduce ourselves to the scalar case. Suppose first $k = 0$. Choose $\phi \in C_0^\infty(\mathbf{R}^m)$ positive with $\int \phi(x)\,dx = 1$. Fix points $(x_0,y_0) \in U \times U$ and define:

$$f_n(x) = n^m\phi(n(x-x_0)) \qquad \text{and} \qquad g_n(y) = n^m\phi(n(y-y_0)).$$

Then if n is large, f_n and g_n have compact support in U. Then:

$$K(x_0, y_0) = \lim_{n\to\infty} \int f_n(x)K(x,y)g_n(y)\,dy\,dx = \lim_{n\to\infty}(f_n, Pg_n)$$

by Lemma 1.1.1. We estimate

$$|(f_n, Pg_n)| \le |P|_{0,0}|f_n|_0|g_n|_0 = |P|_{0,0}|\phi|_0^2$$

to complete the proof in this case.

If $|\alpha| \le k$, $|\beta| \le k$ then:

$$\begin{aligned} D_x^\alpha D_y^\beta K(x,y) &= \lim_{n\to\infty} \int f_n(x)\big\{D_x^\alpha D_y^\beta K(x,y)\big\}g_n(y)\,dy\,dx \\ &= \lim_{n\to\infty} \int \big(D_x^\alpha f_n)K(x,y)(D_y^\beta g_n(y)\big)\,dy\,dx \\ &= \lim_{n\to\infty}(D_x^\alpha f_n, PD_y^\beta g_n). \end{aligned}$$

We use Lemma 1.1.6 to estimate this by

$$\begin{aligned} |D_x^\alpha f_n|_{-k}|PD_y^\beta g_n|_k &\le |f_n|_0|P|_{-k,k}|D_y^\beta g_n|_{-k} \\ &\le |f_n|_0|g_n|_0|P|_{-k,k} = |P|_{-k,k}|\phi|_0^2 \end{aligned}$$

to complete the proof.

1.3. Ellipticity and Pseudo-Differential Operators on Manifolds.

The norms we have given to define the spaces H_s depend upon the Fourier transform. In order to get a more invariant definition which can be used to extend these notions to manifolds, we must consider elliptic pseudo-differential operators.

Let $p \in S^d(U)$ be a square matrix and let U_1 be an open set with $\overline{U}_1 \subset U$. We say that p is elliptic on U_1 if there exists an open subset U_2 with $\overline{U}_1 \subset U_2 \subset \overline{U}_2 \subset U$ and if there exists $q \in S^{-d}$ such that $pq-I \in S^{-\infty}$ and $qp-I \in S^{-\infty}$ over U_2. (To say that $r \in S^{-\infty}$ over U_2 simply means the estimates of section 1.2 hold over U_2. Equivalently, we assume $\phi r \in S^{-\infty}$ for every $\phi \in C_0^\infty(U_2)$). This constant technical fuss over domains will be eliminated very shortly when we pass to considering compact manifolds; the role of U_2 is to ensure uniform estimates over U_1.

It is clear that p is elliptic over U_1 if and only if there exists constants C_0 and C_1 such that $p(x,\xi)$ is invertible for $|\xi| \geq C_0$ and

$$|p(x,\xi)^{-1}| \leq C_1(1+|\xi|)^{-d} \qquad \text{for } |\xi| \geq C_0,\ \ x \in U_2.$$

We define $q = \phi(\xi)p^{-1}(x,\xi)$ where $\phi(\xi)$ is a cut-off function identically 0 near $\xi = 0$ and identically 1 near $\xi = \infty$. We used similar cutoff functions in the proof of Lemma 1.2.8. Furthermore, if $p_0 \in S^{d-1}$, then p is elliptic if and only if $p + p_0$ is elliptic; adding lower order terms does not alter the ellipticity. If p is a polynomial and P is a differential operator, then p is elliptic if and only if the leading symbol $\sigma_L(p) = \sum_{|\alpha|=d} p_\alpha(x)\xi^\alpha$ is invertible for $\xi \neq 0$.

There exist elliptic operators of all orders. Let $\phi(x) \in C_0^\infty$ and define the symbol $p(x,\xi) = \phi(x)(1+|\xi|^2)^{d/2}I$, then this is an elliptic symbol of order d whenever $\phi(x) \neq 0$.

Lemma 1.3.1. *Let $P \in \Psi^d(U)$ be elliptic over U_1 then:*
(a) There exists $Q \in \Psi^{-d}(U)$ such that $PQ - I \sim 0$ and $QP - I \sim 0$ over U_1 (i.e., $\phi(PQ-I)$ and $\phi(QP-I)$ are infinitely smoothing for any $\phi \in C_0^\infty(U_2)$).
(b) P is hypo-elliptic over U_1, i.e., if $f \in H_s$ and if Pf is smooth over U_1 then f is smooth over U_1.
(c) There exists a constant C such that $|f|_d \leq C(|f|_0 + |Pf|_0)$ for $f \in C_0^\infty(U_1)$. (Gärding's inequality).

Proof: We will define Q to have symbol $q_0 + q_1 + \cdots$ where $q_j \in S^{-d-j}$. We try to solve the equation

$$\sigma(PQ-I) \sim \sum_{\alpha,j} d_\xi^\alpha p \cdot D_x^\alpha q_j/\alpha! - I \sim 0.$$

When we decompose this sum into elements of S^{-k}, we conclude we must solve

$$\sum_{|\alpha|+j=k} d_\xi^\alpha p \cdot D_x^\alpha q_j/\alpha! = \begin{cases} I & \text{if } k=0 \\ 0 & \text{if } k \neq 0. \end{cases}$$

We define $q_0 = q$ and then solve the equation inductively to define:

$$q_k = -q \cdot \sum_{\substack{|\alpha|+j=k \\ j<k}} d_\xi^\alpha p \cdot D_x^\alpha q_j/\alpha!\,.$$

This defines Q so $\sigma(PQ-I) \sim 0$ over U_2. Similarly we could solve $\sigma(Q_1P - I) \sim 0$ over U_2. We now compute $\sigma(Q-Q_1) = \sigma(Q-Q_1PQ)+\sigma(Q_1PQ-Q_1) = \sigma((I-Q_1P)Q)+\sigma(Q_1(QP-I)) \sim 0$ over U_2 so that in fact Q and Q_1 agree modulo infinitely smoothing operators. This proves (a).

Let $f \in H_s$ with Pf smooth over U_1, and choose $\phi \in C_0(U_1)$. We compute:

$$\phi f = \phi(I-QP)f + \phi QPf.$$

As $\phi(I-QP) \sim 0$, $\phi(I-QP)f$ is smooth. Since Pf is smooth over U_1, ϕQPf is smooth since Q is pseudo-local. Thus ϕf is smooth which proves (b).

Finally, we choose $\phi \in C_0^\infty(U_2)$ to be identically 1 on U_1. Then if $f \in C_0^\infty(U_1)$,

$$|f|_d = |\phi f|_d = |\phi(I-QP)f + \phi QPf|_d \le |\phi(I-QP)f|_d + |\phi QPf|_d.$$

We estimate the first norm by $C|f|_0$ since $\phi(I-QP)$ is an infinitely smoothing operator. We estimate the second norm by $C|Pf|_0$ since ϕQ is a bounded map from L^2 to H^d. This completes the proof.

We note (c) is immediate if $d<0$ since $|_d \le |_0$. If $d>0$, $|f|_0 + |Pf|_0 \le C(|f|_d)$ so this gives a equivalent norm on H^d.

We now consider the effect of changes of coordinates on our class of pseudo-differential operators. Let $h\colon U \to \widetilde{U}$ be a diffeomorphism. We define $h^*\colon C^\infty(\widetilde{U}) \to C^\infty(U)$ by $h^*f(x) = f(h(x))$. If P is a linear operator on $C^\infty(U)$, we define h_*P acting on $C^\infty(\widetilde{U})$ by $(h_*P)f = (h^{-1})^*P(h^*f)$. The fundamental lemma we shall need is the following:

LEMMA 1.3.2. *Let $h\colon U \to \widetilde{U}$ be a diffeomorphism. Then:*
(a) If $P \in \Psi^d(U)$ then $h_*P \in \Psi^d(\widetilde{U})$. Let $p = \sigma(P)$ and define $h(x) = x_1$ and $dh(x)^t\xi_1 = \xi$. Let $p_1(x_1,\xi_1) = p(x,\xi)$ then $\sigma(h_*P) - p_1 \in S^{d-1}$.
(b) Let U_1 be an open subset with $\overline{U}_1 \subset U$. There exists a constant C such that $|h^*f|_d \le C|f|_d$ for all $f \in C_0^\infty(h(U_1))$. In other words, the Sobolev spaces are invariant.

PROOF: The first step is to localize the problem. Let $\{\phi_i\}$ be a partition of unity and let $P_{ij} = \phi_i P \phi_j$ so $P = \sum_{i,j} P_{ij}$. If the support of ϕ_i is disjoint from the support of ϕ_j, then P_{ij} is an infinitely smoothing operator with a smooth kernel $K_{ij}(x,y)$ by Lemma 1.2.6. Therefore h_*P_{ij} is also given by a smooth kernel and is a pseudo-differential operator by Lemma 1.2.4. Consequently, we may restrict attention to pairs (i,j) such that the supports of ϕ_i and ϕ_j intersect. We assume henceforth P is defined by a symbol $p(x,\xi,y)$ where p has arbitrarily small support in (x,y).

We first suppose h is linear to motivate the constructions of the general case. Let $h(x) = hx$ where h is a constant matrix. We equate:

$$hx = x_1, \qquad hy = y_1, \qquad h^t\xi_1 = \xi$$

and define

$$p_1(x_1,\xi_1,y_1) = p(x,\xi,y).$$

(In the above, h^t denotes the matrix transpose of h). If $f \in C_0^\infty(\widetilde{U})$, we compute:

$$\begin{aligned}(h_*P)f(x_1) &= \int e^{i(x-y)\cdot\xi}\, p(x,\xi,y)f(hy)\,dy\,d\xi\\ &= \int e^{ih^{-1}(x_1-y_1)\cdot\xi}\, p(h^{-1}x_1,\xi,h^{-1}y_1)f(y_1)\\ &\qquad\qquad\times|\det(h)|^{-1}\,dy_1\,d\xi.\end{aligned}$$

We now use the identities $h^{-1}(x_1-y_1)\cdot\xi = (x_1-y_1)\cdot\xi_1$ and $|\det(h)|\,d\xi_1 = d\xi$ to write:

$$\begin{aligned}(h_*P)f(x_1) &= \int e^{i(x_1-y_1)\cdot\xi_1}\, p(h^{-1}x_1,h^t\xi_1,h^{-1}y_1)f(y_1)\,dy_1\,d\xi_1\\ &= \int e^{i(x_1-y_1)\cdot\xi_1}\, p_1(x_1,\xi_1,y_1)f(y_1)\,dy_1\,d\xi_1.\end{aligned}$$

This proves that (h_*P) is a pseudo-differential operator on $\widetilde{U}$. Since we don't need to localize in this case, we compute directly that

$$\sigma(h_*P)(x_1,\xi_1) = p(h^{-1}x_1,h^t\xi_1).$$

We regard (x,ξ) as giving coordinates for T^*M when we expand any covector in the form $\sum\xi_i\,dx^i$. This is exactly the transformation for the cotangent space so we may regard σP as being invariantly defined on $T^*\mathbf{R}^m$.

If h is not linear, the situation is somewhat more complicated. Let

$$\begin{aligned}x-y = h^{-1}(x_1)-h^{-1}(y_1) &= \int_0^1 \frac{d}{dt}\{h^{-1}(tx_1+(1-t)y_1)\}\,dt\\ &= \int_0^1 d(h^{-1})(tx_1+(1-t)y_1)\cdot(x_1-y_1)\,dt = T(x_1,y_1)(x_1-y_1)\end{aligned}$$

where $T(x_1, y_1)$ is a square matrix. If $x_1 = y_1$, then $T(x_1, y_1) = d(h^{-1})$ is invertible since h is a diffeomorphism. We localize using a partition of unity to suppose henceforth the supports are small enough so $T(x_1, y_1)$ is invertible for all points of interest.

We set $\xi_1 = T(x_1, y_1)^t \xi$ and compute:

$$\begin{aligned}(h_*P)(f)(x_1) &= \int e^{i(x-y)\cdot\xi}\, p(x,\xi,y) f(hy)\,dy\,d\xi \\ &= \int e^{iT(x_1,y_1)(x_1-y_1)\cdot\xi}\, p(h^{-1}x_1,\xi,h^{-1}y_1) f(y_1) J\,dy_1\,d\xi \\ &= \int e^{i(x_1-y_1)\cdot\xi_1}\, p_1(x_1,\xi_1,y_1) f(y_1) \\ &\qquad\qquad \times J|\det T(x_1,y_1)|^{-1}\,dy_1\,d\xi_1\end{aligned}$$

where $J = |\det(dh^{-1})| = |\det T(y_1, y_1)|$. By Lemma 1.2.2, this defines a pseudo-differential operator of order d such that $\sigma(h_*P) = p_1$ modulo S^{d-1} which completes the proof of (a). Since $|dh|$ is uniformly bounded on U_1, $|f|_0 \le C|h^*f|_0$ and $|h^*f|_0 \le C|f|_0$. If P is elliptic, then h_*P is elliptic of the same order. For $d > 0$, choose P elliptic of order d and compute:

$$|h^*f|_d \le C(|h^*f|_0 + |Ph^*f|_0) \le C(|f|_0 + |(h_*P)f|_0) \le C|f|_d$$

which completes the proof of (b) if $d \ge 0$. The result for $d \le 0$ follows by duality using Lemma 1.1.6(b).

We introduce the spaces S^d/S^{d-1} and define $\sigma_L(P)$ to be the element defined by $\sigma(P)$ in this quotient. Let P and Q be pseudo-differential operators of order d_1 and d_2 Then PQ is a pseudo-differential operator of order $d_1 + d_2$ and $\sigma_L(PQ) = \sigma_L(P)\sigma_L(Q)$ since the remaining terms in the asymptotic series are of lower order. Similarly $\sigma_L(P^*) = \sigma_L(P)^*$. If we define (x, ξ) as coordinates for $T^*(\mathbf{R}^m)$ by representing a cotangent vector at a point x in the form $\sum \xi_i dx^i$, then Lemma 1.3.2 implies $\sigma_L(P)$ is invariantly defined on $T^*(\mathbf{R}^m)$. If $P = \sum_\alpha p_\alpha D_x^\alpha$ is a differential operator there is a natural identification of $\sum_{|\alpha|=d} p_\alpha \xi^\alpha$ with the image of p in S^d/S^{d-1} so this definition of the leading symbol agrees with that given earlier.

We now extend the results of section 1.2 to manifolds. Let M be a smooth compact Riemannian manifold without boundary. Let m be the dimension of M and let dvol or sometimes simply dx denote the Riemanian measure on M. In Chapter 2, we will use the notation $|\,\mathrm{dvol}\,|$ to denote this measure in order to distinguish between measures and m-forms, but we shall not bother with this degree of formalism here. We restrict to scalars first. Let $C^\infty(M)$ be the space of smooth functions on M and let $P\colon C^\infty(M) \to C^\infty(M)$ be a linear operator. We say that P is a pseudo-differential operator of order d and write $P \in \Psi_d(M)$ if for every open chart

U on M and for every $\phi, \psi \in C_0^\infty(U)$, the localized operator $\phi P \psi \in \Psi_d(U)$. We say that P is elliptic if $\phi P \psi$ is elliptic where $\phi\psi(x) \neq 0$. If $Q \in \Psi_d(U)$, we let $P = \phi Q \psi$ for $\phi, \psi \in C_0^\infty(U)$. Lemma 1.3.2 implies P is a pseudo-differential operator on M so there exists operators of all orders on M. We define:

$$\Psi(M) = \bigcup_d \Psi_d(M) \qquad \text{and} \qquad \Psi_{-\infty}(M) = \bigcap_d \Psi_d(M)$$

to be the set of all pseudo-differential operators on M and the set of infinitely smoothing operators on M.

In any coordinate system, we define $\sigma(P)$ to the symbol of the operator $\phi P \phi$ where $\phi = 1$ near the point in question; this is unique modulo $S^{-\infty}$. The leading symbol is invariantly defined on T^*M, but the total symbol changes under the same complicated transformation that the total symbol of a differential operator does under coordinate transformations. Since we shall not need this transformaton law, we omit the statement; it is implicit in the computations performed in Lemma 1.3.2.

We define $L^2(M)$ using the L^2 inner product

$$(f, g) = \int_M f(x)\overline{g}(x)\, dx, \qquad |f|_0^2 = (f, f).$$

We let $L^2(M)$ be the completion of $C^\infty(M)$ in this norm. Let $P\colon C^\infty(M) \to C^\infty(M)$. We let P^* be defined by $(Pf, g) = (f, P^* g)$ if such a P^* exists. Lemmas 1.3.2 and 1.2.3 imply that:

LEMMA 1.3.3.
(a) If $P \in \Psi_d(M)$, then $P^* \in \Psi_d(M)$ and $\sigma_L(P^*) = \sigma_L(P)^*$. In any coordinate chart, $\sigma(P^*)$ has a asymptotic expansion given by Lemma 1.2.3(a).
(b) If $P \in \Psi_d(M)$ and $Q \in \Psi_e(M)$, then $PQ \in \Psi_{d+e}(M)$ and $\sigma_L(PQ) = \sigma_L(P)\sigma_L(Q)$. In any coordinate chart $\sigma(PQ)$ has an asymptotic expansion given in Lemma 1.2.3(b).

We use a partition of unity to define the Sobolev spaces $H_s(M)$. Cover M by a finite number of coordinate charts U_i with diffeomorphisms $h_i\colon O_i \to U_i$ where the O_i are open subsets of $\mathbf{R}^m$ with compact closure. If $f \in C_0^\infty(U_i)$, we define:

$$|f|_s^{(i)} = |h_i^* f|_s$$

where we shall use the superscript ${}^{(i)}$ to denote the localized norm. Let $\{\phi_i\}$ be a partition of unity subordinate to this cover and define:

$$|f|_s = \sum |\phi_i f|_s^{(i)}.$$

If $\psi \in C^\infty(M)$, we note that $|\phi_i \psi f|_s^{(i)} \le C|\phi_i f|_s^{(i)}$ since multiplication by ψ defines a ΨDO of order 0. Suppose $\{U'_j, O'_j, h'_j, \phi'_j\}$ is another possible choice to define $|'_s$. We estimate:

$$|\phi'_j f|_s'^{(j)} \le \sum_i |\phi'_j \phi_i f|_s'^{(j)}.$$

Since $\phi'_j \phi_i f \in C_0^\infty(U_i \cap U'_j)$, we can use Lemma 1.3.2 (b) to estimate $|_s'^{(j)}$ by $|_s^{(i)}$ so

$$|\phi'_j f|_s'^{(j)} \le \sum_i C|\phi'_j \phi_i f|_s^{(i)} \le C|\phi_i f|_s^{(i)} \le C|f|_s$$

so that $|f|'_s \le C|f|_s$. Similarly $|f|_s \le C|f|'_s$. This shows these two norms are equivalent so $H_s(M)$ is defined independent of the choices made.

We note that:

$$\sum_i \{|\phi_i f|_s^{(i)}\}^2 \le |f|_s^2 \le C \sum_i \{|\phi_i f|_s^{(i)}\}^2$$

so by using this equivalent norm we conclude the $H_s(M)$ are topologically Hilbert spaces. If $\{\psi_i\}$ are given subordinate to the cover U_i with $\psi_i \ge 0$ and $\psi = \sum_i \psi_i > 0$, we let $\phi_i = \psi_i/\psi$ and compute:

$$\begin{aligned}\sum_i |\psi_i f|_s^{(i)} &= \sum_i |\psi \phi_i f|_s^{(i)} \le C \sum_i |\phi_i f|_s^{(i)} = C|f|_s \\ \sum_i |\phi_i f|_s^{(i)} &= \sum_i |\psi^{-1} \psi_i f|_s^{(i)} \le C \sum_i |\psi_i f|_s^{(i)}\end{aligned}$$

to see the norm defined by $\sum_i |\psi_i f|_s^{(i)}$ is equivalent to the norm $|f|_s$ as well.

Lemma 1.1.5 implies that $|f|_t \le |f|_s$ if $t < s$ and that the inclusion of $H_s(M) \to H_t(M)$ is compact. Lemma 1.1.7 implies given $s > t > u$ and $\varepsilon > 0$ we can estimate:

$$|f|_t \le \varepsilon |f|_s + C(\varepsilon)|f|_u.$$

We assume the coordinate charts U_i are chosen so the union $U_i \cup U_j$ is also contained in a larger coordinate chart for all (i, j). We decompose $p \in \Psi_d(M)$ as $P = \sum_{i,j} P_{i,j}$ for $P_{i,j} = \phi_i P \phi_j$. By Lemma 1.2.1 we can estimate:

$$|\phi_i P \phi_j f|_s^{(i)} \le C|\phi_j f|_{s+d}^{(j)}$$

so $|Pf|_s \le C|f|_{s+d}$ and P extends to a continuous map from $H_{s+d}(M) \to H_s(M)$ for all s.

We define:

$$|f|_{\infty,k} = \sum_i |\phi_i f|^{(i)}_{\infty,k}$$

as a measure of the sup norm of the k^{th} derivatives of f. This is independent of particular choices made. Lemma 1.1.4 generalizes to:

$$|f|_{\infty,k} \le C|f|_s \qquad \text{for } s > \frac{m}{2} + k.$$

Thus $H_s(M)$ is a subset of $C^k(M)$ in this situation.

We choose $\psi_i \in C^\infty(U_i)$ with $\sum_i \psi_i^2 = 1$ then:

$$\begin{aligned} |(f,g)| = |\sum_i (\psi_i f, \psi_i g)| &\le \sum_i |(\psi_i f, \psi_i g)| \\ &\le C \sum_i |\psi_i f|^{(i)}_s |\psi_i g|^{(i)}_{-s} \le C|f|_s |g|_{-s}. \end{aligned}$$

Thus the L^2 inner product gives a continuous map $H_s(M) \times H_{-s}(M) \to \mathbf{C}$.

Lemma 1.3.4.
(a) The natural inclusion $H_s \to H_t$ is compact for $s > t$. Furthermore, if $s > t > u$ and if $\varepsilon > 0$, then $|f|_t \le \varepsilon |f|_s + C(\varepsilon)|f|_u$.
(b) If $s > k + \frac{m}{2}$ then $H_s(M)$ is contained in $C^k(M)$ and we can estimate $|f|_{\infty,k} \le C|f|_s$.
(c) If $P \in \Psi_d(M)$ then $P: H_{s+d}(M) \to H_s(M)$ is continuous for all s.
(d) The pairing $H_s(M) \times H_{-s}(M) \to \mathbf{C}$ given by the L^2 inner product is a perfect pairing.

Proof: We have proved every assertion except the fact (d) that the pairing is a perfect pairing. We postpone this proof briefly until after we have discussed elliptic ΨDO's.

