 first draft: winter 2000

CHAPTER 6

TWO-COLORED WALLPAPER PATTERNS

6.0 Business as usual?

6.0.1 Consistency with color. All concepts and methods pertaining to two-colored border patterns discussed in detail in chapter 5 extend appropriately to two-colored wallpaper patterns. Once again, and due to induced color inconsistencies, **coloring may only preserve or decrease symmetry**. As an example, the following coloring of the **cm** pattern in figure 4.27 eliminates **both** its reflection and its glide reflection by way of color inconsistency:

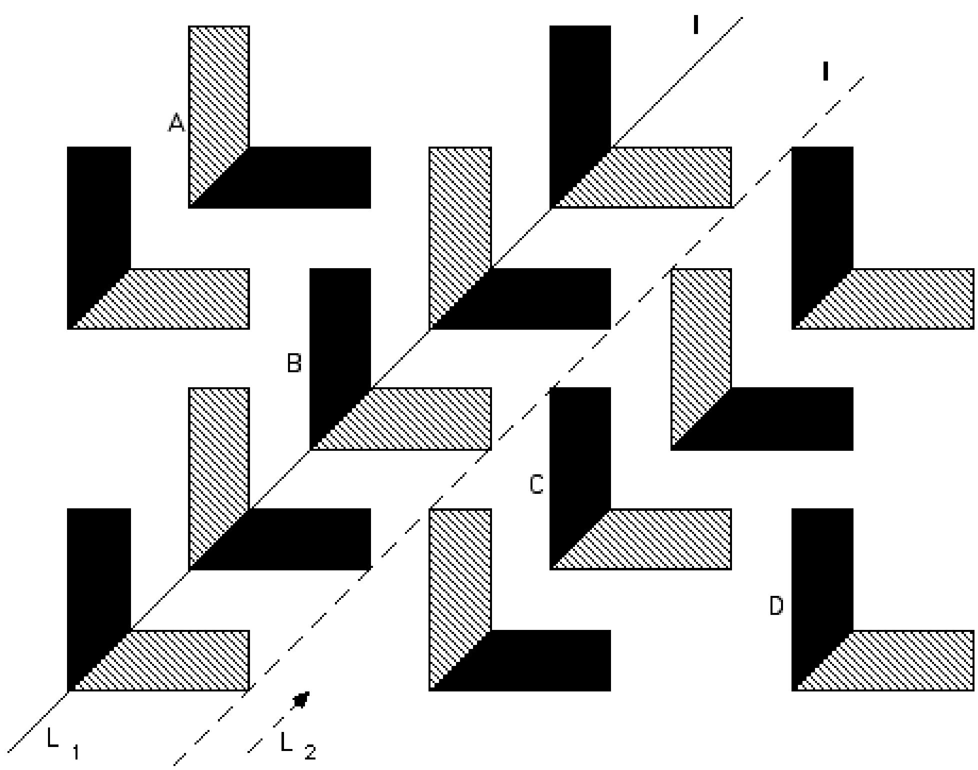


Fig. 6.1

Indeed, the reflection axis $\mathbf{L_1}$ reverses colors as it maps B to itself but preserves colors as it maps A to C; and the shown **upward** glide reflection along $\mathbf{L_2}$ reverses colors as it maps B to C but preserves colors as it maps A to D. (An important lesson drawn out of this example that you should keep in mind throughout this chapter is this: whenever you check a reflection or glide reflection axis for consistency with color, make sure that you look **both** at motifs on or **'near'** that axis and at motifs **'far'** from that axis -- "far" and "near" depending on the fundamental (repeated) region's size.)

6.0.2 The smallest rotation angle. As in the case of one-colored wallpaper patterns (chapter 4), the most important step in classifying two-colored wallpaper patterns is the determination of the pattern's smallest rotation angle; again, coloring may eliminate certain rotations by rendering them inconsistent with color, and it is appropriate to state here that **coloring may only preserve or increase the smallest rotation angle**. As an example, the following two colorings (figures 6.2 & 6.3) of the **p4g** pattern in figure 4.57 do increase the **smallest rotation angle consistent with color** from 90^0 to 180^0 (color-reversing) and 360^0 (none), respectively; and this change most definitely affects our **visual perceptions** of these 'new' wallpaper patterns:

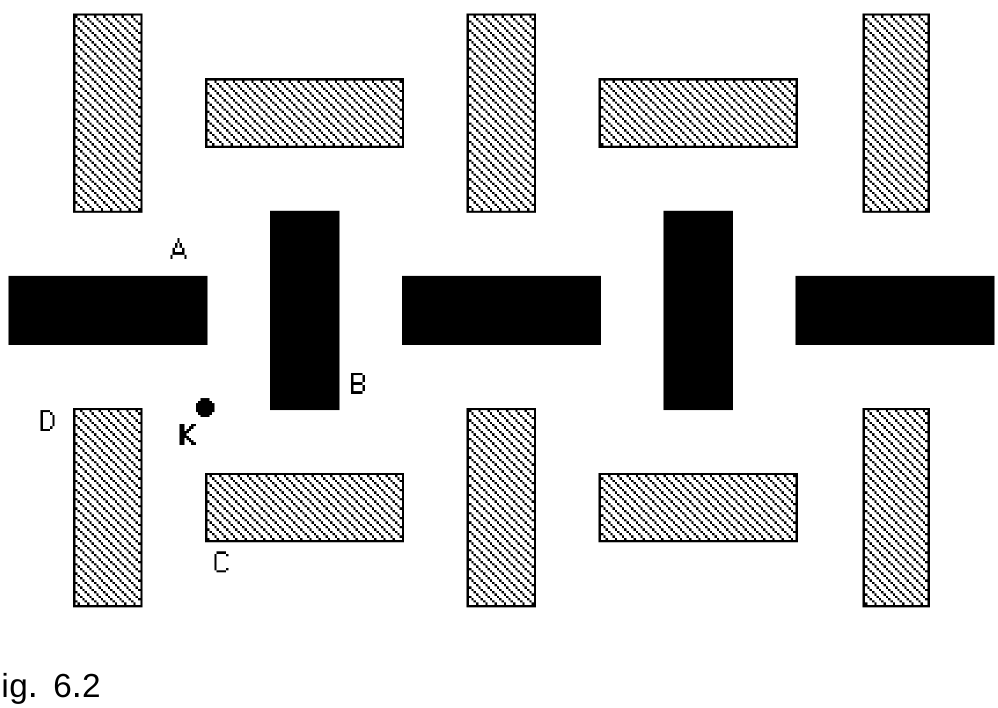


Fig. 6.2

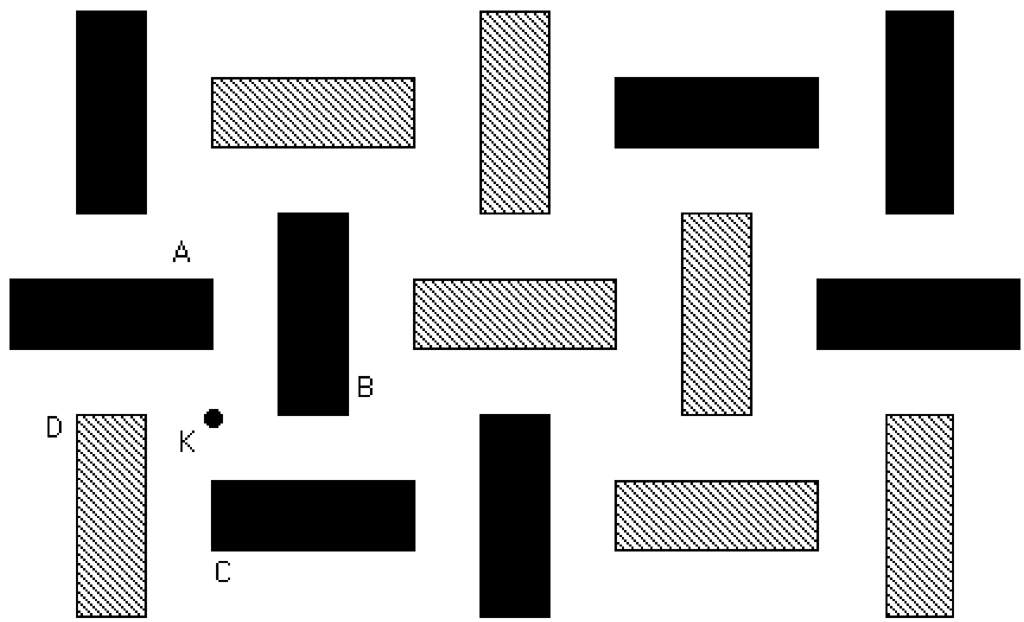
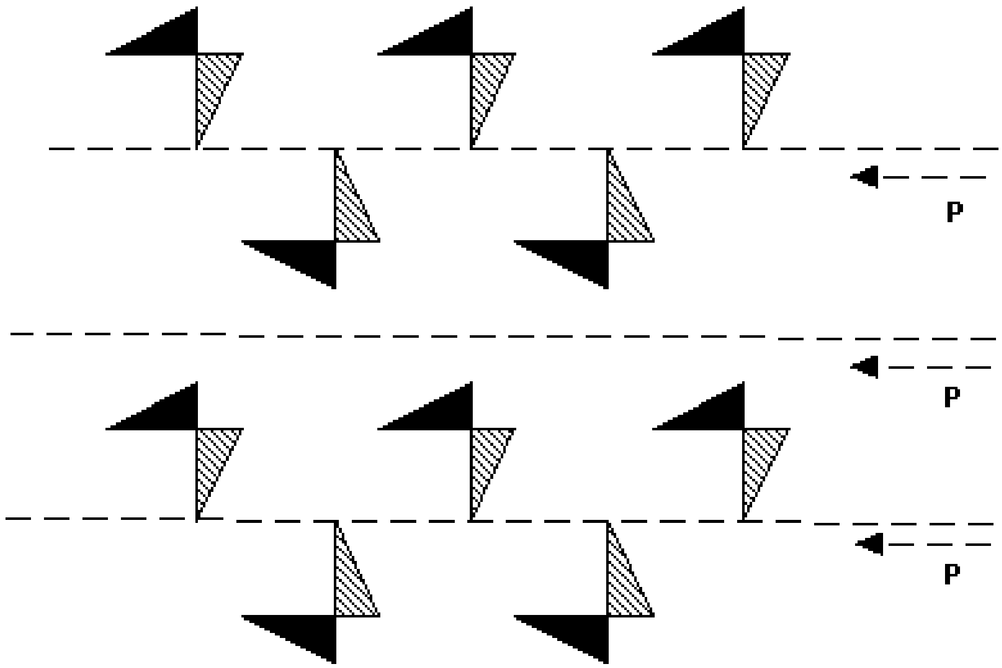


Fig. 6.3

In the pattern of figure 6.2, clockwise 90^0 rotation about **K** maps black A to black B but black B to grey C (inconsistent), while 180^0 rotation about **K** maps **all** black units to grey ones and vice versa (consistent): that is, the initial 90^0 rotation is gone but the induced 180^0 rotation -- recall (4.0.3) that applying a 90^0 rotation **twice** trivially generates a 180^0 rotation -- survives. And in the pattern of figure 6.3 clockwise 90^0 rotation about **K** maps black A to black B but black C to grey D (inconsistent), while 180^0 rotation about **K** maps black A to black C but black B to grey D (inconsistent): that is, **both** the 90^0 and 180^0 rotations about **K** have been rendered color-inconsistent by the original **p4g** pattern's coloring -- which has in fact 'destroyed' **all** rotation centers, twofold and fourfold alike. In a nutshell, the pattern in figure 6.2 is a two-colored **180^0** pattern, while the pattern in figure 6.3 is a two-colored **360^0** pattern.

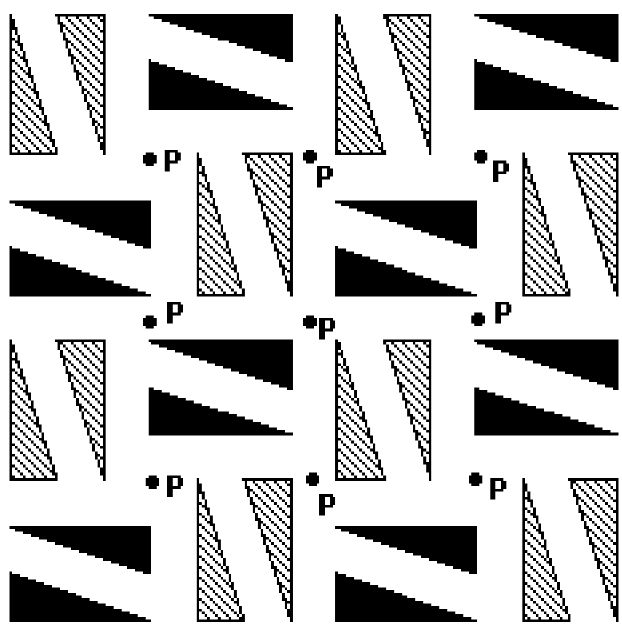
6.0.3 <u>When the two colors are 'inseparable'.</u> As in chapter 5, it is possible for a 'two-colored looking' pattern to be classifiable as one-colored because it has **no color-reversing isometry**. Here are two such 'exotic' examples featuring **color-preserving translation** -- present in **all** two-colored patterns, hence not mentioned -- together with **color-preserving glide reflection** (a

pg, figure 6.4) or **color-preserving half turn** (a **p2**, figure 6.5); of course one may view these two patterns as unions of two **'equal and disjoint'**, black and grey, **pg** and **p2** patterns, respectively.



pg

Fig. 6.4



p2

Fig. 6.5

6.0.4 Symmetry plans and types. Tricky two-colored wallpaper patterns such as the ones presented so far, as well as easier ones, are not that difficult to classify once the smallest rotation angle consistent with color has been determined. Indeed there are **symmetry plans** available at the end of most sections, focused on those symmetry elements that are essential for classification purposes; all notation introduced in 5.2.3 and employed in section 5.9 remains intact. As in chapter 4, little attention is paid to the crystallographic notation's mysteries: simply try to comprehend symmetry plans instead of memorizing **sixty three** type names!

In each of the next seventeen sections we will be looking not only at all possible ways of coloring each one of the seventeen wallpaper types in two colors, but also at all possible two-colored types sharing the same isometries (be them color-preserving or color-reversing) with each of the seventeen parent types. For example, the two-colored patterns in figures 6.2 & 6.3 are no longer associated with the **p4g** parent type, but rather with 180^0 and 360^0 parent types to be determined -- stay tuned! Moreover, the number of two-colored possibilities associated with each parent type will be not only artistically and empirically determined, but also **mathematically justified and predicted**: and it is precisely through this 'prediction process' that you will begin to understand the mathematical structure of the seventeen wallpaper pattern types, and how their isometries **interact** with each other, effectively building each type's symmetry and **'personality'**!

Here is the number of two-colored types associated with each of the seventeen parent types, **including** in each case the one-colored parent type itself (which is justified by our discussion in 6.0.3):

p1:	2	**pmg**:	6	**p3**:	1
pg:	3	**pmm**:	6	**p31m**:	2
pm:	6	**cmm**:	6	**p3m1**:	2
cm:	4	**p4**:	3	**p6**:	2
p2:	3	**p4g**:	4	**p6m**:	4
pgg:	3	**p4m**:	6		

As promised above, the grand total is **63**: the journey begins!

6.1 p1 types (p1, p'_b1)

6.1.1 One direction is not enough! A two-colored wallpaper pattern must by definition have translation consistent with color in **two**, therefore (4.1.1) **infinitely many**, directions. Notice here, as in 5.1.2, the existence of color-preserving translation in **all** two-colored patterns (already mentioned in 6.0.3): since the successive application of any translation that leaves the pattern invariant produces a **double** translation that also leaves the pattern invariant, the **R** x **R** = **P** rule of 5.6.2 allows us to get a color-preserving translation out of every color-reversing translation.

On the other hand, color-reversing translation in **one** direction (with no other color-consistent translations in sight) does **not** make a wallpaper pattern! Combining ideas from section 4.1 (figures 4.12 & 4.13), we use **vertical p′111** border patterns to built the following **'non-pattern'** that has vertical color-reversing translation (hence black and grey **in perfect balance** with each other):

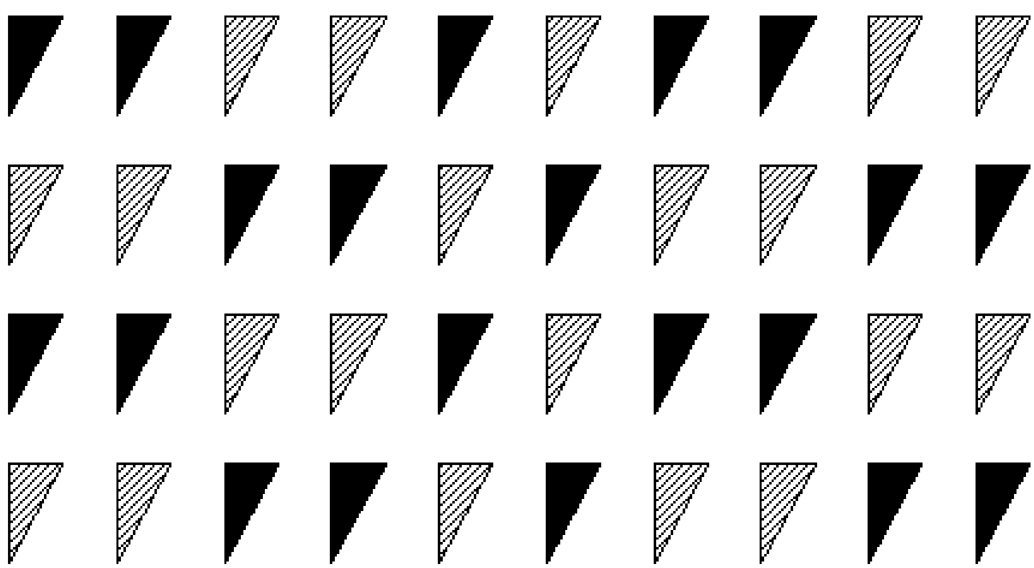
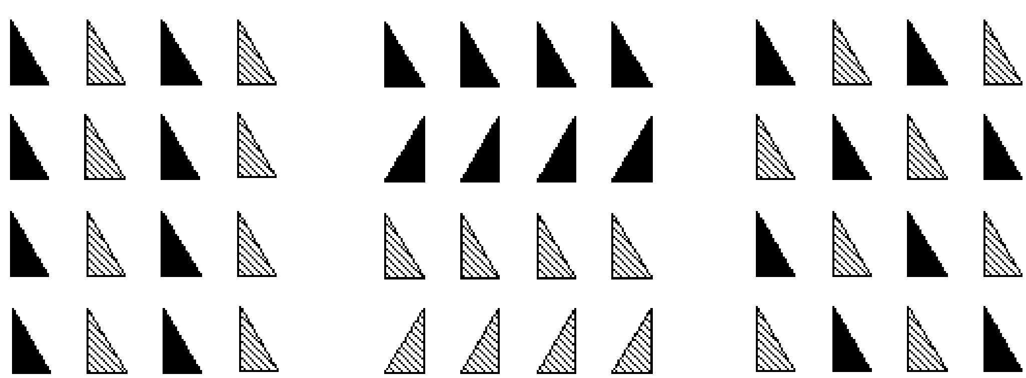


Fig. 6.6

6.1.2 Infinitely many color-reversing translations. Leaving the 'non-pattern' of figure 6.6 behind us, let's have a look at the two-colored wallpaper pattern of figure 6.1: it clearly has no rotation, and we have already pointed out in 6.0.1 that all its reflections and glide reflections are gone due to color inconsistency. In view of our discussion in 6.0.3, you have every right to ask: is it 'truly' two-colored? That is, does it have any isometries that **swap** black and grey? Yes, if you remember to think of **translations**! Indeed, you can easily see that there is an 'obvious' horizontal **color-reversing translation** and three less obvious **'diagonal'** color-reversing translations, mapping A to B, C, and D, respectively; and you can probably see by now that there exist such translations in **infinitely many directions**. This property of the pattern in figure 6.1 should not surprise you in view of our discussions in 4.1.1, 6.1.1, and the **R** x **P** = **R** rule of 5.6.2: every two-colored wallpaper pattern that has color-reversing translation in one direction must have color-reversing translation in infinitely many directions.

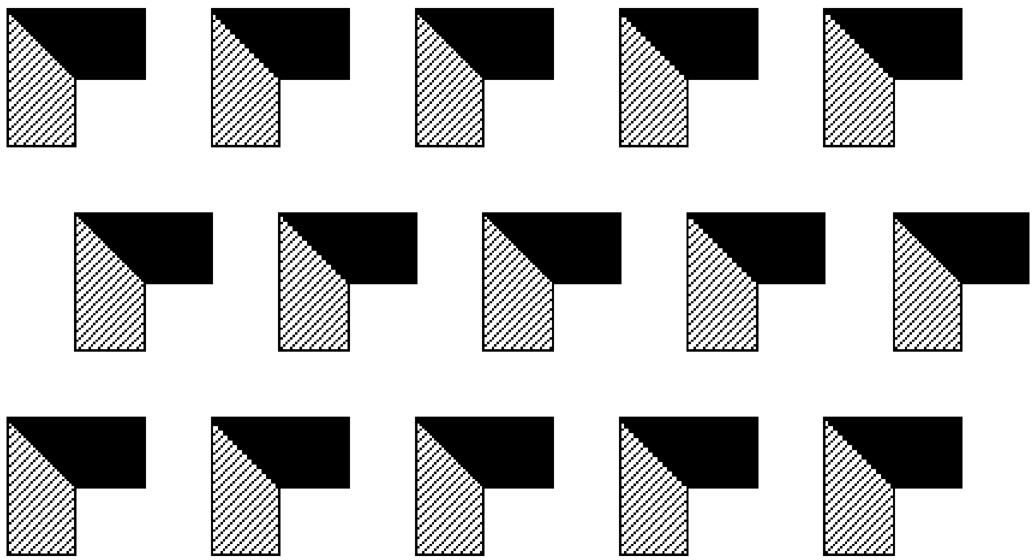
The observation we just made holds true for every two-colored wallpaper pattern: all patterns you are going to see in this chapter have color-reversing translation in **either none or infinitely many directions**. Patterns that have nothing but color-reversing translation (and color-preserving translation, of course) are known as $\mathbf{p'_b1}$ patterns. Here are three examples of such patterns **simpler** in underlying structure than the one in figure 6.1:



$\mathbf{p'_b1}$ $\mathbf{p'_b1}$ $\mathbf{p'_b1}$

Fig. 6.7

6.1.3 <u>No color-reversing translations.</u> The only wallpaper pattern type simpler than the $\mathbf{p'_b1}$ is the one that has the only isometry **common to all** wallpaper patterns (**color-preserving translation**) and nothing else: this is the **p1** type, familiar of course from section 4.1. But how about a **p1** pattern that, just like the **p2** and **pg** patterns in 6.0.3, looks like a ‘genuine’ two-colored pattern, having black and grey in perfect balance with each other? Here is such an example:



p1

Fig. 6.8

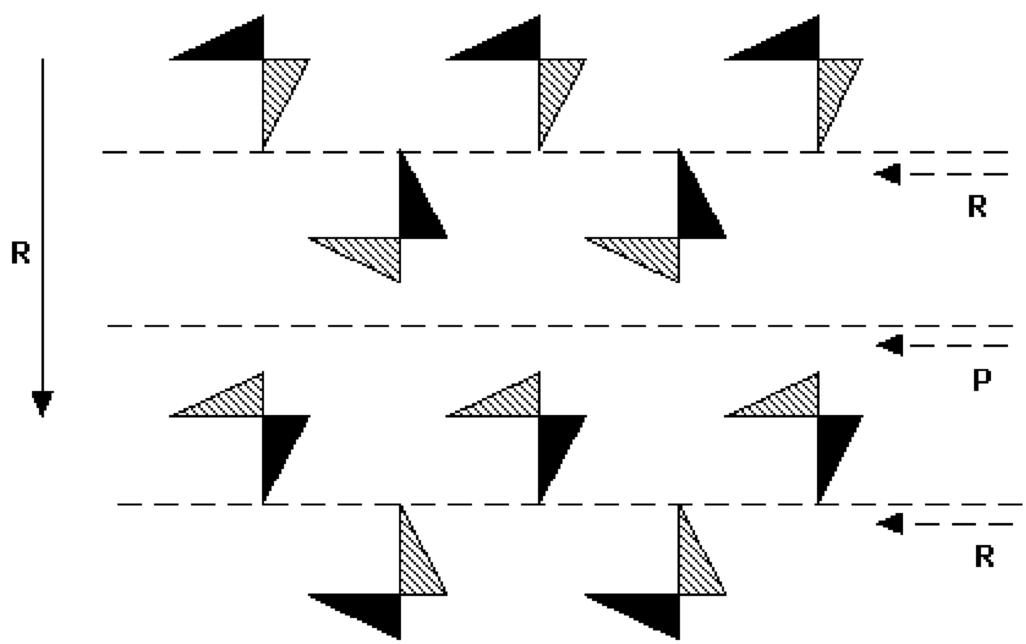
We leave it to you to compare this pattern to the $\mathbf{p'_b1}$ pattern of figure 6.1 and verify its **p1** classification: notice in particular that there are no ‘underlying’ reflections or glide reflections; or, if you wish, they were dead before they were born, ruled out by **structure** and **position** rather than inconsistency with color.

