INVARIANCE THEORY, THE HEAT EQUATION, AND THE ATIYAH-SINGER INDEX THEOREM

by Peter B. Gilkey

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INTRODUCTION

This book treats the Atiyah-Singer index theorem using heat equation methods. The heat equation gives a local formula for the index of any elliptic complex. We use invariance theory to identify the integrand of the index theorem for the four classical elliptic complexes with the invariants of the heat equation. Since the twisted signature complex provides a sufficiently rich family of examples, this approach yields a proof of the Atiyah-Singer theorem in complete generality. We also use heat equation methods to discuss Lefschetz fixed point formulas, the Gauss-Bonnet theorem for a manifold with smooth boundary, and the twisted eta invariant. We shall not include a discussion of the signature theorem for manifolds with boundary.

The first chapter reviews results from analysis. Sections 1.1 through 1.7 represent standard elliptic material. Sections 1.8 through 1.10 contain the material necessary to discuss Lefschetz fixed point formulas and other topics.

Invariance theory and differential geometry provide the necessary link between the analytic formulation of the index theorem given by heat equation methods and the topological formulation of the index theorem contained in the Atiyah-Singer theorem. Sections 2.1 through 2.3 are a review of characteristic classes from the point of view of differential forms. Section 2.4 gives an invariant-theoretic characterization of the Euler form which is used to give a heat equation proof of the Gauss-Bonnet theorem. Sections 2.5 and 2.6 discuss the Pontrjagin forms of the tangent bundle and the Chern forms of the coefficient bundle using invariance theory.

The third chapter combines the results of the first two chapters to prove the Atiyah-Singer theorem for the four classical elliptic complexes. We first present a heat equation proof of the Hirzebruch signature theorem. The twisted spin complex provides a unified way of discussing the signature, Dolbeault, and de Rham complexes. In sections 3.2–3.4, we discuss the half-spin representations, the spin complex, and derive a formula for the $\hat{A}$ genus. We then discuss the Riemann-Roch formula for an almost complex manifold in section 3.5 using the SPIN_c complex. In sections 3.6–3.7 we give a second derivation of the Riemann-Roch formula for holomorphic Kaehler manifods using a more direct approach. In the final two sections we derive the Atiyah-Singer theorem in its full generality.

The final chapter is devoted to more specialized topics. Sections 4.1–4.2 deal with elliptic boundary value problems and derive the Gauss-Bonnet theorem for manifolds with boundary. In sections 4.3–4.4 we discuss the twisted eta invariant on a manifold without boundary and we derive the Atiyah-Patodi-Singer twisted index formula. Section 4.5 gives a brief discussion of Lefschetz fixed point formulas using heat equation methods. In section 4.6 we use the eta invariant to calculate the K-theory of spherical space forms. In section 4.7, we discuss Singer's conjecture for the Euler form and related questions. In section 4.8, we discuss the local formulas for the invariants of the heat equation which have been derived by several authors, and in section 4.9 we apply these results to questions of spectral geometry.

The bibliography at the end of this book is not intended to be exhaustive but rather to provide the reader with a list of a few of the basic papers which have appeared. We refer the reader to the bibliography of Berger and Berard for a more complete list of works on spectral geometry.

This book is organized into four chapters. Each chapter is divided into a number of sections. Each Lemma or Theorem is indexed according to this subdivision. Thus, for example, Lemma 1.2.3 is the third Lemma of section 2 of Chapter 1.

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Chapter 3. The Index Theorem

Chapter 4. Generalized Index Theorems and Special Topics