The sum of two elliptic operators need not be elliptic. However, the sum of two elliptic operators with positive symbols is elliptic. Let P_i have symbol $(1+|\xi_i|^2)^{d/2}$ on U_i and let ϕ_i be a partition of unity. $P = \sum_i \phi_i P_i \phi_i$; this is an elliptic ΨDO of order d for any d, so elliptic operators exist. We let P be an elliptic ΨDO of order d and let ψ_i be identically 1 on the support of ϕ_i. We use these functions to construct $Q \in \Psi_{-d}(M)$ so $PQ - I \in \Psi_{-\infty}(M)$ and $QP - I \in \Psi_{-\infty}(M)$. In each coordinate chart, let $P_i = \psi_i P \psi_i$ then $P_i - P_j \in \Psi_{-\infty}$ on the support of $\phi_i\phi_j$. We construct Q_i as the formal inverse to P_i on the support of ϕ_i, then $Q_i - Q_j \in \Psi_{-\infty}$ on the support of $\phi_i\phi_j$ since the formal inverse is unique. Modulo $\Psi_{-\infty}$ we have $P = \sum_i \phi_i P \sim \sum_i \phi_i P_i$. We define $Q = \sum_j Q_j \phi_j$ and note Q has the desired properties.

It is worth noting we could also construct the formal inverse using a Neumann series. By hypothesis, there exists q so $qp - I \in S^{-1}$ and $pq - I \in$

S^{-1}, where $p = \sigma_L(P)$. We construct Q_1 using a partition of unity so $\sigma_L(Q_1) = q$. We let Q^r and Q^l be defined by the formal series:

$$Q^l = Q_1\left\{\sum_k (-1)^k (PQ_1 - I)^k\right\}$$
$$Q^r = \left\{\sum_k (-1)^k (Q_1 P - I)^k\right\} Q_1$$

to construct formal left and right inverses so $Q = Q^r = Q^l$ modulo $\Psi_{-\infty}$.

Let P be elliptic of order $d > 0$. We estimate:

$$|f|_d \leq |(QP - I)f|_d + |QPf|_d \leq C|f|_0 + C|Pf|_0 \leq C|f|_d$$

so we could define H_d using the norm $|f|_d = |f|_0 + |Pf|_0$. We specialize to the following case. Let Q be elliptic of order $d/2$ and let $P = Q^*Q + 1$. Then Q^*Q is self-adjoint and non-negative so we can estimate $|f|_0 \leq |Pf|_0$ and we can define $|f|_d = |Pf|_0$ in this case. Consequently:

$$|f|_d^2 = (Pf, Pf) = (f, P^*Pf) = (f, g),$$

for $g = P^*Pf$. Since $|P^*Pf|_{-d} \leq C|f|_d$, we conclude:

$$|f|_d = (Pf, Pf)/|f|_d \leq C(f, g)/|g|_{-d} \leq C \sup_h |(f, h)|/|h|_{-d}.$$

Since the pairing of H_d with H_{-d} is continuous, this proves $H_d = H^*_{-d}$. Topologically these are Hilbert spaces so we see dually that $H^*_d = H_{-d}$. This completes the proof of Lemma 1.3.4. We also note that we have proved:

LEMMA 1.3.5. *Let $P \in \Psi_d$ be elliptic. Then there exists $Q \in \Psi_{-d}$ so $PQ - I \in \Psi_{-\infty}$ and $QP - I \in \Psi_{-\infty}$. P is hypoelliptic. If $d > 0$, we can define h_d by using the norm $|f|_0 + |Pf|_0$ and define H_{-d} by duality.*

If V is a vector bundle, we cover M by coordinate charts U_i over which V is trivial. We use this cover to define $H_s(V)$ using a partition of unity. We shall always assume V has a given fiber metric so $L^2(V)$ is invariantly defined. $P: C^\infty(V) \to C^\infty(W)$ is a ΨDO of order d if $\phi P \psi$ is given by a matrix of d^{th} order ΨDO's for $\phi, \psi \in C_0^\infty(U)$ for any coordinate chart U over which V and W are trivial. Lemmas 1.3.4 and 1.3.5 generalize immediately to this situation.

1.4. Fredholm Operators and the Index of a Fredholm Operator.

Elliptic ΨDO's are invertible modulo $\Psi_{-\infty}$. Lemma 1.3.4 will imply that elliptic ΨDO's are invertible modulo compact operators and that such operators are Fredholm. We briefly review the facts we shall need concerning Fredholm and compact operators.

Let H be a Hilbert space and let $\mathrm{END}(H)$ denote the space of all bounded linear maps $T\colon H \to H$. There is a natural norm on $\mathrm{END}(H)$ defined by:

$$|T| = \sup_{x \in H} \frac{|Tx|}{|x|}$$

where the sup ranges over $x \neq 0$. $\mathrm{END}(H)$ becomes a Banach space under this norm. The operations of addition, composition, and taking adjoint are continuous. We let $\mathrm{GL}(H)$ be the subset of $\mathrm{END}(H)$ consisting of maps T which are 1-1 and onto. The inverse boundedness theorem shows that if $T \in \mathrm{END}(H)$ is 1-1 and onto, then there exists $\varepsilon > 0$ such that $|Tx| \geq \varepsilon |x|$ so T^{-1} is bounded as well. The Neuman series:

$$(1-z)^{-1} = \sum_{k=0}^{\infty} z^k$$

converges for $|z| < 1$. If $|I - T| < 1$, we may express $T = I - (I - T)$. If we define

$$S = \sum_{k=0}^{\infty} (I - T)^k$$

then this converges in $\mathrm{END}(H)$ to define an element $S \in \mathrm{END}(H)$ so $ST = TS = I$. Furthermore, this shows $|T|^{-1} \leq (1 - |I - T|)^{-1}$ so $\mathrm{GL}(H)$ contains an open neighborhood of I and the map $T \to T^{-1}$ is continuous there. Using the group operation on $\mathrm{GL}(H)$, we see that $\mathrm{GL}(H)$ is a open subset of $\mathrm{END}(H)$ and is a topological group.

We say that $T \in \mathrm{END}(H)$ is compact if T maps bounded sets to pre-compact sets—i.e., if $|x_n| \leq C$ is a bounded sequence, then there exists a subsequence x_{n_k} so $Tx_{n_k} \to y$ for some $y \in H$. We let $\mathrm{COM}(H)$ denote the set of all compact maps.

LEMMA 1.4.1. *$\mathrm{COM}(H)$ is a closed 2-sided $*$-ideal of* $\mathrm{END}(H)$.

PROOF: It is clear the sum of two compact operators is compact. Let $T \in \mathrm{END}(H)$ and let $C \in \mathrm{COM}(H)$. Let $\{x_n\}$ be a bounded sequence in H then $\{Tx_n\}$ is also a bounded sequence. By passing to a subsequence, we may assume $Cx_n \to y$ and $CTx_n \to z$. Since $TCx_n \to Ty$, this implies CT and TC are compact so $\mathrm{COM}(H)$ is a ideal. Next let $C_n \to C$ in $\mathrm{END}(H)$, and let x_n be a bounded sequence in H. Choose a subsequence

x_n^1 so $C_1x_n^1 \to y^1$. We choose a subsequence of the x_n^1 so $C_2x_n^2 \to y^2$. By continuing in this way and then using the diagonal subsequence, we can find a subsequence we denote by x_n^n so $C_k(x_n^n) \to y^k$ for all k. We note $|Cx_n^n - C_kx_n^n| \le |C - C_k|c$. Since $|C - C_k| \to 0$ this shows the sequence Cx_n^n is Cauchy so C is compact and $\mathrm{COM}(H)$ is closed. Finally let $C \in \mathrm{COM}(H)$ and suppose $C^* \notin \mathrm{COM}(H)$. We choose $|x_n| \le 1$ so $|C^*x_n - C^*x_m| \ge \varepsilon > 0$ for all n, m. We let $y_n = C^*x_n$ be a bounded sequence, then $(Cy_n - Cy_m, x_n - x_m) = |C^*x_n - C^*x_m|^2 \ge \varepsilon^2$. Therefore $\varepsilon^2 \le |Cy_n - Cy_m||x_n - x_m| \le 2|Cy_n - Cy_m|$ so Cy_n has no convergent subsequence. This contradicts the assumption $C \in \mathrm{COM}(H)$ and proves $C^* \in \mathrm{COM}(H)$.

We shall assume henceforth that H is a separable infinite dimensional space. Although any two such Hilbert spaces are isomorphic, it is convenient to separate the domain and range. If E and F are Hilbert spaces, we define $\mathrm{HOM}(E, F)$ to be the Banach space of bounded linear maps from E to F with the operator norm. We let $\mathrm{ISO}(E, F)$ be the set of invertible maps in $\mathrm{HOM}(E, F)$ and let $\mathrm{COM}(E, F)$ be the closed subspace of $\mathrm{HOM}(E, F)$ of compact maps. If we choose a fixed isomorphism of E with F, we may identify $\mathrm{HOM}(E, F) = \mathrm{END}(E)$, $\mathrm{ISO}(E, F) = \mathrm{GL}(E)$, and $\mathrm{COM}(E, F) = \mathrm{COM}(E)$. $\mathrm{ISO}(E, F)$ is a open subset of $\mathrm{HOM}(E, F)$ and the operation of taking the inverse is a continuous map from $\mathrm{ISO}(E, F)$ to $\mathrm{ISO}(F, E)$. If $T \in \mathrm{HOM}(E, F)$, we define:

$$\begin{aligned} \mathrm{N}(T) &= \{\, e \in E : T(E) = 0 \,\} && \text{(the null space)} \\ \mathrm{R}(T) &= \{\, f \in F : f = T(e) \text{ for some } e \in E \,\} && \text{(the range).} \end{aligned}$$

$\mathrm{N}(T)$ is always closed, but $\mathrm{R}(T)$ need not be. If $\perp$ denotes the operation of taking orthogonal complement, then $\mathrm{R}(T)^\perp = \mathrm{N}(T^*)$. We let $\mathrm{FRED}(E, F)$ be the subset of $\mathrm{HOM}(E, F)$ consisting of operators invertible modulo compact operators:

$$\begin{aligned} \mathrm{FRED}(E, F) = \{\, T \in \mathrm{HOM}(E, F) : \exists S_1, S_2 \in \mathrm{HOM}(F, E) \text{ so} \\ S_1T - I \in \mathrm{COM}(E) \text{ and } TS_2 - I \in \mathrm{COM}(F) \,\}. \end{aligned}$$

We note this condition implies $S_1 - S_2 \in \mathrm{COM}(F, E)$ so we can assume $S_1 = S_2$ if we like. The following lemma provides another useful characterization of $\mathrm{FRED}(E, F)$:

LEMMA 1.4.2. *The following are equivalent:*
(a) $T \in \mathrm{FRED}(E, F)$;
(b) $T \in \mathrm{END}(E, F)$ *has* $\dim \mathrm{N}(T) < \infty$, $\dim \mathrm{N}(T^*) < \infty$, $\mathrm{R}(T)$ *is closed, and* $\mathrm{R}(T^*)$ *is closed.*

PROOF: Let $T \in \mathrm{FRED}(E, F)$ and let $x_n \in \mathrm{N}(T)$ with $|x_n| = 1$. Then $x_n = (I - S_1T)x_n = Cx_n$. Since C is compact, there exists a convegent subsequence. This implies the unit sphere in $\mathrm{N}(T)$ is compact so $\mathrm{N}(T)$ is finite dimensional. Next let $y_n = Tx_n$ and $y_n \to y$. We may assume without loss of generality that $x_n \in \mathrm{N}(T)^\perp$. Suppose there exists a constant C so $|x_n| \le C$. We have $x_n = S_1 y_n + (I - S_1 T)x_n$. Since $S_1 y_n \to S_1 y$ and since $(I - S_1 T)$ is compact, we can find a convergent subsequence so $x_n \to x$ and hence $y = \lim_n y_n = \lim_n Tx_n = Tx$ is in the range of T so $\mathrm{R}(T)$ will be closed. Suppose instead $|x_n| \to \infty$. If $x_n' = x_n/|x_n|$ we have $Tx_n' = y_n/|x_n| \to 0$. We apply the same argument to find a subsequence $x_n' \to x$ with $Tx = 0$, $|x| = 1$, and $x \in \mathrm{N}(T)^\perp$. Since this is impossible, we conclude $\mathrm{R}(T)$ is closed (by passing to a subsequence, one of these two possibilities must hold). Since $T^* S_1^* - I = C_1^*$ and $S_2^* T^* - I = C_2^*$ we conclude $T^* \in \mathrm{FRED}(F, E)$ so $\mathrm{N}(T^*)$ is finite dimensional and $\mathrm{R}(T^*)$ is closed. This proves (a) implies (b).

Conversely, suppose $\mathrm{N}(T)$ and $\mathrm{N}(T^*)$ are finite dimensional and that $\mathrm{R}(T)$ is closed. We decompose:

$$E = \mathrm{N}(T) \oplus \mathrm{N}(T)^\perp \qquad F = \mathrm{N}(T^*) \oplus \mathrm{R}(T)$$

where $T\colon \mathrm{N}(T)^\perp \to \mathrm{R}(T)$ is 1-1 and onto. Consequently, we can find a bounded linear operator S so $ST = I$ on $\mathrm{N}(T)^\perp$ and $TS = I$ on $\mathrm{R}(T)$. We extend S to be zero on $\mathrm{N}(T^*)$ and compute

$$ST - I = \pi_{\mathrm{N}(T)} \qquad \text{and} \qquad TS - I = \pi_{\mathrm{N}(T^*)}$$

where π denotes orthogonal projection on the indicated subspace. Since these two projections have finite dimensional range, they are compact which proves $T \in \mathrm{FRED}(E, F)$.

If $T \in \mathrm{FRED}(E, F)$ we shall say that T is Fredholm. There is a natural law of composition:

LEMMA 1.4.3.
(a) If $T \in \mathrm{FRED}(E, F)$ then $T^* \in \mathrm{FRED}(F, E)$.
(b) If $T_1 \in \mathrm{FRED}(E, F)$ and $T_2 \in \mathrm{FRED}(F, G)$ then $T_2 T_1 \in \mathrm{FRED}(E, G)$.

PROOF: (a) follows from Lemma 1.4.2. If $S_1 T_1 - I \in C(E)$ and $S_2 T_2 - I \in C(F)$ then $S_1 S_2 T_2 T_1 - I = S_1(S_2 T_2 - I)T_1 + (S_1 T_1 - I) \in C(E)$. Similarly $T_2 T_1 S_1 S_2 - I \in C(E)$.

If $T \in \mathrm{FRED}(E, F)$, then we define:

$$\mathrm{index}(T) = \dim \mathrm{N}(T) - \dim \mathrm{N}(T^*).$$

We note that $\mathrm{ISO}(E, F)$ is contained in $\mathrm{FRED}(E, F)$ and that $\mathrm{index}(T) = 0$ if $T \in \mathrm{ISO}(E, F)$.

LEMMA 1.4.4.
(a) $\operatorname{index}(T) = -\operatorname{index}(T^*)$.
(b) If $T \in \mathrm{FRED}(E, F)$ *and* $S \in \mathrm{FRED}(F, G)$ *then*

$$\operatorname{index}(ST) = \operatorname{index}(T) + \operatorname{index}(S).$$

(c) $\mathrm{FRED}(E, F)$ *is a open subset of* $\mathrm{HOM}(E, F)$.
(d) $\operatorname{index}\colon \mathrm{FRED}(E, F) \to \mathbf{Z}$ *is continuous and locally constant.*

PROOF: (a) is immediate from the definition. We compute

$$\begin{aligned} \mathrm{N}(ST) &= \mathrm{N}(T) \oplus T^{-1}(\mathrm{R}(T) \cap \mathrm{N}(S)) \\ \mathrm{N}(T^* S^*) &= \mathrm{N}(S^*) \oplus (S^*)^{-1}(\mathrm{R}(S^*) \cap \mathrm{N}(T^*)) \\ &= \mathrm{N}(S^*) \oplus (S^*)^{-1}(\mathrm{R}(T)^\perp \cap \mathrm{N}(S)^\perp) \end{aligned}$$

so that

$$\begin{aligned} \operatorname{index}(ST) &= \dim \mathrm{N}(T) + \dim(\mathrm{R}(T) \cap \mathrm{N}(S)) - \dim \mathrm{N}(S^*) \\ &\qquad - \dim(\mathrm{R}(T)^\perp \cap \mathrm{N}(S)^\perp) \\ &= \dim \mathrm{N}(T) + \dim(\mathrm{R}(T) \cap \mathrm{N}(S)) + \dim(\mathrm{R}(T)^\perp \cap \mathrm{N}(S)) \\ &\qquad - \dim \mathrm{N}(S^*) - \dim(\mathrm{R}(T)^\perp \cap \mathrm{N}(S)^\perp) \\ &\qquad\quad - \dim(\mathrm{R}(T)^\perp \cap \mathrm{N}(S)) \\ &= \dim \mathrm{N}(T) + \dim \mathrm{N}(S) - \dim \mathrm{N}(S^*) - \dim(\mathrm{R}(T)^\perp) \\ &= \dim \mathrm{N}(T) + \dim \mathrm{N}(S) - \dim \mathrm{N}(S^*) - \dim \mathrm{N}(T^*) \\ &= \operatorname{index}(S) + \operatorname{index}(T) \end{aligned}$$

We prove (c) and (d) as follows. Fix $T \in \mathrm{FRED}(E, F)$. We decompose:

$$E = \mathrm{N}(T) \oplus \mathrm{N}(T)^\perp \qquad \text{and} \qquad F = \mathrm{N}(T^*) \oplus \mathrm{R}(T)$$

where $T\colon \mathrm{N}(T)^\perp \to \mathrm{R}(T)$ is 1-1 onto. We let $\pi_1\colon E \to \mathrm{N}(T)$ be orthogonal projection. We define $E_1 = \mathrm{N}(T^*) \oplus E$ and $F_1 = \mathrm{N}(T) \oplus F$ to be Hilbert spaces by requiring the decompositon to be orthogonal. Let $S \in \mathrm{HOM}(E, F)$, we define $S_1 \in \mathrm{HOM}(E_1, F_1)$ by:

$$S_1(f_0 \oplus e) = \pi_1(e) \oplus (f_0 + S_1(e)).$$

It is clear $|S_1 - S_1'| = |S - S'|$ so the map $S \to S_1$ defines a continuous map from $\mathrm{HOM}(E, F) \to \mathrm{HOM}(E_1, F_1)$. Let $i_1\colon E \to E_1$ be the natural inclusion and let $\pi_2\colon F_1 \to F$ be the natural projection. Since $\mathrm{N}(T)$ and $\mathrm{N}(T^*)$ are finite dimensional, these are Fredholm maps. It is immediate from the definition that if $S \in \mathrm{HOM}(E, F)$ then

$$S = \pi_2 S_1 i_1.$$

If we let $T = S$ and decompose $e = e_0 \oplus e_1$ and $f = f_0 \oplus f_1$ for $e_0 \in \mathrm{N}(T)$ and $f_0 \in \mathrm{N}(T^*)$, then:

$$T_1(f_0 \oplus e_0 \oplus e_1) = e_0 \oplus f_0 \oplus Te_1$$

so that $T_1 \in \mathrm{ISO}(E_1, F_1)$. Since $\mathrm{ISO}(E_1, F_1)$ is an open subset, there exists $\varepsilon > 0$ so $|T - S| < \varepsilon$ implies $S_1 \in \mathrm{ISO}(E_1, F_1)$. This implies S_1 is Fredholm so $S = \pi_2 S_1 i_1$ is Fredholm and $\mathrm{FRED}(E, F)$ is open. Furthermore, we can compute $\mathrm{index}(S) = \mathrm{index}(\pi_2) + \mathrm{index}(S_1) + \mathrm{index}(i_1) = \mathrm{index}(\pi_2) + \mathrm{index}(i_1) = \dim \mathrm{N}(T) - \dim \mathrm{N}(T^*) = \mathrm{index}(T)$ which proves index is locally constant and hence continuous. This completes the proof of the lemma.

We present the following example of an operator with index 1. Let ϕ_n be orthonormal basis for L^2 as $n \in \mathbf{Z}$. Define the one sided shift

$$T\phi_n = \begin{cases} \phi_{n-1} & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ \phi_n & \text{if } n < 0 \end{cases}$$

then T is surjective so $\mathrm{N}(T^*) = \{0\}$. Since $\mathrm{N}(T)$ is one dimensional, $\mathrm{index}(T) = 1$. Therefore $\mathrm{index}(T^n) = n$ and $\mathrm{index}((T^*)^n) = -n$. This proves $\mathrm{index}\colon \mathrm{FRED}(E, F) \to \mathbf{Z}$ is surjective. In the next chapter, we will give several examples of differential operators which have non-zero index.

If we specialize to the case $E = F$ then $\mathrm{COM}(E)$ is a closed two-sided ideal of $\mathrm{END}(E)$ so we can pass to the quotient algebra $\mathrm{END}(E)/\mathrm{COM}(E)$. If $\mathrm{GL}(\mathrm{END}(E)/\mathrm{COM}(E))$ denotes the group of invertible elements and if $\pi\colon \mathrm{END}(E) \to \mathrm{END}(E)/\mathrm{COM}(E)$ is the natural projection, then $\mathrm{FRED}(E)$ is π^{-1} of the invertible elements. If C is compact and T Fredholm, $T + tC$ is Fredholm for any t. This implies $\mathrm{index}(T) = \mathrm{index}(T + tC)$ so we can extend $\mathrm{index}\colon \mathrm{GL}(\mathrm{END}(E)/\mathrm{COM}(E)) \to \mathbf{Z}$ as a surjective group homomorphism.

Let $P\colon C^\infty(V) \to C^\infty(W)$ be a elliptic ΨDO of order d. We construct an elliptic ΨDO S of order $-d$ with $S\colon C^\infty(W) \to C^\infty(V)$ so that $SP - I$ and $PS - I$ are infinitely smoothing operators. Then

$$P\colon H_s(V) \to H_{d-s}(W) \qquad \text{and} \qquad S\colon H_{d-s}(W) \to H_s(V)$$

are continuous. Since $SP - I\colon H_s(V) \to H_t(V)$ is continuous for any t, it is compact. Similarly $PS - I$ is a compact operator so both P and S are Fredholm. If $f \in \mathrm{N}(P)$, then f is smooth by Lemma 1.3.5. Consequently, $\mathrm{N}(P)$ and $\mathrm{N}(P^*)$ are independent of the choice of s and $\mathrm{index}(P)$ is invariantly defined. Furthermore, if P_τ is a smooth 1-parameter family of such operators, then $\mathrm{index}(P_\tau)$ is independent of the parameter τ by Lemma 1.4.4. In particular, $\mathrm{index}(P)$ only depends on the homotopy type of the leading symbol of P. In Chapter 3, we will give a topological formula for $\mathrm{index}(P)$ in terms of characteristic classes.

We summarize our conclusions about $\mathrm{index}(P)$ in the following

LEMMA 1.4.5. *Let* $P\colon C^\infty(V) \to C^\infty(W)$ *be an elliptic* Ψ*DO of order* d *over a compact manifold without boundary. Then:*
(a) $\mathrm{N}(P)$ *is a finite dimensional subset of* $C^\infty(V)$.
(b) $P\colon H_s(V) \to H_{s-d}(W)$ *is Fredholm.* P *has closed range.* $\mathrm{index}(P)$ *does not depend on the particular* s *chosen.*
(c) $\mathrm{index}(P)$ *only depends on the homotopy type of the leading symbol of* P *within the class of elliptic* Ψ*DO's of order* d.