6.2 pg types (pg, p'_b1g, pg′)

6.2.1 <u>Those elusive glide reflections.</u> While the pattern in figure 6.2 clearly has vertical and horizontal color-preserving reflections and in-between color-reversing glide reflections, as well as 180^0

rotations of both kinds, the corresponding isometries of the pattern in figure 6.3 are **all** inconsistent with color; the only consistent with color isometries that the pattern in figure 6.3 seems to have are **translations**, and in particular vertical and horizontal color-reversing translations. So, are we to conclude that the pattern in question is a $\mathbf{p'_b1}$? Well, as in every context in life, some knowledge of **‘history’** can only help. Going back to the pattern’s progenitor in figure 4.57, we see the standard **p4g** ‘diagonal’ glide reflection: we leave it to you to check that the particular **NW-SE** axis shown in figure 4.57 (passing through **bottoms** of vertical rectangles) provides a **color-preserving** glide reflection; and that the NW-SE glide reflection axes right next to it (passing through **tops** of vertical rectangles) provide **color-reversing** glide reflections. So the pattern in figure 6.3 is not a $\mathbf{p'_b1}$, but rather what is known as a $\mathbf{p'_b1g}$: color-reversing translation ($\mathbf{p'_b1}$) plus glide reflection, **both** color-preserving and color-reversing (**g**).

6.2.2 <u>Let’s change those triangles a little!</u> A slight modification of the **pg** pattern in figure 6.4 yields another example of a $\mathbf{p'_b1g}$:



$\mathbf{p'_b1g}$

Fig. 6.9

The visual difference between the ‘two kinds’ of glide reflection axes is much more clear than the one in the example discussed in 6.2.1, so it is even less surprising that one kind of axes preserve colors while the other kind reverse colors.

6.2.3 Hunting for the third type. As we have seen in 5.2.1 and 5.5.1, it is possible to have **both kinds** of vertical reflection axes or half turn centers **reverse** colors in a two-colored border pattern. Therefore it is very reasonable to expect to have patterns in the **pg** family where **all** glide reflection axes **reverse** colors. Could a coloring of the familiar **p4g** pattern of figure 4.57 produce such an example? Well, a closer look at the NW-SE glide reflection of the $\mathbf{p'_b1g}$ pattern in figure 6.3 suggests this attempt:

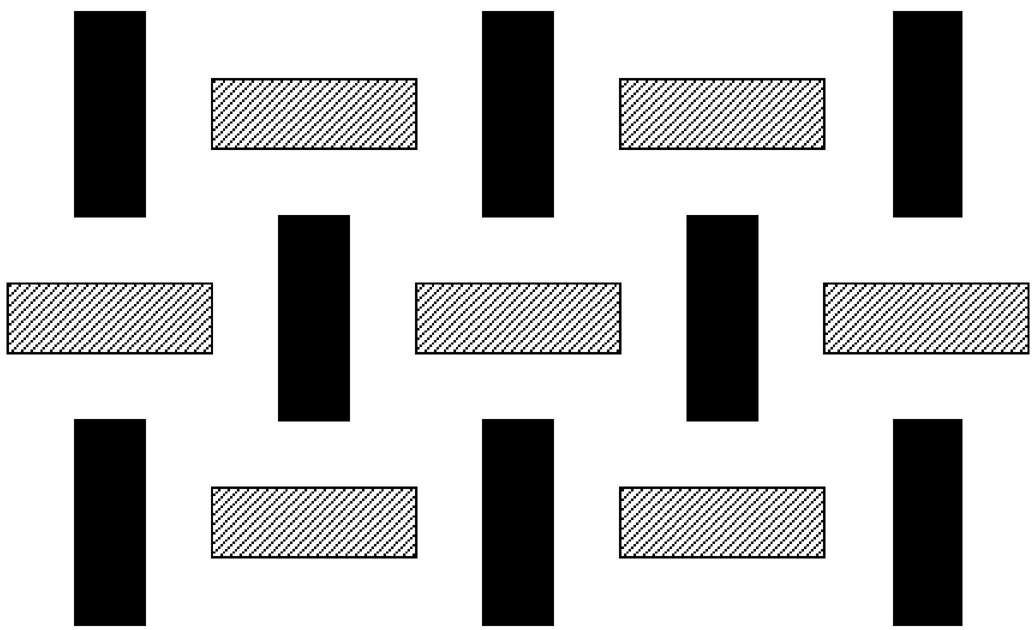
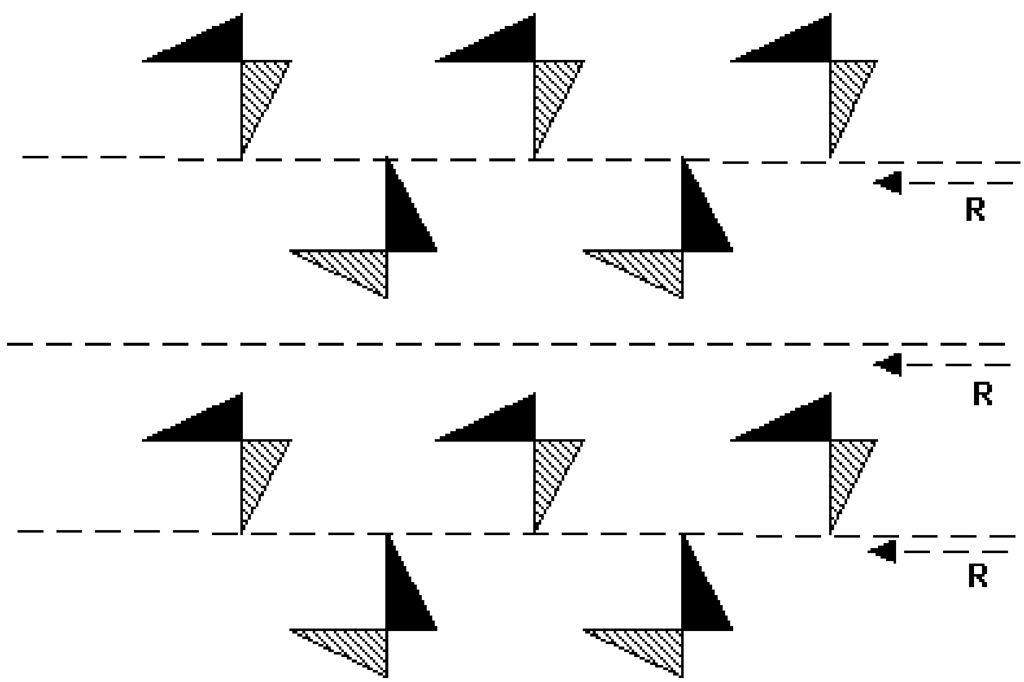


Fig. 6.10

Indeed all NW-SE glide reflections reverse colors. But so do the NE-SW glide reflections (which were in fact inconsistent with color in figure 6.3)! Could such a pattern ever belong to the **pg** family? As we will point out in sections 7.2, 7.9, and 7.10, and as you may already have observed in chapter 4, whenever a pattern has reflection and/or glide reflection in **two distinct directions** it **must** also have **rotation**: indeed our pattern above has color-reversing **90^0** rotation -- not to mention its color-preserving reflections and glide reflections -- and it belongs to the **p4g** family (see 6.11.2).

The lesson drawn out of this example is that some times we get **more** symmetry than desired, especially when we try to **'hide'** a rich underlying structure by way of coloring. This is a lesson worth remembering, but what about our original quest for a **pg** kind of pattern with **color-reversing** glide reflection **only**? Well, perhaps it is time to be less adventurous, avoid 'structural traps', and look for a more down-to-earth example; not that it is the simplest way out, but, once again, a modification of the **pg** pattern in figure 6.4 works:



pg′

Fig. 6.11

Patterns such as the one in figure 6.11 are known as **pg′**.

6.2.4 Examples. These colorings should be compared to the **p1** colorings employed in figure 6.7:

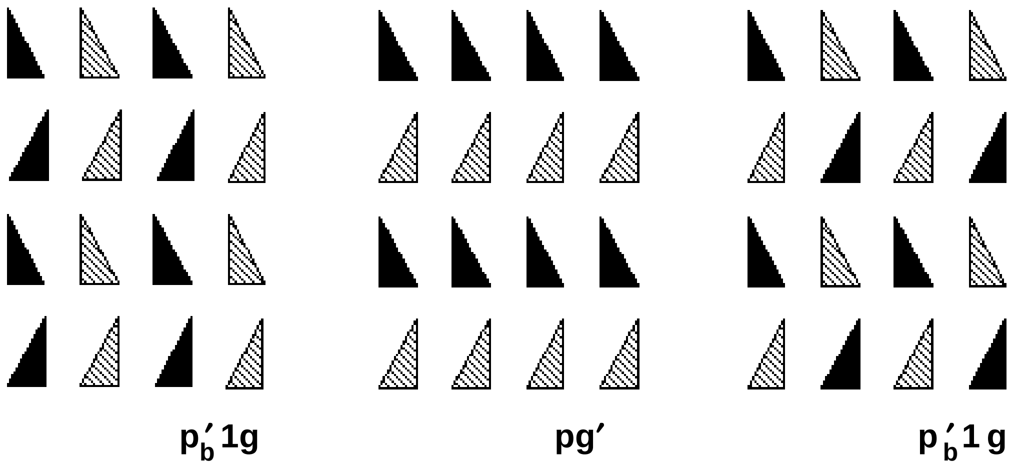


Fig. 6.12

6.2.5 <u>Are there any more **pg** types?</u> The three look-alikes in figures 6.4, 6.9, and 6.11 represent the three types (**pg**, $\mathbf{p_b'1g}$, **pg′**, respectively) discussed so far in this section: they all have glide reflection in a single direction, and what makes them distinct is the effect of their glide reflections on color. Could there be other such types? Well, in the absence of rotations and reflections, the only other isometry that could split the three types into subtypes is **translation**. At first it looks like we could have three × two = six cases: **three** possibilities for glide reflection (color-preserving only (**PP**) or both color-preserving and color-reversing (**PR**) or color-reversing only (**RR**)), and **two** possibilities for translation (color-preserving only (**PP**) or both color-preserving and color-reversing (**PR**), see section 6.1).

But a pattern's glide reflections and translations are **not independent** of each other: as we will prove in section 7.4, and as you can see in figure 6.13 right below, a glide reflection (mapping A to B) and a translation (mapping B to C) **combined** produce another glide reflection (mapping A to C) **parallel** to the first one (G × T = G); moreover (see figure 6.21 further below), the **combination** of two **parallel** glide reflections of **opposite vectors** is a translation **perpendicular** to their axes (G × G = T).

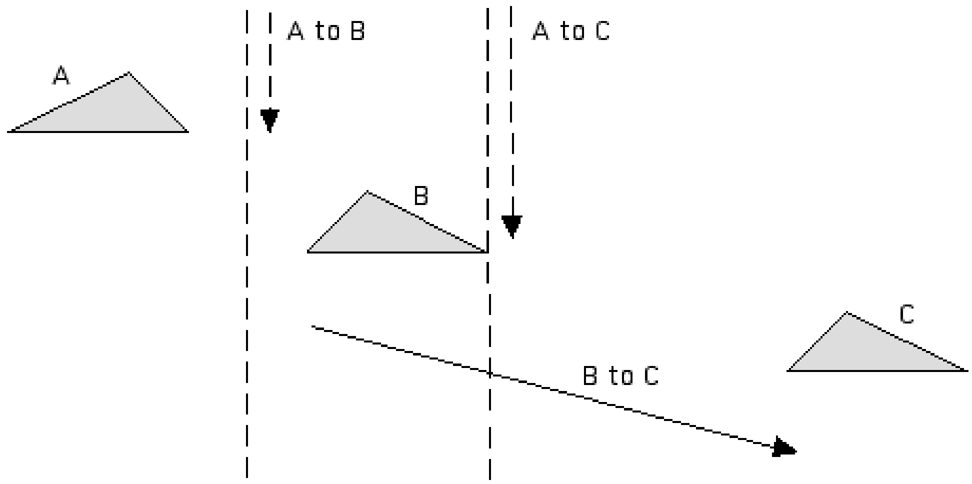


Fig. 6.13

In view of these facts, the multiplication rules of 5.6.2 analyse the six cases mentioned above as follows:

G(**PP**) × T(**PP**) = G(**PP**), G(**PP**) × G(**PP**) = T(**PP**): **pg**
G(**PP**) × T(**PR**) = G(**PR**), G(**PP**) × G(**PP**) = T(**PP**): impossible
G(**PR**) × T(**PP**) = G(**PR**), G(**PR**) × G(**PR**) = T(**PR**): impossible
G(**PR**) × T(**PR**) = G(**PR**), G(**PR**) × G(**PR**) = T(**PR**): $\mathbf{p'_b1g}$
G(**RR**) × T(**PP**) = G(**RR**), G(**RR**) × G(**RR**) = T(**PP**): **pg′**
G(**RR**) × T(**PR**) = G(**PR**), G(**RR**) × G(**RR**) = T(**PP**): impossible

So, there are no more types in the **pg** family after all. Using the examples of this section you can certainly confirm that the only member of the **pg** family that has both kinds of translations (color-preserving and color-reversing) is the one that has both kinds of glide reflections ($\mathbf{p'_b1g}$), just as the above equations indicate. And an important byproduct of the entire discussion, quite useful to remember throughout this chapter, is this: in the presence of (glide) reflections, **translations play no role** at all when it comes to classifying two-colored wallpaper patterns; indeed a pattern with (glide) reflection has translations of both kinds **if and only if** it has both kinds of (glide) reflections! (Recall (1.4.8) that every reflection may be viewed as a **special case** of glide reflection.)

6.2.6 Symmetry plans. We capture the structure of the three **pg** types as follows:

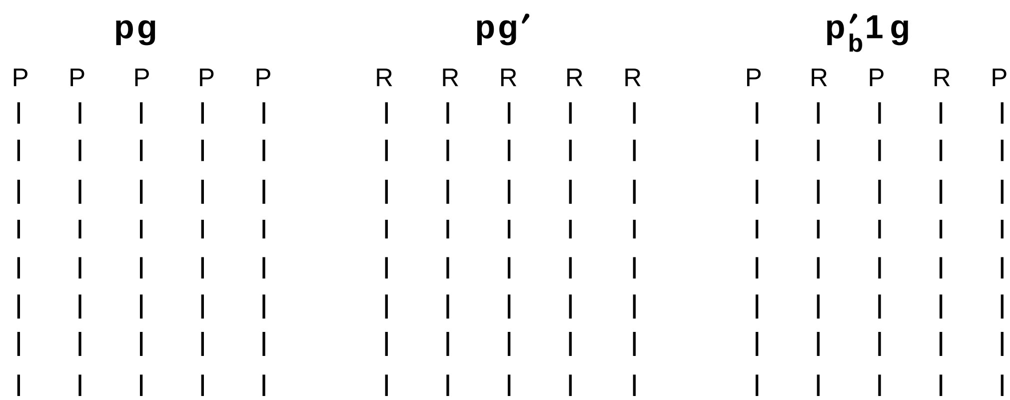
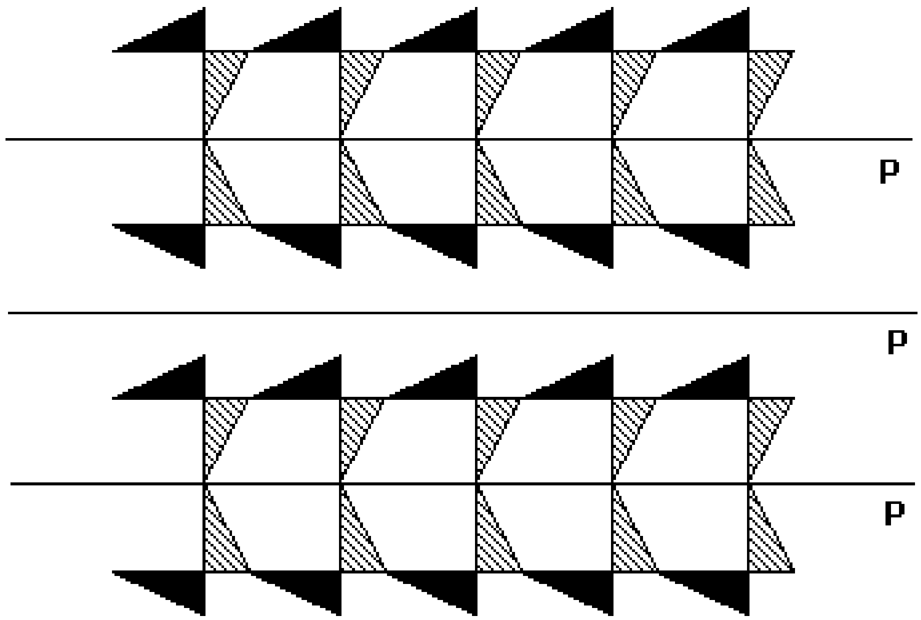


Fig. 6.14

In these symmetry plans glide reflection vectors are not shown for the sake of simplicity, but you **must** indicate them in your work!

6.3 pm types (pm, pm′, p'_b1m, p′m, p'_bg, c′m)

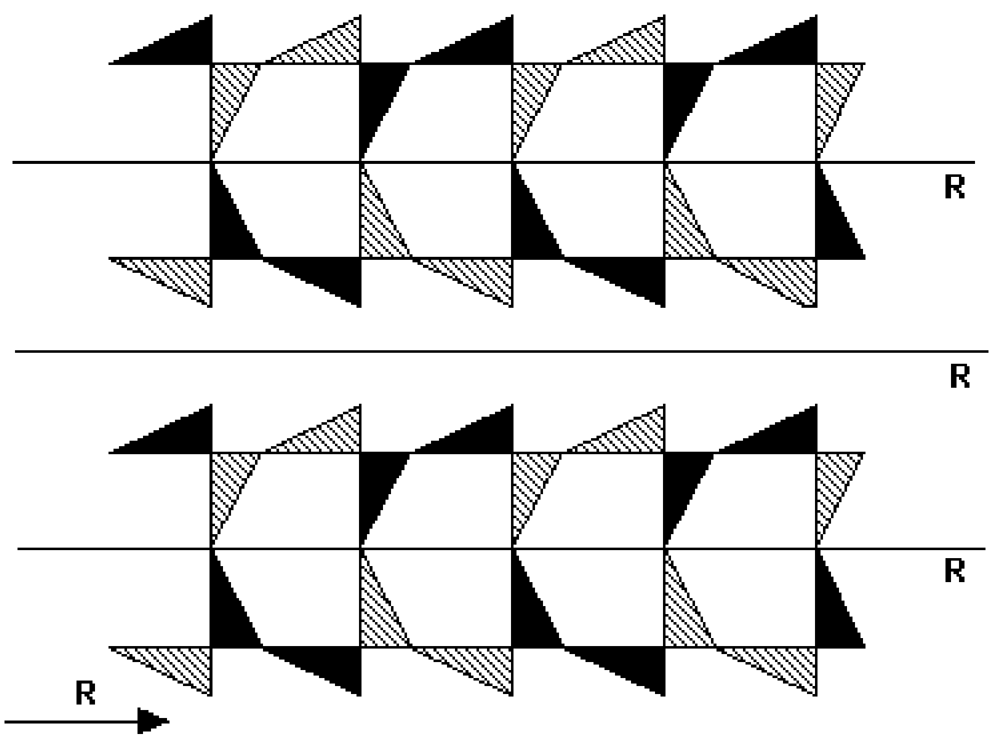
6.3.1 Upgrading the glide reflection to reflection. Employing an old idea from 2.7.2 -- where we viewed a **pma2** pattern as ‘half’ of a **pmm2** pattern -- in the opposite direction, we are now ‘doubling’ the three **pg**-like patterns in figures 6.4, 6.9, and 6.11 into **pm**-like patterns by **‘fattening’** the glide reflections into reflections; that is, we reflect the pattern across every glide reflection axis without gliding the image. This process is bound to produce **six** two-colored **pm**-like patterns having **reflection** in **one** direction: indeed as we reflect across the glide reflection axes we have the **option** of a color effect either opposite to or same as that of the glide reflection (see also 6.3.2 and 6.3.5), so we end up with three × two = six **pm** types. We illustrate the process in the following six figures, indicating in each case the ‘original’ **pg**-like pattern and providing the name for the ‘new’ **pm**-like pattern. Make sure you can rediscover the old **pg**-like pattern inside the richer structure of the new pattern; there is more than mere nostalgia in our call: the old glide reflection is alive and well, **‘hidden’** under the new reflection and ready to play an important role in the classification process!



pg → **pm**

Fig. 6.15

All reflections and hidden glide reflections preserve colors, so the new pattern is classified as a **pm**, despite being two-colored.



$\mathbf{pg} \to \mathbf{p'_b g}$

Fig. 6.16

Reflections reverse colors, hidden glide reflections preserve colors (**g**); there exists **color-reversing translation** along the reflection axes ($\mathbf{p'_b}$). Such patterns are known as $\mathbf{p'_b g}$.

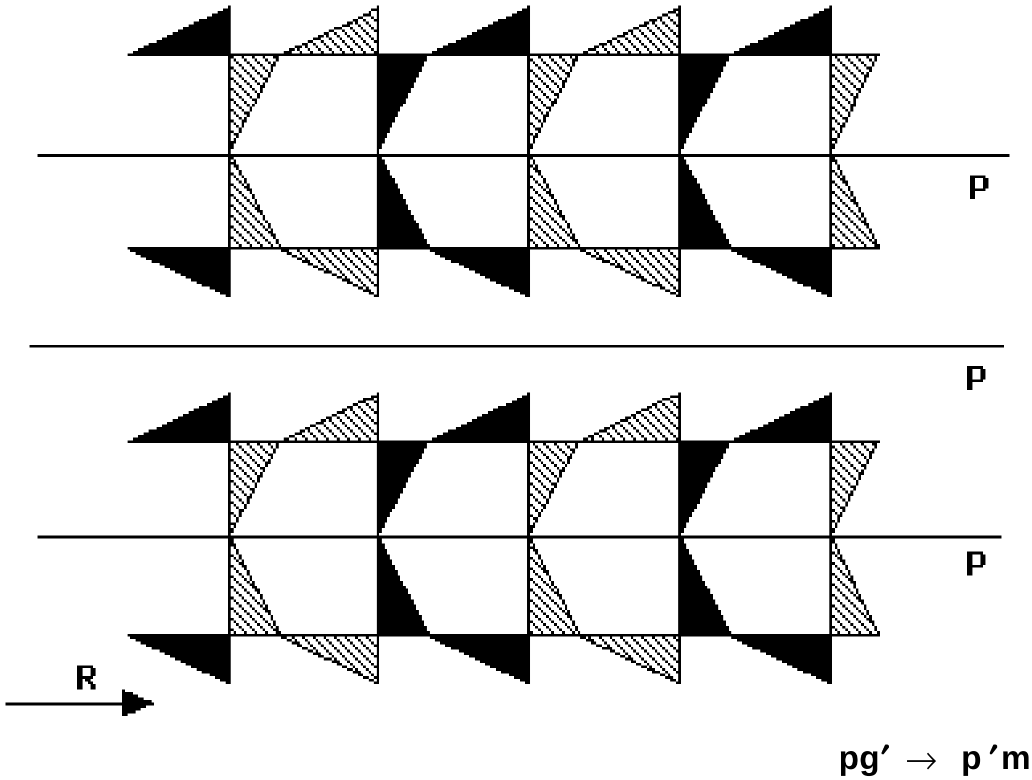
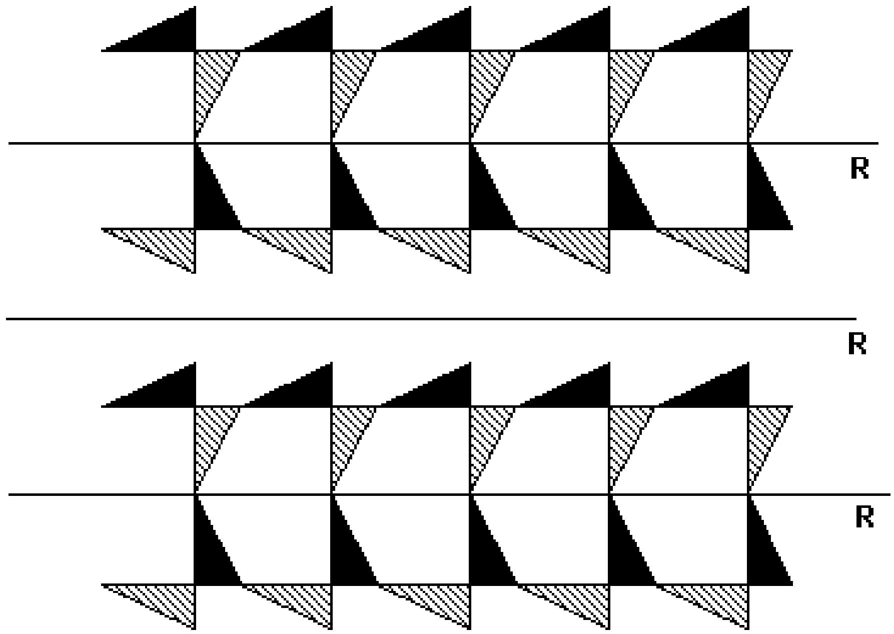


Fig. 6.17

Once again we get color-reversing translation along the reflection axes (**p′**), all of which preserve colors (**m**): the new pattern is known as **p′m**.

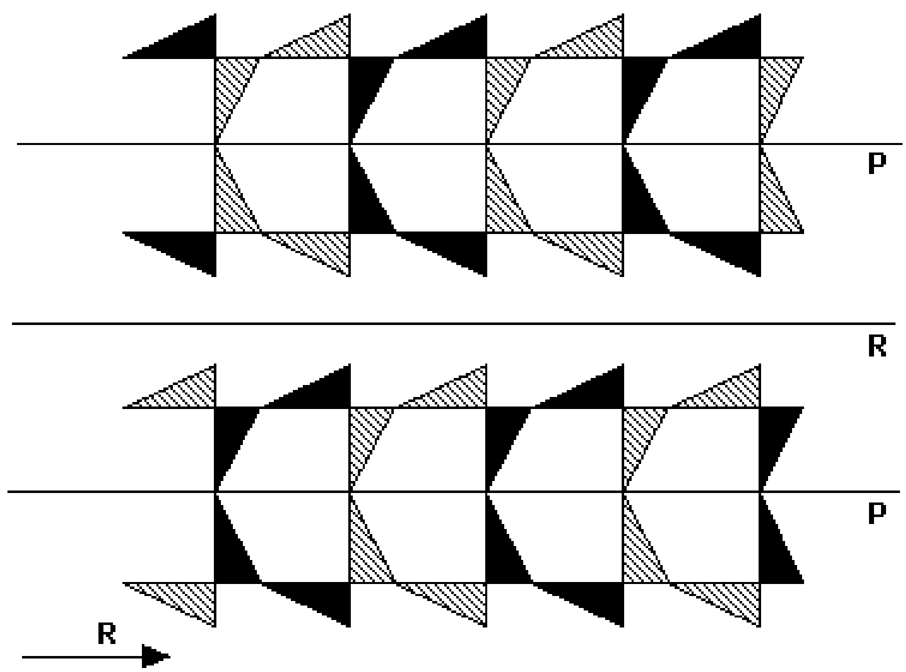
Comparing the two patterns in figures 6.15 & 6.17 we see that they are similar not only in name, but in structure as well; in fact the only thing that makes them **distinct** is that the **p′m** has color-reversing translation while the **pm** doesn't. But didn't we promise back in 6.2.5 that "in the presence of (glide) reflection translation will play no role in the classification process"? Well, there is indeed another, more subtle way of distinguishing between **pm** and **p′m**, and that is their **hidden** ('old') **glide reflection**, which of course preserves colors in the case of the **pm** (an 'offspring' of **pg**) but reverses colors in the case of the **p′m** (an 'offspring' of **pg′**)!



pg′ → pm′

Fig. 6.18

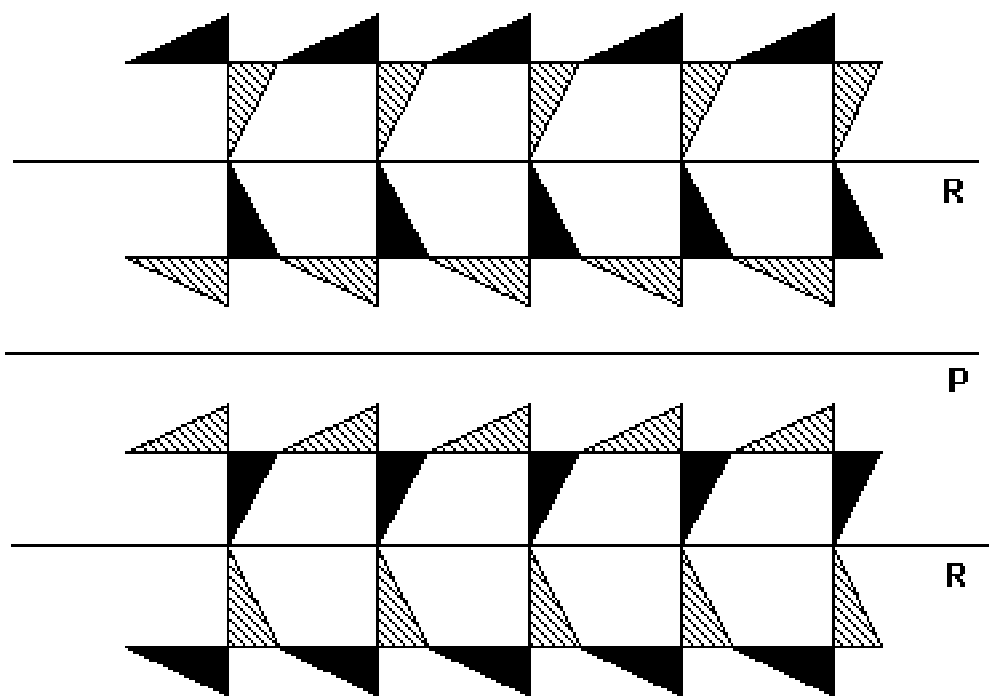
All reflections and hidden glide reflections in this **pm′** pattern do reverse colors (**m′**). Notice the absence of color-reversing translation.



p'_b1g → c′m

Fig. 6.19

Things started getting a bit complicated! Unlike the previous four types, this pattern has **both** color-preserving and color-reversing reflection, and likewise both color-preserving and color-reversing hidden glide reflection; notice that each reflection and hidden glide reflection associated with it have **opposite** effect on color. And, for the first time, we get color-reversing translation in directions **both** parallel and perpendicular to that of the reflection. **Visually**, the effect of all this is a feeling that every other **column** in our pattern has been **shifted** (like in the case of the **cm** patterns of section 4.4), hence its somewhat unexpected name (**c′m**).



p'_b1g $\rightarrow$ **p'_b1m**

Fig. 6.20

Just as in the case of the other ‘offspring’ of **p'_b1g** we just discussed (**c′m**, figure 6.19), this new pattern, known as **p'_b1m**, has **both** color-preserving and color-reversing reflections. Unlike in the case of the **c′m**, however, the hidden glide reflection of the **p'_b1m** always has the same effect on color as the corresponding reflection.

6.3.2 Are there any other types? The process employed in 6.3.1

produced six **pm**-like two-colored wallpaper patterns out of the three **pg**-like patterns of section 6.2. We must ask: could there be any more types in the **pm** family, 'unrelated' perhaps to **pg** types? Well, looking back at the new types we constructed, we can **fully** describe them in terms of the effect on color (**R** or **P**) of their 'two kinds' of reflection (R) **and** hidden glide reflection (G) as follows:

pm:	R(**PP**)/G(**PP**)
$\mathbf{p'_bg}$:	R(**RR**)/G(**PP**)
p′m:	R(**PP**)/G(**RR**)
pm′:	R(**RR**)/G(**RR**)
c′m:	R(**PR**)/G(**RP**)
$\mathbf{p'_b1m}$:	R(**PR**)/G(**PR**)

It becomes clear that the only possible extra types we could get would be of a form like R(**PR**)/G(**RR**) or R(**RR**)/G(**PR**), etc. That is, we 'need' types where the hidden glide reflection has the same effect on color as the corresponding reflection in the case of **every other** reflection axis, and the opposite effect on color of that of the corresponding reflection in the case of all other reflection axes. In other words, we 'need' situations like the one pictured right below:

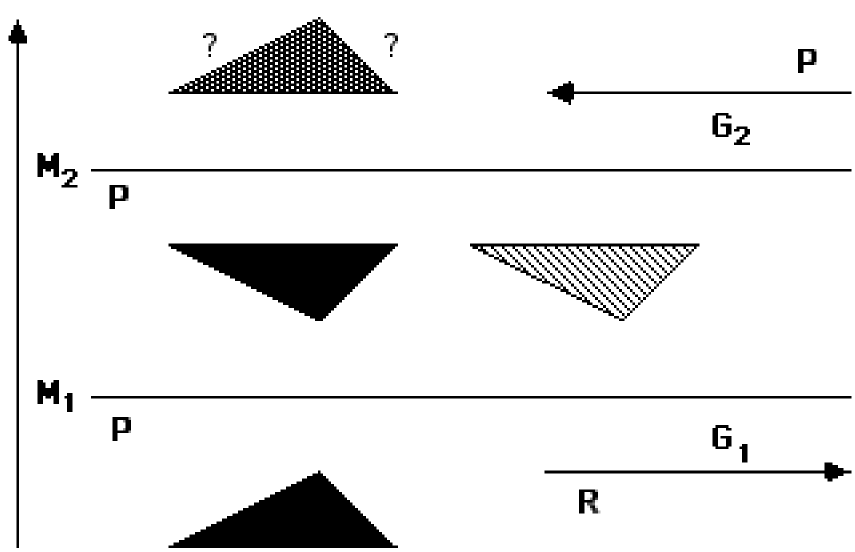


Fig. 6.21

But this is an impossible situation! Indeed the bottom reflection (**M**$_1$) followed by the top one (**M**$_2$) produce the shown **vertical** translation which must be **color-preserving** ($\mathbf{P} \times \mathbf{P} = \mathbf{P}$); but the **same translation** is produced by combining the corresponding hidden glide reflections ($\mathbf{G_1}$ followed by $\mathbf{G_2}$) of the shown **opposite vectors**, hence it has to be **color-reversing** ($\mathbf{P} \times \mathbf{R} = \mathbf{R}$), too!

The **contradiction** we have arrived at shows that there cannot possibly be any **pm**-like two-colored wallpaper patterns other than the **six** types already derived in 6.3.1.

6.3.3 Examples. You should pay special attention to the sixth example, which should belong to the **cm** family but is in fact a $\mathbf{p'_b1m}$ because its in-between glide reflection is **inconsistent** with color:

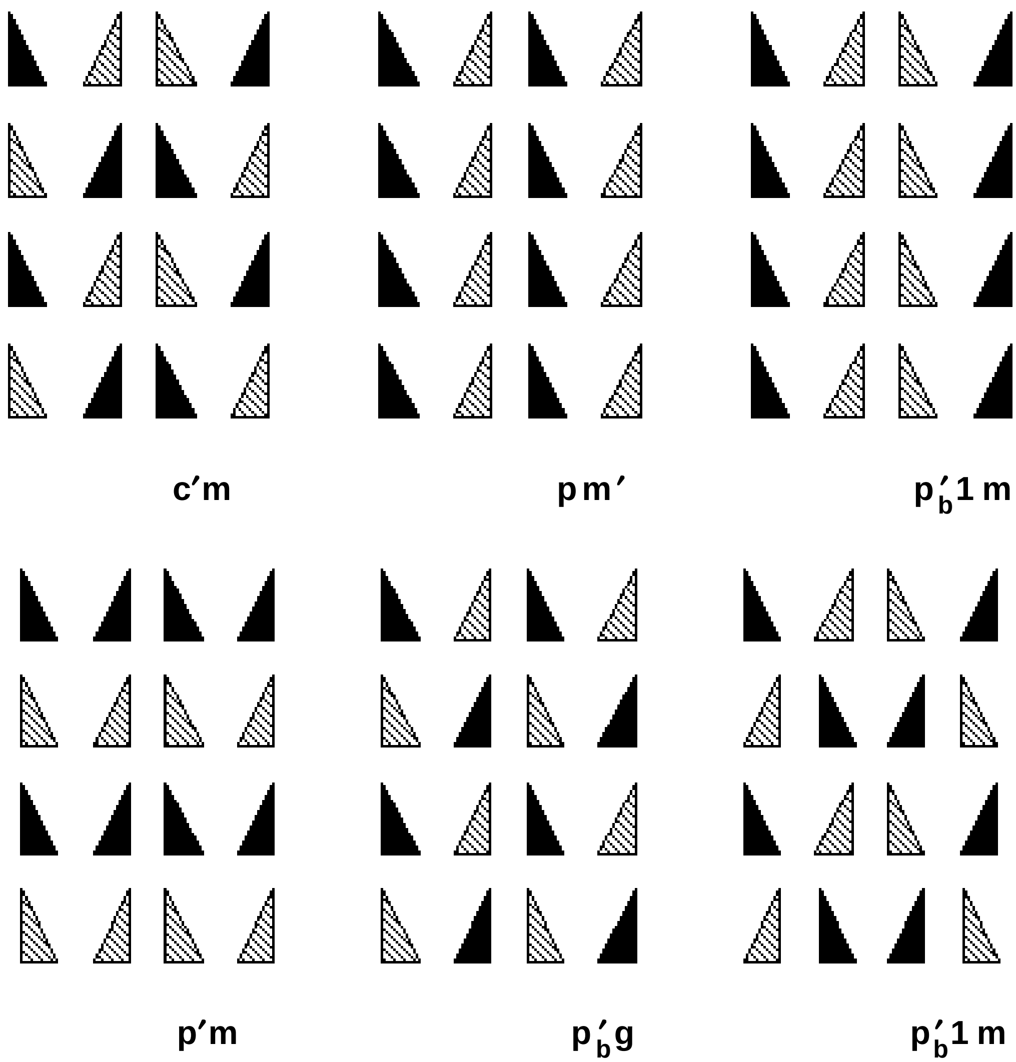


Fig. 6.22

6.3.4 Translations and hidden glide reflections revisited. The examples in 6.3.1 and 6.3.3 make ‘visually obvious’ the fact that there always exists a glide reflection employing the same axis as any given reflection. Such a ‘hidden’ glide reflection exists because a translation **parallel** to the reflection axis is **always** there (just as in the case of **p1m1** and **pmm2** border patterns); and it is easy to see that the hidden glide reflection’s minimal gliding vector is always **equal** to the minimal translation vector along the reflection axis.

But **why** should such a parallel translation be there, after all? The double application of every glide reflection produces a parallel translation of vector **twice** as long as the gliding vector (2.4.2, 5.4.1), but why should a ‘vectorless’ reflection carry the obligation to produce a translation **parallel** to itself? This is best explained through a ‘proof without words’:

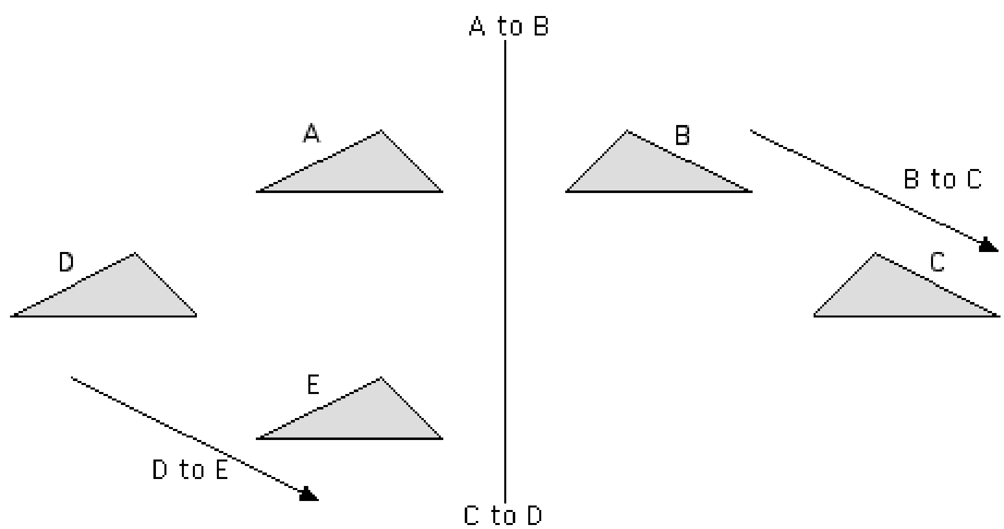


Fig. 6.23

[Since every wallpaper pattern has translations in **at least two** directions, pick one in a direction **non-perpendicular** to that of the reflection axis; then subsequent application of reflection (A to B), translation (B to C), reflection (C to D), and translation (D to E) produces a parallel to the reflection translation (mapping A to E)!]

Once we know that a translation vector parallel to the reflection axis **exists**, then it is possible to pick the **minimal** such vector -- recall that wallpaper patterns do **not** have arbitrarily small translations (4.0.4) -- which is easily shown to be the minimal gliding vector of a (hidden) glide reflection along the reflection axis. You should verify all these ideas for the examples in 6.3.1 and 6.3.3; you may in particular verify that the vertical translation guaranteed by the process in figure 6.23 is actually **twice as long** as the pattern's minimal vertical translation.

6.3.5 Symmetry plans. Even though we classified the **pm** types looking at their reflections and hidden glide reflections, we prefer to provide their symmetry plans based on reflections and parallel to them **translations**. It is of course easy to see that there exists a **color-reversing** translation parallel to the reflection axis if and only if the reflection and the corresponding hidden glide reflection have **opposite** effect on color.

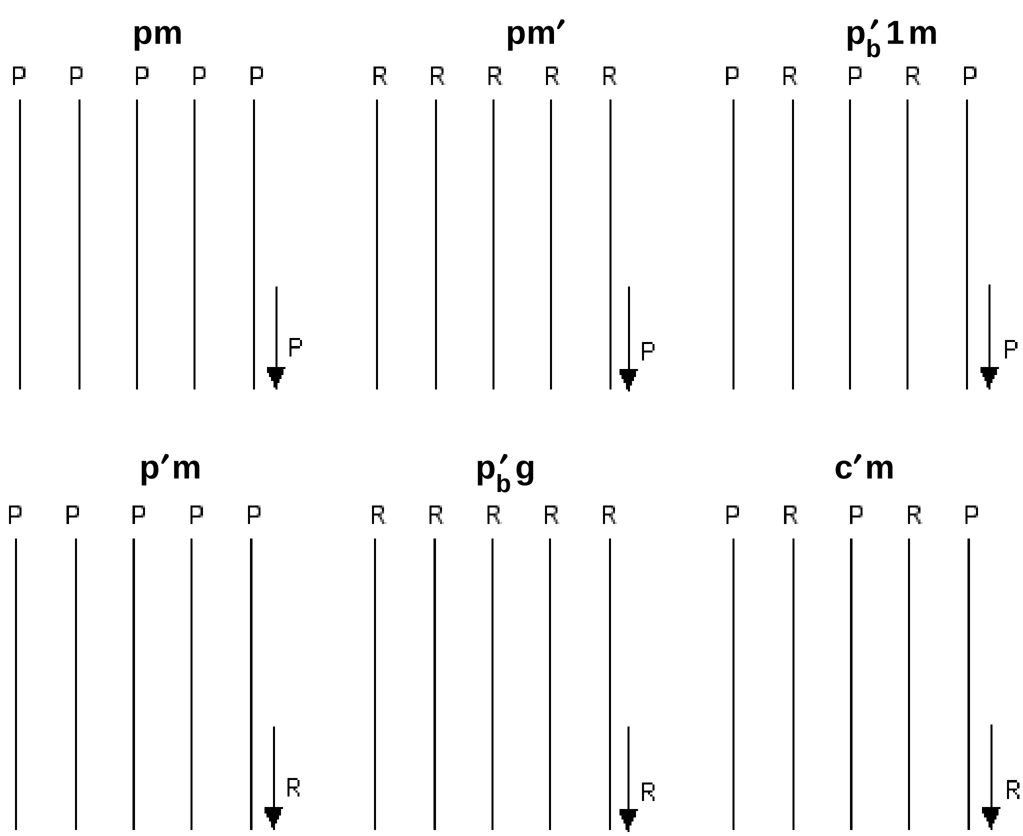
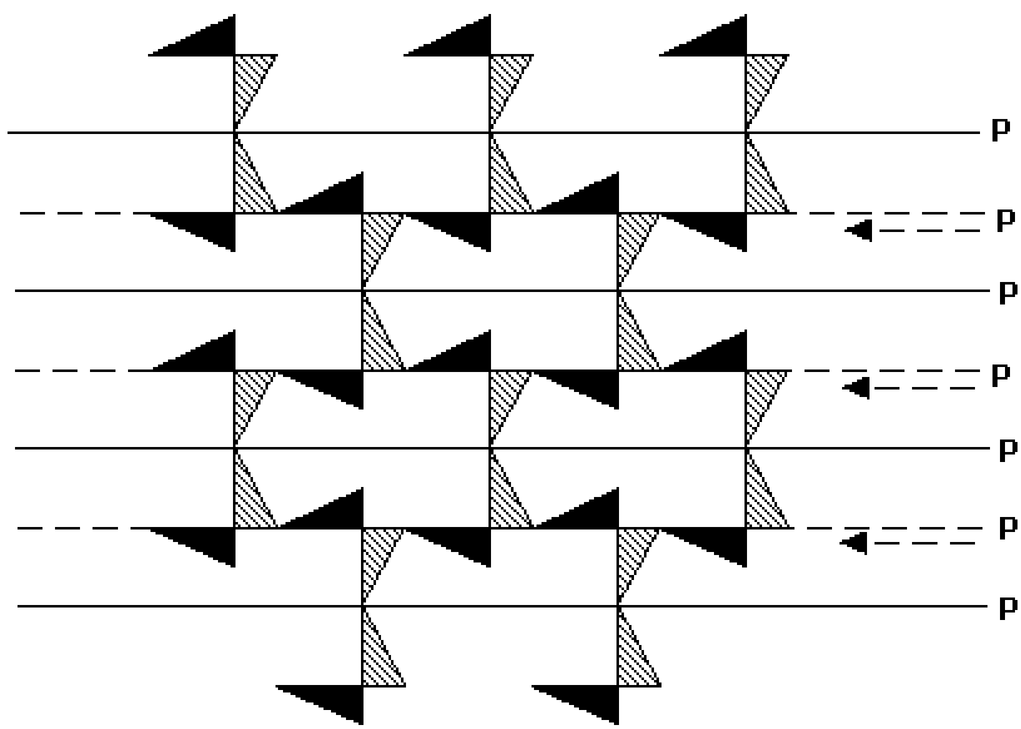


Fig. 6.24

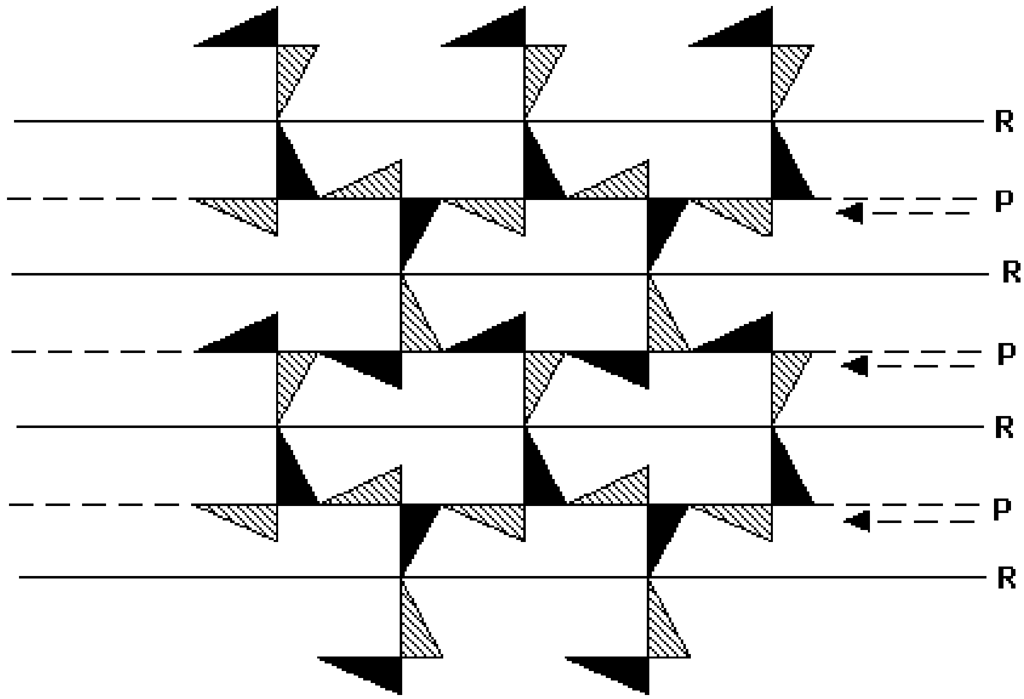
6.4 cm types (cm, cm′, $p'_c g$, $p'_c m$)