1.5. Elliptic Complexes, The Hodge Decomposition Theorem, And Poincaré Duality.

Let V be a graded vector bundle. V is a collection of vector bundles $\{V_j\}_{j\in\mathbf{Z}}$ such that $V_j \neq \{0\}$ for only a finite number of indices j. We let P be a graded ΨDO of order d. P is a collection of d^{th} order pseudo-differential operators $P_j: C^\infty(V_j) \to C^\infty(V_{j+1})$. We say that (P,V) is a complex if $P_{j+1}P_j = 0$ and $\sigma_L P_{j+1} \sigma_L P_j = 0$ (the condition on the symbol follows from $P^2 = 0$ for differential operators). We say that (P,V) is elliptic if:

$$\mathrm{N}(\sigma_L P_j)(x,\xi) = \mathrm{R}(\sigma_L P_{j-1})(x,\xi) \qquad \text{for } \xi \neq 0$$

or equivalently if the complex is exact on the symbol level.

We define the cohomology of this complex by:

$$H^j(V,P) = \mathrm{N}(P_j)/\,\mathrm{R}(P_{j-1}).$$

We shall show later in this section that $H^j(V,P)$ is finite dimensional if (P,V) is an elliptic complex. We then define

$$\operatorname{index}(P) = \sum_j (-1)^j \dim H^j(V,P)$$

as the Euler characteristic of this elliptic complex.

Choose a fixed Hermitian inner product on the fibers of V. We use that inner product together with the Riemannian metric on M to define $L^2(V)$. We define adjoints with respect to this structure. If (P,V) is an elliptic complex, we construct the associated self-adjoint Laplacian:

$$\Delta_j = (P^*P)_j = P_j^*P_j + P_{j-1}P_{j-1}^*: C^\infty(V_j) \to C^\infty(V_j).$$

If $p_j = \sigma_L(P_j)$, then $\sigma_L(\Delta_j) = p_j^*p_j + p_{j-1}p_{j-1}^*$. We can also express the condition of ellipticity in terms of Δ_j :

Lemma 1.5.1. *Let (P,V) be a d^{th} order partial differential complex. Then (P,V) is elliptic if and only if Δ_j is an elliptic operator of order $2d$ for all j.*

Proof: We suppose that (P,V) is elliptic; we must check $\sigma_L(\Delta_j)$ is non-singular for $\xi \neq 0$. Suppose $(p_j^*p_j + p_{j-1}p_{j-1}^*)(x,\xi)v = 0$. If we dot this equation with v, we see $p_jv \cdot p_jv + p_{j-1}^*v \cdot p_{j-1}^*v = 0$ so that $p_jv = p_{j-1}^*v = 0$. Thus $v \in \mathrm{N}(p_j)$ so $v \in \mathrm{R}(p_{j-1})$ so we can write $v = p_{j-1}w$. Since $p_{j-1}^*p_{j-1}w = 0$, we dot this equation with w to see $p_{j-1}w \cdot p_{j-1}w = 0$ so $v = p_{j-1}w = 0$ which proves Δ_j is elliptic. Conversely, let $\sigma_L(\Delta_j)$ be non-singular for $\xi \neq 0$. Since (P,V) is a complex, $\mathrm{R}(p_{j-1})$ is a subset of $\mathrm{N}(p_j)$.

Conversely, let $v \in \mathrm{N}(p_j)$. Since $\sigma_L(\Delta_j)$ is non-singular, we can express $v = (p_j^* p_j + p_{j-1}p_{j-1}^*)w$. We apply p_j to conclude $p_j p_j^* p_j w = 0$. We dot this equation with $p_j w$ to conclude $p_j p_j^* p_j w \cdot p_j w = p_j^* p_j w \cdot p_j^* p_j w = 0$ so $p_j^* p_j w = 0$. This implies $v = p_{j-1}p_{j-1}^* w \in \mathrm{R}(p_{j-1})$ which completes the proof.

We can now prove the following:

Theorem 1.5.2 (Hodge decomposition theorem). *Let (P, V) be a d^{th} order ΨDO elliptic complex. Then*
(a) We can decompose $L^2(V_j) = \mathrm{N}(\Delta_j) \oplus \mathrm{R}(P_{j-1}) \oplus \mathrm{R}(P_j^*)$ as an orthogonal direct sum.
(b) $\mathrm{N}(\Delta_j)$ is a finite dimensional vector space and there is a natural isomorphism of $H^j(P, V) \simeq \mathrm{N}(\Delta_j)$. The elements of $\mathrm{N}(\Delta_j)$ are smooth sections to V_j.

Proof: We regard $\Delta_j\colon H_{2d}(V_j) \to L^2(V_j)$. Since this is elliptic, it is Fredholm. This proves $\mathrm{N}(\Delta_j)$ is finite dimensional. Since Δ_j is hypoelliptic, $\mathrm{N}(\Delta_j)$ consists of smooth sections to V. Since Δ_j is self adjoint and Fredholm, $\mathrm{R}(\Delta_j)$ is closed so we may decompose $L^2(V_j) = \mathrm{N}(\Delta_j) \oplus \mathrm{R}(\Delta_j)$. It is clear $\mathrm{R}(\Delta_j)$ is contained in the span of $\mathrm{R}(P_{j-1})$ and $\mathrm{R}(P_j^*)$. We compute the L^2 inner product:

$$(P_j^* f, P_{j-1} g) = (f, P_j P_{j-1} g) = 0$$

since $P_j P_{j-1} = 0$. This implies $\mathrm{R}(P_{j-1})$ and $\mathrm{R}(P_j^*)$ are orthogonal. Let $f \in \mathrm{N}(\Delta_j)$, we take the L^2 inner product with f to conclude

$$0 = (\Delta_j f, f) = (P_j f, P_j f) + (P_{j-1}^* f, P_{j-1}^* f)$$

so $\mathrm{N}(\Delta_j) = \mathrm{N}(P_j) \cap \mathrm{N}(P_{j-1}^*)$. This implies $\mathrm{R}(\Delta_j)$ contains the span of $\mathrm{R}(P_{j-1})$ and $\mathrm{R}(P_j^*)$. Since these two subspaces are orthogonal and $\mathrm{R}(\Delta_j)$ is closed, $\mathrm{R}(P_{j-1})$ and $\mathrm{R}(P_j^*)$ are both closed and we have an orthogonal direct sum:

$$L^2(V_j) = \mathrm{N}(\Delta_j) \oplus \mathrm{R}(P_{j-1}) \oplus \mathrm{R}(P_j^*).$$

This proves (a).

The natural inclusion of $\mathrm{N}(\Delta_j)$ into $\mathrm{N}(P_j)$ defines a natural map of $\mathrm{N}(\Delta_j) \to H^j(P, V) = \mathrm{N}(P_j)/\mathrm{R}(P_{j-1})$. Since $\mathrm{R}(P_{j-1})$ is orthogonal to $\mathrm{N}(\Delta_j)$, this map is injective. If $f \in C^\infty(V_j)$ and $P_j f = 0$, we can decompose $f = f_0 \oplus \Delta_j f_1$ for $f_0 \in \mathrm{N}(\Delta_j)$. Since f and f_0 are smooth, $\Delta_j f_1$ is smooth so $f_1 \in C^\infty(V_j)$. $P_j f = P_j f_0 + P_j \Delta_j f_1 = 0$ implies $P_j \Delta_j f_1 = 0$. We dot this equation with $P_j f_1$ to conclude:

$$0 = (P_j f_1, P_j P_j^* P_j f_1 + P_j P_{j-1} P_{j-1}^* f_1) = (P_j^* P_j f_1, P_j^* P_j f_1)$$

so $P_j^* P_j f_1 = 0$ so $\Delta_j f_1 = P_{j-1}P_{j-1}^* f_1 \in \mathrm{R}(P_{j-1})$. This implies f and f_0 represent the same element of $H^j(P, V)$ so the map $\mathrm{N}(\Delta_j) \to H^j(P, V)$ is surjective. This completes the proof.

To illustrate these concepts, we discuss briefly the de Rham complex. Let T^*M be the cotangent space of M. The exterior algebra $\Lambda(T^*M)$ is the universal algebra generated by T^*M subject to the relation $\xi \wedge \xi = 0$ for $\xi \in T^*M$. If $\{e_1, \dots, e_m\}$ is a basis for T^*M and if $I = \{1 \le i_1 < i_2 < \dots < i_p \le m\}$, we define $e_I = e_{i_1} \wedge \dots \wedge e_{i_p}$. The $\{e_I\}$ form a basis for $\Lambda(T^*M)$ which has dimension 2^m. If we define $|I| = p$, then $\Lambda^p(T^*M)$ is the span of the $\{e_I\}_{|I|=p}$; this is the bundle of p-forms. A section of $C^\infty(\Lambda^p T^*M)$ is said to be a smooth p-form over M.

Let $x = (x_1, \dots, x_m)$ be local coordinates on M and let $\{dx_1, \dots, dx_m\}$ be the corresponding frame for T^*M. If $f \in C^\infty(M) = C^\infty(\Lambda^0(T^*M))$, we define:

$$df = \sum_k \frac{\partial f}{\partial x_k}\, dx_k.$$

If $y = (y_1, \dots, y_m)$ is another system of local coordinates on M, the identity:

$$dy_j = \sum_k \frac{\partial y_j}{\partial x_k}\, dx_k$$

means that d is well defined and is independent of the coordinate system chosen. More generally, we define $d(f\,dx_I) = df \wedge dx_I$ so that, for example,

$$d\Bigl(\sum_j f_j\, dx^j\Bigr) = \sum_{j<k} \left\{ \frac{\partial f_k}{\partial x_j} - \frac{\partial f_j}{\partial x_k} \right\} dx_j \wedge dx_k.$$

Again this is well defined and independent of the coordinate system. Since mixed partial derivatives commute, $d^2 = 0$ so

$$d\colon C^\infty(\Lambda^p(T^*M)) \to C^\infty(\Lambda^{p+1}(T^*M))$$

forms a complex.

Let $\xi \in T^*M$ and let $\operatorname{ext}(\xi)\colon \Lambda^p(T^*M) \to \Lambda^{p+1}(T^*M)$ be defined by exterior multiplication, i.e., $\operatorname{ext}(\xi)\omega = \xi \wedge \omega$. If we decompose $\xi = \sum \xi_j\, dx_j$ relative to a local coordinate frame, then $df = \sum_j \partial f/\partial x_j\, dx_j$ implies $\sigma(d) = i \operatorname{ext}(\xi)$; the symbol of exterior differentiation is exterior multiplication up to a factor of i. Fix $\xi \ne 0$ and choose a basis $\{e_1, \dots, e_m\}$ for T^*M such that $\xi = e_1$. Then

$$\operatorname{ext}(\xi)e_I = \begin{cases} 0 & \text{if } i_1 = 1 \\ e_J \text{ for } J = \{1, i_1, \dots, i_p\} & \text{if } i_1 > 1. \end{cases}$$

From this it is clear that $\mathrm{N}(\operatorname{ext}(\xi)) = \mathrm{R}(\operatorname{ext}(\xi))$ so the de Rham complex is an elliptic complex.

A Riemannian metric on M defines fiber metrics on $\Lambda^p(T^*M)$. If $\{e_i\}$ is an orthonormal local frame for TM we take the dual frame $\{e_i^*\}$ for T^*M. For notational simplicity, we will simply denote this frame again by $\{e_i\}$.

The corresponding $\{e_I\}$ define an orthonormal local frame for $\Lambda(T^*M)$. We define interior multiplication $\operatorname{int}(\xi)\colon \Lambda^p(T^*M) \to \Lambda^{p-1}(T^*M)$ to be the dual of exterior multiplication. Then:

$$\operatorname{int}(e_1)e_I = \begin{cases} e_j \quad \text{for } J = \{i_2, \dots, i_p\} & \text{id } i_1 = 1 \\ 0 & \text{if } i_1 > 1 \end{cases}$$

so

$$\operatorname{int}(e_1)\operatorname{ext}(e_1) + \operatorname{ext}(e_1)\operatorname{int}(e_1) = I.$$

If $|\xi|^2$ denotes the length of the covector ξ, then more generally:

$$(i \operatorname{ext}(\xi) - i \operatorname{int}(\xi))^2 = |\xi|^2 I.$$

If $\delta\colon C^\infty(\Lambda^p(T^*M)) \to C^\infty(\Lambda^{p-1}(T^*M))$ is the adjoint of d, then $\sigma_L\delta = i \operatorname{int}(\xi)$. We let $\delta d + d\delta = (d+\delta)^2 = \Delta$. $\sigma_L\Delta = |\xi|^2$ is elliptic. The de Rham theorem gives a natural isomorphism from the cohomology of M to $H^*(\Lambda, d)$:

$$H^p(M; \mathbf{C}) = \mathrm{N}(d_p)/\,\mathrm{R}(d_{p-1})$$

where we take closed modulo exact forms. The Hodge decomposition theorem implies these groups are naturally isomorphic to the harmonic p-forms $\mathrm{N}(\Delta_p)$ which are finite dimensional. The Euler-Poincaré characteristic $\chi(M)$ is given by:

$$\chi(M) = \sum(-1)^p \dim H^p(M; \mathbf{C}) = \sum(-1)^p \dim \mathrm{N}(\Delta_p) = \operatorname{index}(d)$$

is therefore the index of an elliptic complex.

If M is oriented, we let dvol be the oriented volume element. The Hodge $*$ operator $*\colon \Lambda^p(T^*M) \to \Lambda^{m-p}(T^*M)$ is defined by the identity:

$$\omega \wedge *\omega = (\omega \cdot \omega)\,\mathrm{dvol}$$

where "$\cdot$" denotes the inner product defined by the metric. If $\{e_i\}$ is an oriented local frame for T^*M, then $\mathrm{dvol} = e_1 \wedge \cdots \wedge e_m$ and

$$*(e_1 \wedge \cdots \wedge e_p) = e_{p+1} \wedge \cdots \wedge e_m.$$

The following identities are immediate consequences of Stoke's theorem:

$$** = (-1)^{p(m-p)} \qquad \text{and} \qquad \delta = (-1)^{mp+m+1} * d * .$$

Since $\Delta = (d+\delta)^2 = d\delta+\delta d$ we compute $*\Delta = \Delta *$ so $*\colon \mathrm{N}(\Delta_p) \to \mathrm{N}(\Delta_{m-p})$ is an isomorphism. We may regard $*$ as an isomorphism $*\colon H^p(M;\mathbf{C}) \to H^{m-p}(M;\mathbf{C})$; in this description it is Poincaré duality.

The exterior algebra is *not* very suited to computations owing to the large number of signs which enter in the discussion of $*$. When we discuss the signature and spin complexes in the third chapter, we will introduce Clifford algebras which make the discussion of Poincaré duality much easier.

It is possible to "roll up" the de Rham complex and define:

$$(d+\delta)_{\mathrm{e}}\colon C^\infty(\Lambda^{\mathrm{e}}(T^*M)) \to C^\infty(\Lambda^{\mathrm{o}}(T^*M))$$

where

$$\Lambda^{\mathrm{e}}(T^*M) = \bigoplus_{2k} \Lambda^{2k}(T^*M) \qquad \text{and} \qquad \Lambda^{\mathrm{o}}(T^*M) = \bigoplus_{2k+1} \Lambda^{2k+1}(T^*M)$$

denote the differential forms of even and odd degrees. $(d+\delta)$ is an elliptic operator since $(d+\delta)_{\mathrm{e}}^*(d+\delta)_{\mathrm{e}} = \Delta$ is elliptic since $\dim \Lambda^{\mathrm{e}} = \dim \Lambda^{\mathrm{o}}$. (In this representation $(d+\delta)_{\mathrm{e}}$ is not self-adjoint since the range and domain are distinct, $(d+\delta)_{\mathrm{e}}^* = (d+\delta)_{\mathrm{o}}$). It is clear $\operatorname{index}(d+\delta)_{\mathrm{e}} = \dim \mathrm{N}(\Delta^{\mathrm{e}}) = \dim \mathrm{N}(\Delta^{\mathrm{o}}) = \chi(M)$. We can always "roll up" any elliptic complex to form an elliptic complex of the same index with two terms. Of course, the original elliptic complex does not depend upon the choice of a fiber metric to define adjoints so there is some advantage in working with the full complex occasionally as we shall see later.

We note finally that if m is even, we can always find a manifold with $\chi(M)$ arbitrary so there exist lots of elliptic operators with non-zero index. We shall see that $\operatorname{index}(P) = 0$ if m is odd (and one must consider pseudo-differential operators to get a non-zero index in that case).

We summarize these computations for the de Rham complex in the following:

Lemma 1.5.3. *Let $\Lambda(M) = \Lambda(T^*M)$ be the complete exterior algebra.*
(a) $d\colon C^\infty(\Lambda^p(T^*M)) \to C^\infty(\Lambda^{p+1}(T^*M))$ is an elliptic complex. The symbol is $\sigma_L(d)(x,\xi) = i \operatorname{ext}(\xi)$.
(b) If δ is the adjoint, then $\sigma_L(\delta)(x,\xi) = -i \operatorname{int}(\xi)$.
(c) If $\Delta_p = (d\delta+\delta d)_p$ is the associated Laplacian, then $\mathrm{N}(\Delta_p)$ is finite dimensional and there are natural identifications:

$$\mathrm{N}(\Delta_p) \simeq \mathrm{N}(d_p)/\,\mathrm{R}(d_{p-1}) \simeq H^p(M;\mathbf{C}).$$

(d) $\operatorname{index}(d) = \chi(M)$ is the Euler-Poincaré characteristic of M.
(e) If M is oriented, we let $*$ be the Hodge operator. Then

$$** = (-1)^{p(m-p)} \qquad \text{and} \qquad \delta = (-1)^{mp+m+1} * d * .$$

Furthermore, $*\colon \mathrm{N}(\Delta_p) \simeq \mathrm{N}(\Delta_{m-p})$ gives Poincaré duality.

In the next chapter, we will discuss the Gauss-Bonnet theorem which gives a formula for $\operatorname{index}(d) = \chi(M)$ in terms of curvature.

1.6. The Heat Equation.

Before proceeding with our discussion of the index of an elliptic operator, we must discuss spectral theory. We restrict ourselves to the context of a compact self-adjoint operator to avoid unnecessary technical details. Let $T \in \mathrm{COM}(H)$ be a self-adjoint compact operator on the Hilbert space H. Let

$$\mathrm{spec}(T) = \{ \lambda \in \mathbf{C} : (T - \lambda) \notin \mathrm{GL}(H) \}.$$

Since $\mathrm{GL}(H)$ is open, $\mathrm{spec}(T)$ is a closed subset of C. If $|\lambda| > |T|$, the series

$$g(\lambda) = \sum_{n=0}^{\infty} T^n / \lambda^{n+1}$$

converges to define an element of $\mathrm{END}(H)$. As $(T - \lambda) g(\lambda) = g(\lambda)(T - \lambda) = -I$, $\lambda \notin \mathrm{spec}(T)$. This shows $\mathrm{spec}(T)$ is bounded.

Since T is self-adjoint, $\mathrm{N}(T - \bar{\lambda}) = \{0\}$ if $\lambda \notin \mathbf{R}$. This implies $\mathrm{R}(T - \lambda)$ is dense in H. Since $Tx \cdot x$ is always real, $|(T - \lambda)x \cdot x| \geq \mathrm{im}(\lambda)|\,|x|^2$ so that $|(T - \lambda)x| \geq \mathrm{im}(\lambda)|\,|x|$. This implies $\mathrm{R}(T - \lambda)$ is closed so $T - \lambda$ is surjective. Since $T - \lambda$ is injective, $\lambda \notin \mathrm{spec}(T)$ if $\lambda \in \mathbf{C} - \mathbf{R}$ so $\mathrm{spec}(T)$ is a subset of $\mathbf{R}$. If $\lambda \in [-|T|, |T|]$, we define $E(\lambda) = \{ x \in H : Tx = \lambda x \}$.

Lemma 1.6.1. *Let $T \in \mathrm{COM}(H)$ be self-adjoint. Then*

$$\dim\{E(-|T|) \oplus E(|T|)\} > 0.$$

Proof: If $|T| = 0$ then $T = 0$ and the result is clear. Otherwise choose $|x_n| = 1$ so that $|Tx_n| \to |T|$. We choose a subsequence so $Tx_n \to y$. Let $\lambda = |T|$, we compute:

$$\begin{aligned} |T^2 x_n - \lambda^2 x_n|^2 &= |T^2 x_n|^2 + |\lambda^2 x_n|^2 - 2\lambda^2 T^2 x_n \cdot x_n \\ &\leq 2\lambda^4 - 2\lambda^2 |Tx_n|^2 \to 0. \end{aligned}$$

Since $Tx_n \to y$, $T^2 x_n \to Ty$. Thus $\lambda^2 x_n \to Ty$. Since $\lambda^2 \neq 0$, this implies $x_n \to x$ for $x = Ty/\lambda^2$. Furthermore $|T^2 x - \lambda^2 x| = 0$. Since $(T^2 - \lambda^2) = (T - \lambda)(T + \lambda)$, either $(T + \lambda)x = 0$ so $x \in E(-\lambda) \neq \{0\}$ or $(T - \lambda)(y_1) = 0$ for $y_1 = (T + \lambda)x \neq 0$ so $E(\lambda) \neq \{0\}$. This completes the proof.

If $\lambda \neq 0$, the equation $Tx = \lambda x$ implies the unit disk in $E(\lambda)$ is compact and hence $E(\lambda)$ is finite dimensional. $E(\lambda)$ is T-invariant. Since T is self-adjoint, the orthogonal complement $E(\lambda)$ is also T-invariant. We take an orthogonal decomposition $H = E(|T|) \oplus E(-|T|) \oplus H_1$. T respects this decomposition; we let T_1 be the restriction of T to H_1. Clearly $|T_1| \leq |T|$. If we have equality, then Lemma 1.6.1 implies there exists $x \neq 0$ in H_1

so $T_1 x = \pm|T|x$, which would be false. Thus $|T_1| < |T|$. We proceed inductively to decompose:

$$H = E(\lambda_1) \oplus E(-\lambda_1) \oplus \cdots \oplus E(\lambda_n) \oplus E(-\lambda_n) \oplus H_n$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ and where $|T_n| < \lambda_n$. We also assume $E(\lambda_n) \oplus E(-\lambda_n) \neq \{0\}$. We suppose that the λ_n do not converge to zero but converge to ε positive. For each n, we choose x_n so $|x_n| = 1$ and $Tx_n = \pm\lambda_n x_n$. Since T is self-adjoint, this is an orthogonal decomposition; $|x_j - x_k| = \sqrt{2}$. Since T is compact, we can choose a convergent subsequence $\lambda_n x_n \to y$. Since $\lambda_n \to \varepsilon$ positive, this implies $x_n \to x$. This is impossible so therefore the $\lambda_n \to 0$. We define $H_0 = \bigcap_n H_n$ as a closed subset of H. Since $|T| < \lambda_n$ for all n on H_0, $|T| = 0$ so $T = 0$ and $H_0 = E(0)$. This defines a direct sum decomposition of the form:

$$H = \bigoplus_k E(\mu_k) \oplus E(0)$$

where the $\mu_k \in \mathbf{R}$ are the non-zero subspace of the $E(\lambda_n)$ and $E(-\lambda_n)$. We construct a complete orthonormal system $\{\phi_n\}_{n=1}^\infty$ for H with either $\phi_n \in E(0)$ or $\phi_n \in E(\mu_k)$ for some k. Then $T\phi_n = \lambda_n \phi_n$ so ϕ_n is an eigenvector of T. This proves the spectral decomposition for self-adjoint compact operators:

Lemma 1.6.2. *Let $T \in \mathrm{COM}(H)$ be self-adjoint. We can find a complete orthonormal system for H consisting of eigenvectors of T.*

We remark that this need not be true if T is self-adjoint but not compact or if T is compact but not self-adjoint.