6.4.1 Playing that old game again. As we have seen in 4.4.3, every **cm** pattern can be seen as a **pm** pattern every other row of which has been **shifted**. Therefore it is reasonable to assume that the application of that process to the two-colored **pm**-like patterns of 6.3.1 -- shifting columns rather than rows, of course -- is bound to produce two-colored **cm**-like patterns. This turns out to be largely true, with a couple of exceptions: the ‘standard’ shifting process leads from the **c′m** and the **p'_b1m** ‘back’ to **p'_b1g** (due to induced **color inconsistencies**). We illustrate all this in the following six figures:



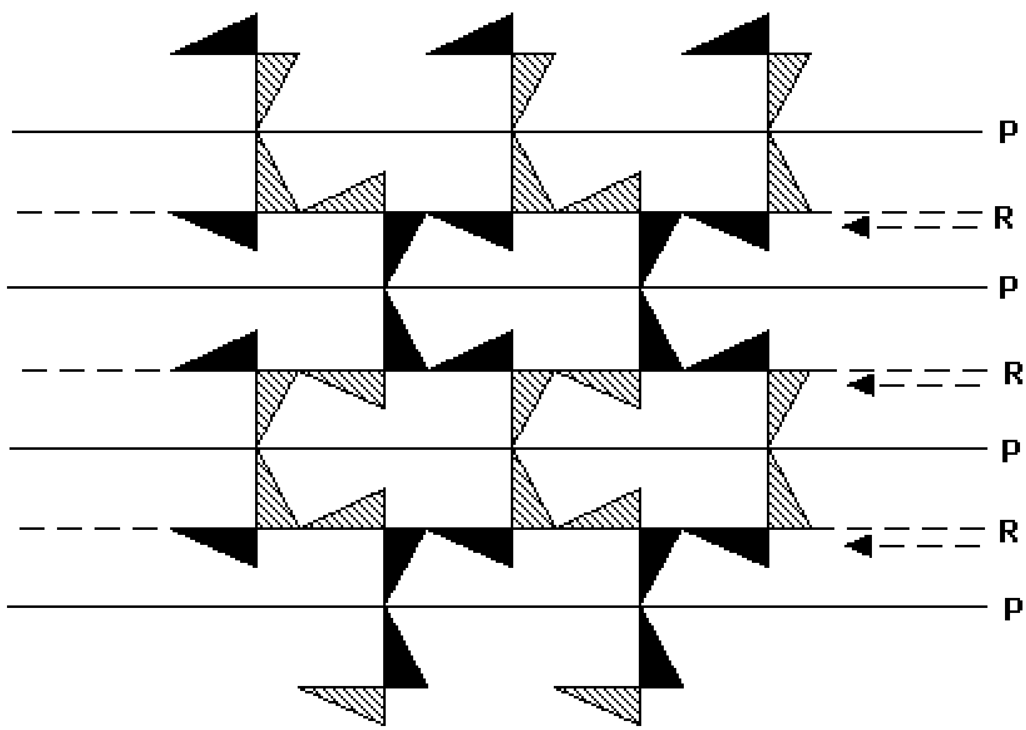
pm → cm

Fig. 6.25



$p'_b g \rightarrow p'_c g$

Fig. 6.26



$p'm \rightarrow p'_c m$

Fig. 6.27

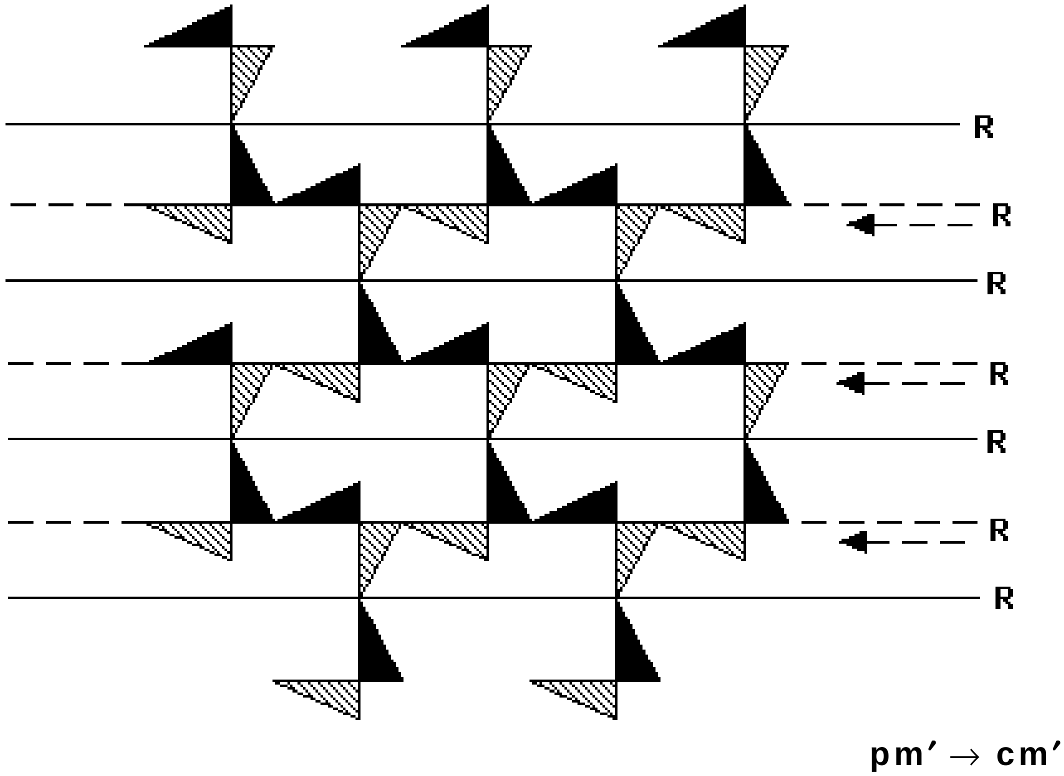


Fig. 6.28

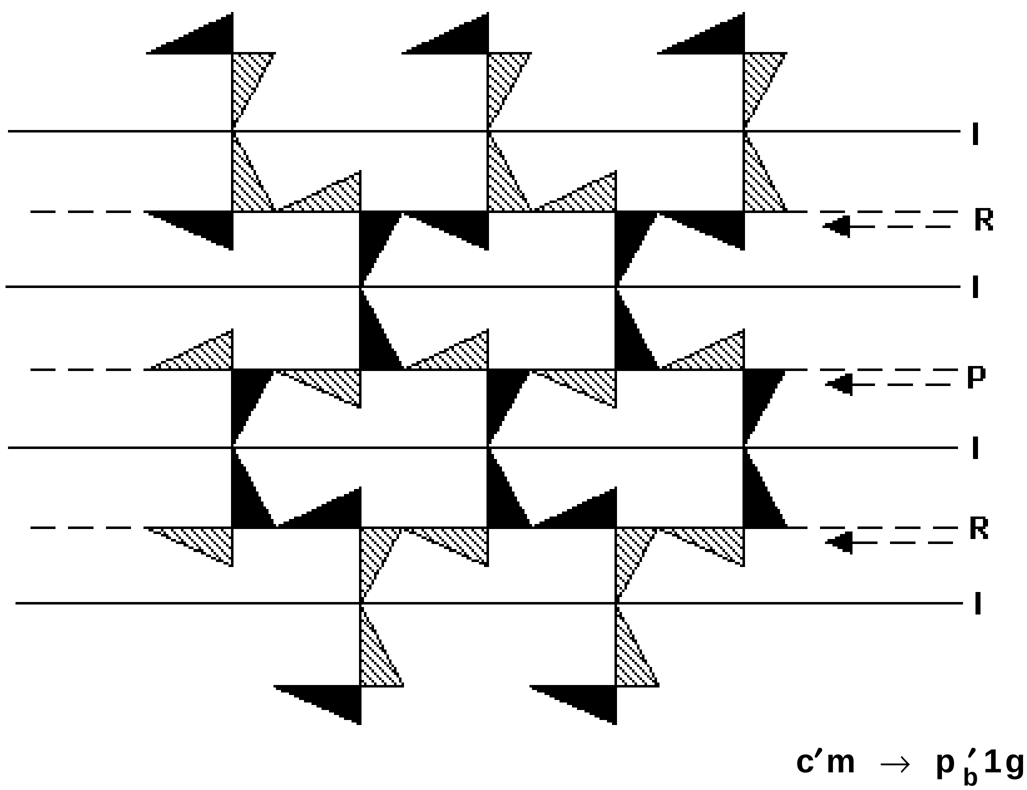
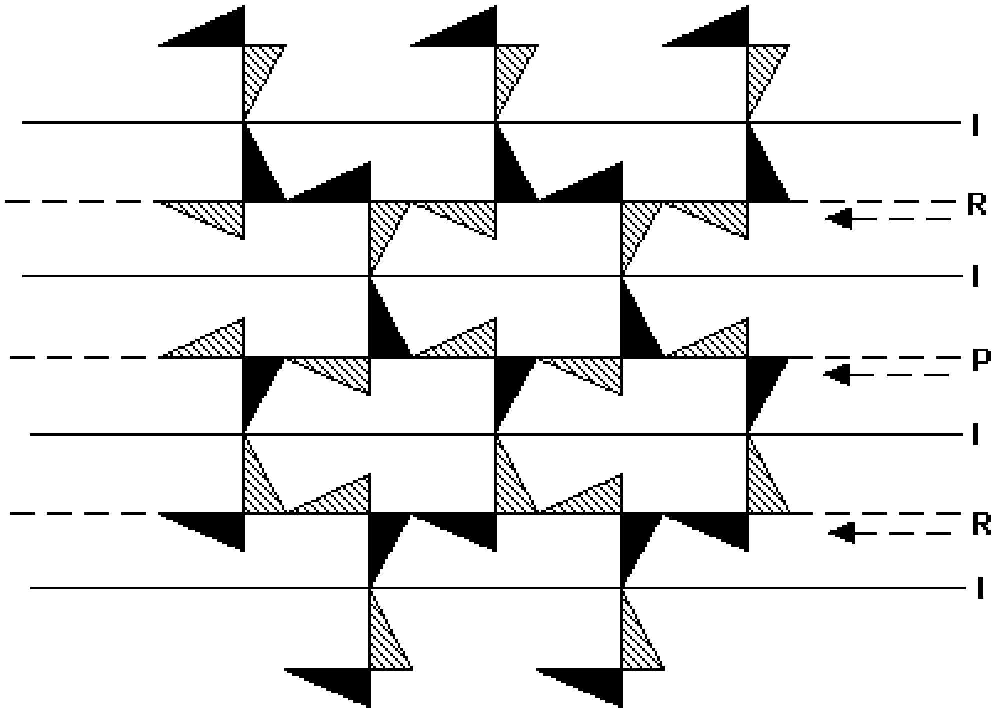


Fig. 6.29



$\mathbf{p_b'1m} \rightarrow \mathbf{p_b'1g}$

Fig. 6.30

6.4.2 <u>Another ‘game’ to consider.</u> Let us revisit that $\mathbf{p_b'1}$ pattern in figure 6.1: since its underlying structure (before coloring) is that of a **cm**, it is reasonable to assume that some other colorings **may** produce **new cm**-like two-colored patterns. In figures 6.31-6.34 below we present a few failed attempts toward such additional **cm** types (some of which involve color **inconsistencies** leading this time to patterns belonging to the **p1** and **pm** groups rather than the **pg** group):

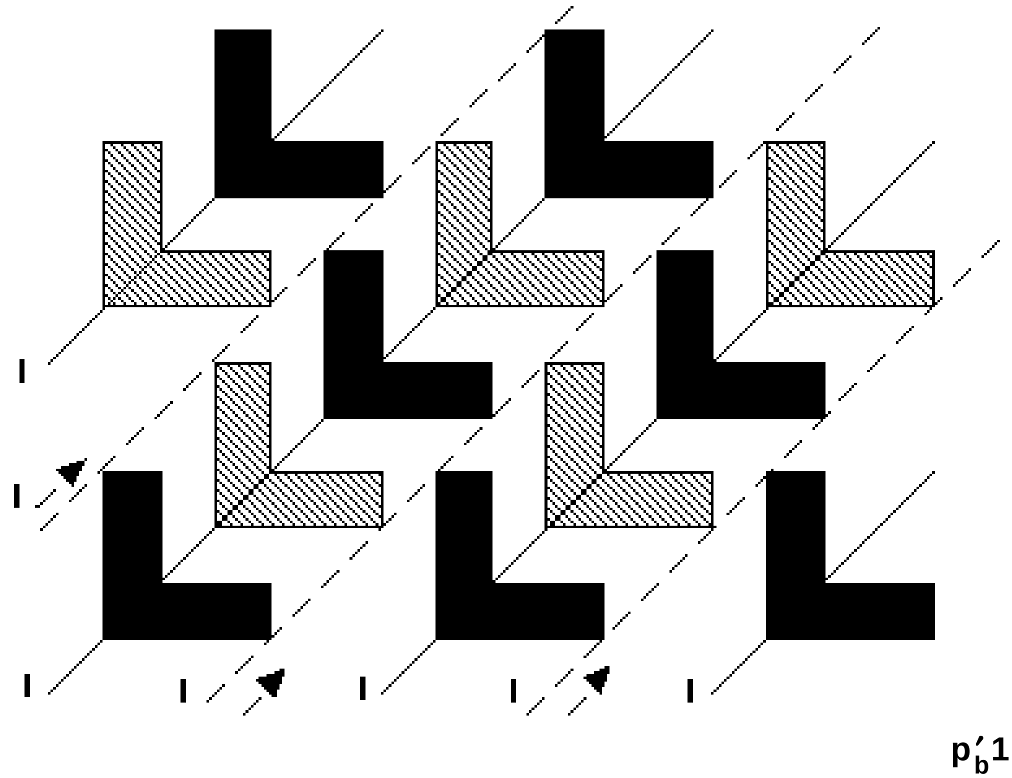


Fig. 6.31

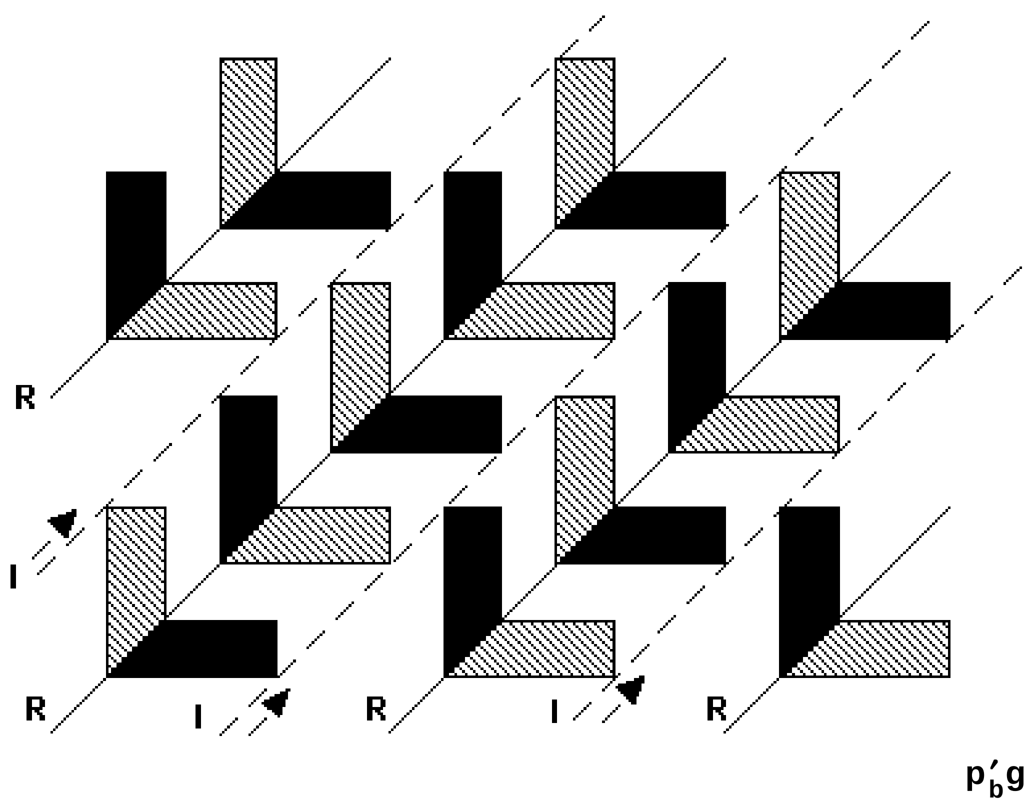


Fig. 6.32

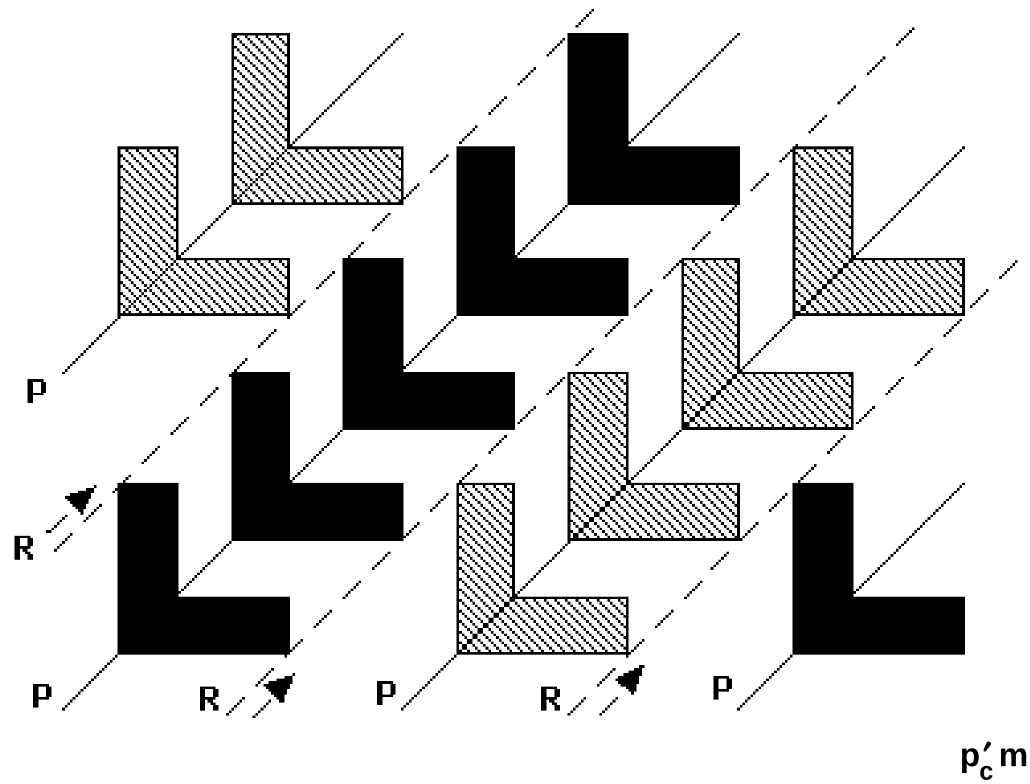


Fig. 6.33

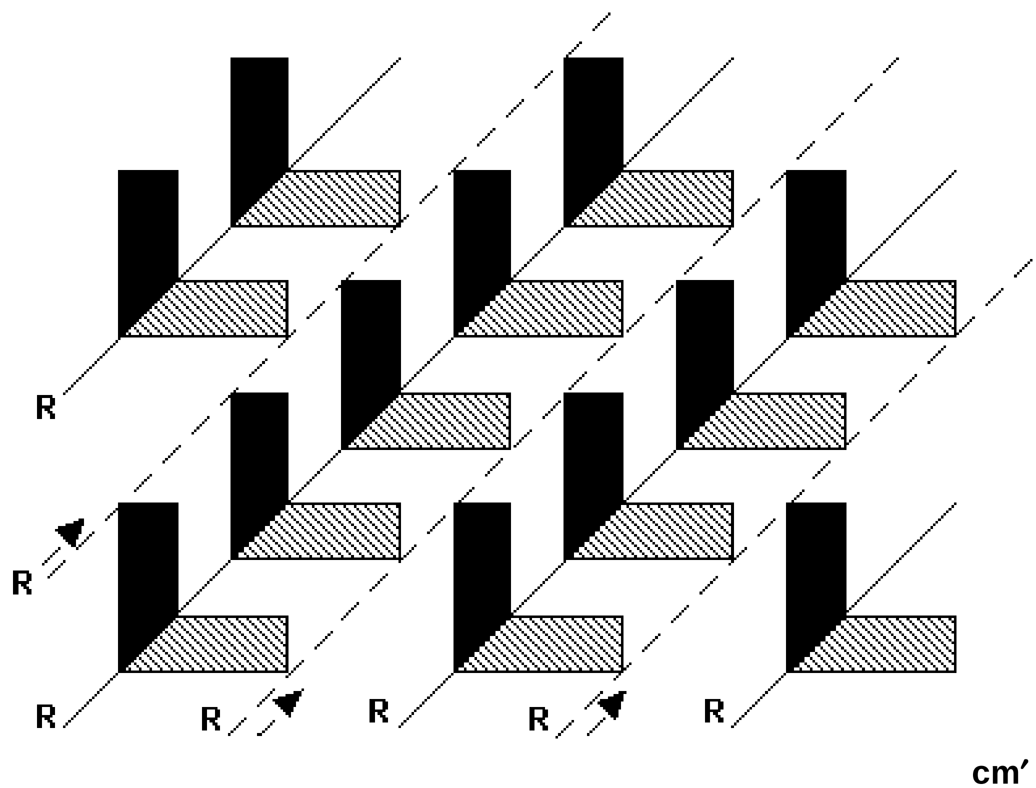


Fig. 6.34

So, while our first two colorings induced inconsistencies, yielding types **'lower'** than **cm**, the last two colorings provided **cm** types already known to us. Again, this begs the question: are there any other **cm**-like two-colored patterns besides the ones we have so far 'discovered'? The answer follows easily from the facts discussed right below in 6.4.3 and 6.4.4.

6.4.3 No 'essential' hidden glide reflections. The reason we got six rather than just three **pm**-like patterns in section 6.3 was the possibility of using a reflection axis for a (hidden) glide reflection of **opposite** effect on color. This is not quite possible in the case of a **cm**-like pattern ... simply because there cannot possibly be color-reversing translations **parallel** to reflection axes in such patterns!

To establish our claim above, let us first recall that a double application of a glide reflection yields a **color-preserving** translation parallel to it (5.4.1). Next, observe that the **smallest** possible translation vector parallel to the glide reflection axis has length equal to **2g**, where **g** > 0 is the length of the **shortest** glide reflection vector. To establish this observation we argue **by contradiction**, employing yet another 'proof without words':

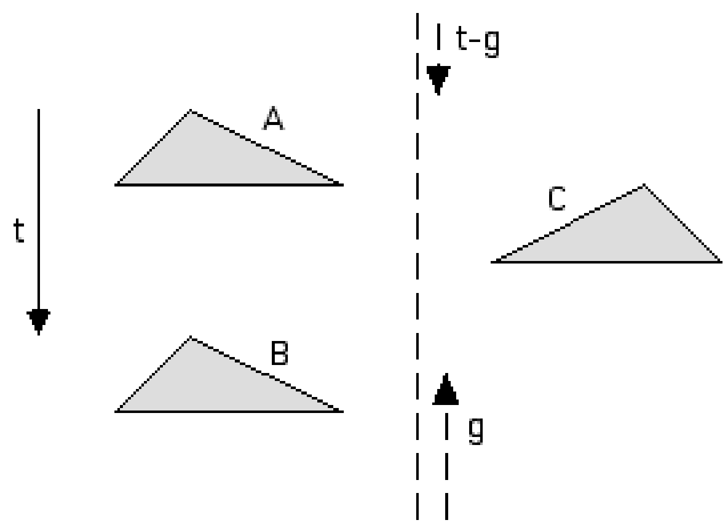


Fig. 6.35

[Assume that there is a **downward** translation vector of length **t** strictly **smaller** than **2g** (mapping A to B); apply then an **upward** glide reflection of length **g** (mapping B to C): the result is a downward glide reflection (mapping A to C) of length **t**–**g**, which is **shorter** than **g**, contradiction. (In case you like **inequalities** and **absolute values**, it's all a consequence of $0 < \mathbf{t} < \mathbf{2g} \Rightarrow |\mathbf{t}-\mathbf{g}| < \mathbf{g}$!)]