We can use this lemma to prove the following:

Lemma 1.6.3. *Let $P: C^\infty(V) \to C^\infty(V)$ be an elliptic self-adjoint ΨDO of order $d > 0$.*
(a) We can find a complete orthonormal basis $\{\phi_n\}_{n=1}^\infty$ for $L^2(V)$ of eigenvectors of P. $P\phi_n = \lambda_n \phi_n$.
(b) The eigenvectors ϕ_n are smooth and $\lim_{n\to\infty} |\lambda_n| = \infty$.
(c) If we order the eigenvalues $|\lambda_1| \leq |\lambda_2| \leq \cdots$ then there exists a constant $C > 0$ and an exponent $\delta > 0$ such that $|\lambda_n| \geq Cn^\delta$ if $n > n_0$ is large.
Remark: The estimate (c) can be improved to show $|\lambda_n| \sim n^{d/m}$ but the weaker estimate will suffice and is easier to prove.

Proof: $P: H_d(V) \to L^2(V)$ is Fredholm. $P: \mathrm{N}(P)^\perp \cap H_d(M) \to \mathrm{N}(P)^\perp \cap L^2(V)$ is 1-1 and onto; by Gärding's inequality $P\phi \in L^2$ implies $\phi \in H_d$. We let S denote the inverse of this map and extend S to be zero on the finite dimensional space $\mathrm{N}(P)$. Since the inclusion of $H_d(M)$ into $L^2(V)$ is compact, S is a compact self-adjoint operator. S is often referred to as the Greens operator. We find a complete orthonormal system $\{\phi_n\}$ of eigenvectors of S. If $S\phi_n = 0$ then $P\phi_n = 0$ since $\mathrm{N}(S) = \mathrm{N}(P)$. If $S\phi_n = \mu_n\phi_n \neq 0$ then $P\phi_n = \lambda_n\phi_n$ for $\lambda_n = \mu_n^{-1}$. Since the $\mu_n \to 0$, the $|\lambda_n| \to \infty$. If k is an integer such that $dk > 1$, then $P^k - \lambda_n^k$ is elliptic. Since $(P^k - \lambda_n^k)\phi_n = 0$, this implies $\phi_n \in C^\infty(V)$. This completes the proof of (a) and (b).

By replacing P with P^k we replace λ_n by λ_n^k. Since $kd > \frac{m}{2}$ if k is large, we may assume without loss of generality that $d > \frac{m}{2}$ in the proof of (c). We define:

$$|f|_{\infty,0} = \sup_{x\in M} |f(x)| \qquad \text{for } f \in C^\infty(V).$$

We estimate:

$$|f|_{\infty,0} \leq C|f|_d \leq C(|Pf|_0 + |f|_0).$$

Let $F(a)$ be the space spanned by the ϕ_j where $|\lambda_j| \leq a$ and let $n = \dim F(a)$. We estimate $n = n(a)$ as follows. We have

$$|f|_{\infty,0} \leq C(1+a)|f|_0 \qquad \text{on } F(a).$$

Suppose first $V = M \times \mathbf{C}$ is the trivial line bundle. Let c_j be complex constants, then this estimate shows:

$$\left|\sum_{j=1}^n c_j\phi_j(x)\right| \leq C(1+a)\left\{\sum_{j=1}^n |c_j|^2\right\}^{1/2}.$$

If we take $c_j = \bar\phi_j(x)$ then this yields the estimate:

$$\sum_{j=1}^n \phi_j(x)\bar\phi_j(x) \leq C(1+a)\left\{\sum_{j=1}^n \phi_j(x)\bar\phi_j(x)\right\}^{1/2},$$

i.e.,

$$\sum_{j=1}^n \phi_j(x)\bar\phi_j(x) \leq C^2(1+a)^2.$$

We integrate this estimate over M to conclude

$$n \leq C^2(1+a)^2 \operatorname{vol}(M)$$

or equivalently

$$C_1(n - C_2)^{1/2} \leq a = |\lambda_n|$$

from which the desired estimate follows.

If $\dim V = k$, we choose a local orthonormal frame for V to decompose ϕ_j into components ϕ_j^u for $1 \le u \le k$. We estimate:

$$\left|\sum_{j=1}^n c_j^u \phi_j^u(x)\right| \le C(1+a)\left\{\sum_{j=1}^n |c_j^u|^2\right\}^{1/2} \qquad \text{for } u = 1, \dots, k.$$

If we let $c_j^u = \bar\phi_j^u(x)$, then summing over u yields:

$$\left|\sum_{j=1}^n \phi_j \bar\phi_j(x)\right| \le kC(1+a)\left|\sum_{j=1}^n \phi_j \bar\phi_j(x)\right|^{1/2}$$

since each term on the left hand side of the previous inequality can be estimated seperately by the right hand side of this inequality. This means that the constants which arise in the estimate of (c) depend on k, but this causes no difficulty. This completes the proof of (c).

The argument given above for (c) was shown to us by Prof. B. Allard (Duke University) and it is a clever argument to avoid the use of the Schwarz kernel theorem.

Let $P: C^\infty(V) \to C^\infty(V)$ be an elliptic ΨDO of order $d > 0$ which is self-adjoint. We say P has positive definite leading symbol if there exists $p(x,\xi): T^*M \to \mathrm{END}(V)$ such that $p(x,\xi)$ is a positive definite Hermitian matrix for $\xi \ne 0$ and such that $\sigma P - p \in S^{d-1}$ in any coordinate system. The spectrum of such a P is not necessarily non-negative, but it is bounded from below as we shall show. We construct Q_0 with leading symbol $\sqrt{p}$ and let $Q = Q_0^* Q_0$. Then by hypothesis $P - Q \in S^{d-1}$. We compute:

$$\begin{aligned}
(Pf,f) &= ((P-Q)f,f) + (Qf,f)\\
|((P-Q)f,f)| &\le C|f|_{d/2}|(P-Q)f|_{d/2} \le C|f|_{d/2}|f|_{d/2-1}\\
(Qf,f) &= (Q_0f,Q_0f) \qquad \text{and} \qquad |f|_{d/2}^2 < C|f|_0^2 + C|Q_0f|_0^2.
\end{aligned}$$

We use this to estimate for any $\varepsilon > 0$ that:

$$\begin{aligned}
|((P-Q)f,f)| &\le C|f|_{d/2}(|f|_{d/2-1}) \le \varepsilon|f|_{d/2}^2 + C(\varepsilon)|f|_{d/2}|f|_0\\
&\le 2\varepsilon|f|_{d/2}^2 + C(\varepsilon)|f|_0^2\\
&\le 2C\varepsilon|Q_0f|_0^2 + C(\varepsilon)|f|_0^2.
\end{aligned}$$

Choose ε so $2C\varepsilon \le 1$ and estimate:

$$(Pf,f) \ge (Q_0f,Q_0f) - |((P-Q)f,f)| \ge -C(\varepsilon)|f|_0^2.$$

This implies:

LEMMA 1.6.4. *Let $P: C^\infty(V) \to C^\infty(V)$ be an elliptic ΨDO of order $d > 0$ which is self-adjoint with positive definite leading symbol. Then* $\operatorname{spec}(P)$ *is contained in* $[-C, \infty)$ *for some constant* C.

We fix such a P henceforth. The heat equation is the partial differential equation:

$$\left(\frac{d}{dt} + P\right) f(x,t) = 0 \qquad \text{for } t \geq 0 \text{ with } f(x,0) = f(x).$$

At least formally, this has the solution $f(x,t) = e^{-tP} f(x)$. We decompose $f(x) = \sum c_n \phi_n(x)$ for $c_n = (f, \phi_n)$ in a generalized Fourier series. The solution of the heat equation is given by:

$$f(x,t) = \sum_n e^{-t\lambda_n} c_n \phi_n(x).$$

Proceeding formally, we define:

$$K(t,x,y) = \sum_n e^{-t\lambda_n} \phi_n(x) \otimes \bar{\phi}_n(y): V_y \to V_x$$

so that:

$$\begin{aligned} e^{-tP} f(x) &= \int_M K(t,x,y) f(y) \operatorname{dvol}(y) \\ &= \sum_n e^{-t\lambda_n} \phi_n(x) \int_M f(y) \cdot \phi_n(y) \operatorname{dvol}(y). \end{aligned}$$

We regard $K(t,x,y)$ as an endomorphism from the fiber of V over y to the fiber of V over x.

We justify this purely formal procedure using Lemma 1.3.4. We estimate that:

$$|\phi_n|_{\infty,k} \leq C(|\phi_n|_0 + |P^j \phi|_0) = C(1 + |\lambda_n|^j) \qquad \text{where } jd > k + \tfrac{m}{2}.$$

Only a finite number of eigenvalues of P are negative by Lemma 1.6.4. These will not affect convergence questions, so we may assume $\lambda > 0$. Estimate:

$$e^{-t\lambda} \lambda^j \leq t^{-j} C(j) e^{-t\lambda/2}$$

to compute:

$$|K(t,x,y)|_{\infty,k} \leq t^{-j(k)} C(k) \sum_n e^{-t\lambda_n/2}.$$

Since $|\lambda| \geq C n^\delta$ for $\delta > 0$ and n large, the series can be bounded by

$$\sum_{n>0} e^{-tn^{\delta}/2}$$

which converges. This shows $K(t,x,y)$ is an infinitely smooth function of (t,x,y) for $t>0$ and justifies all the formal procedures involved. We compute

$$\mathrm{Tr}_{L^2} e^{-tP} = \sum e^{-t\lambda_n} = \int_M \mathrm{Tr}_{V_x} K(t,x,x)\ \mathrm{dvol}(x).$$

We can use this formula to compute the index of Q:

Lemma 1.6.5. *Let $Q\colon C^\infty(V) \to C^\infty(W)$ be an elliptic ΨDO of order $d>0$. Then for $t>0$, e^{-tQ^*Q} and e^{-tQQ^*} are in $\Psi_{-\infty}$ with smooth kernel functions and*

$$\mathrm{index}(Q) = \mathrm{Tr}\, e^{-tQ^*Q} - \mathrm{Tr}\, e^{-tQQ^*} \qquad \text{for any } t>0.$$

Proof: Since e^{-tQ^*Q} and e^{-tQQ^*} have smooth kernel functions, they are in $\Psi_{-\infty}$ so we must only prove the identity on $\mathrm{index}(Q)$. We define $E_0(\lambda) = \{\,\phi \in L^2(V) : Q^*Q\phi = \lambda\phi\,\}$ and $E_1(\lambda) = \{\,\phi \in L^2(W) : QQ^*\phi = \lambda\phi\,\}$. These are finite dimensional subspaces of smooth sections to V and W. Because $Q(Q^*Q) = (QQ^*)Q$ and $Q^*(QQ^*) = (Q^*Q)Q^*$, Q and Q^* define maps:

$$Q\colon E_0(\lambda) \to E_1(\lambda) \qquad \text{and} \qquad Q^*\colon E_1(\lambda) \to E_0(\lambda).$$

If $\lambda \neq 0$, $\lambda = Q^*Q\colon E_0(\lambda) \to E_1(\lambda) \to E_0(\lambda)$ is an isomorphism so $\dim E_0(\lambda) = \dim E_1(\lambda)$. We compute:

$$\begin{aligned}\mathrm{Tr}\, e^{-tQ^*Q} - \mathrm{Tr}\, e^{-tQQ^*} &= \sum_\lambda e^{-t\lambda}\{\dim E_0(\lambda) - \dim E_1(\lambda)\}\\ &= e^{-t0}\{\dim E_0(0) - \dim E_1(0)\}\\ &= \mathrm{index}(Q)\end{aligned}$$

which completes the proof.

If (Q,V) is a elliptic complex, we use the same reasoning to conclude $e^{-t\Delta_i}$ is in $\Psi_{-\infty}$ with a smooth kernel function. We define $E_i(\lambda) = \{\,\phi \in L^2(V_i) : \Delta_i\phi = \lambda\phi\,\}$. Then $Q_i\colon E_i(\lambda) \to E_{i+1}(\lambda)$ defines an acyclic complex (i.e., $\mathrm{N}(Q_i) = \mathrm{R}(Q_{i-1})$) if $\lambda \neq 0$ so that $\sum (-1)^i \dim E_i(\lambda) = 0$ for $\lambda \neq 0$. Therefore

$$\mathrm{index}(Q) = \sum_i (-1)^i\, \mathrm{Tr}(e^{-t\Delta_i})$$

and Lemma 1.6.5 generalizes to the case of elliptic complexes which are not just two term complexes.

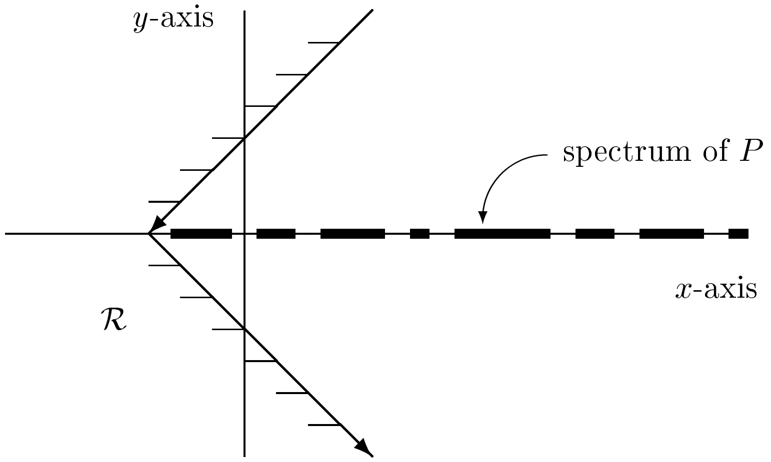
Let P be an elliptic ΨDO of order $d>0$ which is self-adjoint with positive definite leading symbol. Then $\mathrm{spec}(P)$ is contained in $[-C,\infty)$ for some constant C. For $\lambda \notin \mathrm{spec}(P)$, $(P-\lambda)^{-1} \in \mathrm{END}(L^2(V))$ satisfies:

$$|(P-\lambda)^{-1}| = \mathrm{dist}(\lambda, \mathrm{spec}(P))^{-1} = \inf_n |\lambda - \lambda_n|^{-1}.$$

The function $|(P-\lambda)^{-1}|$ is a continuous function of $\lambda \in \mathbf{C} - \operatorname{spec}(P)$. We use the Riemann integral with values in the Banach space $\operatorname{END}(L^2(V))$ to define:

$$e^{-tP} = \frac{1}{2\pi i}\int_\gamma e^{-t\lambda}(P-\lambda)^{-1}\,d\lambda$$

where γ is the path about $[-C,\infty)$ given by the union of the appropriate pieces of the straight lines $\operatorname{Re}(\lambda+C+1) = \pm\operatorname{Im}(\lambda)$ pictured below. We let $\mathcal{R}$ be the closed region of $\mathbf{C}$ consisting of γ together with that component of $\mathbf{C}-\gamma$ which does not contain $[-C,\infty)$.



We wish to extend $(P-\lambda)^{-1}$ to H_s. We note that

$$|\lambda-\mu|^{-1} \le C \qquad \text{for } \lambda \in \mathcal{R},\ \mu \in \operatorname{spec}(P)$$

so that $|(P-\lambda)^{-1}f|_0 < C|f|_0$. We use Lemma 1.3.5 to estimate:

$$\begin{aligned}|(P-\lambda)^{-1}f|_{kd} &\le C\{|P^k(P-\lambda)^{-1}f|_0 + |(P-\lambda)^{-1}f|_0\}\\ &\le C\{|P^{k-1}f|_0 + |\lambda P^{k-1}(P-\lambda)^{-1}f|_0 + |f|_0\}\\ &\le C\{|f|_{kd-d} + |\lambda|\,|(P-\lambda)^{-1}f|_{kd-d}\}.\end{aligned}$$

If $k=1$, this implies $|(P-\lambda)^{-1}f|_d \le C(1+|\lambda|)|f|_0$. We now argue by induction to estimate:

$$|(P-\lambda)^{-1}f|_{kd} \le C(1+|\lambda|)^{k-1}|f|_{kd-d}.$$

We now interpolate. If $s>0$, choose k so $kd \ge s > kd-d$ and estimate:

$$\begin{aligned}|(P-\lambda)^{-1}f|_s &\le C|(P-\lambda)^{-1}f|_{kd} \le C(1+|\lambda|)^{k-1}|f|_{kd-d}\\ &\le C(1+|\lambda|)^{k-1}|f|_s.\end{aligned}$$

Similarly, if $s < 0$, we use duality to estimate:

$$|((P-\lambda)^{-1}f,g)| = |(f,(P-\bar{\lambda})^{-1}g)| \le C(1+|\lambda|)^{k-1}|f|_s|g|_{-s}$$

which by Lemma 1.3.4 shows

$$|(P-\lambda)^{-1}f|_s < C(1+|\lambda|)^{k-1}|f|_s$$

in this case as well.

LEMMA 1.6.6. *Let P be a elliptic ΨDO of order $d > 0$ which is self-adjoint. We suppose the leading symbol of P is positive definite. Then:*
(a) Given s there exists $k = k(s)$ and $C = C(s)$ so that

$$|(P-\lambda)^{-1}f|_s \le C(1+|\lambda|)^k|f|_s$$

for all $\lambda \in \mathcal{R}$.
(b) Given j there exists $k = k(j,d)$ so $(P^k-\lambda)^{-1}$ represents a smoothing operator with C^j kernel function which is of trace class for any $\lambda \in \mathcal{R}$.

PROOF: (a) follows from the estimates previously. To prove (b) we let

$$K_k(x,y,\lambda) = \sum_n \frac{1}{\lambda_n^k-\lambda}\phi_n(x)\otimes\phi_n(y)$$

be the kernel of $(P^k-\lambda)^{-1}$. The region $\mathcal{R}$ was chosen so $|\tilde{\lambda}-\lambda| \ge \varepsilon|\tilde{\lambda}|$ for some $\varepsilon > 0$ and $\tilde{\lambda} \in \mathbf{R}^+$. Therefore $|(\lambda_n^k-\lambda)|^{-1} \le \varepsilon|\lambda_n^k|^{-1} \le \varepsilon^{-1}n^{-k\delta}$ by Lemma 1.6.3. The convergence of the sum defining K_k then follows using the same arguments as given in the proof that e^{-tP} is a smoothing operator.

This technical lemma will be used in the next subsection to estimate various error terms which occur in the construction of a parametrix.

1.7. Local Formula for The Index of an Elliptic Operator.

In this subsection, let $P: C^\infty(V) \to C^\infty(V)$ be a self-adjoint elliptic partial differential operator of order $d > 0$. Decompose the symbol $\sigma(P) = p_d + \cdots + p_0$ into homogeneous polymonials p_j of order j in $\xi \in T^*M$. We assume p_d is a positive definite Hermitian matrix for $\xi \neq 0$. Let the curve γ and the region $\mathcal{R}$ be as defined previously.

The operator $(P-\lambda)^{-1}$ for $\lambda \in \mathcal{R}$ is not a pseudo-differential operator. We will approximate $(P-\lambda)^{-1}$ by a pseudo-differential operator $R(\lambda)$ and then use that approximation to obtain properties of $\exp(-tP)$. Let U be an open subset of $\mathbf{R}^m$ with compact closure. Fix $d \in \mathbf{Z}$. We make the following definition to generalize that given in section 1.2:

Definition. *$q(x,\xi,\lambda) \in S^k(\lambda)(U)$ is a symbol of order k depending on the complex parameter $\lambda \in \mathcal{R}$ if*

(a) *$q(x,\xi,\lambda)$ is smooth in $(x,\xi,\lambda) \in \mathbf{R}^m \times \mathbf{R}^m \times \mathcal{R}$, has compact x-support in U and is holomorphic in λ.*

(b) *For all (α,β,γ) there exist constants $C_{\alpha,\beta,\gamma}$ such that:*

$$|D_x^\alpha D_\xi^\beta D_\lambda^\gamma q(x,\xi,\gamma)| \leq C_{\alpha,\beta,\gamma}(1 + |\xi| + |\lambda|^{1/d})^{k-|\beta|-d|\gamma|}\,.$$

We say that $q(x,\xi,\gamma)$ is homogeneous of order k in (ξ,λ) if

$$q(x,t\xi,t^d\lambda) = t^k q(x,\xi,\gamma) \qquad \text{for } t \geq 1.$$

We think of the parameter λ as being of order d. It is clear if q is homogeneous in (ξ,λ), then it satisfies the decay conditions (b). Since the p_j are polynomials in ξ, they are regular at $\xi = 0$ and define elements of $S^j(\lambda)$. If the p_j were only pseudo-differential, we would have to smooth out the singularity at $\xi = 0$ which would destroy the homogeneity in (ξ,λ) and they would not belong to $S^j(\lambda)$; equivalently, $d_\xi^\alpha p_j$ will not exhibit decay in λ for $|\alpha| > j$ if p_j is not a polynomial. Thus the restriction to differential operators is an essential one, although it is possible to discuss some results in the pseudo-differential case by taking more care with the estimates involved.

We also note that $(p_d - \lambda)^{-1} \in S^{-d}(\lambda)$ and that the spaces $S^*(\lambda)$ form a symbol class closed under differentiation and multiplication. They are suitable generalizations of ordinary pseudo-differential operators discussed earlier.

We let $\Psi_k(\lambda)(U)$ be the set of all operators $Q(\lambda): C_0^\infty(U) \to C_0^\infty(U)$ with symbols $q(x,\xi,\gamma)$ in $S^k(\lambda)$ having x-support in U; we let $q = \sigma Q$. For any fixed λ, $Q(\lambda) \in \Psi_k(U)$ is an ordinary pseudo-differential operator of order k. The new features arise from the dependence on the parameter λ. We extend the definition of $\sim$ given earlier to this wider class by:

$$q \sim \sum_j q_j$$

if for every $k > 0$ there exists $n(k)$ so $n \ge n(k)$ implies $q - \sum_{j\le n} q_j \in S^{-k}(\lambda)$. Lemmas 1.2.1, 1.2.3(b), and 1.3.2 generalize easily to yield the following Lemma. We omit the proofs in the interests of brevity as they are identical to those previously given with only minor technical modifications.

LEMMA 1.7.1.
(a) Let $Q_i \in \Psi_{k_i}(\lambda)(U)$ with symbol q_i. Then $Q_1Q_2 \in \Psi_{k_1+k_2}(\lambda)(U)$ has symbol q where:

$$q \sim \sum_\alpha d_\xi^\alpha q_1 D_x^\alpha q_2/\alpha!\,.$$

(b) Given $n > 0$ there exists $k(n) > 0$ such that $Q \in \Psi_{-k(n)}(\lambda)(U)$ implies $Q(\lambda)$ defines a continuous map from $H_{-n} \to H_n$ and we can estimate the operator norm by:

$$|Q(\lambda)|_{-n,n} \le C(1+|\lambda|)^{-n}.$$

(c) If $h: U \to \widetilde{U}$ is a diffeomorphism, then $h_*: \Psi_k(\lambda)(U) \to \Psi_k(\lambda)(\widetilde{U})$ and

$$\sigma(h_*P) - p(h^{-1}x_1,\,(dh^{-1}(x_1))^t\xi_1,\lambda) \in S^{k-1}(\lambda)(\widetilde{U}).$$

As before, we let $\Psi(\lambda)(U) = \bigcup_n \Psi_n(\lambda)(U)$ be the set of all pseudo-differential operators depending on a complex parameter $\lambda \in \mathcal{R}$ defined over U. This class depends on the order d chosen and also on the region $\mathcal{R}$, but we supress this dependence in the interests of notational clarity. There is no analogue of the completeness of Lemma 1.2.8 for such symbols since we require analyticity in λ. Thus in constructing an approximation to the parametrix, we will always restrict to a finite sum rather than an infinite sum.

Using (c), we extend the class $\Psi(\lambda)$ to compact manifolds using a partition of unity argument. (a) and (b) generalize suitably. We now turn to the question of ellipticity. We wish to solve the equation:

$$\sigma(R(\lambda)(P-\lambda)) - I \sim 0.$$

Inductively we define $R(\lambda)$ with symbol $r_0 + r_1 + \cdots$ where $r_j \in S^{-d-j}(\lambda)$. We define $p'_j(x,\xi,\lambda) = p_j(x,\xi)$ for $j < d$ and $p'_d(x,\xi,\lambda) = p_d(x,\xi) - \lambda$. Then $\sigma(P-\lambda) = \sum_{j=0}^d p'_j$. We note that $p'_j \in S^j(\lambda)$ and that ${p'_d}^{-1} \in S^{-d}(\lambda)$ so that $(P-\lambda)$ is elliptic in a suitable sense. The essential feature of this construction is that the parameter λ is absorbed in the leading symbol and not treated as a lower order perturbation.

The equation $\sigma(R(\lambda)(P-\lambda)) \sim I$ yields:

$$\sum_{\alpha,j,k} d_\xi^\alpha r_j \cdot D_x^\alpha p'_k/\alpha! \sim I.$$

We decompose this series into terms homogeneous of order $-n$ to write:

$$\sum_n \sum_{|\alpha|+j+d-k=n} d_\xi^\alpha r_j \cdot D_x^\alpha p'_k/\alpha! \sim I$$

where $j, k \geq 0$ and $k \leq d$. There are no terms with $n < 0$, i.e., positive order. If we investigate the term with $n = 0$, we arrive at the condition $r_0 p'_d = I$ so $r_0 = (p_d - \lambda)^{-1}$ and inductively:

$$r_n = -r_0 \sum_{\substack{|\alpha|+j+d-k=n \\ j<n}} d_\xi^\alpha r_j D_x^\alpha p'_k/\alpha!\,.$$

If $k = d$ then $|\alpha| > 0$ so $D_x^\alpha p'_k = D_x^\alpha p_k$ in this sum. Therefore we may replace p'_k by p_k and write

$$r_n = -r_0 \sum_{\substack{|\alpha|+j+d-k=n \\ j<n}} d_\xi^\alpha r_j D_x^\alpha p_k/\alpha!\,.$$

In a similar fashion, we can define $\widetilde{R}(\lambda)$ so $\sigma((P-\lambda)\widetilde{R}(\lambda) - I) \sim 0$. This implies $\sigma(R(\lambda) - \widetilde{R}(\lambda)) \sim 0$ so $R(\lambda)$ provides a formal left and right inverse.

Since such an inverse is unique modulo lower order terms, $R(\lambda)$ is well defined and unique modulo lower order terms in any coordinate system. We define $R(\lambda)$ globally on M using a partition of unity argument. To avoid convergence questions, we shall let $R(\lambda)$ have symbol $r_0 + \cdots + r_{n_0}$ where n_0 is chosen to be very large. $R(\lambda)$ is unique modulo $\Psi_{-n_0-d}(\lambda)$. For notational convenience, we supress the dependence of $R(\lambda)$ upon n_0.

$R(\lambda)$ gives a good approximation to $(P-\lambda)^{-1}$ in the following sense:

LEMMA 1.7.2. *Let $k > 0$ be given. We can choose $n_0 = n_0(k)$ so that*

$$|\{(P-\lambda)^{-1} - R(\lambda)\}f|_k \leq C_k(1+|\lambda|)^{-k}|f|_{-k} \qquad \text{for } \lambda \in \mathcal{R},\ f \in C^\infty(V).$$

Thus $(P-\lambda)^{-1}$ is approximated arbitrarily well by the parametrix $R(\lambda)$ in the operator norms as $\lambda \to \infty$.

PROOF: We compute:

$$\begin{aligned} |\{(P-\lambda)^{-1} - R(\lambda)\}f|_k &= |(P-\lambda)^{-1}\{I - (P-\lambda)R(\lambda)\}f|_k \\ &\leq C_k(1+|\lambda|^\nu)|(I-(P-\lambda)R(\lambda))f|_k \end{aligned}$$

by Lemma 1.6.6. Since $I - (P-\lambda)R(\lambda) \in S^{-n_0}$, we use Lemma 1.7.1 to complete the proof.

We define $E(t) = \frac{1}{2\pi i}\int_\gamma e^{-t\lambda}R(\lambda)\,d\lambda$. We will show shortly that this has a smooth kernel $K'(t,x,y)$. Let $K(t,x,y)$ be the smooth kernel of e^{-tP}. We will use Lemma 1.2.9 to estimate the difference between these two kernels. We compute:

$$E(t) - e^{-tP} = \frac{1}{2\pi i} \int_\gamma e^{-t\lambda} (R(\lambda) - (P - \lambda)^{-1})\, d\lambda.$$

We assume $0 < t < 1$ and make a change of variables to replace $t\lambda$ by λ. We use Cauchy's theorem to shift the resulting path $t\gamma$ inside $\mathcal{R}$ back to the original path γ where we have uniform estimates. This expresses:

$$E(t) - e^{-tP} = \frac{1}{2\pi i} \int_\gamma e^{-\lambda} (R(t^{-1}\lambda) - (P - t^{-1}\lambda)) t^{-1}\, d\lambda$$

We estimate therefore:

$$\begin{aligned} |E(t) - e^{-tP}|_{-k,k} &\le C_k \int_\gamma |e^{-\lambda} (1 + t^{-1}|\lambda|)^{-k}\, d\lambda| \\ &\le C_k t^k \end{aligned}$$

provided n_0 is large enough. Lemma 1.2.9 implies

LEMMA 1.7.3. *Let k be given. If n_0 is large enough, we can estimate:*

$$|K(t, x, y) - K'(t, x, y)|_{\infty,k} \le C_k t^k \qquad \text{for } 0 < t < 1.$$

This implies that K' approximates K to arbitrarily high jets as $t \to 0$.

We now study the operator $E(t)$. We define

$$e_n(t, x, \xi) = \frac{1}{2\pi i} \int_\gamma e^{-t\lambda} r_n(x, \xi, \lambda)\, d\lambda,$$

then $E(t)$ is a ΨDO with symbol $e_0 + \cdots + e_{n_0}$. If we integrate by parts in λ we see that we can express

$$e_n(t, x, \xi) = \frac{1}{2\pi i} \frac{1}{t^k} \int_\gamma e^{-t\lambda} \frac{d^k}{d\lambda^k} r_n(x, \xi, \lambda)\, d\lambda.$$

Since $\dfrac{d^k}{d\lambda^k} r_n$ is homogeneous of degree $-d - n - kd$ in (ξ, λ), we see that $e_n(t, x, \xi) \in S^{-\infty}$ for any $t > 0$. Therefore we can apply Lemma 1.2.5 to conclude $E_n(t)$ which has symbol $e_n(t)$ is represented by a kernel function defined by:

$$K_n(t, x, y) = \int e^{i(x-y)\cdot\xi}\, e_n(t, x, \xi)\, d\xi.$$

We compute:

$$K_n(t,x,x) = \frac{1}{2\pi i}\iint_\gamma e^{-t\lambda} r_n(x,\xi,\lambda)\,d\lambda\,d\xi.$$

We make a change of variables to replace λ by $t^{-1}\lambda$ and ξ by $t^{-1/d}\xi$ to compute:

$$\begin{aligned} K_n(t,x,x) &= t^{-\frac{m}{d}-1}\frac{1}{2\pi i}\iint_\gamma e^{-\lambda} r_n(x, t^{\frac{-1}{d}}\xi, t^{-1}\lambda)\,d\lambda\,d\xi \\ &= t^{-\frac{m}{d}-1+\frac{n+d}{d}}\frac{1}{2\pi i}\iint_\gamma e^{-\lambda} r_n(x,\xi,\lambda)\,d\lambda\,d\xi \\ &= t^{\frac{n-m}{d}} e_n(x) \end{aligned}$$

where we let this integral define $e_n(x)$.

Since $p_d(x,\xi)$ is assumed to be positive definite, d must be even since p_d is a polynomial in ξ. Inductively we express r_n as a sum of terms of the form:

$$r_0^{j_1} q_1 r_0^{j_2} q_2 \dots r_0^{j_k}$$

where the q_k are polynomials in (x,ξ). The sum of the degrees of the q_k is odd if n is odd and therefore $r_n(x,-\xi,\lambda) = -r_n(x,\xi,\lambda)$. If we replace ξ by $-\xi$ in the integral defining $e_n(x)$, we conclude $e_n(x) = 0$ if n is odd.

LEMMA 1.7.4. *Let P be a self-adjoint elliptic partial differential operator of order $d > 0$ such that the leading symbol of P is positive definite for $\xi \neq 0$. Then:*
(a) If we choose a coordinate system for M near a point $x \in M$ and choose a local frame for V, we can define $e_n(x)$ using the complicated combinatorial recipe given above. $e_n(x)$ depends functorially on a finite number of jets of the symbol $p(x,\xi)$.
(b) If $K(t,x,y)$ is the kernel of e^{-tP} then

$$K(t,x,x) \sim \sum_{n=0}^{\infty} t^{\frac{n-m}{d}} e_n(x) \qquad \text{as } t \to 0^+$$

i.e., given any integer k there exists $n(k)$ such that:

$$\left| K(t,x,x) - \sum_{n\le n(k)} t^{\frac{n-m}{d}} e_n(x) \right|_{\infty,k} < C_k t^k \qquad \text{for } 0<t<1.$$

(c) $e_n(x) \in \mathrm{END}(V,V)$ is invariantly defined independent of the coordinate system and local frame for V.

(d) $e_n(x) = 0$ *if* n *is odd.*

Proof: (a) is immediate. We computed that

$$K'(t,x,x) = \sum_{n=0}^{n_0} t^{\frac{n-m}{d}} e_n(x)$$

so (b) follows from Lemma 1.7.3. Since $K(t,x,x)$ does not depend on the coordinate system chosen, (c) follows from (b). We computed (d) earlier to complete the proof.

We remark that this asymptotic representation of $K(t,x,x)$ exists for a much wider class of operators P. We refer to the literature for further details. We shall give explicit formulas for e_0, e_2, and e_4 in section 4.8 for certain examples arising in geometry.

The invariants $e_n(x) = e_n(x,P)$ are sections to the bundle of endomorphisms, $\mathrm{END}(V)$. They have a number of functorial properties. We sumarize some of these below.

Lemma 1.7.5.
(a) Let $P_i\colon C^\infty(V_i) \to C^\infty(V_i)$ *be elliptic self-adjoint partial differential operators of order* $d > 0$ *with positive definite leading symbol. We form* $P = P_1 \oplus P_2\colon C^\infty(V_1 \oplus V_2) \to C^\infty(V_1 \oplus V_2)$*. Then* P *is an elliptic self-adjoint partial differential operator of order* $d > 0$ *with positive definite leading symbol and* $e_n(x, P_1 \oplus P_2) = e_n(x,P_1) \oplus e_n(x,P_2)$*.*
(b) Let $P_i\colon C^\infty(V_i) \to C^\infty(V_i)$ *be elliptic self-adjoint partial differential operators of order* $d > 0$ *with positive definite leading symbol defined over different manifolds* M_i*. We let*

$$P = P_1 \otimes 1 + 1 \otimes P_2\colon C^\infty(V_1 \otimes V_2) \to C^\infty(V_1 \otimes V_2)$$

over $M = M_1 \times M_2$*. Then* P *is an elliptic self-adjoint partial differential operator of order* $d > 0$ *with positive definite leading symbol over* M *and*

$$e_n(x,P) = \sum_{p+q=n} e_p(x_1,P_1) \otimes e_q(x_2,P_2).$$

(c) Let $P\colon C^\infty(V) \to C^\infty(V)$ *be an elliptic self-adjoint partial differential operator of order* $d > 0$ *with positive definite leading symbol. We decompose the total symbol of* P *in the form:*

$$p(x,\xi) = \sum_{|\alpha| \le d} p_\alpha(x)\xi^\alpha.$$

Fix a local frame for V *and a system of local coordinates on* M*. We let indices* a, b *index the local frame and let* $p_\alpha = p_{ab\alpha}$ *give the components*

of the matrix p_α. *We introduce formal variables* $p_{ab\alpha/\beta} = D_x^\beta p_{ab\alpha}$ *for the jets of the symbol of* P. *Define* $\mathrm{ord}(p_{ab\alpha/\beta}) = |\beta| + d - |\alpha|$. *Then* $e_n(x,P)$ *can be expressed as a sum of monomials in the* $\{p_{ab\alpha/\beta}\}$ *variables which are homogeneous of order* n *in the jets of the symbol as discussed above and with coefficients which depend smoothly on the leading symbol* $\{p_{ab\alpha}\}_{|\alpha|=d}$.
(d) If the leading symbol of P *is scalar, then the invariance theory can be simplified. We do not need to introduce the components of the symbol explicitly and can compute* $e_n(x,P)$ *as a non-commutative polynomial which is homogeneous of order* n *in the* $\{p_{\alpha/\beta}\}$ *variables with coefficients which depend smoothly upon the leading symbol.*
Remark: The statement of this lemma is somewhat technical. It will be convenient, however, to have these functorial properties precisely stated for later reference. This result suffices to study the index theorem. We shall give one more result which generalizes this one at the end of this section useful in studying the eta invariant. The objects we are studying have a bigrading; one grading comes from counting the number of derivatives in the jets of the symbol, while the other measures the degree of homogeneity in the (ξ, λ) variables.

Proof: (a) and (b) follow from the identities:

$$\begin{aligned} e^{-t(P_1\oplus P_2)} &= e^{-tP_1} \oplus e^{-tP_2} \\ e^{-t(P_1\otimes 1 + 1\otimes P_2)} &= e^{-tP_1} \otimes e^{-tP_2} \end{aligned}$$

so the kernels satisfy the identities:

$$\begin{aligned} K(t,x,x,P_1\oplus P_2) &= K(t,x,x,P_1) \oplus K(t,x,x,P_2) \\ K(t,x,x,P_1\otimes 1 + 1\otimes P_2) &= K(t,x_1,x_1,P_1) \otimes K(t,x_2,x_2,P_2). \end{aligned}$$

We equate equal powers of t in the asymptotic series:

$$\begin{aligned} &\sum t^{\frac{n-m}{d}} e_n(x, P_1 \oplus P_2) \\ &\qquad\qquad \sim \sum t^{\frac{n-m}{d}} e_n(x,P_1) \oplus \sum t^{\frac{n-m}{d}} e_n(x,P_2) \\ &\sum t^{\frac{n-m}{d}} e_n(x, P_1\otimes 1 + 1\otimes P_2) \\ &\qquad\qquad \sim \left\{\sum t^{\frac{p-m_1}{d}} e_p(x_1,P_1)\right\} \otimes \left\{\sum t^{\frac{q-m_2}{d}} e_q(x_2,P_2)\right\} \end{aligned}$$

to complete the proof of (a) and (b). We note that the multiplicative property (b) is a direct consequence of the identity $e^{-t(a+b)} = e^{-ta}e^{-tb}$; it was for this reason we worked with the heat equation. Had we worked instead with the zeta function to study $\mathrm{Tr}(P^{-s})$ the corresponding multiplicative property would have been much more difficult to derive.

We prove (c) as follows: expand $p(x,\xi) = \sum_j p_j(x,\xi)$ into homogeneous polynomials where $p_j(x,\xi) = \sum_{|\alpha|=j} p_\alpha(x)\xi^\alpha$. Suppose for the moment that $p_d(x,\xi) = p_d(x,\xi)I_V$ is scalar so it commutes with every matrix. The approximate resolvant is given by:

$$\begin{aligned} r_0 &= (p_d(x,\xi)-\lambda)^{-1} \\ r_n &= -r_0 \sum_{\substack{|\alpha|+j+d-k=n \\ j<n}} d_\xi^\alpha r_j D_x^\alpha p_k/\alpha!\,. \end{aligned}$$

Since p_d is scalar, r_0 is scalar so it and all its derivatives commute with any matrix. If we assume inductively r_j is of order j in the jets of the symbol, then $d_\xi^\alpha r_j D_x^\alpha p_k/\alpha!$ is homogeneous of order $j+|\alpha|+d-k=n$ in the jets of the symbol. We observe inductively we can decompose r_n in the form:

$$r_n = \sum_{\substack{n=dj+d-|\alpha| \\ |\alpha|\le n}} r_0^j r_{n,j,\alpha}(x)\xi^\alpha$$

where the $r_{n,j,\alpha}(x)$ are certain non-commutative polynomials in the jets of the total symbol of P which are homogeneous of order n in the sense we have defined.

The next step in the proof of Lemma 1.7.4 was to define:

$$\begin{aligned} e_n(t,x,\xi) &= \frac{1}{2\pi i}\int_\gamma e^{-t\lambda} r_n(x,\xi,\lambda)\,d\lambda \\ &= \frac{1}{2\pi i}\sum_{j,\alpha} r_{n,j,\alpha}(x)\xi^\alpha \int_\gamma e^{-t\lambda} r_0^j\,d\lambda \\ e_n(x) &= \frac{1}{2\pi i}\sum_{j,\alpha} r_{n,j,\alpha}(x)\int\left(\int_\gamma e^{-\lambda} r_0^j\,d\lambda\right)\xi^\alpha\,d\xi. \end{aligned}$$

We note again that $e_n = 0$ if n is odd since the resulting function of ξ would be odd. The remaining coefficients of $r_{n,j,\alpha}(x)$ depend smoothly on the leading symbol p_d. This completes the proof of the lemma.

If the leading symbol is not scalar, then the situation is more complicated. Choose a local frame to represent the $p_\alpha = p_{ab\alpha}$ as matrices. Let $h(x,\xi,\lambda) = \det(p_d(x,\xi)-\lambda)^{-1}$. By Cramer's rule, we can express $r_0 = r_{0ab} = \{(p_d-\lambda)^{-1}\}_{ab}$ as *polynomials* in the $\{h,\lambda,\xi,p_{ab\alpha}(x)\}$ variables. We noted previously that r_n was a sum of terms of the form $r_0 q_0 r_0 q_1 \dots r_0 q_k r_0$. The matrix components of such a product can in turn be decomposed as a sum of terms $h^v\tilde q_v$ where $\tilde q_v$ is a polynomial in the $(\lambda,\xi,p_{ab\alpha/\beta})$ variables. The same induction argument which was used in the scalar case shows the $\tilde q_v$ will be homogeneous of order n in the jets of the symbol. The remainder of the argument is the same; performing the $d\lambda\, d\xi$ integral yields a smooth function of the leading symbol as a coefficient of such a term, but this function is not in general rational of course. The scalar case is much simpler as we don't need to introduce the components of the matrices explicitly (although the frame dependence is still there of course) since r_0 can be commuted. This is a technical point, but one often useful in making specific calculations.

We define the scalar invariants

$$a_n(x,P) = \operatorname{Tr} e_n(x,P)$$

where the trace is the fiber trace in V over the point x. These scalar invariants $a_n(x,P)$ inherit suitable functorial properties from the functorial properties of the invariants $e_n(x,P)$. It is immediate that:

$$\begin{aligned} \operatorname{Tr}_{L^2} e^{-tP} &= \int_M \operatorname{Tr}_{V_x} K(t,x,x)\, \mathrm{dvol}(x) \\ &\sim \sum_{n=0}^{\infty} t^{\frac{m-n}{d}} \int_M a_n(x,P)\, \mathrm{dvol}(x). \\ &\sim \sum_{n=0}^{\infty} t^{\frac{m-n}{d}} a_n(P) \end{aligned}$$

where $a_n(P) = \int_M a_n(x,P)\, \mathrm{dvol}(x)$ is the integrated invariant. This is a spectral invariant of P which can be computed from local information about the symbol of P.

Let (P,V) be an elliptic complex of differential operators and let Δ_i be the associated Laplacians. We define:

$$a_n(x,P) = \sum_i (-1)^i \operatorname{Tr} e_n(x,\Delta_i)$$

then Lemma 1.6.5 and the remark which follows this lemma imply

$$\operatorname{index}(P) = \sum_i (-1)^i \operatorname{Tr} e^{-t\Delta_i} \sim \sum_{n=0}^{\infty} t^{\frac{n-m}{d}} \int_M a_n(x,P)\, \mathrm{dvol}(x).$$

Since the left hand side does not depend on the parameter t, we conclude:

THEOREM 1.7.6. *Let (P,V) be a elliptic complex of differential operators.*
(a) $a_n(x,P)$ can be computed in any coordinate system and relative to any local frames as a complicated combinatorial expression in the jets of P and of P^* up to some finite order. $a_n = 0$ if n is odd.

(b)

$$\int_M a_n(x,P)\,\mathrm{dvol}(x) = \begin{cases} \mathrm{index}(P) & \text{if } n = m \\ 0 & \text{if } n \neq m\,. \end{cases}$$

We note as an immediate consequence that $\mathrm{index}(P) = 0$ if m is odd. The local nature of the invariants a_n will play a very important role in our discussion of the index theorem. We will develop some of their functorial properties at that point. We give one simple example to illustrate this fact.

Let $\pi\colon M_1 \to M_2$ be a finite covering projection with fiber F. We can choose the metric on M_1 to be the pull-back of a metric on M_2 so that π is an isometry. Let d be the operator of the de Rham complex. $a_n(x,d) = a_n(\pi x, d)$ since a_n is locally defined. This implies:

$$\chi(M_1) = \int_{M_1} a_m(x,d)\,\mathrm{dvol}(x) = |F| \int_{M_2} a_m(x,d)\,\mathrm{dvol}(x) = |F|\chi(M_2)$$

so the Euler-Poincare characteristic is multiplicative under finite coverings. A similar argument shows the signature is multiplicative under orientation preserving coverings and that the arithmetic genus is multiplicative under holomorphic coverings. (We will discuss the signature and arithmetic genus in more detail in Chapter 3).

We conclude this section with a minor generalization of Lemma 1.7.5 which will be useful in discussing the eta invariant in section 1.10:

Lemma 1.7.7. *Let $P\colon C^\infty(V) \to C^\infty(V)$ be an elliptic self-adjoint partial differential operator of order $d > 0$ with positive definite leading symbol and let Q be an auxilary partial differential operator on $C^\infty(V)$ of order $a \geq 0$. Qe^{-tP} is an infinitely smoothing operator with kernel $QK(t,x,y)$. There is an asymptotic expansion on the diagonal:*

$$\{QK(t,x,y)\}|_{x=y} \sim \sum_{n=0}^{\infty} t^{(n-m-a)/d} e_n(x,Q,P).$$

The e_n are smooth local invariants of the jets of the symbols of P and Q and $e_n = 0$ if $n + a$ is odd. If we let $Q = \sum q_\alpha D_x^\alpha$, we define $\mathrm{ord}(q_\alpha) = a - |\alpha|$. If the leading symbol of P is scalar, we can compute $e_n(x,Q,P)$ as a non-commutative polynomial in the variables $\{q_\alpha, p_{\alpha/\beta}\}$ which is homogeneous of order n in the jets with coefficients which depend smoothly on the $\{p_\alpha\}_{|\alpha|=d}$ variables. This expression is <u>linear</u> *in the $\{q_\alpha\}$ variables and does not involve the higher jets of these variables. If the leading symbol of P is not scalar, there is a similar expression for the matrix components of e_n in the matrix components of these variables. $e_n(x,Q,P)$ is additive and multiplicative in the sense of Lemma 1.7.5(a) and (b) with respect to direct sums of operators and tensor products over product manifolds.*

Remark: In fact it is not necessary to assume P is self-adjoint to define e^{-tP} and to define the asymptotic series. It is easy to generalize the techniques we have developed to prove this lemma continues to hold true if we only asume that $\det(p_d(x,\xi)-\lambda)\neq 0$ for $\xi\neq 0$ and $\mathrm{Im}(\lambda)\leq 0$. This implies that the spectrum of P is pure point and contained in a cone about the positive real axis. We omit the details.