6.4.4 <u>No (glide) reflections of both kinds.</u> You may already have noticed another feature common to all two-colored **cm**-like patterns presented so far: in each example, all reflections have the **same effect on color**; and, likewise, all glide reflections have the same effect on color. This is **not** a coincidence! As figure 6.36 indicates, every two adjacent reflection axes -- therefore **all** reflection axes -- in a **cm**-like pattern **must** have the same effect on color:

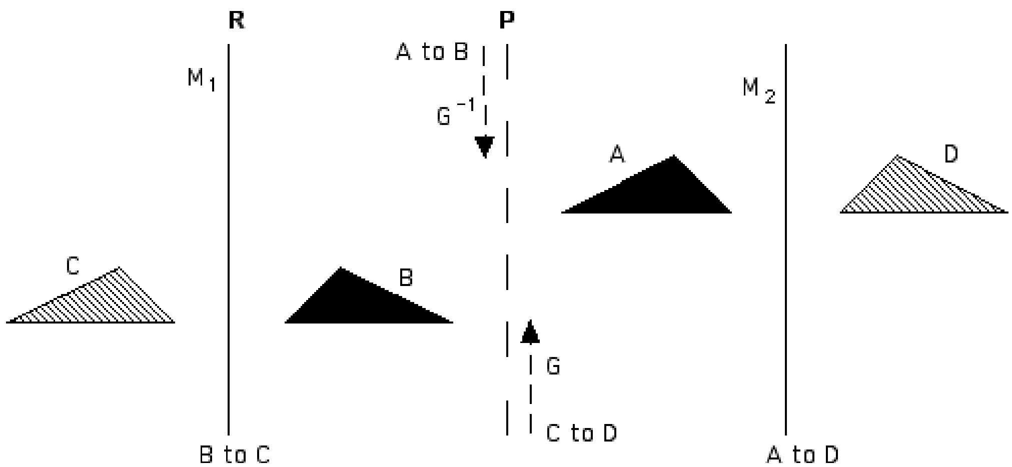


Fig. 6.36

[Assume that $\mathbf{M_1}$ **reverses colors** and that **G** preserves colors, the other three possibilities being treated in a very similar manner. Then $\mathbf{M_2}$ is the outcome of successive applications of $\mathbf{G^{-1}}$ (**downward** glide reflection), $\mathbf{M_1}$, and **G** (**upward** glide reflection). Employing the notation of 4.0.4, we may write $\mathbf{M_2 = G*M_1*G^{-1}}$, so that the 'multiplication rule' of 5.6.2 yields $\mathbf{P} \times \mathbf{R} \times \mathbf{P} = \mathbf{R}$ and therefore $\mathbf{M_2}$ **must reverse colors** (as figure 6.36 demonstrates).]

There is a similar argument (and picture) demonstrating the same fact for glide reflections: **all** axes have the **same** effect on color. At this point you may recall our 'innocent' comments in 4.4.6 to the effect that all reflection and glide reflection axes in a **cm** pattern **'look the same'**. It's a bit deeper than that: every two adjacent reflection axes ($\mathbf{M_1}$, $\mathbf{M_2}$) are **conjugates** (4.0.4) of each other by way of some **'in-between'** glide reflection (**G**); in simpler terms, there exists a glide reflection (**G**) **mapping** one ($\mathbf{M_1}$) to the other ($\mathbf{M_2}$), and that has the consequences discussed above (as well as in 4.0.4 & 4.0.5 and even 4.11.2). Similar facts hold true for the glide reflections of every **cm** pattern: every two adjacent glide reflection axes are mirrored to each other by some **'in-between'** reflection).

Putting everything discussed in the preceeding paragpaphs together we arrive at a **conjecture**: whenever $I_2 = I[I_1]$, where I, I_1, I_2 are isometries leaving a two-colored pattern invariant, I_1 and I_2 must have the same effect on color. As already indicated, our conjecture is not that difficult to prove -- via $I_2 = I{*}I_1{*}I^{-1}$ and the 'multiplication rules' of 5.6.2 -- so we will not delve into the details. But, please, remember this **important** fact that we will be using throughout this chapter: every two isometries of a **two-colored** pattern **mapped** to each other by a **third isometry** (and its inverse) **must** have the same effect on color (by way of being **conjugates** of each other).

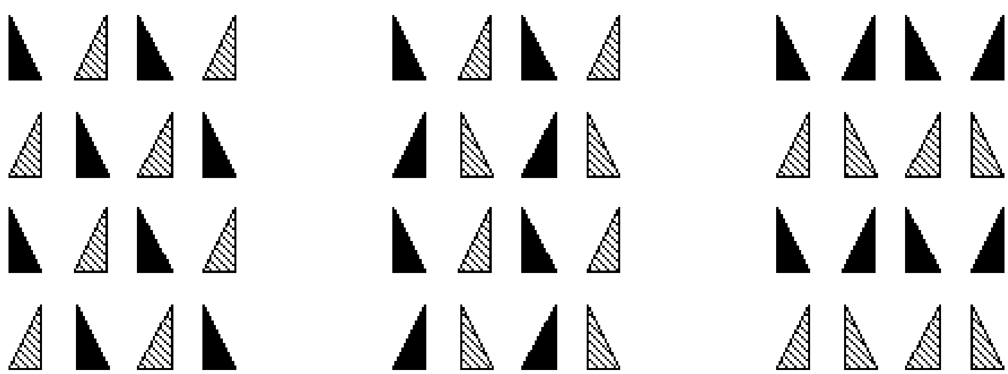
The observation made here is in fact important enough to be assigned a name of its own, **Conjugacy Principle**; a principle that not only will help us to classify and understand wallpaper patterns from here on, but has already been employed in less pronounced ways: for example, it does lie behind the fact that **every other** reflection axis in a **pm**-like two-colored pattern (or glide reflection axis in a **pg**-like two-colored pattern) has the same effect on color! (Couldn't it be called the **Mapping Principle**, instead? Well, we prefer "Conjugacy Principle" because it resonates with the crucial role played by the **abstract algebraic structure** of wallpaper patterns -- a structure not discussed here, but rather prominent in the literature...) Beyond the Conjugacy Principle (studied in detail in section 8.0), (glide) reflections are further analysed in section 8.1.

6.4.5 Only four **cm** types! It is now easy to show that there are no **cm**-like two-colored patterns other than the ones already described in this section. Indeed if we view a **cm** two-colored pattern as a **'merge'** of a **pm** pattern (reflections) and a **pg** pattern (glide reflections), we see that there are only **two** possibilities for each **'partner'**: only **pm**, **pm′** for **pm** (6.4.4 rules out **p'_b1m** and **c′m**, while 6.4.3 rules out **p′m** and **p'_bg**) and only **pg**, **pg′** for **pg** (**p'_b1g** is ruled out by 6.4.4). But two × two = four, and we can certainly write down the new (**cm**) types as **'products'** of the old ones (**pm**, **pg**):

cm = **pm** × **pg**, **cm′** = **pm′** × **pg′**, **p'_cg** = **pm′** × **pg**, **p'_cm** = **pm** × **pg′**

Of course this 'multiplication' was first introduced in section 5.7, where we viewed **pmm2**s as 'products' of **pm11**s and **p1m1**s.

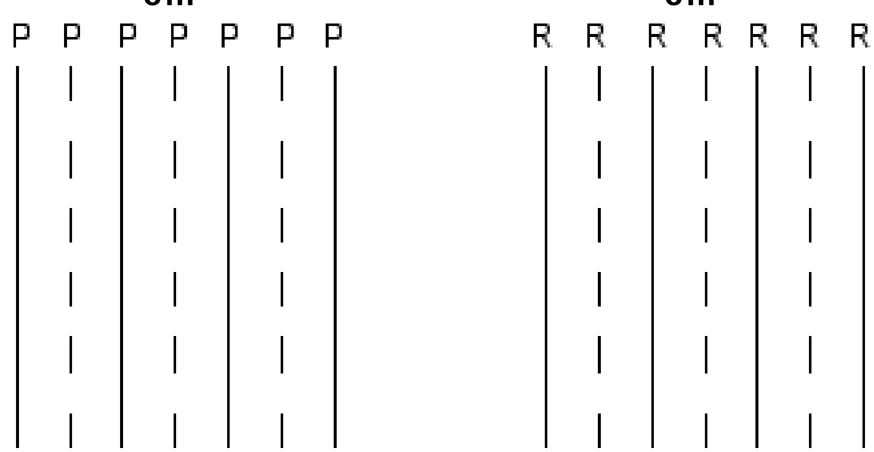
6.4.6 Further examples and symmetry plans.



cm′ **p'_cg** **p'_cm**

Fig. 6.37

cm **cm′**



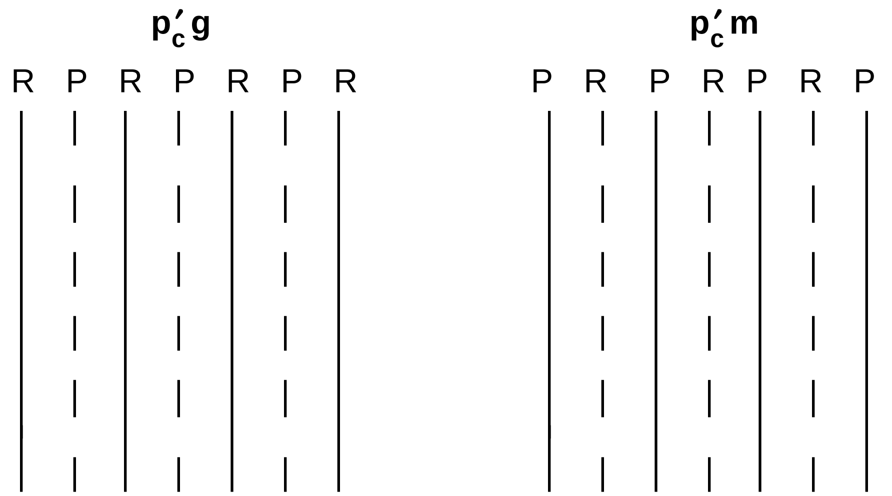
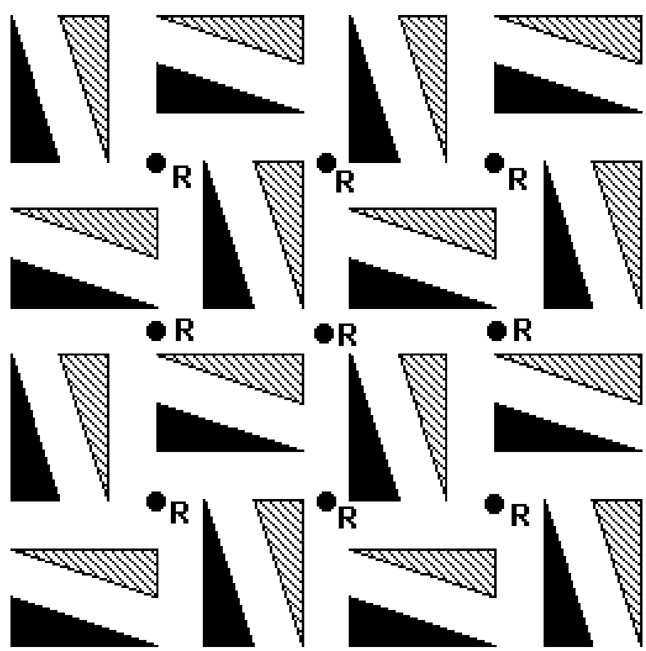


Fig. 6.38

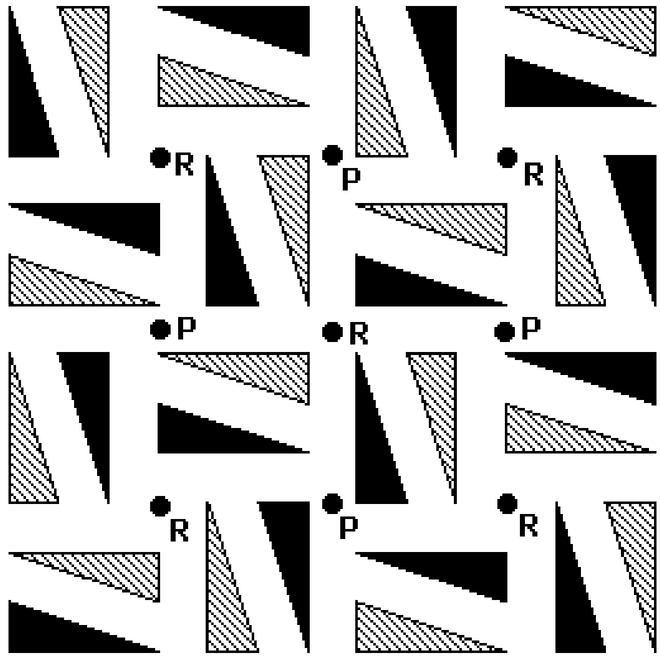
6.5 p2 types (p2, p2′, $p_b'2$)

6.5.1 A good place to start! ‘Experimenting’ a bit with the **p2** pattern in figure 6.5, we get a couple of ‘genuine’ two-colored ones:



p2′

Fig. 6.39

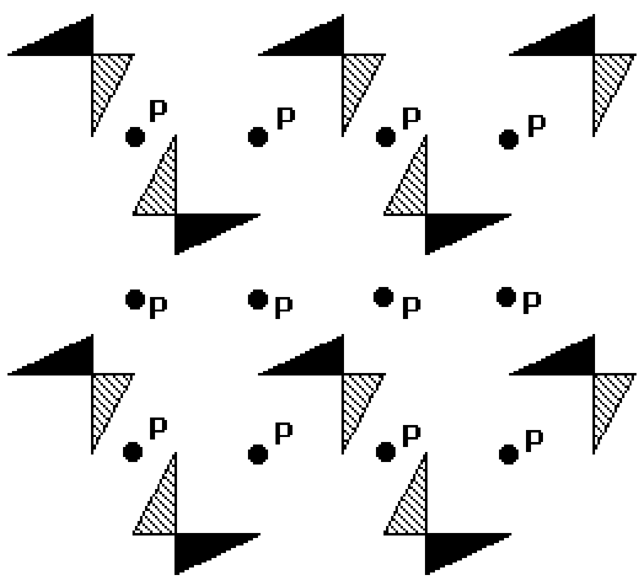


p'_b2

Fig. 6.40

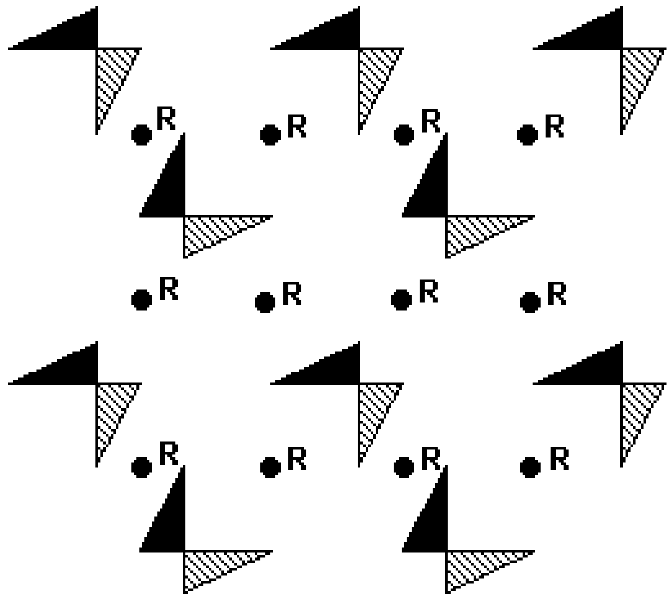
The first type (**p2′**) has color-reversing half turns **only**, the second (**p'_b2**) has **both** color-preserving and color-reversing ones.

6.5.2 From **pg** to **p2**. Let's revisit those 'root' patterns of 6.2:



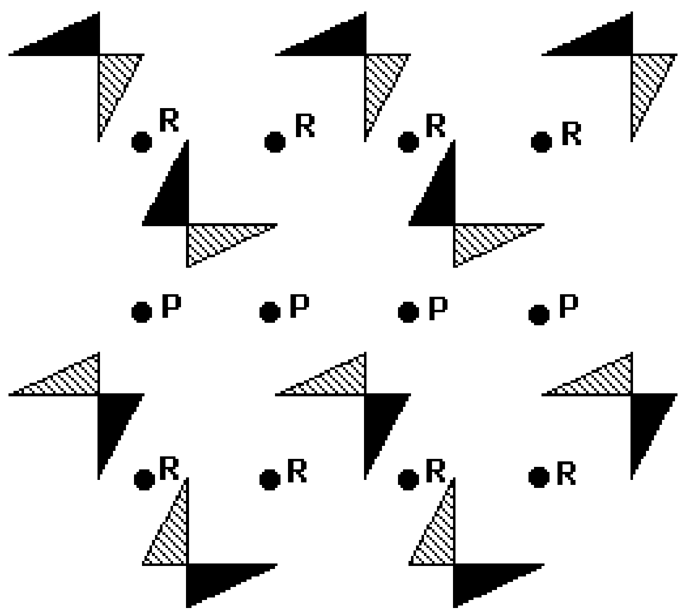
pg → p2

Fig. 6.41



$pg' \to p2'$

Fig. 6.42



$p'_b1g \to p'_b2$

Fig. 6.43

What happened? By applying a 'secret' **vertical** reflection to **every other row** of a **pg**-like pattern, we end up -- in this case anyway -- with a **p2**-like pattern! This incident suggests a strong analogy between the two types, which we discuss right below.

6.5.3 Half turns and translations. Are there any more **p2**-like two-colored patterns? The answer is “no”, and it strongly relies on figure 6.44, inspired in turn by figure 6.13:

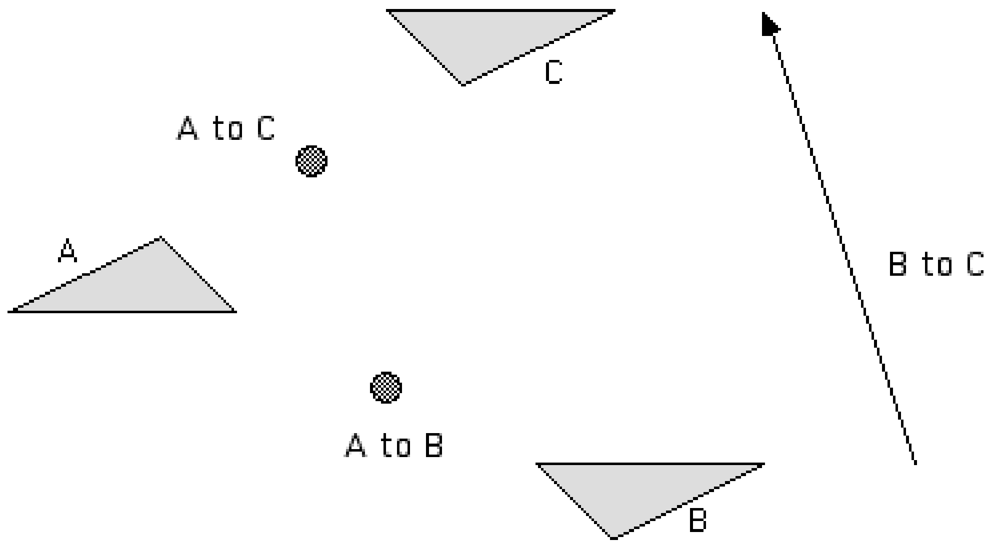


Fig. 6.44

What’s the story here? Well, go back to section 6.2 for a moment and recall how we established the correlation between glide reflections of **both** kinds (color-preserving and color-reversing) and color-reversing translations: it all follows from the fact that the combination of a translation and a glide reflection is another glide reflection (figure 6.13); and that correlation proves (6.2.5) that there exist precisely three two-colored patterns in the **pg** family. In **exactly** the same way, figure 6.44 shows that the combination of a 180^0 rotation (mapping A to B) and a translation (mapping B to C) is another 180^0 rotation (mapping A to C). It follows, for example, that we cannot have a pattern with color-preserving half turns **only** and color-reversing translations: **P** $\times$ **R** = **R**, etc. (For a complete analysis of why there can only be three **p2** types follow 6.2.5 **case by case**, with 6.4.4 (Conjugacy Principle) in mind, replacing glide reflections by half turns.)

A few additional comments are in order. First of all, the fact illustrated in figure 6.44 (rotation $\times$ translation = rotation) holds true for **arbitrary** rotations, not just for 180^0 rotations: a rigorous proof will be given in section 7.6. More to the point, the close analogy between **pg** and **p2** is **also** based on the fact that, just as the combination of two **parallel** glide reflections is a translation (figure 6.21), the combination of two **180^0** rotations is indeed a translation: use the two half turns of figure 6.44 ‘in the reverse’ to see how rotating B to A and then A to C is equivalent to translating B to C! (Yes, this **rotation × rotation = translation** equation requires the two angles to be **180^0** or at least **equal** to each other **and** of **opposite** orientation; see figure 6.99 or 7.5.2 for details.)

6.5.4 Symmetry plans. Nothing but rotation centers this time:

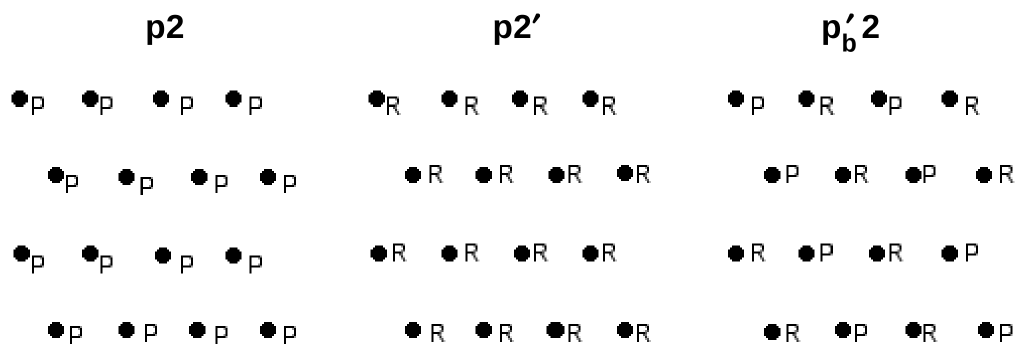


Fig. 6.45

6.5.5 Further examples. First our usual two-colored triangles:

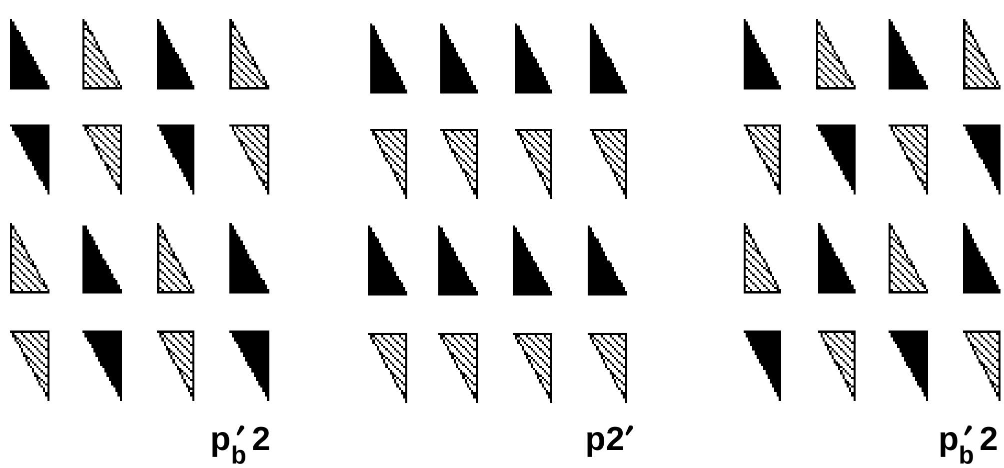
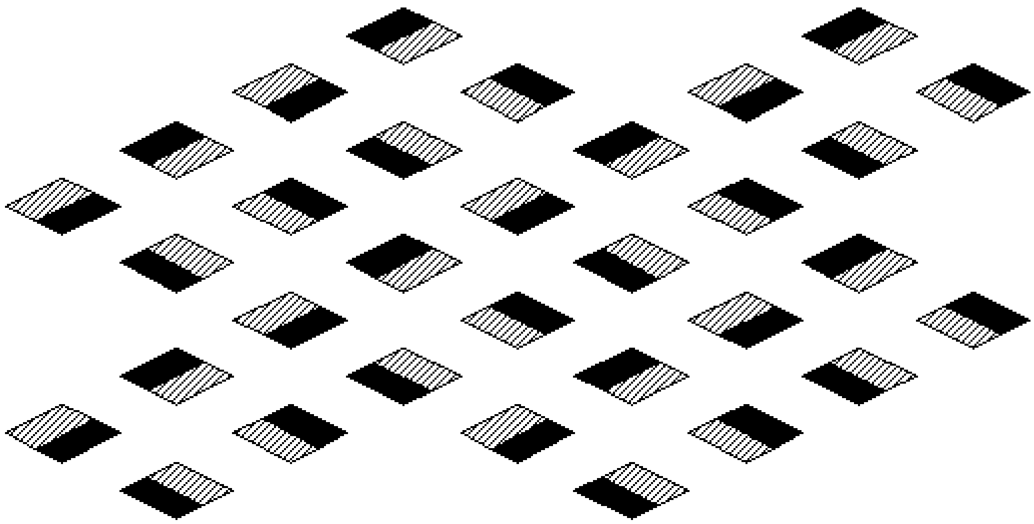


Fig. 6.46

You should probably compare figure 6.46 to figure 6.12!