PROOF: Qe^{-tP} has smooth kernel $QK(t,x,y)$ where Q acts as a differential operator on the x variables. One representative piece is:

$$\begin{aligned} q_\alpha(x)D_x^\alpha K_n(t,x,y) &= q_\alpha(x)D_x^\alpha \int e^{i(x-y)\cdot\xi}\, e_n(t,x,\xi)\,d\xi \\ &= q_\alpha(x)\sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!\,\gamma!}\int e^{i(x-y)\cdot\xi}\,\xi^\beta D_x^\gamma e_n(t,x,\xi)\,d\xi. \end{aligned}$$

We define $q_{\beta,\gamma}(x) = q_\alpha(x)\dfrac{\alpha!}{\beta!\,\gamma!}$ where $\beta+\gamma=\alpha$, then a representative term of this kernel has the form:

$$q_{\beta,\gamma}(x)\int e^{i(x-y)\cdot\xi}\,\xi^\beta D_x^\gamma e_n(t,x,\xi)\,d\xi$$

where $q_{\beta,\gamma}(x)$ is homogeneous of order $a-|\beta|-|\gamma|$ in the jets of the symbol of Q.

We suppose the leading symbol is scalar to simplify the computations; the general case is handled similarly using Cramer's rule. We express

$$e_n = \frac{1}{2\pi i}\int e^{-t\lambda} r_n(x,\xi,\lambda)\,d\lambda$$

where r_n is a sum of terms of the form $r_{n,j,\alpha}(x)\xi^\alpha r_0^j$. When we differentiate such a term with respect to the x variables, we change the order in the jets of the symbol, but do not change the (ξ,λ) degree of homogeneity. Therefore $QK(t,x,y)$ is a sum of terms of the form:

$$q_{\beta,\gamma}(x)\int\left\{\int e^{-t\lambda} r_0^j(x,\xi,\lambda)\,d\lambda\right\} e^{i(x-y)\cdot\xi}\,\xi^\beta\xi^\delta\tilde r_{n,j,\delta}(x)\,d\xi$$

where:

$$\begin{aligned} &r_{n,j,\delta} \text{ is of order } n+|\gamma| \text{ in the jets of the symbol,} \\ &-d-n = -jd+|\delta|. \end{aligned}$$

We evaluate on the diagonal to set $x=y$. This expression is homogeneous of order $-d-n+|\beta|$ in (ξ,λ), so when we make the appropriate change of variables we calculate this term becomes:

$$t^{(n-m-|\beta|)/d}q_{\beta,\gamma}(x)\tilde r_{n,j,\delta}(x)\cdot\left\{\int\left\{e^{-\lambda}r_0^j(x,\xi,\lambda)\,d\lambda\right\}\xi^\beta\xi^\delta\,d\xi\right\}.$$

This is of order $n+|\gamma|+a-|\beta|-|\gamma| = n+a-|\beta| = \nu$ in the jets of the symbol. The exponent of t is $(n-m-|\beta|)/d = (\nu-a+|\beta|-m-|\beta|)/d = (\nu-a-m)/d$. The asymptotic formula has the proper form if we index by the order in the jets of the symbol ν. It is linear in q and vanishes if $|\beta|+|\delta|$ is odd. As d is even, we compute:

$$\nu + a = n + a - |\beta| + a = jd - d - |\delta| + 2a - |\beta| \equiv |\delta| + |\beta| \mod 2$$

so that this term vanishes if $\nu + a$ is odd. This completes the proof.

1.8. Lefschetz Fixed Point Theorems.

Let $T: M \to M$ be a continuous map. T^* defines a map on the cohomology of M and the Lefschetz number of T is defined by:

$$L(T) = \sum (-1)^p \operatorname{Tr}(T^* \text{ on } H^p(M; \mathbf{C})).$$

This is always an integer. We present the following example to illustrate the concepts involved. Let $M = S^1 \times S^1$ be the two dimensional torus which we realize as $\mathbf{R}^2$ modulo the integer lattice $\mathbf{Z}^2$. We define $T(x,y) = (n_1 x + n_2 y, n_3 x + n_4 y)$ where the n_i are integers. Since this preserves the integer lattice, this defines a map from M to itself. We compute:

$$\begin{aligned}
&T^*(1) = 1, \qquad T^*(dx \wedge dy) = (n_1 n_4 - n_2 n_3) dx \wedge dy \\
&T^*(dx) = n_1\, dx + n_2\, dy, \qquad T^*(dy) = n_3\, dx + n_4\, dy \\
&L(T) = 1 + (n_1 n_4 - n_2 n_3) - (n_1 + n_4).
\end{aligned}$$

Of course, there are many other interesting examples.

We computed $L(T)$ in the above example using the de Rham isomorphism. We let $T^* = \Lambda^p(dT): \Lambda^p(T^*M) \to \Lambda^p(T^*M)$ to be the pull-back operation. It is a map from the fiber over $T(x)$ to the fiber over x. Since $dT^* = T^*d$, T^* induces a map on $H^p(M, \mathbf{C}) = \ker(d_p)/\operatorname{image}(d_{p-1})$. If T is the identity map, then $L(T) = \chi(M)$. The perhaps somewhat surprising fact is that Lemma 1.6.5 can be generalized to compute $L(T)$ in terms of the heat equation.

Let $\Delta_p = (d\delta + \delta d)_p: C^\infty(\Lambda^p T^*M) \to C^\infty(\Lambda^p T^*M)$ be the associated Laplacian. We decompose $L^2(\Lambda^p T^*M) = \bigoplus_\lambda E_p(\lambda)$ into the eigenspaces of Δ_p. We let $\pi(p,\lambda)$ denote orthogonal projection on these subspaces, and we define $T^*(p,\lambda) = \pi(p,\lambda)T^*: E_p(\lambda) \to E_p(\lambda)$. It is immediate that:

$$\operatorname{Tr}(T^* e^{-t\Delta_p}) = \sum_\lambda e^{-t\lambda} \operatorname{Tr}(T^*(p,\lambda)).$$

Since $dT^* = T^*d$ and $d\pi = \pi d$, for $\lambda \neq 0$ we get a chain map between the exact sequences:

$$\begin{array}{ccccccc}
\cdots & E_{p-1}(\lambda) & \xrightarrow{d} & E_p(\lambda) & \xrightarrow{d} & E_{p+1}(\lambda) & \cdots \\
 & \downarrow{\scriptstyle T^*} & & \downarrow{\scriptstyle T^*} & & \downarrow{\scriptstyle T^*} & \\
\cdots & L^2(\Lambda^{p-1}) & \xrightarrow{d} & L^2(\Lambda^p) & \xrightarrow{d} & L^2(\Lambda^{p+1}) & \cdots \\
 & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & \\
\cdots & E_{p-1}(\lambda) & \xrightarrow{d} & E_p(\lambda) & \xrightarrow{d} & E_{p+1}(\lambda) & \cdots
\end{array}$$

Since this diagram commutes and since the two rows are long exact sequences of finite dimensional vector spaces, a standard result in homological algebra implies:

$$\sum_p (-1)^p \operatorname{Tr}(T^*(p,\lambda)) = 0 \qquad \text{for } \lambda \neq 0.$$

It is easy to see the corresponding sum for $\lambda = 0$ yields $L(T)$, so Lemma 1.6.5 generalizes to give a heat equation formula for $L(T)$.

This computation was purely formal and did not depend on the fact that we were dealing with the de Rham complex.

LEMMA 1.8.1. *Let (P,V) be an elliptic complex over M and let $T: M \to M$ be smooth. We assume given linear maps $V_i(T): V_i(Tx) \to V_i(x)$ so that $P_i(V_i(T)) = V_i(T)P_i$. Then T induces a map on $H^p(P,V)$. We define*

$$L(T)_P = \sum_p (-1)^p \operatorname{Tr}(T \text{ on } H^p(P,V)).$$

Then we can compute $L(T)_P = \sum_p \operatorname{Tr}(V_i(T)e^{-t\Delta_p})$ where $\Delta_p = (P^*P + PP^*)_p$ is the associated Laplacian.

The $V_i(T)$ are a smooth linear action of T on the bundles V_i. They will be given by the representations involved for the de Rham, signature, spin, and Dolbeault complexes as we shall discuss in Chapter 4. We usually denote $V_i(T)$ by T^* unless it is necessary to specify the action involved.

This Lemma implies Lemma 1.6.5 if we take $T = I$ and $V_i(T) = I$. To generalize Lemma 1.7.4 and thereby get a local formula for $L(T)_P$, we must place some restrictions on the map T. We assume the fixed point set of T consists of the finite disjoint union of smooth submanifolds N_i. Let $\nu(N_i) = T(M)/T(N_i)$ be the normal bundle over the submanifold N_i. Since dT preserves $T(N_i)$, it induces a map dT_ν on the bundle $\nu(N_i)$. We suppose $\det(I - dT_\nu) \neq 0$ as a non-degeneracy condition; there are no normal directions left fixed infinitesimally by T.

If T is an isometry, this condition is automatic. We can construct a non-example by defining $T(z) = z/(z+1): S^2 \to S^2$. The only fixed point is at $z = 0$ and $dT(0) = I$, so this fixed point is degenerate.

If K is the kernel of e^{-tP}, we pull back the kernel to define $T^*(K)(t,x,y) = T^*(x)K(t,Tx,y)$. It is immediate T^*K is the kernel of T^*e^{-tP}.

LEMMA 1.8.2. *Let P be an elliptic partial differential operator of order $d > 0$ which is self-adjoint and which has a positive definite leading symbol for $\xi \neq 0$. Let $T: M \to M$ be a smooth non-degenerate map and let $T^*: V_{T(x)} \to V_x$ be a smooth linear action. If K is the kernel of e^{-tP} then $\operatorname{Tr}(T^*e^{-tP}) = \int_M \operatorname{Tr}(T^*K)(t,x,x)\,\operatorname{dvol}(x)$. Furthermore:*
(a) If T has no fixed points, $|\operatorname{Tr}(T^*e^{-tP})| \leq C_n t^{-n}$ as $t \to 0^+$ for any n.

(b) If the fixed point set of T consists of submanifolds N_i of dimension m_i, we will construct scalar invariants $a_n(x)$ which depend functorially upon a finite number of jets of the symbol and of T. The $a_n(x)$ are defined over N_i and

$$\operatorname{Tr}(T^* e^{-tP}) \sim \sum_i \sum_{n=0}^{\infty} t^{\frac{n-m_i}{d}} \int_{N_i} a_n(x)\ \mathrm{dvol}_i(x).$$

$\mathrm{dvol}_i(x)$ *denotes the Riemannian measure on the submanifold.*
It follows that if T has no fixed points, then $L(T)_P = 0$.

PROOF: Let $\{e_n, r_n, K_n\}$ be as defined in section 1.7. The estimates of Lemma 1.7.3 show that $|T^*K - \sum_{n\leq n_0} T^*K_n|_{\infty,k} \leq C(k)t^{-k}$ as $t \to 0^+$ for any k if $n_0 = n_0(k)$. We may therefore replace K by K_n in proving (a) and (b). We recall that:

$$e_n(t,x,\xi) = \frac{1}{2\pi i}\int_\gamma e^{-t\lambda} r_n(x,\xi,\lambda)\, d\lambda.$$

We use the homogeneity of r_n to express:

$$\begin{aligned} e_n(t,x,\xi) &= \frac{1}{2\pi i}\int_\gamma e^{-\lambda} r_n(x,\xi,t^{-1}\lambda)t^{-1}\, d\lambda \\ &= t^{\frac{n}{d}} \frac{1}{2\pi i}\int_\gamma e^{-\lambda} r_n(x,t^{\frac{1}{d}}\xi,\lambda)\, d\lambda \\ &= t^{\frac{n}{d}} e_n(x,t^{\frac{1}{d}}\xi) \end{aligned}$$

where we define $e_n(x,\xi) = e_n(1,x,\xi) \in S^{-\infty}$. Then:

$$\begin{aligned} K_n(t,x,y) &= \int e^{i(x-y)\cdot\xi}\, e_n(t,x,\xi)\, d\xi \\ &= t^{\frac{n-m}{d}} \int e^{i(x-y)\cdot t^{-\frac{1}{d}}\xi}\, e_n(x,\xi)\, d\xi. \end{aligned}$$

This shows that:

$$T^*K_n(t,x,x) = t^{\frac{n-m}{d}} \int T^*(x) e^{i(Tx-x)\cdot t^{-1/d}\xi}\, e_n(Tx,\xi)\, d\xi.$$

We must study terms which have the form:

$$\int e^{i(Tx-x)\cdot t^{-1/d}\xi}\ \operatorname{Tr}(T^*(x)e_n(Tx,\xi))\, d\xi\, dx$$

where the integral is over the cotangent space $T^*(M)$. We use the method of stationary phase on this highly oscillatory integral. We first bound

$|Tx - x| \geq \varepsilon > 0$. Using the argument developed in Lemma 1.2.6, we integrate by parts to bound this integral by $C(n, k, \varepsilon)t^k$ as $t \to 0$ for any k. If T has no fixed points, this proves (a). There is a slight amount of notational sloppiness here since we really should introduce partitions of unity and coordinate charts to define $Tx - x$, but we supress these details in the interests of clarity.

We can localize the integral to an arbitrarily small neighborhood of the fixed point set in proving (b). We shall assume for notational simplicity that the fixed point set of T consists of a single submanifold N of dimension m_1. The map dT_ν on the normal bundle has no eigenvalue 1. We identify ν with the span of the generalized eigenvectors of dT on $T(M)|_N$ which correspond to eigenvalues not equal to 1. This gives a direct sum decomposition over N:

$$T(M) = T(N) \oplus \nu \qquad \text{and} \qquad dT = I \oplus dT_\nu.$$

We choose a Riemannian metric for M so this splitting is orthogonal. We emphasize that these are bundles over N and not over the whole manifold M.

We describe the geometry near the fixed manifold N using the normal bundle ν. Let $y = (y_1, \dots, y_{m_1})$ be local coordinates on N and let $\{\vec{s}_1, \dots, \vec{s}_{m-m_1}\}$ be a local orthonormal frame for ν. We use this local orthonormal frame to introduce fiber coordinates $z = (z_1, \dots, z_{m-m_1})$ for ν by decomposing any $\vec{s} \in \nu$ in the form:

$$\vec{s} = \sum_j z_j \vec{s}_j(y).$$

We let $x = (y, z)$ be local coordinates for ν. The geodesic flow identifies a neighborhood of the zero section of the bundle ν with a neighborhood of N in M so we can also regard $x = (y, z)$ as local coordinates on M.

We decompose

$$T(x) = (T_1(x), T_2(x))$$

into tangential and fiber coordinates. Because the Jacobian matrix has the form:

$$dT(y, 0) = \begin{pmatrix} I & 0 \\ 0 & dT_\nu \end{pmatrix}$$

we conclude that $T_1(x) - y$ must vanish to second order in z along N.

We integrate $\mathrm{Tr}(T^* K_n)(t, x, x)$ along a small neighborhood of the zero section of ν. We shall integrate along the fibers first to reduce this to an integral along N. We decompose $\xi = (\xi_1, \xi_2)$ corresponding to the decomposition of $x = (y, z)$. Let

$$D(\nu) = \{\, (y, z) : |z| \leq 1 \,\}$$

be the unit disk bundle of the normal bundle. Let $U = T^*(D(\nu))$ be the cotangent bundle of the unit disk bundle of the normal bundle. We assume the metric chosen so the geodesic flow embeds $D(\nu)$ in M. We parametrize U by $\{\,(y, z, \xi_1, \xi_2) : |z| \le 1\,\}$. Let $s = t^{\frac{1}{d}}$. Modulo terms which vanish to infinite order in s, we compute:

$$\begin{aligned}
\mathcal{I} &\stackrel{\text{def}}{=} \int_M \operatorname{Tr}((T^* K_n)(t, x, x))\, dx \\
&= s^{n-m} \int_U e^{i(T_1(y,z)-y)\cdot\xi_1 s^{-1}}\, e^{i(T_2(y,z)-z)\cdot\xi_2 s^{-1}} \\
&\qquad\qquad\qquad \times \operatorname{Tr}(T^*(x) e_n(Tx, \xi_1, \xi_2))\, d\xi_1\, d\xi_2\, dz\, dy.
\end{aligned}$$

The non-degeneracy assumption on T means the phase function $\overline{w} = T_2(y, z) - z$ defines a non-degenerate change of variables if we replace (y, z) by $(y, \overline{w})$. This transforms the integral into the form:

$$\begin{aligned}
\mathcal{I} = s^{n-m} &\int_{\overline{U}} e^{i(T_1(y,\overline{w})-y)\cdot\xi_1 s^{-1}}\, e^{i\overline{w}\cdot\xi_2 s^{-1}} \\
&\times |\det(I - dT_2)|^{-1} \operatorname{Tr}(T^*(y, \overline{w}) e_n(T(y, \overline{w}), \xi_1, \xi_2)\, d\xi_1\, d\xi_2\, d\overline{w}\, dy,
\end{aligned}$$

where $\overline{U}$ is the image of U under this change. We now make another change of coordinates to let $w = s^{-1}\overline{w}$. As $s \to 0$, $s^{-1}\overline{U}$ will converge to $T^*(v)$. Since $d\overline{w} = s^{m-m_1}\, dw$ this transforms the integral I to the form:

$$\begin{aligned}
\mathcal{I} = s^{n-m_1} &\int_{s^{-1}\overline{U}} e^{i(T_1(y,sw)-y)\cdot\xi_1 s^{-1}}\, e^{iw\cdot\xi_2}\, |\det(I - dT_2)^{-1}|(y, sw) \\
&\qquad \times \operatorname{Tr}(T^*(y, sw) e_n(T(y, sw), \xi_1, \xi_2))\, d\xi_1\, d\xi_2\, dw\, dy.
\end{aligned}$$

Since the phase function $T_1(y, \overline{w}) - y$ vanishes to second order at $\overline{w} = 0$, the function $(T_1(y, sw) - y)s^{-1}$ is regular at $s = 0$. Define:

$$\begin{aligned}
e_n'(y, w, \xi_1, \xi_2, s) = e^{i(T_1(y,sw)-y)\cdot\xi_1 s^{-1}}\, &|\det(I - dT_2)(y, sw)| \\
&\times \operatorname{Tr}(T^*(y, sw) e_n(T(y, sw), \xi_1, \xi_2)).
\end{aligned}$$

This vanishes to infinite order in ξ_1, ξ_2 at ∞ and is regular at $s = 0$. To complete the proof of Lemma 1.8.2, we must evaluate:

$$s^{(n-m_1)} \int_{s^{-1}\overline{U}} e_n'(y, w, \xi_1, \xi_2, s) e^{iw\cdot\xi_2}\, d\xi_1\, d\xi_2\, dw\, dy.$$

We expand e_n' in a Taylor series in s centered at $s = 0$. If we differentiate e_n' with respect to s a total of k times and evaluate at $s = 0$, then the exponential term disappears and we are left with an expression which is polynomial in w and of degree at worst $2k$. It still vanishes to infinite order in (ξ_1, ξ_2). We decompose:

$$e'_n = \sum_{j \le j_0} s^j \sum_{|\alpha| \le 2j} c_{j,\alpha}(y, \xi_1, \xi_2) w^\alpha + s^{j_0} \varepsilon(y, w, \xi_1, \xi_2, s)$$

where ε is the remainder term.

We first study a term of the form:

$$s^{n+j-m_1} \int_{|w|<s^{-1}} c_{j,\alpha}(y, \xi_1, \xi_2) w^\alpha e^{i\xi_2 \cdot w} \, d\xi_1 \, d\xi_2 \, dw \, dy.$$

Since $c_{j,\alpha}$ vanishes to infinite order in ξ, the $d\xi$ integral creates a function which vanishes to infinite order in w. We can let $s \to 0$ to replace the domain of integration by the normal fiber. The error vanishes to infinite order in s and gives a smooth function of y. The dw integral just yields the inverse Fourier transform with appropriate terms and gives rise to asymptotics of the proper form. The error term in the Taylor series grows at worst polynomially in w and can be bounded similarly. This completes the proof. If there is more than one component of the fixed point set, we sum over the components since each component makes a separate contibution to the asymptotic series.

The case of isolated fixed points is of particular interest. Let $d = 2$ and $\sigma_L(P) = |\xi|^2 I_V$. We compute the first term at a fixed point:

$$r_0(x, \xi, \lambda) = (|\xi|^2 - \lambda)^{-1} \qquad e_0(x, \xi, t) = e^{-t|\xi|^2}$$

$$\int \operatorname{Tr}(T^* K_0)(t, x, x) \operatorname{dvol}(x) = \int \operatorname{Tr}(T^*(x)) e^{i(Tx-x)\cdot\xi} \, e^{-t|\xi|^2} \, d\xi \, dx.$$

We assume $T(0) = 0$ is an isolated non-degenerate fixed point. We let $y = Tx - x$ be a change of variables and compute the first term:

$$\int \operatorname{Tr}(T^*(y)) e^{iy\cdot\xi} \, |\det(I - dT(y))|^{-1} e^{-t|\xi|^2} \, d\xi \, dy.$$

We make a change of variables $\xi \to \xi t^{-1/2}$ and $y \to y t^{1/2}$ to express the first term:

$$\int \operatorname{Tr}(T^*(yt^{1/2})) e^{iy\cdot\xi} \, |\det(I - dT(yt^{1/2})| e^{-|\xi|^2} \, d\xi \, dy.$$

The $d\xi$ integral just gives $e^{-|y|^2}$ so this becomes

$$\int \operatorname{Tr}(T^*(yt^{1/2})) |\det(I - dT(yt^{1/2}))|^{-1} e^{-|y|^2} \, dy.$$

We expand this in a Taylor series at $t = 0$ and evaluate to get

$$\operatorname{Tr} T^*(0) \, |\det(I - dT(O))|^{-1}.$$

This proves:

LEMMA 1.8.3. *Let P be a second order elliptic partial differential operator with leading symbol $|\xi|^2 I$. Let $T\colon M \to M$ be smooth with non-degenerate isolated fixed points. Then:*

$$\operatorname{Tr} T^* e^{-tP} = \sum_i \operatorname{Tr}(T^*)\,|\det(I-dT)|^{-1}(x_i)$$

summed over the fixed point set.

We combine Lemmas 1.8.1 and 1.8.2 to constuct a local formula for $L(T)_P$ to generalize the local formula for index(P) given by Theorem 1.7.6; we will discuss this further in the fourth chapter.