And now a $\mathbf{p'_b2}$ example that many students misclassify thinking that it has glide reflection:



$\mathbf{p'_b2}$

Fig. 6.47

In an echo of the discussion in 4.5.1, we would like to point out that the half turn centers in figure 6.47 form **parallelograms** rather than rectangles (figures 6.41, 6.42, 6.43, 6.46) or squares (figures 6.5, 6.39, 6.40). This observation both **justifies** the 'general' arrangement (in parallelograms) of half turn centers in the **p2** symmetry plans (6.5.4) and **rules out** (4.8.2) the glide reflection in figure 6.47!

6.6 pgg types (pgg, pgg′, pg′g′)

6.6.1 From one to two directions. Let's now apply a 'secret' vertical **glide** reflection (6.5.2) of **both** kinds (color-preserving and color-reversing) to **every row** of a **pg** or **pg′**:

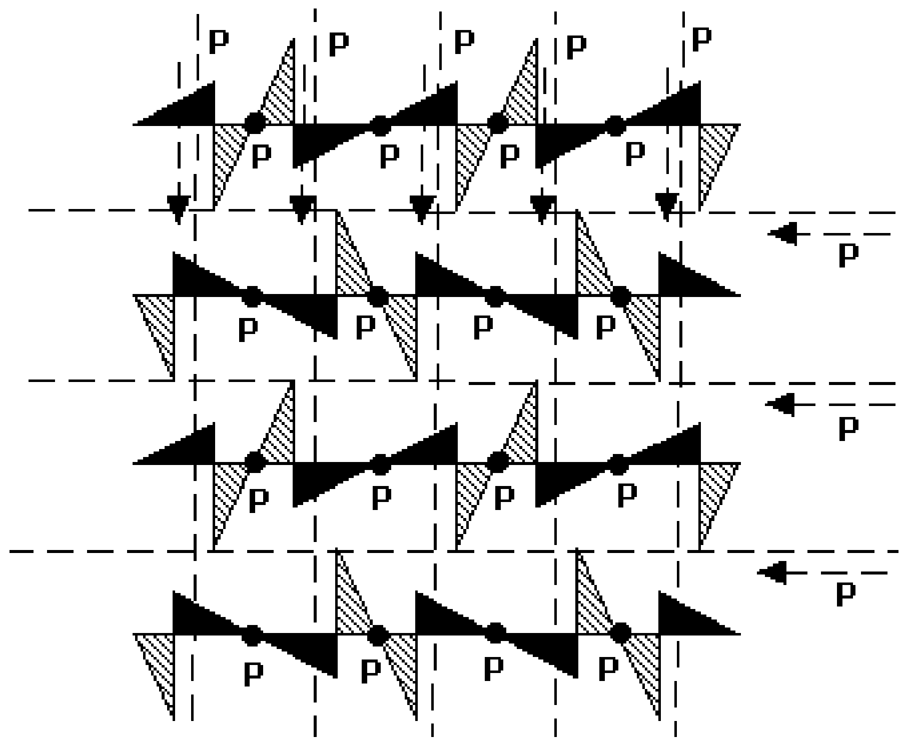


Fig. 6.48

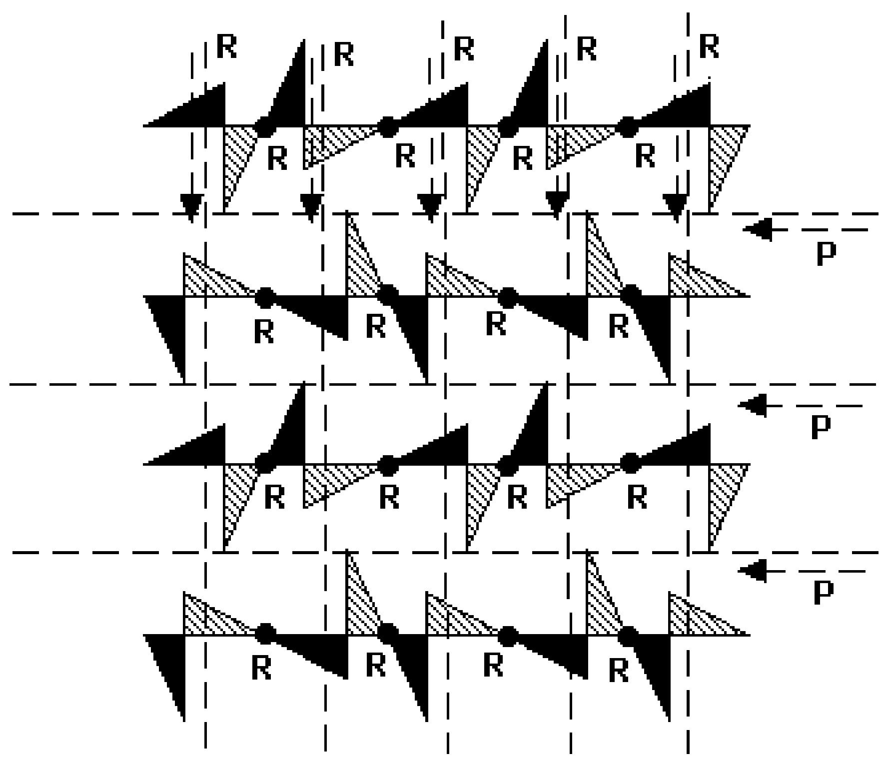
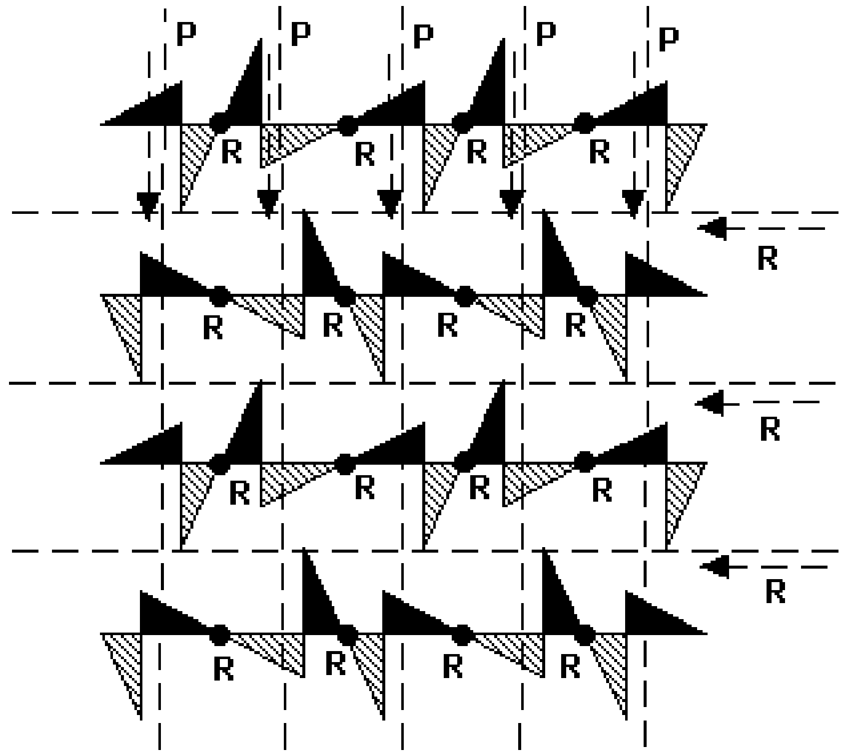
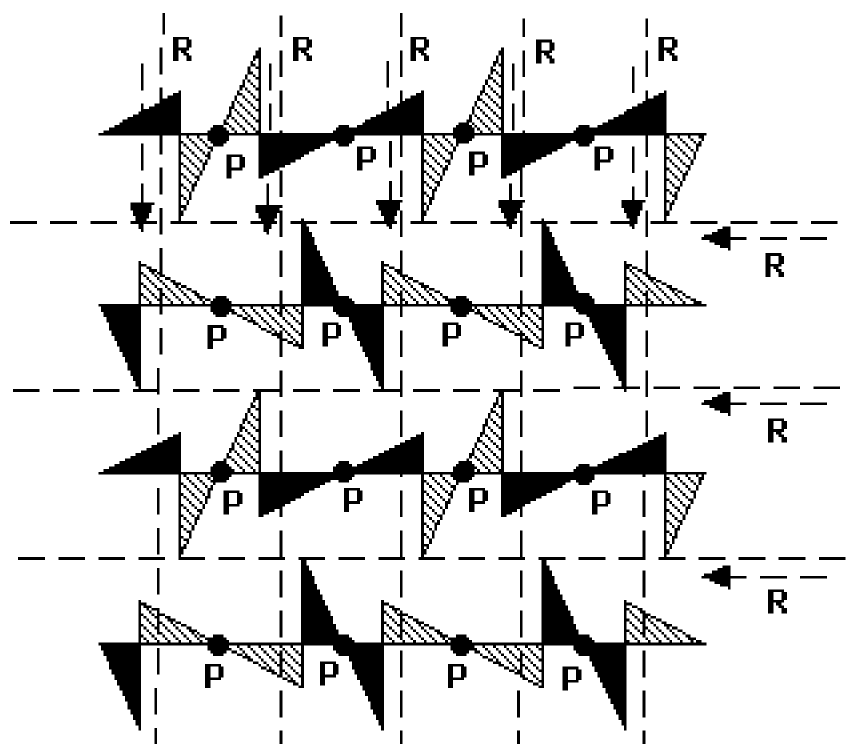


Fig. 6.49



pg′ → p g g ′

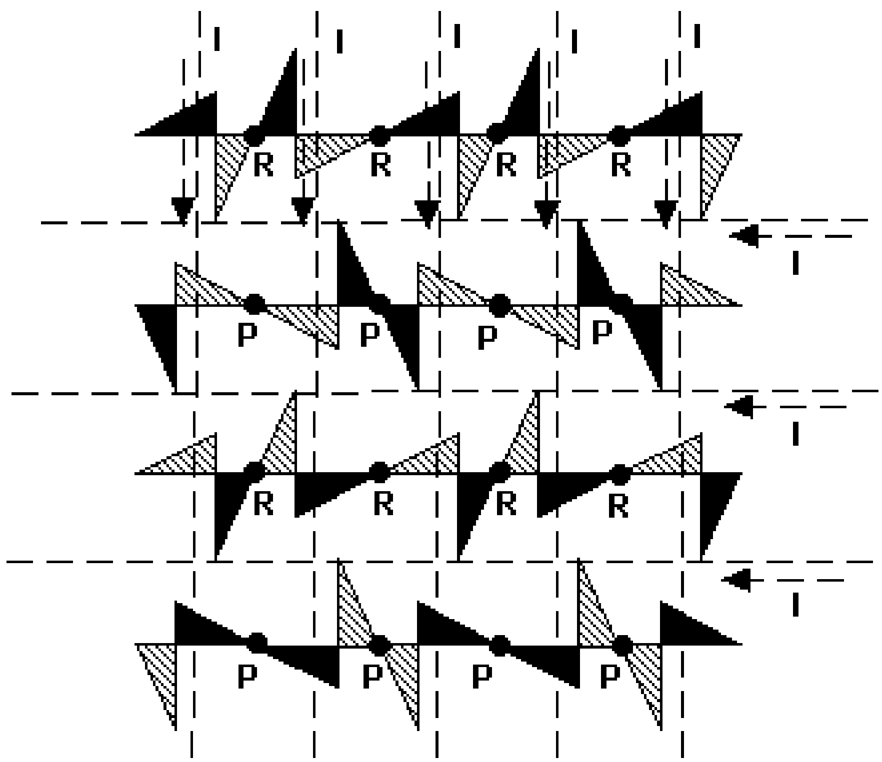
Fig. 6.50



pg′ → p g ′g ′

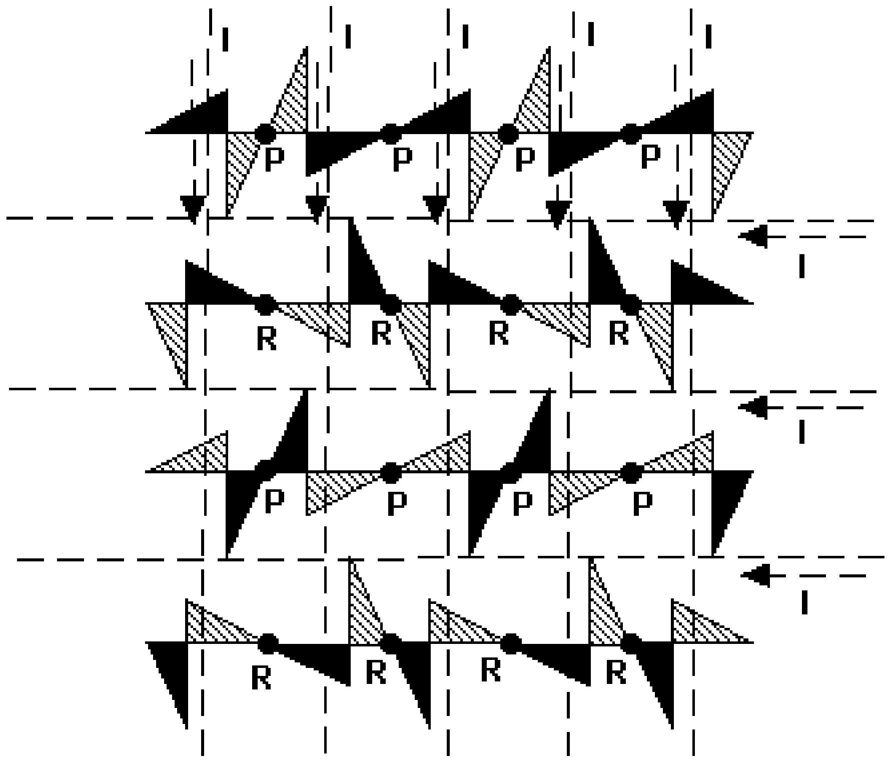
Fig. 6.51

So far so good: we obtained four two-colored patterns (from the 'root' **pg** and **pg′** patterns of figures 6.4 & 6.11 always) having glide reflection in **two** perpendicular directions, welcome additions to the **pgg** family; two of them (figures 6.49 & 6.50) **look** distinct but are the same mathematically (**pgg′**), with color-preserving glide reflection in one direction and color-reversing glide reflection in the other direction. But there will be 'casualties' caused by **color inconsistencies** as we apply this process to the $\mathbf{p'_b1g}$: right below you find two $\mathbf{p'_b2}$ patterns with color-inconsistent glide reflection (mappable in fact to each other by either color-preserving horizontal glide reflection or color-reversing vertical glide reflection):



$\mathbf{p'_b1g \rightarrow p'_b2}$

Fig. 6.52



$p'_b1g \rightarrow p'_b2$

Fig. 6.53

6.6.2 Only three types indeed! Looking at the **pgg**-like patterns obtained so far, we notice that none of them has glide reflection of both kinds in the same direction: such a situation is indeed **impossible** because of what figures 6.54 & 6.55 tell us:

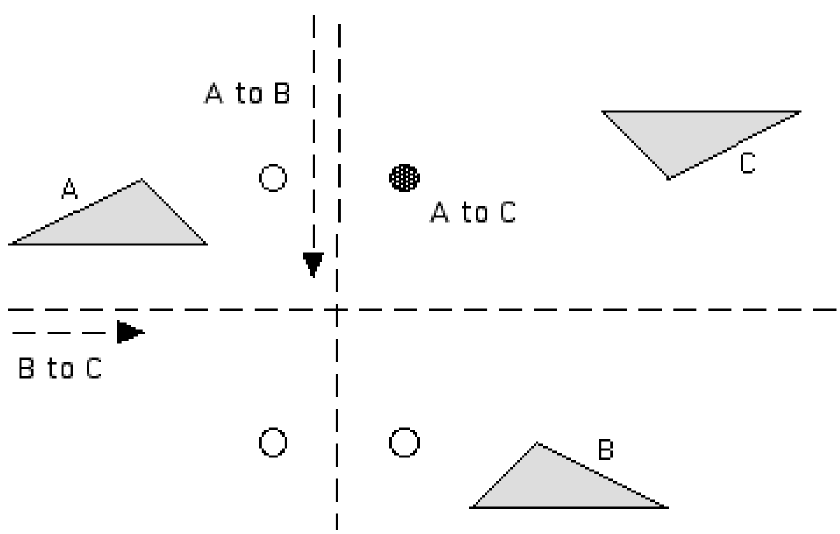


Fig. 6.54

This is a demonstration of a significant fact: the combination of two **perpendicular** glide reflections (mapping A to B and B to C) produces a **half turn** (mapping A to C)! We will go through a rigorous proof of a generalization of this in section 7.10, but you should try to verify an important aspect of this new theorem: depending on **which way** the gliding vector of each of the two glide reflections goes (north-south versus south-north and west-east versus east-west), as well as the **order** in which the two glide reflections are combined, we get **four** plausible centers (and half turns) out of **eight** actual possibilities; in our case, a **'symbolic' rule** yields (west-east) $\times$ (north-south) = northeast. (Notice also that the **distances** of the rotation center from the glide reflection axes are equal to **half** the length of the corresponding gliding vectors; as an important **special case**, the composition of two perpendicular **reflections** is a half turn centered at their intersection point. These facts throw new light into sections 2.6 (**pma2** border patterns) and 2.7 (**pmm2** border patterns), as well as several sections in chapter 4!)

Now figure 6.55, together with the preceding remarks, shows why color-preserving and color-reversing glide reflection axes of a **pgg**-like pattern **cannot** coexist in the same direction:

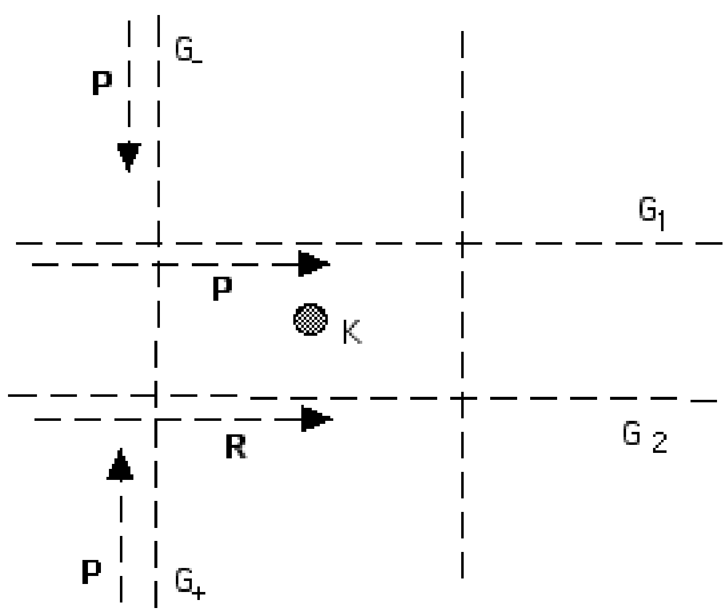


Fig. 6.55

Is the half turn (at) K color-preserving or color-reversing? In view of $\mathbf{K = G_1{*}G_+}$ (G applied upwards (**P**) followed by G_1 (**P**)) and $\mathbf{K = G_2{*}G_-}$ (G applied downwards (**P**) followed by G_2 (**R**)) we conclude that the half turn at K must be **both** color-preserving and color-reversing; that is, the situation featured in figure 6.55 ('mixed' horizontal axes) is **impossible**.

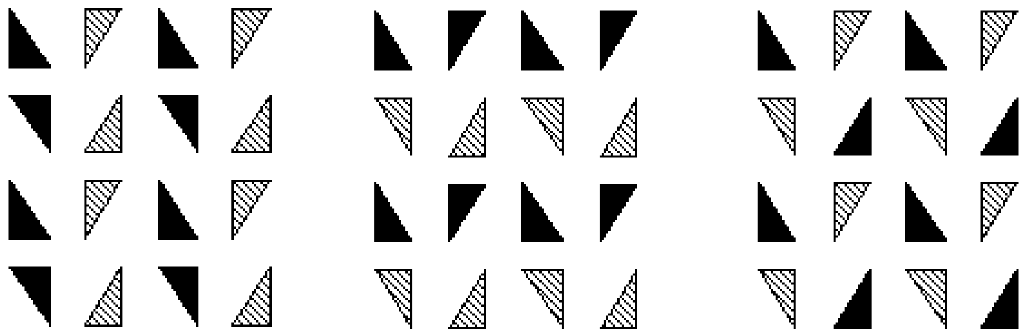
We conclude that each of the two **pg**-like 'factors' of a **pgg**-like pattern could be either a **pg** or a **pg′**, but **not** a $\mathbf{p'_b1g}$. This should allow for four possibilities, but since the outcome of this 'multiplication' is **not** affected by the **order** of 'factors', we are down to **three** types:

pgg = **pg** × **pg**, **pgg′** = **pg** × **pg′** = **pg′** × **pg**, **pg′g′** = **pg′** × **pg′**

6.6.3 Another way of looking at it. The discussion in 6.6.2 was very useful in terms of analysing the **structure** of the **pgg** pattern, but it is certainly not the easiest way to see that any two of its glide reflections parallel to each other must have the same effect on color. Indeed that follows **at once** from our **Conjugacy Principle** (6.4.4): every two **adjacent** parallel axes are mapped to each other by **any** half turn center lying **half way** between them! It might be a good idea for you to see how the Conjugacy Principle works in this **special case,** though: you should be able to provide your own proof, arguing in the spirit of figure 6.36.

In another direction now, let's revisit the **pgg** example of 4.8.3 and figure 4.43. We state there, with the Conjugacy Principle in mind (4.11.2), that it **appears** that there are **two** kinds of glide reflection axes in **both** directions: our reservations are now further justified by the impossibility of coloring that pattern in such a way that any two parallel glide reflections would have opposite effect on color!

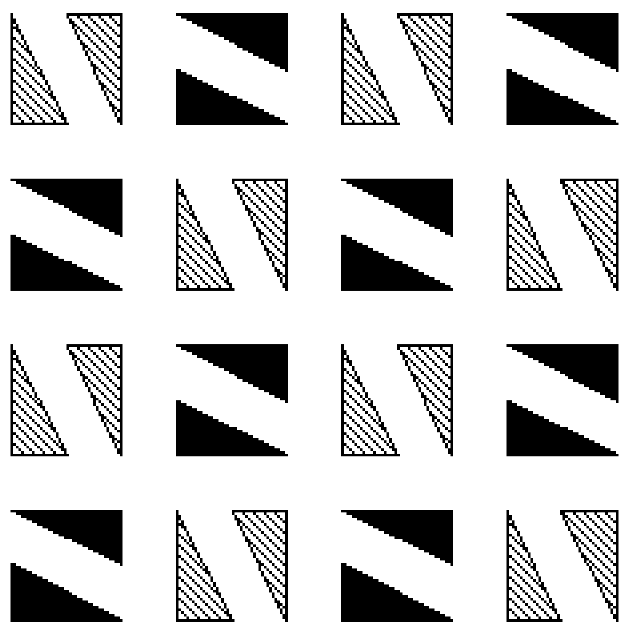
6.6.4 Further examples. First, three **pgg**-like 'triangles' that you should compare to the **p2**-like patterns of figure 6.46:



pg′g′ **pgg′** **pgg′**

Fig. 6.56

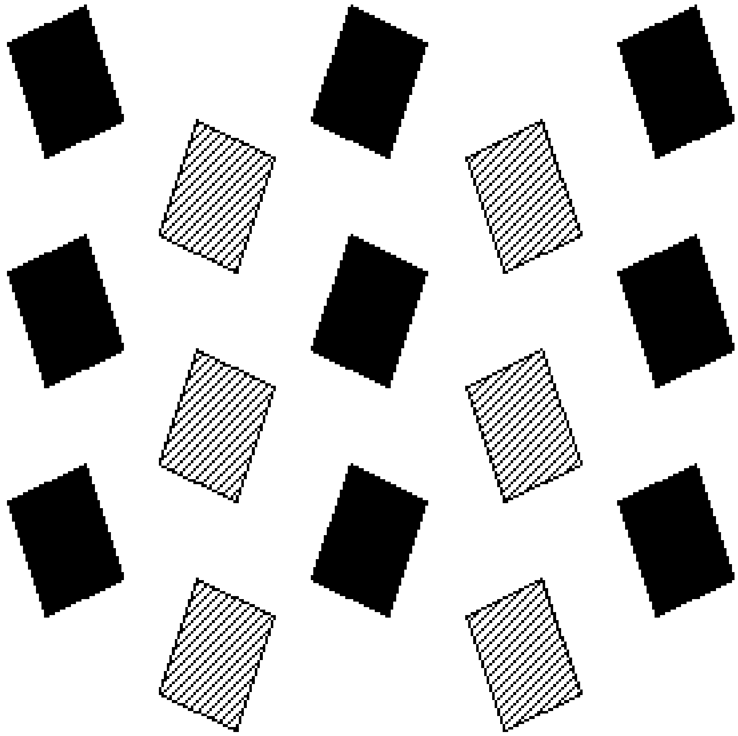
The ‘proximity’ between the two families (**pgg** and **p2**) is further outlined through the following curious example of a **pg′g′** that is a **close relative** of the **p2** example in figure 6.5:



pg′g′

Fig. 6.57

This is an example that many would classify as a **p2**: after all, the rotations of **both** the **pg′g′** and the **p2** are color-preserving **only**. Well, the advice offered in 4.8.2 remains valid: after you locate **all** (hopefully!) the rotation centers, check for ‘**in-between’ diagonal** glide reflection! Instead of applying this ‘squaring’ process to the **p2′** in figure 6.39 for a **pgg′**, we offer a fancy **pmg**-generated **pgg′**:



pgg′

Fig. 6.58

This pattern (**Laurie Beitchman**, Fall 1999) consists of **two pmg**s of distinct colors; its vertical and horizontal glide reflections reverse and preserve colors, respectively. Again, you may opt to find the glide reflection axes **after** you get **all** the half turn centers!