We can use Lemma 1.8.3 to prove the classical Lefschetz fixed point formula for the de Rham complex. Let $T\colon V \to V$ be a linear map, then it is easily computed that:

$$\sum_p (-1)^p \operatorname{Tr} \Lambda^p(T) = \det(I-T).$$

We compute:

$$\begin{aligned} L(T) &= \sum_p (-1)^p \operatorname{Tr}(T^* e^{-t\Delta_p}) \\ &= \sum_{p,i} (-1)^p \operatorname{Tr}(\Lambda^p dT)\,|\det(I-dT)|^{-1}(x_i) \\ &= \sum_i \det(I-dT)\,|\det(I-dT)|^{-1}(x_i) \\ &= \sum_i \operatorname{sign}\det(I-dT)(x_i) \end{aligned}$$

summed over the fixed point set of T. This proves:

THEOREM 1.8.4 (CLASSICAL LEFSCHETZ FIXED POINT FORMULA). *Let $T\colon M \to M$ be smooth with isolated non-degenerate fixed points. Then:*

$$\begin{aligned} L(T) &= \sum_p (-1)^p \operatorname{Tr}(T^* \text{ on } H^p(M;\mathbf{C})) \\ &= \sum_i \operatorname{sign}\det(I-dT)(x_i) \end{aligned}$$

summed over the fixed point set.

Remark: We can generalize Lemma 1.8.2 to study $\operatorname{Tr}(T^*Qe^{-tP})$ where Q is an auxilary differential operator of order a. Just as in lemma 1.7.7 we may obtain an asymptotic series:

$$\operatorname{Tr}(T^*Qe^{-tP}) \sim \sum_i \sum_{n=0}^{\infty} t^{(n-m_i-a)/d}\, a_n(x,Q,P)\operatorname{dvol}_i(x).$$

We shall omit the details as the additional terms created by the operator Q are exactly the same as those given in the proof of Lemma 1.7.7. Each term a_n is homogeneous of order n in the jets of the symbols of (Q, P) and of the map T in a suitable sense.

1.9. Elliptic Boundary Value Problems.

In section 1.7 we derived a local formula for the index of an elliptic partial-differential complex using heat equation methods. This formula will lead to a heat equation proof of the Atiyah-Singer index theorem which we shall discuss later. Unfortunately, it is not known at present how to give a heat equation proof of the Atiyah-Bott index theorem for manifolds with boundary in full generality. We must adopt a much stronger notion of ellipticity to deal with the analytic problems involved. This will yield a heat equation proof of the Gauss-Bonnet theorem for manifolds with boundary which we shall discuss in the fourth chapter.

Let M be a smooth compact manifold with smooth boundary dM and let $P: C^\infty(V) \to C^\infty(V)$ be a partial differential operator of order $d > 0$. We let $p = \sigma_L(P)$ be the leading symbol of P. We assume henceforth that p is self-adjoint and elliptic—i.e., $\det p(x, \xi) \neq 0$ for $\xi \neq 0$. Let $\mathbf{R}_\pm$ denote the non-zero positive/negative real numbers. It is immediate that

$$\det\{p(x,\xi) - \lambda\} \neq 0 \qquad \text{for } (\xi, \lambda) \neq (0,0) \in T^*(M) \times \{\mathbf{C} - \mathbf{R}_+ - \mathbf{R}_-\},$$

since p is self-adjoint and elliptic.

We fix a fiber metric on V and a volume element on M to define the global inner product $(\cdot,\cdot)$ on $L^2(V)$. We assume that P is formally self-adjoint:

$$(Pf, g) = (f, Pg)$$

for f and g smooth section with supports disjoint from the boundary dM. We must impose boundary conditions to make P self-adjoint. For example, if $P = -\partial^2/\partial x^2$ on the line segment $[0, A]$, then P is formally self-adjoint and elliptic, but we must impose Neumann or Dirichlet boundary conditions to ensure that P is self-adjoint with discrete spectrum.

Near dM we let $x = (y, r)$ where $y = (y_1, \dots, y_{m-1})$ is a system of local coordinates on dM and where r is the normal distance to the boundary. We assume $dM = \{x : r(x) = 0\}$ and that $\partial/\partial r$ is the inward unit normal. We further normalize the choice of coordinate by requiring the curves $x(r) = (y_0, r)$ for $r \in [0, \delta)$ are unit speed geodesics for any $y_0 \in dM$. The inward geodesic flow identifies a neighborhood of dM in M with the collar $dM \times [0, \delta)$ for some $\delta > 0$. The collaring gives a splitting of $T(M) = T(dM) \oplus T(\mathbf{R})$ and a dual splitting $T^*(M) = T^*(dM) \oplus T^*(\mathbf{R})$. We let $\xi = (\zeta, z)$ for $\zeta \in T^*(dM)$ and $z \in T^*(\mathbf{R})$ reflect this splitting.

It is convenient to discuss boundary conditions in the context of graded vector bundles. A graded bundle U over M is a vector bundle U together with a fixed decomposition

$$U = U_0 \oplus \cdots \oplus U_{d-1}$$

into sub-bundles U_j where $U_j = \{0\}$ is permitted in this decomposition. We let $W = V \otimes 1^d = V \oplus \cdots \oplus V$ restricted to dM be the bundle of

Cauchy data. If $W_j = V|_{dM}$ is the $(j+1)^{\text{st}}$ factor in this decomposition, then this defines a natural grading on W; we identify W_j with the bundle of $j^{\text{ th}}$ normal derivatives. The restriction map:

$$\underline{\gamma}: C^\infty(V) \to C^\infty(W)$$

defined by:

$$\underline{\gamma}(f) = (f_0, \ldots, f_{d-1}) \qquad \text{where } f_j = D_r^j f|_{dM} = (-i)^j \left.\frac{\partial^j f}{\partial r^j}\right|_{dM}$$

assigns to any smooth section its Cauchy data.

Let W' be an auxiliary graded vector bundle over dM. We assume that $\dim W = d \cdot \dim V$ is even and that $2 \dim W' = \dim W$. Let $B: C^\infty(W) \to C^\infty(W')$ be a tangential differential operator over dM. Decompose $B = B_{ij}$ for

$$B_{ij}: C^\infty(W_i) \to C^\infty(W'_j)$$

and assume that

$$\operatorname{ord}(B_{ij}) \le j - i.$$

It is natural to regard a section to $C^\infty(W_i)$ as being of order i and to define the graded leading symbol of B by:

$$\sigma_g(B)_{ij}(y,\zeta) = \begin{cases} \sigma_L(B_{ij})(y,\zeta) & \text{if } \operatorname{ord}(B_{ij}) = j - i \\ 0 & \text{if } \operatorname{ord}(B_{ij}) < j - i. \end{cases}$$

We then regard $\sigma_g(B)$ as being of graded homogeneity 0.

We let P_B be the operator P restricted to those $f \in C^\infty(V)$ such that $B\underline{\gamma} f = 0$. For example, let $P = -\partial^2/\partial x^2$ on the interval $[0, A]$. To define Dirichlet boundary conditions at $x = 0$, we would set:

$$W' = \mathbf{C} \oplus 0 \qquad \text{and} \quad B_{0,0} = 1,\ B_{1,1} = B_{0,1} = B_{1,0} = 0$$

while to define Neumann boundary conditions at $x = 0$, we would set:

$$W' = 0 \oplus \mathbf{C} \qquad \text{and} \quad B_{1,1} = 1,\ B_{0,0} = B_{0,1} = B_{1,0} = 0.$$

To define the notion of ellipticity we shall need, we consider the ordinary differential equation:

$$p(y, 0, \zeta, D_r) f(r) = \lambda f(r) \qquad \text{with } \lim_{r\to\infty} f(r) = 0$$

where

$$(\zeta, \lambda) \ne (0,0) \in T^*(dM) \times \mathbf{C} - \mathbf{R}_+.$$

We say that (P,B) is elliptic with respect to $\mathbf{C}-\mathbf{R}_+$ if $\det(p(x,\xi)-\lambda)\neq 0$ on the interior for all $(\zeta,\lambda)\neq(0,0)\in T^*(M)\times\mathbf{C}-\mathbf{R}_+$ and if on the boundary there always exists a unique solution to this ordinary differential equation such that $\sigma_g(B)(y,\zeta)\underline{\gamma}f=f'$ for any prescribed $f'\in W'$. In a similar fashion, we define ellipticity with respect to $\mathbf{C}-\mathbf{R}_+-\mathbf{R}_-$ if these conditions hold for $\lambda\in\mathbf{C}-\mathbf{R}_+-\mathbf{R}_-$.

Again, we illustrate these notions for the operator $P=-\partial^2/\partial x^2$ and a boundary condition at $x=0$. Since $\dim M=1$, there is no dependence on ζ and we must simply study the ordinary differential equation:

$$-f''=\lambda f \qquad \text{with } \lim_{r\to\infty} f(r)=0 \quad (\lambda\neq 0).$$

If $\lambda\in\mathbf{C}-\mathbf{R}_+$, then we can express $\lambda=\mu^2$ for $\operatorname{Im}(\mu)>0$. Solutions to the equation $-f''=\lambda f$ are of the form $f(r)=ae^{i\mu r}+be^{-i\mu r}$. The decay at ∞ implies $b=0$ so $f(r)=ae^{i\mu r}$. Such a function is uniquely determined by either its Dirichlet or Neumann data at $r=0$ and hence P is elliptic with respect to either Neumann or Dirichlet boundary conditions.

We assume henceforth that P_B is self-adjoint and that (P,B) is elliptic with respect to either the cone $\mathbf{C}-\mathbf{R}_+$ or the cone $\mathbf{C}-\mathbf{R}_+-\mathbf{R}_-$. It is beyond the scope of this book to develop the analysis required to discuss elliptic boundary value problems; we shall simply quote the required results and refer to the appropriate papers of Seeley and Greiner for further details. Lemmas 1.6.3 and 1.6.4 generalize to yield:

LEMMA 1.9.1. *Let $P\colon C^\infty(V)\to C^\infty(V)$ be an elliptic partial differential operator of order $d>0$. Let B be a boundary condition. We assume (P,B) is self-adjoint and elliptic with respect to $\mathbf{C}-\mathbf{R}_+-\mathbf{R}_-$.*
(a) We can find a complete orthonormal system $\{\phi_n\}_{n=1}^\infty$ for $L^2(V)$ with $P\phi_n=\lambda_n\phi_n$.
(b) $\phi_n\in C^\infty(V)$ and satisfy the boundary condition $B\underline{\gamma}\phi_n=0$.
(c) $\lambda_n\in\mathbf{R}$ and $\lim_{n\to\infty}|\lambda_n|=\infty$. If we order the λ_n so $|\lambda_1|\leq|\lambda_2|\cdots$ then there exists n_0 and $\delta>0$ so that $|\lambda_n|>n^\delta$ for $n>n_0$.
(d) If (P,B) is elliptic with respect to the cone $\mathbf{C}-\mathbf{R}_+$, then the λ_n are bounded from below and $\operatorname{spec}(P_B)$ is contained in $[-C,\infty)$ for some C.

For example, if $P=-\dfrac{\partial^2}{\partial x^2}$ on $[0,\pi]$ with Dirichlet boundary conditions, then the spectral resolution becomes $\left\{\sqrt{\frac{2}{\pi}}\sin(nx)\right\}_{n=1}^\infty$ and the corresponding eigenvalues are n^2.

If (P,B) is elliptic with respect to the cone $\mathbf{C}-\mathbf{R}_+$, then e^{-tP_B} is a smoothing operator with smooth kernel $K(t,x,x')$ defined by:

$$K(t,x,x')=\sum_n e^{-t\lambda_n}\phi_n(x)\otimes\bar{\phi}_n(x').$$

Lemma 1.7.4 generalizes to yield:

LEMMA 1.9.2. *Let (P, B) be elliptic with respect to the cone $\mathbf{C} - \mathbf{R}_+$, and be of order $d > 0$.*
(a) e^{-tP_B} is an infinitely smoothing operator with smooth kernel $K(t, x, x')$.
(b) On the interior we define $a_n(x, P) = \operatorname{Tr} e_n(x, P)$ be as in Lemma 1.7.4. On the boundary we define $a_n(y, P, B)$ using a complicated combinatorial recipe which depends functorially on a finite number of jets of the symbols of P and of B.
(c) As $t \to 0^+$ we have an asymptotic expansion:

$$\begin{aligned}\operatorname{Tr} e^{-tP_B} = \sum_n e^{-t\lambda_n} &= \int_M \operatorname{Tr} K(t, x, x)\,\mathrm{dvol}(x)\\ &\sim \sum_{n=0}^{\infty} t^{\frac{n-m}{d}} \int_M a_n(x, P)\,\mathrm{dvol}(x)\\ &\qquad + \sum_{n=0}^{\infty} t^{\frac{n-m+1}{d}} \int_{dM} a_n(y, P, B)\,\mathrm{dvol}(y).\end{aligned}$$

(d) $a_n(x, P)$ and $a_n(y, P, B)$ are invariantly defined scalar valued functions which do not depend on the coordinate system chosen nor on the local frame chosen. $a_n(x, P) = 0$ if n is odd, but $a_n(y, P, B)$ is in general non-zero for all values of n.

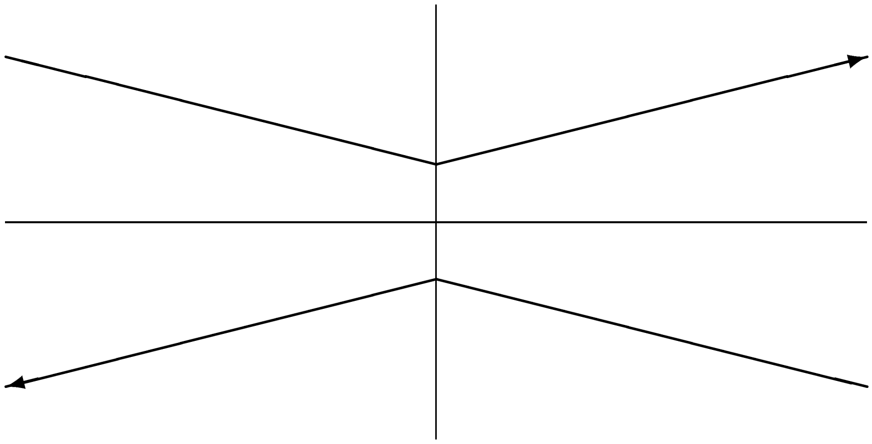
The interior term $a_n(x, P)$ arises from the calculus described previously. We first construct a parametrix on the interior. Since this parametrix will not have the proper boundary values, it is necessary to add a boundary correction term which gives rise to the additional boundary integrands $a_n(y, P, B)$.

To illustrate this asymptotic series, we let $P = -\dfrac{\partial^2}{\partial x^2} - e(x)$ on the interval $[0, A]$ where $e(x)$ is a real potential. Let B be the modified Neumann boundary conditions: $f'(0) + s(0)f(0) = f'(A) + s(A)f(A) = 0$. This is elliptic and self-adjoint with respect to the cone $\mathbf{C} - \mathbf{R}_+$. It can be computed that:

$$\begin{aligned}&\operatorname{Tr} e^{-tP_B}\\ &\quad\sim (4\pi t)^{-1/2} \int \{1 + t \cdot e(x) + t^2 \cdot (e''(x) + 3e(x)^2)/6 + \cdots\}\,dx\\ &\qquad\qquad + \frac{1}{2} + \left(\frac{t}{\pi}\right)^{1/2} \big(s(0) - s(A)\big)\\ &\qquad\qquad\quad + \frac{t}{4} \cdot \big(e(0) + e(A) + 2s^2(0) + 2s^2(A)\big) + \cdots\end{aligned}$$

The terms arising from the interior increase by integer powers of t while the terms from the boundary increase by powers of $t^{1/2}$. In this integral, dx is ordinary unnormalized Lebesgue measure.

In practice, we shall be interested in first order operators which are elliptic with respect to the cone $\mathbf{C}-\mathbf{R}_+-\mathbf{R}_-$ and which have both positive and negative spectrum. An interesting measure of the spectral asymmetry of such an operator is obtained by studying $\operatorname{Tr}(P_B e^{-tP_B^2})$ which will be discussed in more detail in the next section. To study the index of an elliptic operator, however, it suffices to study $e^{-tP_B^2}$; it is necessary to work with P_B^2 of course to ensure that the spectrum is positive so this converges to define a smoothing operator. There are two approaches which are available. The first is to work with the function $e^{-t\lambda^2}$ and integrate over a path of the form:



to define $e^{-tP_B^2}$ directly using the functional calculus. The second approach is to define $e^{-tP_B^2}$ using the operator P^2 with boundary conditions $B\underline{\gamma}f = B\underline{\gamma}Pf = 0$. These two approaches both yield the same operator and give an appropriate asymptotic expansion which generalizes Lemma 1.9.2.

We use the heat equation to construct a local formula for the index of certain elliptic complexes. Let $Q\colon C^\infty(V_1) \to C^\infty(V_2)$ be an elliptic differential operator of order $d > 0$. Let $Q^*\colon C^\infty(V_2) \to C^\infty(V_1)$ be the formal adjoint. Let $B_1\colon C^\infty(W_1) \to C^\infty(W_1')$ be a boundary condition for the operator Q. We assume there exists a boundary condition B_2 for Q^* of the same form so that

$$(Q_{B_1})^* = (Q^*)_{B_2}.$$

We form:

$$P = Q \oplus Q^* \qquad \text{and} \qquad B = B_1 \oplus B_2$$

so $P\colon C^\infty(V_1 \oplus V_2) \to C^\infty(V_1 \oplus V_2)$. Then P_B will be self-adjoint. We assume that (P, B) is elliptic with respect to the cone $\mathbf{C} - \mathbf{R}_+ - \mathbf{R}_-$.

This is a very strong assmption which rules out many interesting cases, but is necessary to treat the index theorem using heat equation methods. We emphasize that the Atiyah-Bott theorem in its full generality does not follow from these methods.

Since $P^2 = Q^*Q + QQ^*$, it is clear that P_B^2 decomposes as the sum of two operators which preserve $C^\infty(V_1)$ and $C^\infty(V_2)$. We let $\mathcal{S}$ be the endomorphism $+1$ on V_1 and -1 on V_2 to take care of the signs; it is clear $P_B^2\mathcal{S} = \mathcal{S}P_B^2$. The same cancellation lemma we have used previously yields:

$$\operatorname{index}(Q_1, B_1) = \dim \mathrm{N}(Q_{B_1}) - \dim \mathrm{N}(Q^*_{B_2}) = \operatorname{Tr}(\mathcal{S}e^{-tP_B^2}).$$

(The fact that this definition of the index agrees with the definition given in the Atiyah-Bott paper follows from the fact that $B: C^\infty(W) \to C^\infty(W_1)$ is surjective). Consequently, the application of Lemma 1.9.2 yields:

THEOREM 1.9.3. *Let $Q: C^\infty(V_1) \to C^\infty(V_2)$ be an elliptic differential operator of order $d > 0$. Let $Q^*: C^\infty(V_2) \to C^\infty(V_1)$ be the formal adjoint and define $P = Q \oplus Q^*: C^\infty(V_1 \oplus V_2) \to C^\infty(V_1 \oplus V_2)$. Let $B = B_1 \oplus B_2$ be a boundary condition such that (P, B) is elliptic with respect to the cone $\mathbf{C} - \mathbf{R}_+ - \mathbf{R}_-$ and so that P_B is self-adjoint. Define $\mathcal{S} = +1$ on V_1 and -1 on V_2 then:*
(a) $\operatorname{index}(Q_{B_1}) = \operatorname{Tr}(\mathcal{S}e^{-tP_B^2})$ *for all t.*
(b) There exist local invariants $a_n(x, Q)$ and $a_n(y, Q, B_1)$ such that:

$$\operatorname{Tr}(\mathcal{S}e^{-tP_B^2}) \sim \sum_{n=0}^{\infty} t^{\frac{n-m}{2d}} \int_M a_n(x, Q)\,\mathrm{dvol}(x) + \sum_{n=0}^{\infty} t^{\frac{n-m+1}{2d}} \int_{dM} a_n(y, Q, B_1)\,\mathrm{dvol}(y).$$

(c)

$$\int_M a_n(x, Q)\,\mathrm{dvol}(x) + \int_{dM} a_{n-1}(y, Q, B_1)\,\mathrm{dvol}(y) = \begin{cases} \operatorname{index}(Q_{B_1}) & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$$

This gives a local formula for the index; there is an analogue of Lemma 1.7.5 giving various functorial properties of these invariants we will discuss in the fourth chapter where we shall discuss the de Rham complex and the Gauss-Bonnet theorem for manifolds with boundary.

We now specialize henceforth to first order operators. We decompose

$$p(x, \xi) = \sum_j e_j(x)\xi_j$$

and near the boundary express:

$$p(y, 0, \zeta, z) = e_0(y) \cdot z + \sum_{j=1}^{m-1} e_j(y)\zeta_j = e_0(y) \cdot z + p(y, 0, \zeta, 0).$$

We study the ordinary differential equation:

$$\{ip_0\partial/\partial r + p(y,0,\zeta,0)\}f(r) = \lambda f(r)$$

or equivalently:

$$-ip_0\{\partial/\partial r + ip_0^{-1}p(y,0,\zeta,0) - ip_0^{-1}\lambda)\}f(r) = 0.$$

With this equation in mind, we define:

$$\tau(y,\zeta,\lambda) = ip_0^{-1}(p(y,0,\zeta,0) - \lambda).$$

Lemma 1.9.4. *Let $p(x,\xi)$ be self-adjoint and elliptic. Then $\tau(y,\zeta,\lambda)$ has no purely imaginary eigenvalues for $(\zeta,\lambda) \neq (0,0) \in T^*(dM) \times (\mathbf{C} - \mathbf{R}_+ - \mathbf{R}_-)$.*

Proof: We suppose the contrary and set $\tau(y,\zeta,\lambda)v = izv$ where z is real and $v \neq 0$. This implies that:

$$(p(y,0,\zeta,0) - \lambda)v = p_0 zv$$

or equivalently that:

$$p(y,0,\zeta,-z)v = \lambda v.$$

Since p is self-adjoint, this implies $\lambda = 0$ so $\zeta \neq 0$ which contradicts the ellipticity of p.

We define bundles $\Pi_\pm(\tau)$ over $T^*(dM)\times\{\mathbf{C}-\mathbf{R}_+-\mathbf{R}_-\}-(0,0)$ to be the span of the generalized eigenvectors of τ which correspond to eigenvalues with positive/negative real part; $\Pi_+ \oplus \Pi_- = V$. The differential equation has the form:

$$\{\partial/\partial r + \tau\}f = 0$$

so the condition $\lim_{r\to\infty} f(r) = 0$ implies that $f(0) \in \Pi_+(\tau)$. Thus $\Pi_+(\tau)$ is the bundle of Cauchy data corresponding to solutions to this ODE. Since $d = 1$, the boundary condition B is just an endomorphism:

$$B\colon V_{|dM} \to W'$$

and we conclude:

Lemma 1.9.5. *Let P be a first order formally self-adjoint elliptic differential operator. Let B be a 0^{th} order boundary condition. Define*

$$\tau(y,\zeta,\lambda) = ip_0^{-1}(p(y,0,\zeta,0) - \lambda).$$

Then τ has no purely imaginary eigenvalues and we define $\Pi_\pm(\tau)$ to be bundles of generalized eigenvectors corresponding to eigenvalues with positive real and negative real parts. (P,B) is elliptic with respect to the cone

$\mathbf{C} - \mathbf{R}_+ - \mathbf{R}_-$ *if and only if* $B\colon \Pi_+(\tau) \to W'$ *is an isomorphism for all* $(\zeta, \lambda) \neq (0,0) \in T^*(dM) \times \{\mathbf{C} - \mathbf{R}_+ - \mathbf{R}_-\}$. P_B *is self-adjoint if and only if* $p_0\,\mathrm{N}(B)$ *is perpendicular to* $\mathrm{N}(B)$.

PROOF: We have checked everything except the condition that P_B be self-adjoint. Since P is formally self-adjoint, it is immediate that:

$$(Pf, g) - (f, Pg) = \int_{dM} (-ip_0 f, g).$$

We know that on the boundary, both f and g have values in $\mathrm{N}(B)$ since they satisfy the boundary condition. Thus this vanishes identically if and only if $(p_0 f, g) = 0$ for all $f, g \in \mathrm{N}(B)$ which completes the proof.

We emphasize that these boundary conditions are much stronger than those required in the Atiyah-Bott theorem to define the index. They are much more rigid and avoid some of the pathologies which can occur otherwise. Such boundary conditions do not necessarily exist in general as we shall see later.

We shall discuss the general case in more detail in Chapter 4. We complete this section by discussing the one-dimensional case case to illustrate the ideas involved. We consider $V = [0,1] \times \mathbf{C}^2$ and let P be the operator:

$$P = -i\frac{\partial}{\partial r}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

At the point 0, we have:

$$\tau(\lambda) = -i\lambda\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus if $\mathrm{Im}(\lambda) > 0$ we have $\Pi_+(\tau) = \left\{\binom{a}{a}\right\}$ while if $\mathrm{Im}(\lambda) < 0$ we have $\Pi_+(\tau) = \left\{\binom{a}{-a}\right\}$. We let B be the boundary projection which projects on the first factor—$\mathrm{N}(B) = \left\{\binom{0}{a}\right\}$; we take the same boundary condition at $x = 1$ to define an elliptic self-adjoint operator P_B. Let $P^2 = -\dfrac{\partial^2}{\partial x^2}$ with boundary conditions: Dirichlet boundary conditions on the first factor and Neumann boundary conditions on the second factor. The index of the problem is -1.

1.10. Eta and Zeta Functions.

We have chosen to work with the heat equation for various technical reasons. However, much of the development of the subject has centered on the zeta function so we shall briefly indicate the relationship between these two in this section. We shall also define the eta invariant which plays an important role in the Atiyah-Singer index theorem for manifolds with boundary.

Recall that Γ is defined by:

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$$

for $\mathrm{Re}(s) > 0$. We use the functional equation $s\Gamma(s) = \Gamma(s+1)$ to extend Γ to a meromorphic function on C with isolated simple poles at $s = 0, -1, -2, \ldots$. Let $P\colon C^\infty(V) \to C^\infty(V)$ be an elliptic self-adjoint partial differential operator of order $d > 0$ with positive definite leading symbol. We showed in section 1.6 that this implies $\mathrm{spec}(P)$ is contained in $[-C, \infty)$ for some constant C. We now assume that P itself is positive definite—i.e., $\mathrm{spec}(P)$ is contained in $[\varepsilon, \infty)$ for some $\varepsilon > 0$.

Proceeding formally, we define P^s by:

$$P^s = \frac{1}{2\pi i} \int_\alpha \lambda^s (P - \lambda)^{-1} \, d\lambda$$

where α is a suitable path in the half-plane $\mathrm{Re}(\lambda) > 0$. The estimates of section 1.6 imply this is smoothing operator if $\mathrm{Re}(s) \ll 0$ with a kernel function given by:

$$L(s, x, y) = \sum_n \lambda_n^s \phi_n(x) \otimes \bar{\phi}_n(y);$$

this converges to define a C^k kernel if $\mathrm{Re}(s) < s_0(k)$.

We use the Mellin transform to relate the zeta and heat kernels.

$$\int_0^\infty t^{s-1} e^{-\lambda t} dt = \lambda^{-s} \int_0^\infty (\lambda t)^{s-1} e^{-\lambda t} d(\lambda t) = \lambda^{-s} \Gamma(s).$$

This implies that:

$$\Gamma(s) \operatorname{Tr}(P^{-s}) = \int_0^\infty t^{s-1} \operatorname{Tr} e^{-tP} \, dt.$$

We define $\zeta(s, P) = \operatorname{Tr}(P^{-s})$; this is holomorphic for $\mathrm{Re}(s) \gg 0$. We decompose this integral into $\int_0^1 + \int_1^\infty$. We have bounded the eigenvalues of

P away from zero. Using the growth estimates of section 1.6 it is immediate that:

$$\int_1^\infty t^{s-1} \operatorname{Tr} e^{-tP}\, dt = r(s,P)$$

defines an entire function of s. If $0 < t < 1$ we use the results of section 1.7 to expand

$$\operatorname{Tr} e^{-tP} = \sum_{n \le n_0} t^{\frac{n-m}{d}} a_n(P) + O\left(t^{\frac{n_0-m}{d}}\right)$$

$$a_n(P) = \int_M a_n(x,P)\, \mathrm{dvol}(x)$$

where $a_n(x,P)$ is a local scalar invariant of the jets of the total symbol of P given by Lemma 1.7.4.

If we integrate the error term which is $O\left(t^{\frac{n_0-m}{d}}\right)$ from 0 to 1, we define a holomorphic function of s for $\mathrm{Re}(s) + \frac{n_0-m}{d} > 0$. We integrate $t^{s-1} t^{\frac{(n-m)}{d}}$ between 0 and 1 to conclude:

$$\Gamma(s) \operatorname{Tr}(P^{-s}) = \Gamma(s)\zeta(s,P) = \sum_{n<n_0} d(sd+n-m)^{-1} a_n(P) + r_{n_0}$$

where r_{n_0} is holomorphic for $\mathrm{Re}(s) > -\frac{(n_0-m)}{d}$. This proves $\Gamma(s)\zeta(s,P)$ extends to a meromorphic function on C with isolated simple poles. Furthermore, the residue at these poles is given by a local formula. Since Γ has isolated simple poles at $s = 0, -1, -2, \dots$ we conclude that:

LEMMA 1.10.1. *Let P be a self-adjoint, positive, elliptic partial differential operator of order $d > 0$ with positive definite symbol. We define:*

$$\zeta(s,P) = \operatorname{Tr}(P^{-s}) = \Gamma(s)^{-1} \int_0^\infty t^{s-1} \operatorname{Tr} e^{-tP}\, dt.$$

This is well defined and holomorphic for $\mathrm{Re}(s) \gg 0$. It has a meromorphic extension to C with isolated simple poles at $s = (m-n)/d$ for $n = 0, 1, 2, \dots$. The residue of ζ at these local poles is $a_n(P)\Gamma((m-n)/d)^{-1}$. If $s = 0, -1, \dots$ is a non-positive integer, then $\zeta(s,P)$ is regular at this value of s and its value is given by $a_n(P) \operatorname{Res}_{s=(m-n)/d} \Gamma(s)$. $a_n(P)$ is the invariant given in the asymptotic expansion of the heat equation. $a_n(P) = \int_M a_n(x,P)\, \mathrm{dvol}(x)$ where $a_n(x,P)$ is a local invariant of the jets of the total symbol of P. a_n vanishes if n is odd.

Remark: If A is an auxilary differential operator of order a, we can define

$$\zeta(s,A,P) = \operatorname{Tr}(AP^{-s}) = \Gamma(s)^{-1} \int_0^\infty t^{s-1} \operatorname{Tr}(Ae^{-tP})\, dt.$$

We apply Lemma 1.7.7 to see this has a meromorphic extension to $\mathbf{C}$ with isolated simple poles at $s = (m + a - n)/d$. The residue at these poles is given by the generalized invariants of the heat equation

$$\frac{a_n(A, P)}{\Gamma\{(m + a - n)/d\}}.$$

There is a similar theorem if $dM \neq \emptyset$ and if B is an elliptic boundary condition as discussed in section 1.9.

If P is only positive semi-definite so it has a finite dimensional space corresponding to the eigenvalue 0, we define:

$$\zeta(s, P) = \sum_{\lambda_n > 0} \lambda_n^{-s}$$

and Lemma 1.10 extends to this case as well with suitable modifications. For example, if $M = S^1$ is the unit circle and if $P = -\partial^2/\partial\theta^2$ is the Laplacian, then:

$$\zeta(s, P) = 2 \sum_{n > 0} n^{-2s}$$

is essentially just the Riemann zeta function. This has a simple isolated pole at $s = 1/2$.

If $Q\colon C^\infty(V_1) \to C^\infty(V_2)$, then the cancellation lemma of section 1.6 implies that:

$$\varepsilon^{-s}\,\mathrm{index}(Q) = \zeta(s, Q^*Q + \varepsilon) - \zeta(s, QQ^* + \varepsilon)$$

for any $\varepsilon > 0$ and for any s. ζ is regular at $s = 0$ and its value is given by a local formula. This gives a local formula for index(Q). Using Lemma 1.10.1 and some functorial properties of the heat equation, it is not difficult to show this local formula is equivalent to the local formula given by the heat equation so no new information has resulted. The asymptotics of the heat equation are related to the values and residues of the zeta function by Lemma 1.10.1.

If we do not assume that P is positive definite, it is possible to define a more subtle invariant which measures the difference between positive and negative spectrum. Let P be a self-adjoint elliptic partial differential operator of order $d > 0$ which is not necessarily positive definite. We define the eta invariant by:

$$\begin{aligned}\eta(s, P) &= \sum_{\lambda_n > 0} \lambda_n^{-s} - \sum_{\lambda_n < 0} (-\lambda_n)^{-s} = \sum_{\lambda_n \neq 0} \mathrm{sign}(\lambda_n)|\lambda_n|^{-s} \\ &= \mathrm{Tr}(P \cdot (P^2)^{-(s+1)/2}).\end{aligned}$$

Again, this is absolutely convergent and defines a holomorphic function if $\mathrm{Re}(s) \gg 0$.

We can also discuss the eta invariant using the heat equation. The identity:

$$\int_0^\infty t^{(s-1)/2} \lambda e^{-\lambda^2 t}\, dt = \Gamma((s+1)/2)\, \mathrm{sign}(\lambda)|\lambda|^{-s}$$

implies that:

$$\eta(s, P) = \Gamma((s+1)/2)^{-1} \int_0^\infty t^{(s-1)/2}\, \mathrm{Tr}\{Pe^{-tP^2}\}\, dt.$$

Again, the asymptotic behavior at $t = 0$ is all that counts in producing poles since this decays exponentially at ∞ assuming P has no zero eigenvector; if $\dim \mathrm{N}(P) > 0$ a seperate argument must be made to take care of this eigenspace. This can be done by replacing P by $P + \varepsilon$ and letting $\varepsilon \to 0$.

Lemma 1.7.7 shows that there is an asymptotic series of the form:

$$\mathrm{Tr}(Pe^{-tP^2}) \sim \sum_{n=0}^\infty t^{\frac{n-m-d}{2d}} a_n(P, P^2)$$

for

$$a_n(P, P^2) = \int_M a_n(x, P, P^2)\, \mathrm{dvol}(x).$$

This is a local invariant of the jets of the total symbol of P.

We substitute this asymptotic expansion into the expression for η to see:

$$\eta(s, P)\Gamma((s+1)/2)^{-1} = \sum_{n \le n_0} \frac{2d}{ds + n - m} a_n(P, P^2) + r_{n_0}$$

where r_{n_0} is holomorphic on a suitable half-plane. This proves η has a suitable meromorphic extension to $\mathbf{C}$ with locally computable residues.

Unfortunately, while it was clear from the analysis that ζ was regular at $s = 0$, it is not immediate that η is regular at $s = 0$ since Γ does not have a pole at $\frac{1}{2}$ to cancel the pole which may be introduced when $n = m$. in fact, if one works with the local invariants involved, $a_n(x, P, P^2) \neq 0$ in general so the local poles are in fact present at $s = 0$. However, it is a fact which we shall discuss later that η is regular at $s = 0$ and we will define

$$\tilde{\eta}(p) = \frac{1}{2}\{\eta(0, p) + \dim \mathrm{N}(p)\} \quad \mod \mathbf{Z}$$

(we reduce modulo $\mathbf{Z}$ since η has jumps in $2\mathbf{Z}$ as eigenvalues cross the origin).

We compute a specific example to illustrate the role of η in measuring spectral asymmetry. Let $P = -i\partial/\partial\theta$ on $C^\infty(S^1)$, then the eigenvalues of p are the integers so $\eta(s,p) = 0$ since the spectrum is symmetric about the origin. Let $a \in \mathbf{R}$ and define:

$$P_a = P - a, \qquad \eta(s, P_a) = \sum_{n \in \mathbf{Z}} \operatorname{sign}\{n-a\}|n-a|^{-s}.$$

We differentiate this with respect to the parameter a to conclude:

$$\frac{d}{da}\eta(s, P(a)) = \sum_{n \in \mathbf{Z}} s((n-a)^2)^{-(s+1)/2}.$$

If we compare this with the Riemann zeta function, then

$$\sum_{n \in \mathbf{Z}} ((n-a)^2)^{-(s+1)/2}$$

has a simple pole at $s = 0$ with residue 2 and therefore:

$$\left.\frac{d}{da}\eta(s, P(a))\right|_{s=0} = 2.$$

Since η vanishes when $a = 0$, we integrate this with respect to a to conclude:

$$\eta(0, P(a)) = 2a + 1 \mod 2\mathbf{Z} \qquad \text{and} \qquad \tilde{\eta}(P(a)) = a + \frac{1}{2}$$

is non-trivial. (We must work modulo $\mathbf{Z}$ since $\operatorname{spec} P(a)$ is periodic with period 1 in a).

We used the identity:

$$\frac{d}{da}\eta(s, P_a) = -s \operatorname{Tr}(\dot{P}_a (P_a^2)^{-(s+1)/2})$$

in the previous computation; it in fact holds true in general:

LEMMA 1.10.2. *Let $P(a)$ be a smooth 1-parameter family of elliptic self-adjoint partial differential operators of order $d > 0$. Assume $P(a)$ has no zero spectrum for a in the parameter range. Then if "$\cdot$" denotes differentiation with respect to the parameter a,*

$$\dot{\eta}(s, P(a)) = -s \operatorname{Tr}(\dot{P}(a)(P(a)^2)^{-(s+1)/2}).$$

If $P(a)$ has zero spectrum, we regard $\eta(s, P(a)) \in \mathbf{C}/\mathbf{Z}$.

PROOF: If we replace P by P^k for an odd positive integer k, then $\eta(s, P^k) = \eta(ks, P)$. This shows that it suffices to prove Lemma 1.10.2 for $d \gg 0$.

By Lemma 1.6.6, $(P-\lambda)^{-1}$ is smoothing and hence of trace class for d large. We can take trace inside the integral to compute:

$$\eta(s,P(a)) = \frac{1}{2\pi i}\left\{\int_{\gamma_1} \lambda^{-s}\operatorname{Tr}((P(a)-\lambda)^{-1})\,d\lambda \right.$$
$$\left. - \int_{\gamma_2} (-\lambda)^{-s}\operatorname{Tr}((P(a)-\lambda)^{-1}\,d\lambda\right\}$$

where γ_1 and γ_2 are paths about the positive/negative real axis which are oriented suitably. We differentiate with respect to the parameter a to express:

$$\frac{d}{da}((P(a)-\lambda)^{-1} = -(P(a)-\lambda)^{-1}\dot{P}(a)(P(a)-\lambda)^{-1}.$$

Since the operators involved are of Trace class,

$$\operatorname{Tr}\left(\frac{d}{da}(P(a)-\lambda)^{-1}\right) = -\operatorname{Tr}(\dot{P}(a)(P(a)-\lambda)^{-2}).$$

We use this identity and bring trace back outside the integral to compute:

$$\eta(s,P(a)) = \frac{1}{2\pi i}\operatorname{Tr}(\dot{P}(a))\left\{\int_{\gamma_1} -\lambda^{-s}(P(a)-\lambda)^{-2}\,d\lambda \right.$$
$$\left. + \int_{\gamma_2} (-\lambda)^{-s}(P(a)-\lambda)^{-2}\,d\lambda\right\}.$$

We now use the identity:

$$\frac{d}{d\lambda}((P(a)-\lambda)^{-1}) = (P(a)-\lambda)^{-2}$$

to integrate by parts in λ in this expression. This leads immediately to the desired formula.

We could also calculate using the heat equation. We proceed formally:

$$\begin{aligned}
\frac{d\eta}{da}(s,P(a)) &= \Gamma\{(s+1)/2\}^{-1}\int_0^\infty t^{(s-1)/2}\operatorname{Tr}\left(\frac{d}{da}(P(a)e^{-tP(a)^2})\right)dt \\
&= \Gamma\{(s+1)/2\}^{-1}\int_0^\infty t^{(s-1)/2}\operatorname{Tr}\left(\frac{dP}{da}(1-2tP^2)e^{-tP^2}\right)dt \\
&= \Gamma\{(s+1)/2\}^{-1}\int_0^\infty t^{(s-1)/2}\left(1+2t\frac{d}{dt}\right)\operatorname{Tr}\left(\frac{dP}{da}e^{-tP^2}\right)dt.
\end{aligned}$$

We now integrate by parts to compute:

$$= \Gamma\{(s+1)/2\}^{-1} \int_0^\infty -st^{(s-1)/2} \operatorname{Tr}\left(\frac{dP}{da} e^{-tP^2}\right) dt$$
$$\sim -s\Gamma\{(s+1)/2\}^{-1} \sum_n \int_0^1 t^{(s-1)/2} t^{(n-m-d)/2d} a_n \left(\frac{dP}{da}, P^2\right) dt$$
$$\sim -s\Gamma\{(s+1)/2\}^{-1} \sum_n 2/(s+(n-m)/d) a_n \left(\frac{dP}{da}, P^2\right).$$

In particular, this shows that $\frac{d\tilde{\eta}}{da}$ is regular at $s=0$ and the value

$$-\Gamma\left(\tfrac{1}{2}\right)^{-1} a_m \left(\frac{dP}{da}, P^2\right)$$

is given by a local formula.

The interchange of order involved in using global trace is an essential part of this argument. $\operatorname{Tr}(\dot{P}(a)(P^2)^{-(s+1)/2})$ has a meromorphic extension to $\mathbf{C}$ with isolated simple poles. The pole at $s=-1/2$ can be present, but is cancelled off by the factor of $-s$ which multiplies the expression. Thus $\frac{d}{da} \operatorname{Res}_{s=0} \eta(s, P(a)) = 0$. This shows the global residue is a homotopy invariant; this fact will be used in Chapter 4 to show that in fact η is regular at the origin. This argument does not go through locally, and in fact it is possible to construct operators in any dimension $m \geq 2$ so that the local eta function is not regular at $s=0$.

We have assumed that $P(a)$ has no zero spectrum. If we supress a finite number of eigenvalues which may cross the origin, this makes no contribution to the residue at $s=0$. Furthermore, the value at $s=0$ changes by jumps of an even integer as eigenvalues cross the origin. This shows $\tilde{\eta}$ is in fact smooth in the parameter a and proves:

LEMMA 1.10.3.
(a) Let P be a self-adjoint elliptic partial differential operator of positive order d. Define:

$$\eta(s,P) = \operatorname{Tr}(P(P^2)^{-(s+1)/2}) = \Gamma\{(s+1)/2\}^{-1} \int_0^\infty t^{(s-1)/2} \operatorname{Tr}(Pe^{-tP^2})\, dt.$$

This admits a meromorphic extension to $\mathbf{C}$ with isolated simple poles at $s=(m-n)/d$. The residue of η at such a pole is computed by:

$$\operatorname{Res}_{s=(m-n)/d} = 2\Gamma\{(m+d-n)/2d\}^{-1} \int_M a_n(x,P,P^2)\, \operatorname{dvol}(x)$$

where $a_n(x,P,P^2)$ is defined in Lemma 1.7.7; it is a local invariant in the jets of the symbol of P.

(b) Let $P(a)$ be a smooth 1-parameter family of such operators. If we assume eigenvalues do not cross the origin, then:

$$\begin{aligned}\frac{d}{da}\eta(s,P(a)) &= -s\operatorname{Tr}\left(\frac{dP}{da}\cdot(P(a)^2)^{-(s+1)/2}\right)\\ &= -2s\Gamma\{(s+1)/2\}^{-1}\int_0^\infty t^{(s-1)/2}\operatorname{Tr}\left(\frac{dP}{da}\cdot e^{-tP^2}\right)dt.\end{aligned}$$

Regardless of whether or not eigenvalues cross the origin, $\tilde\eta(P(a))$ is smooth in $\mathbf{R}$ mod $\mathbf{Z}$ in the parameter a and

$$\begin{aligned}\frac{d}{da}\tilde\eta(P(a)) &= -\Gamma(\tfrac12)^{-1}a_m\left(\frac{dP}{da},P^2\right)\\ &= -\Gamma(\tfrac12)^{-1}\int_M a_m\left(x,\frac{dP}{da},P^2\right)\mathrm{dvol}(x)\end{aligned}$$

is the local invariant in the jets of the symbols of $\left(\frac{dP}{da},P^2\right)$ given by Lemma 1.7.7.

In the example on the circle, the operator $P(a)$ is locally isomorphic to P. Thus the value of η at the origin is *not* given by a local formula. This is a global invariant of the operator; only the derivative is locally given.

It is not necessary to assume that P is a differential operator to define the eta invariant. If P is an elliptic self-adjoint pseudo-differential operator of order $d>0$, then the sum defining $\eta(s,P)$ converges absolutely for $s\gg 0$ to define a holomorphic function. This admits a meromorphic extension to the half-plane $\mathrm{Re}(s)>-\delta$ for some $\delta>0$ and the results of Lemma 1.10.3 continue to apply. This requires much more delicate estimates than we have developed and we shall omit details. The reader is referred to the papers of Seeley for the proofs.

We also remark that it is not necessary to assume that P is self-adjoint; it suffices to assume that $\det(p(x,\xi)-it)\neq 0$ for $(\xi,t)\neq(0,0)\in T^*M\times\mathbf{R}$. Under this ellipticity hypothesis, the spectrum of P is discrete and only a finite number of eigenvalues lie on or near the imaginary axis. We define:

$$\eta(s,P)=\sum_{\mathrm{Re}(\lambda_i)>0}(\lambda_i)^{-s}-\sum_{\mathrm{Re}(\lambda_i)<0}(-\lambda_i)^{-s}$$

$$\tilde\eta(P)=\frac12\left\{\eta(s,P)-\frac1s\operatorname{Res}_{s=0}\eta(s,P)+\dim\mathrm{N}(i\mathbf{R})\right\}_{s=0}\qquad\text{mod }\mathbf{Z}$$

where $\dim\mathrm{N}(i\mathbf{R})$ is the dimension of the finite dimensional vector space of generalized eigenvectors of P corresponding to purely imaginary eigenvalues.

In section 4.3, we will discuss the eta invariant in further detail and use it to define an index with coefficients in a locally flat bundle using secondary characteristic classes.

If the leading symbol of P is positive definite, the asymptotics of $\mathrm{Tr}(e^{-tP})$ as $t \to 0^+$ are given by local formulas integrated over the manifold. Let $A(x) = a_0x^a + \cdots + a_a$ and $B(x) = b_0x^b + \cdots + b_b$ be polynomials where $b_0 > 0$. The operator $A(P)e^{-tB(P)}$ is infinitely smoothing. The asymptotics of $\mathrm{Tr}(A(P)e^{-tB(P)})$ are linear combinations of the invariants $a_N(P)$ giving the asymptotics of $\mathrm{Tr}(e^{-tP})$. Thus there is no new information which results by considering more general operators of heat equation type. If the leading symbol of P is indefinite, one must consider both the zeta and eta function or equivalently:

$$\mathrm{Tr}(e^{-tP^2}) \qquad \text{and} \qquad \mathrm{Tr}(Pe^{-tP^2}).$$

Then if $b_0 > 0$ is even, we obtain enough invariants to calculate the asymptotics of $\mathrm{Tr}(A(P)e^{-tB(P)})$. We refer to Fegan-Gilkey for further details.