6.6.5 <u>Symmetry plans.</u> What follows captures our structural observations on the interplay between axes and centers (6.6.2):

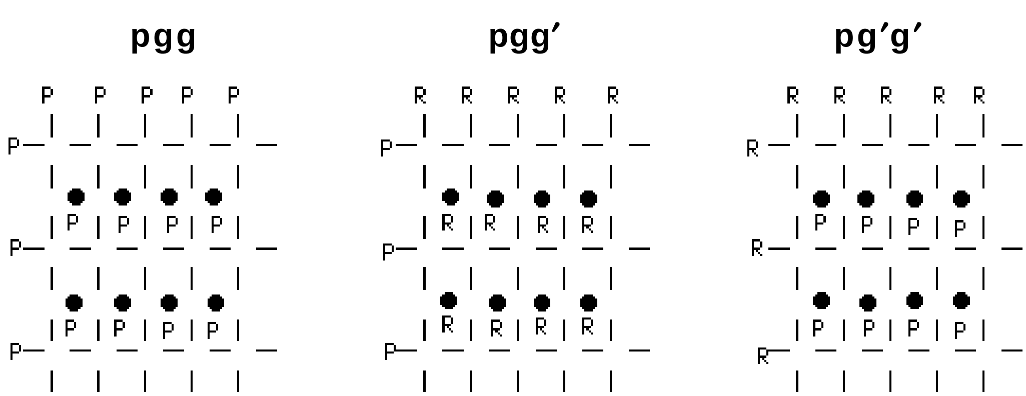


Fig. 6.59

You should compare these **pgg** symmetry plans to the **p2** symmetry plans in figure 6.45: parallelograms have now been **'ruled' by glide reflection** into rectangles, and the real reason is revealed in 8.2.2!

6.7 pmg types (pmg, p'_bmg, pmg′, pm′g, p'_bgg, pm′g′)

6.7.1 How many types at most? By now we know the game well enough to try to **predict** how many two-colored types can **possibly** exist within a family sharing the same symmetrical structure. In the case of the **pmg** (reflection in one direction, glide reflection in a direction perpendicular to that of the reflection), we are dealing with the **'product'** of a 'vertical' **pm** with a 'horizontal' **pg**. So it seems **at first** that we could have up to six × three = eighteen types ... but we also know that several cases will most likely have to be ruled out, as it has happened in the case of the **pgg**.

First, let's not forget the **pmg**'s 180^0 rotation and its centers, located -- special case of figure 6.55! -- **on** glide reflection axes and **half way** between every two adjacent reflection axes: arguing as in 6.6.3 (Conjugacy Principle), we see that all reflection axes of a **pmg** must have the **same effect** on color. (Alternatively, and closer in spirit to 6.4.4, we may appeal to the Conjugacy Principle by way of reflection axes mapped to each other by glide reflections rather than half turns!) This rules out **p'_b1m** and **c′m** for the **pm** 'factor', so we are down to **at most** four × three = twelve **pmg** types.

Next, observe that **'vertical'** hidden glide reflections and associated translations along reflection axes are fully **determined** by the **'horizontal'** glide reflections. Indeed the combination of any two **adjacent** glide reflections -- of **opposite** vectors, as in figure 6.21 -- produces the **shortest possible** (by figure 6.60 below) translation parallel to the reflection. It follows that there exists vertical color-reversing translation if and only if there exists horizontal glide reflection of both kinds.

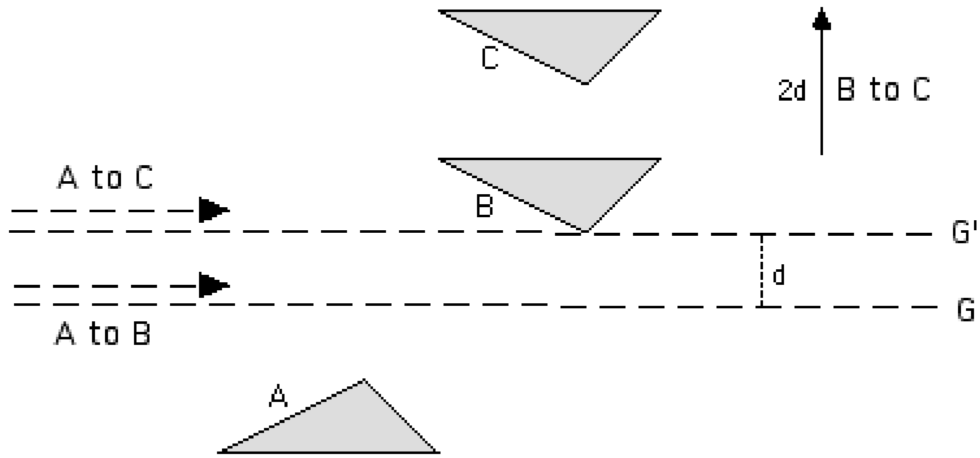


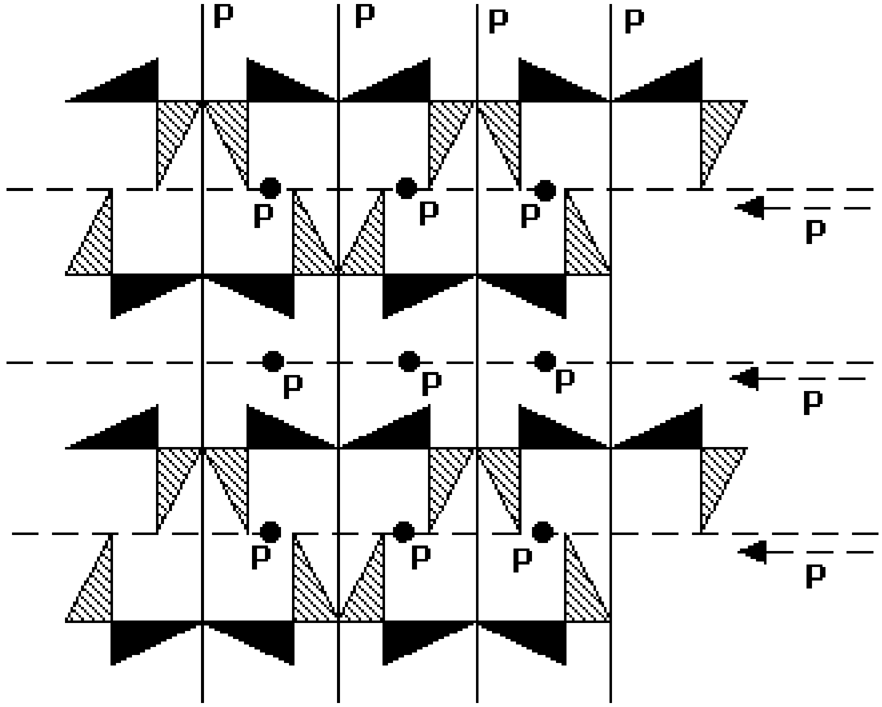
Fig. 6.60

[The combination of a glide reflection G (mapping A to B) and a translation of length 2d **perpendicular** to it (mapping B to C) produces a glide reflection G′ (mapping A to C) **parallel** to G, of **same** gliding vector and at a distance **d** from G; so if d (the distance between any two adjacent horizontal glide reflections) is assumed to be minimal then **2d** (the length of the resulting vertical translation) must be minimal, too.]

Putting everything together, we see that **all that matters** when it comes to the first factor (**pm**) of a **pmg** is whether its reflections preserve (**PP**) or reverse (**RR**) colors: color-reversing translations along reflection axes (and associated hidden glide reflections) **appear** to play no crucial role anymore -- except that, as pointed out above, they do make their presence obvious indirectly, through the **pmg**'s second factor (**pg**)! Anyhow, there can be **at most** two × three = six **pmg**-like two-colored wallpaper patterns: two possibilities for the first factor (**PP**, **RR**) and three possibilities for the second factor (**PP**, **PR**, **RR**); see also 6.7.4.

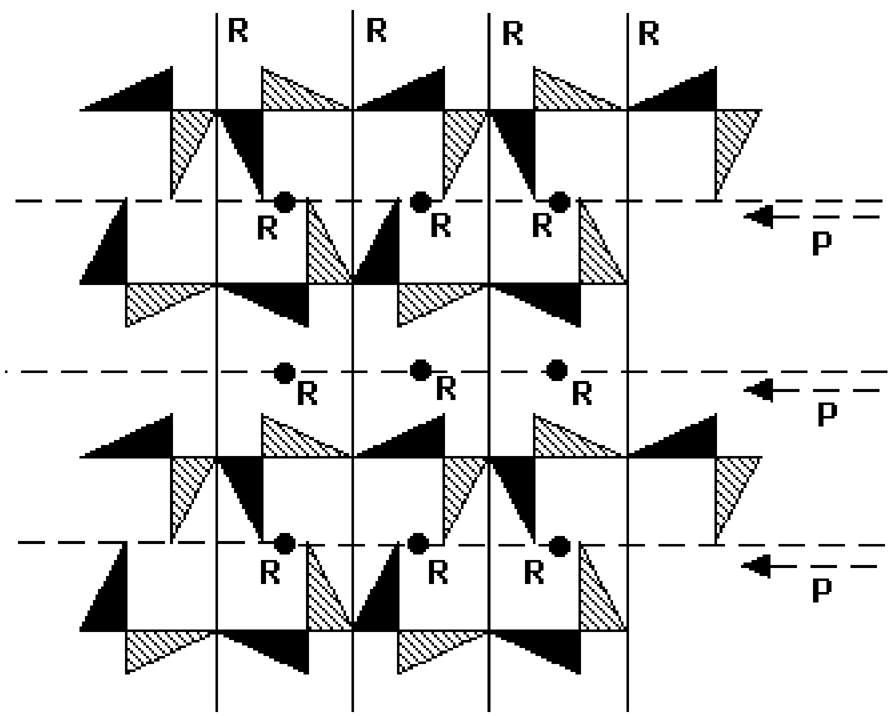
There is no obvious reason to exclude any one of the resulting six possible types; and as we are going to see right below, each of them does show up, predictably perhaps, in concrete examples!

6.7.2 Are they there after all? Applying a ‘secret’ **reflection** to **every row** of our **pg** ‘root’ patterns, we **do** get six **pmg** types:



pg → pmg

Fig. 6.61



pg → pm′g

Fig. 6.62

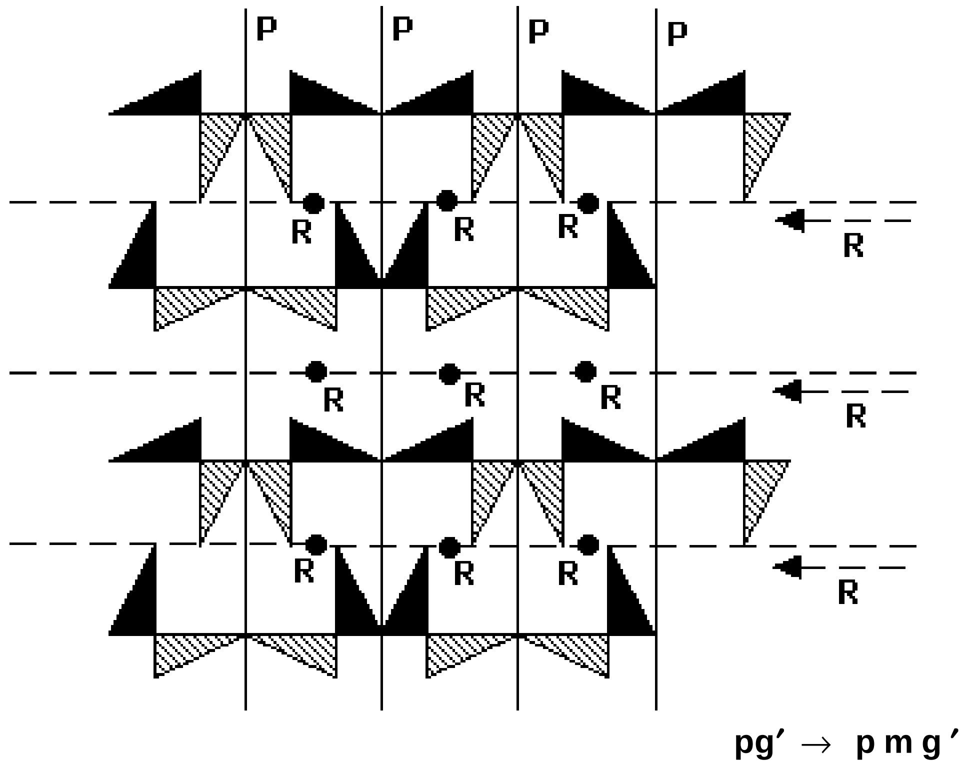


Fig. 6.63

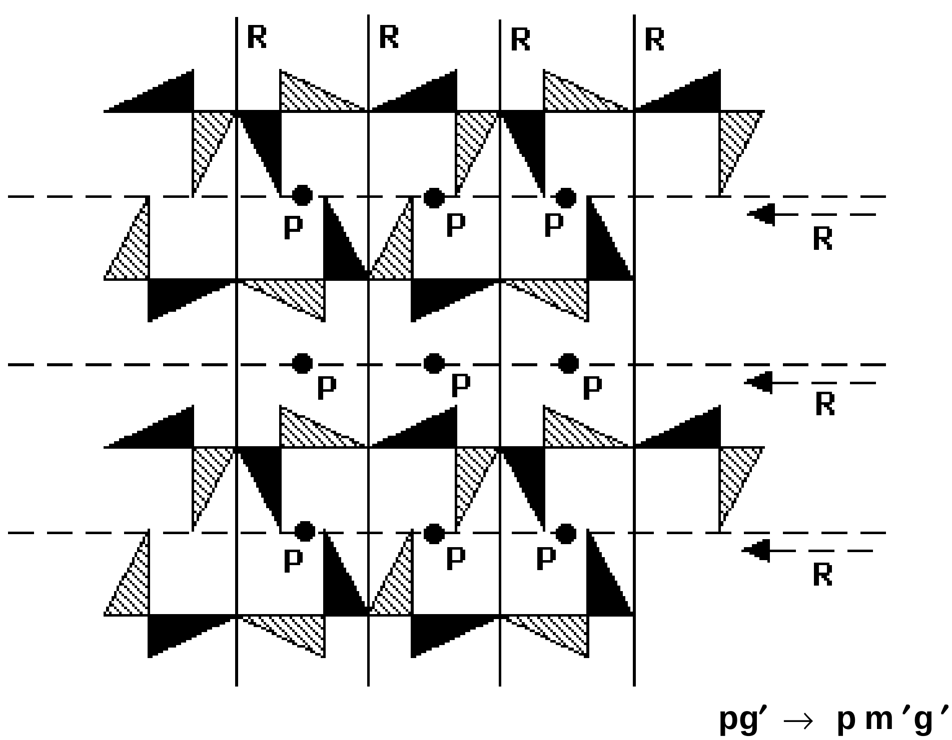
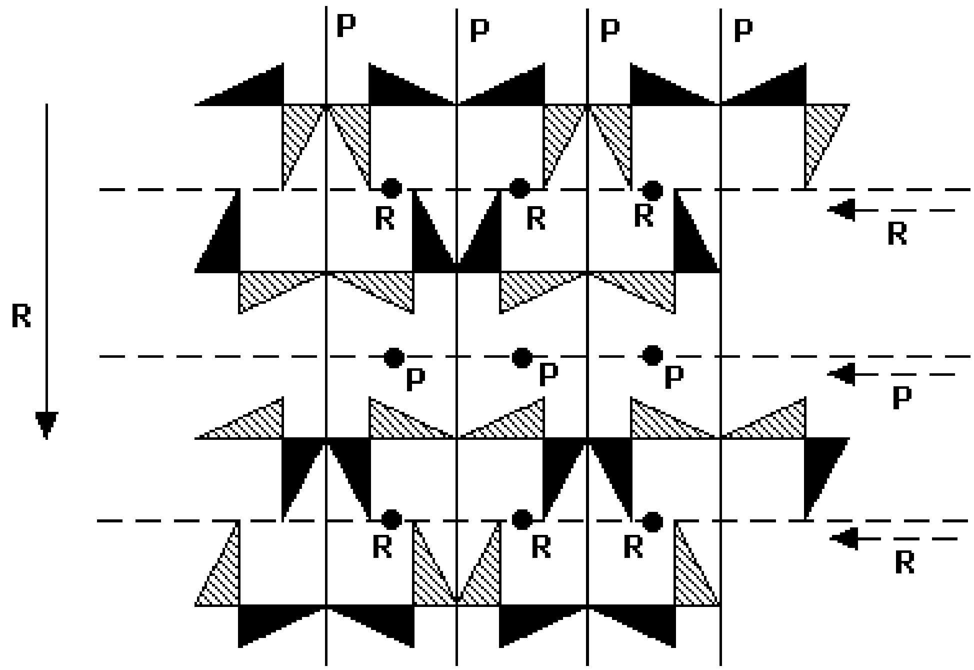


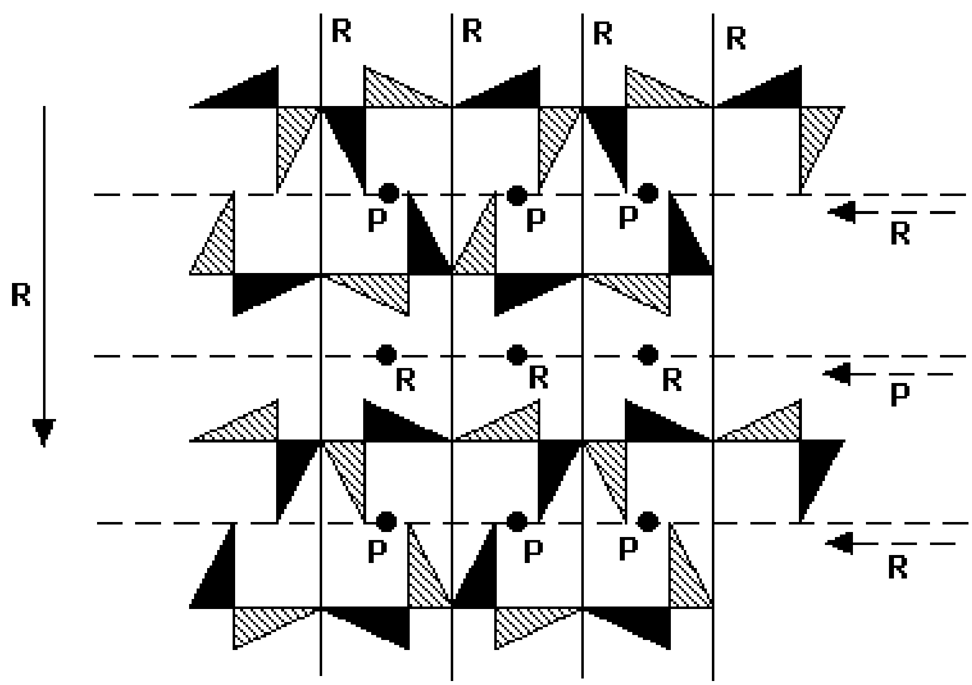
Fig. 6.64

So far there have been no surprises, save perhaps for the total absence of color-reversing translation -- provided in fact by the last two types, offspring of $\mathbf{p_b'1g}$ and rather more interesting:



$\mathbf{p_b'1g \rightarrow p_b'mg}$

Fig. 6.65



$\mathbf{p_b'1g \rightarrow p_b'gg}$

Fig. 6.66

This completes the **pmg** picture. The last two types, coming from the only **pg**-root with color-reversing translation (figure 6.9), have themselves color-reversing translation along the direction of the reflection ($\mathbf{p'_b}$).

6.7.3 Examples. First, five types for six triangular colorings:

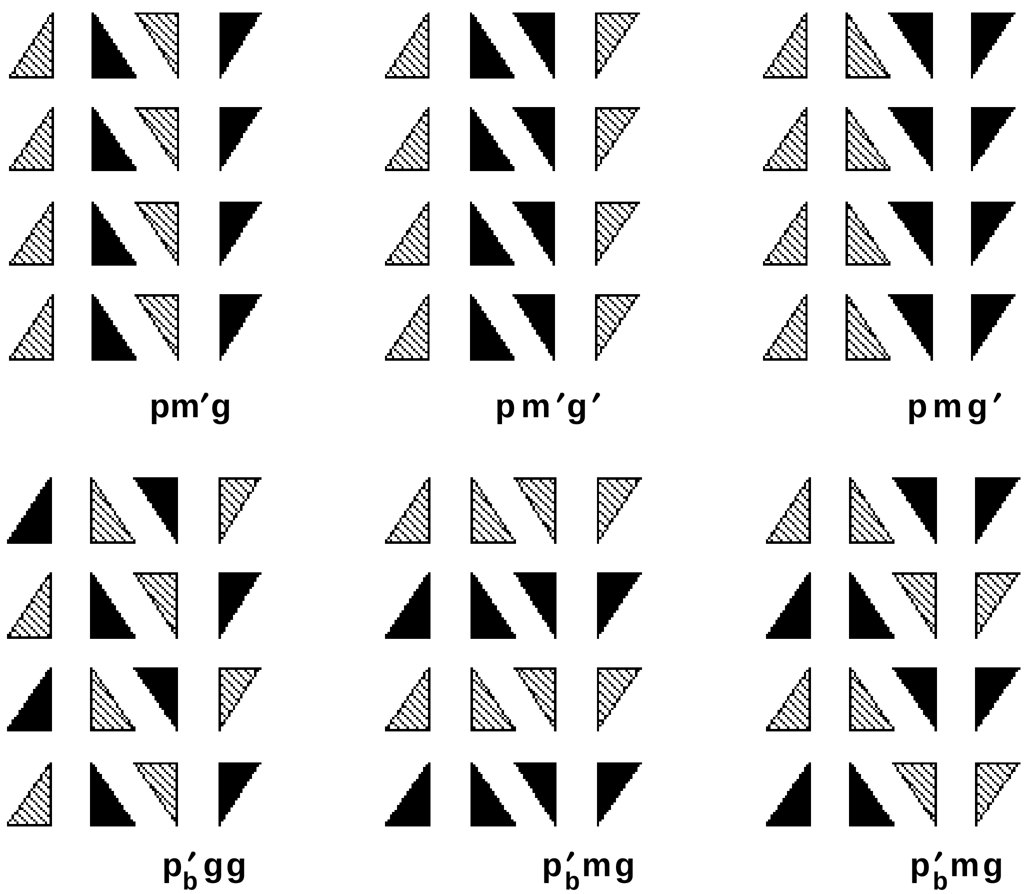


Fig. 6.67

And now an old **p4g** acquaintance (figures 6.2, 6.3, 6.10), revisited and (inconsistently) recolored as a $\mathbf{p'_b mg}$, calling for additional such **pmg**-like creations:

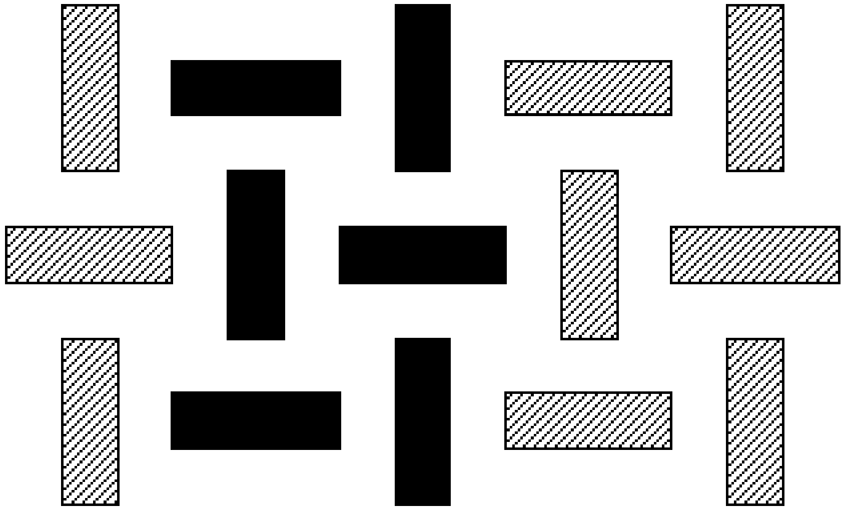


Fig. 6.68

6.7.4 Symmetry plans. Make sure you understand the complex interaction between reflection, glide reflection, and rotation:

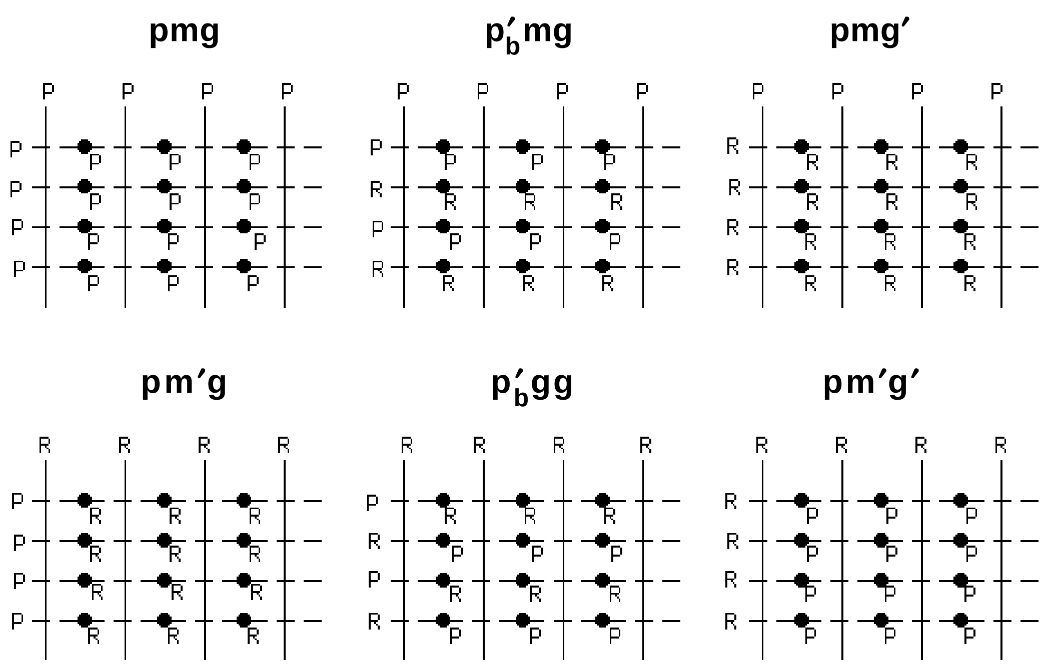


Fig. 6.69

Even though this is not exactly how we classified the **pmg**-like patterns, it is not a bad idea to express the six types as ‘products’ of simpler types; the main difficulty lies with the **pm** 'factor':

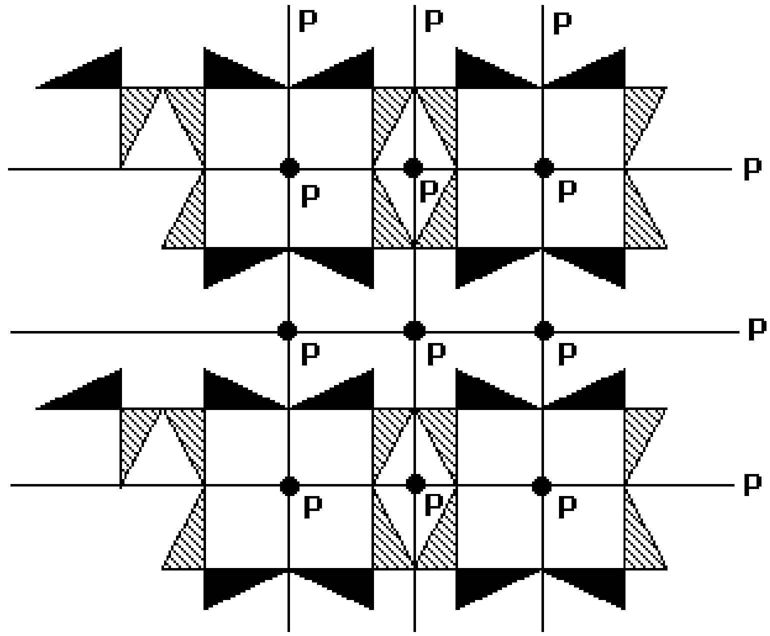
pmg = **pm** × **pg**, $\mathbf{p'_b mg}$ = **p′m** × $\mathbf{p'_b 1g}$, **pmg′** = **pm** × **pg′**,
pm′g = **pm′** × **pg**, $\mathbf{p'_b gg}$ = $\mathbf{p'_b g}$ × $\mathbf{p'_b 1g}$, **pm′g′** = **pm′** × **pg′**

Of course the mysteries of the crystallographic notation and everything else make a bit more sense now, don't you think? (Notice again the role played by the 'vertical' color-reversing translation in determining the first factor in our products: **p′m** or $\mathbf{p'_b g}$ in its presence (associated with a second factor of $\mathbf{p'_b 1g}$), **pm** or **pm′** in its absence (associated with a second factor of **pg** or **pg′**).)

6.8 pmm types (pmm, $\mathbf{p'_b mm}$, pmm′, c′mm, $\mathbf{p'_b gm}$, pm′m′)

6.8.1 An easy guess this time. The **pmm** type may of course be viewed as the 'product' of two **pm**s. It can be shown as in 6.7.1 that it is easier to work with effect on color rather than types, and that we do not need to worry about the **pm**'s hidden glide reflections or color-reversing translations. With three possibilities (section 6.3) for **each** direction of reflection (**PP**, **PR**, **RR**), and the **order** of 'factors' in our 'multiplication' reduced to the **trivial** "vertical reflection versus horizontal reflection" issue, there seem to be **at most** six possible **pmm**-like types: **PP** × **PP**, **PP** × **PR**, **PP** × **RR**, **PR** × **PR**, **PR** × **RR**, **RR** × **RR**. Let's see how many of those we can actually get -- if not all!

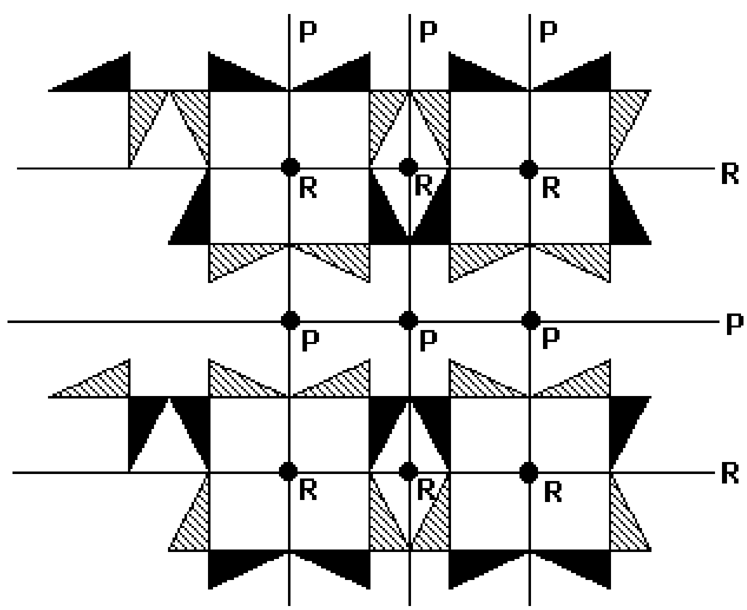
6.8.2 From the **pmg**s to the **pmm**s. Returning to old tricks, we will now try to get as many **pmm** types as possible by **'perfect shiftings'** (4.4.2) of the **pmg** types we created in section 6.7; that is, we shift every other row by **half** the minimal horizontal translation.



pmg → pmm

Fig. 6.70

Somewhat confused? It is not a bad idea to go back to figure 6.61 for a moment and compare the two patterns! Let’s move on:

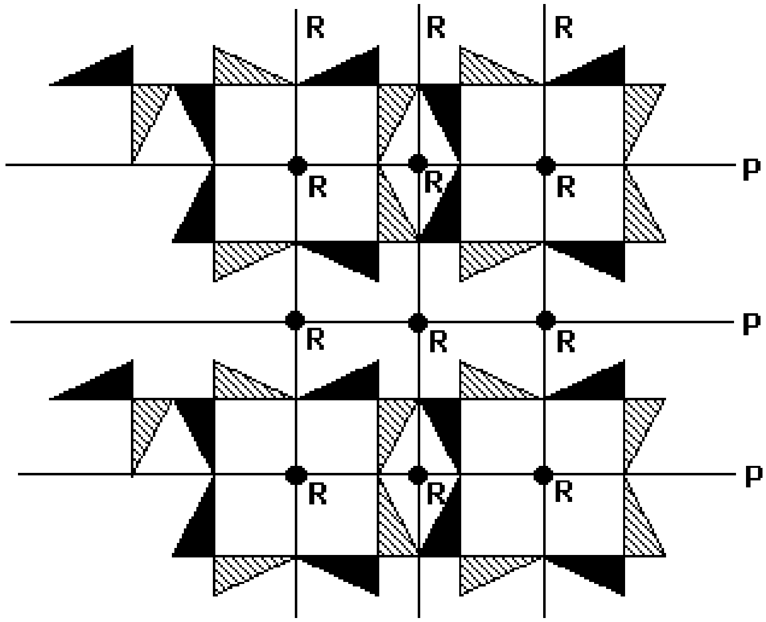


$\mathbf{p'_b mg \rightarrow p'_b mm}$

Fig. 6.71

Notice how the mixed horizontal **glide** reflections of the $\mathbf{p'_b mg}$ (figure 6.65) have turned into the mixed horizontal reflections of the

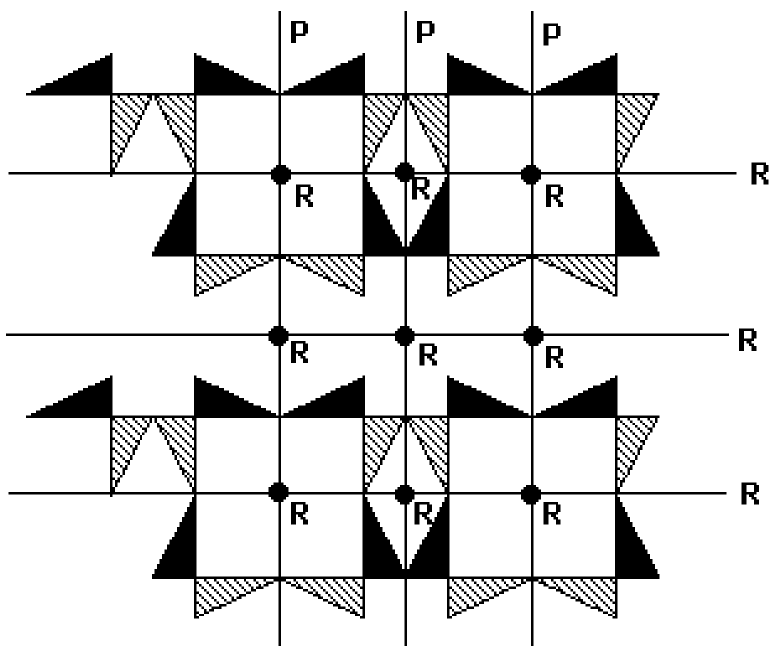
$\mathbf{p'_b mm}$, while all vertical reflections remained color-preserving: we started with **PP** × **PR** and ended up, predictably, with **PP** × **PR**.



pm′g → pmm′

Fig. 6.72

No axis 'lost' its effect on color as figure 6.62 got 'perfectly shifted' into figure 6.72 (**RR** × **PP**). But look at our next step:

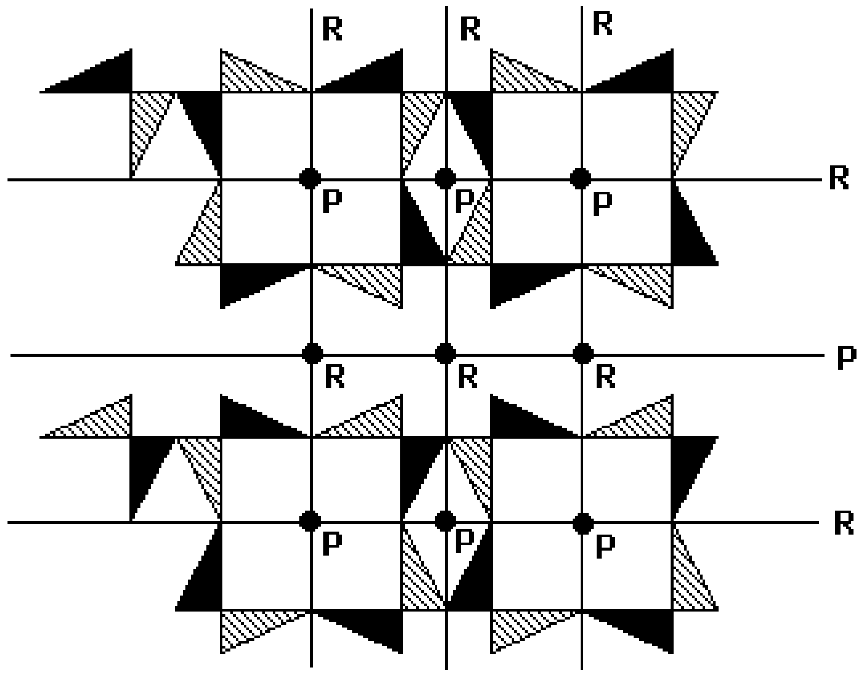


pmg′ → pmm′

Fig. 6.73

The two patterns in figures 6.72 & 6.73 **look** distinct, but mathematically they are the same (**RR** × **PP** versus **PP** × **RR**), even though they are related to two distinct **pmg**-like patterns: indeed the **pmg′** pattern of figure 6.63 (**PP** × **RR**) and the **pm′g** pattern of figure 6.62 (**RR** × **PP**) are **not** the same because the **pmg**'s two 'factors', unlike those of the **pmm**, are **not** equivalent (reflection × glide reflection as opposed to reflection × reflection).

Let's go on to a 'perfect shifting' of figure 6.66:

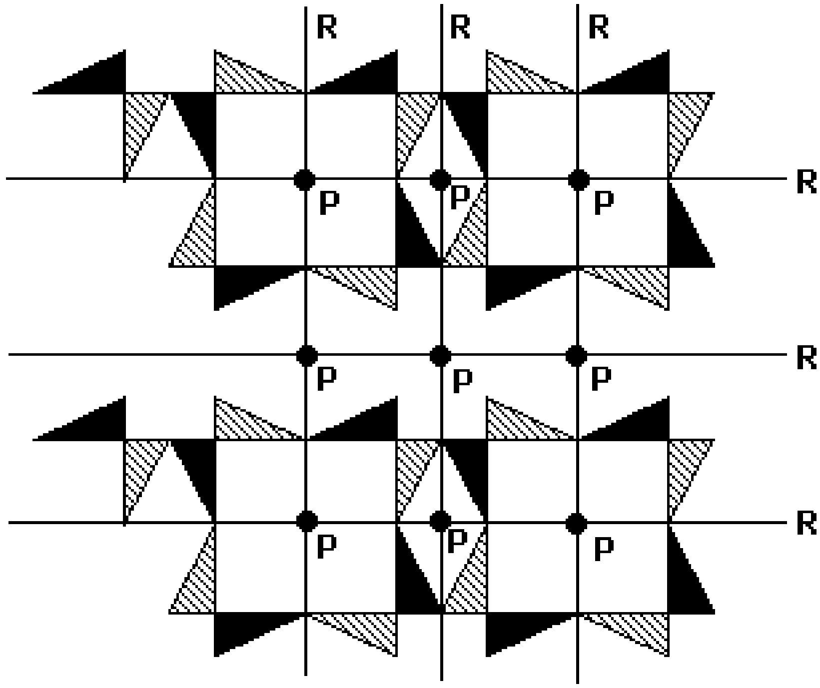


$\mathbf{p'_b gg \rightarrow p'_b gm}$

Fig. 6.74

This time we went from **RR** × **PR** to **PR** × **RR**: again 'no changes' (keeping in mind the equivalence between **RR** × **PR** and **PR** × **RR** in the **pmm** type, in accordance to our observations above on the equivalence between its 'vertical' and 'horizontal' directions).

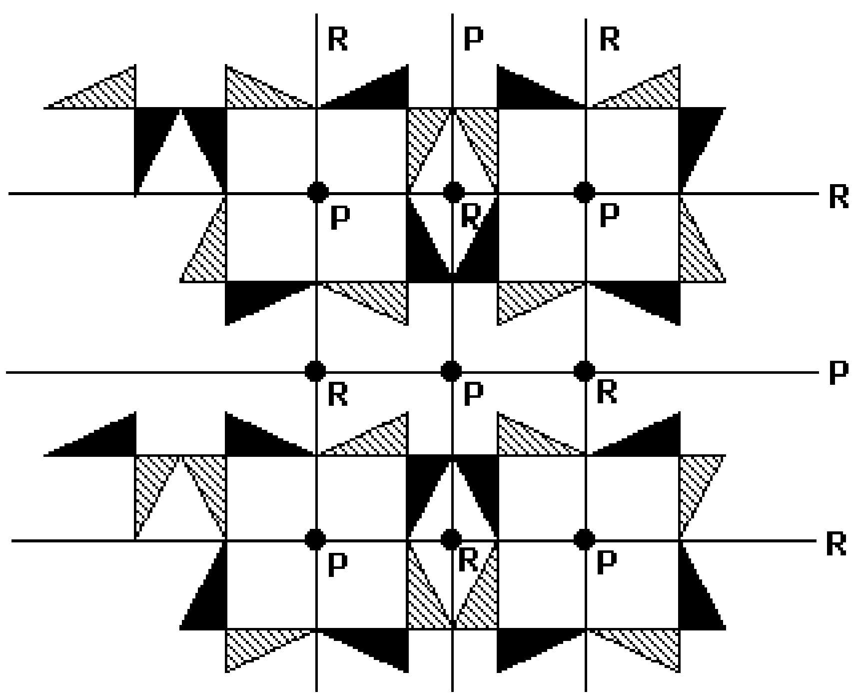
Finally, a 'perfect shifting' of the **pm′g′** pattern of figure 6.64 leads, most predictably, to a **RR** × **RR** **pmm**-like pattern having color-reversing reflections **only**:



$\mathbf{pm'g'} \rightarrow \mathbf{pm'm'}$

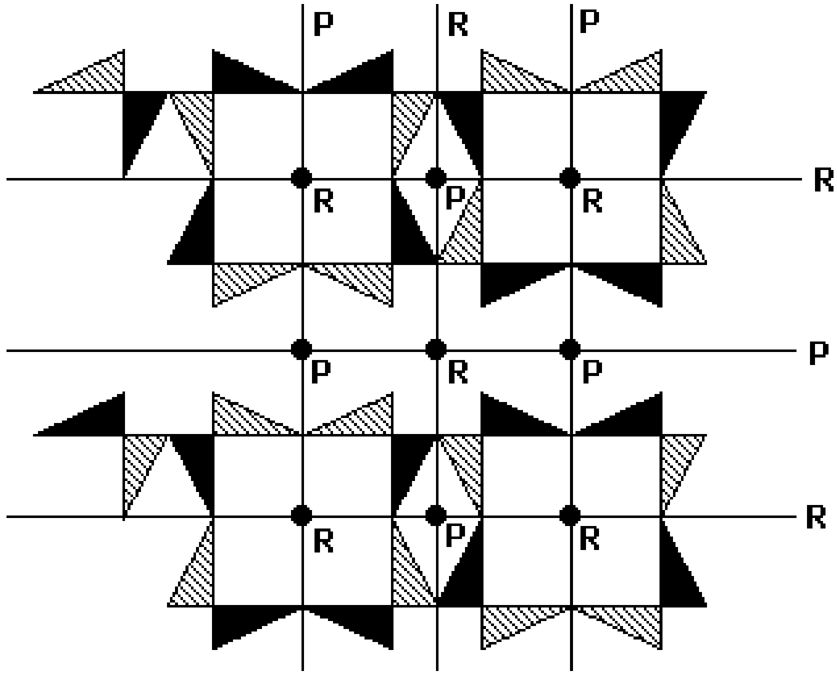
Fig. 6.75

So we did get **five** out of six possible types, missing **PR** × **PR**: does this mean that there is no such **pmm**-like type? Certainly not:



$\mathbf{p'_b gm} \rightarrow \mathbf{c'mm}$

Fig. 6.76

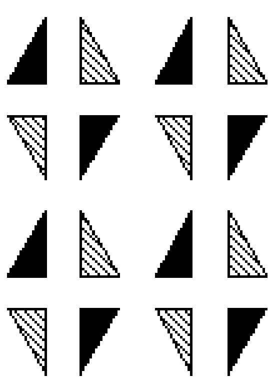


$p'_b mm \rightarrow c'mm$

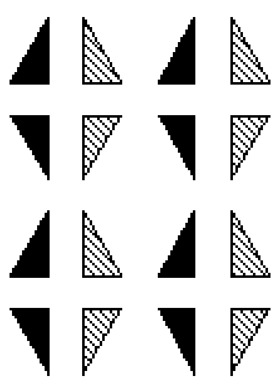
Fig. 6.77

What happened? **Shifting** now **columns** (rather than **rows**), and departing from two **pmm** (rather than **pmg**) types (figures 6.71 & 6.74), we did arrive at **two** 'distinct' representatives (figures 6.77 & 6.76, respectively) of the sought sixth **pmm**-like type!

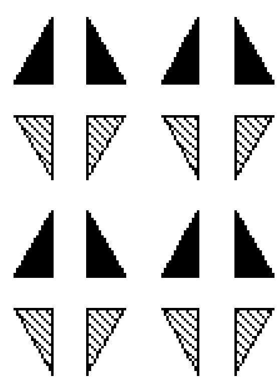
6.8.3 Examples. A larger than usual collection of 'triangular patterns' indicating the **pmm**'s richness; notice how the last four examples have 'dropped' from **cmm** to **pmm** because of coloring.



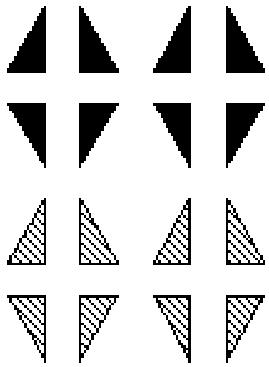
pm′m′



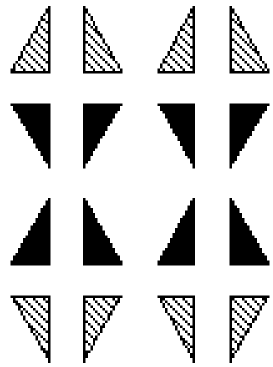
pmm′



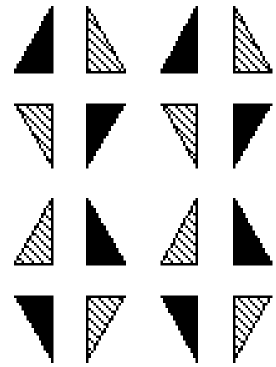
pmm′



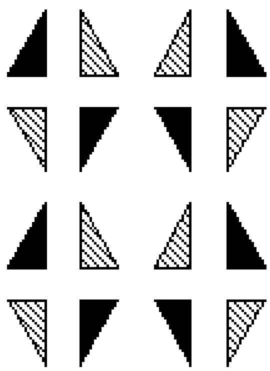
$p'_b mm$



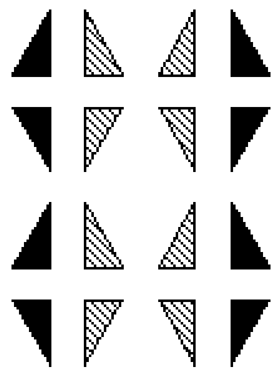
$p'_b mm$



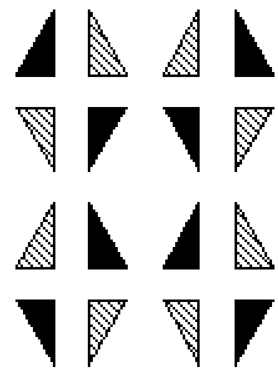
$p'_b gm$



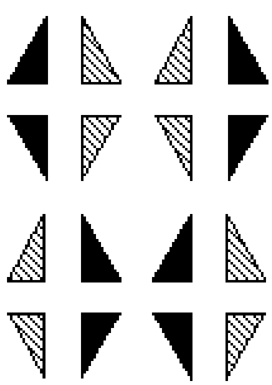
$p'_b gm$



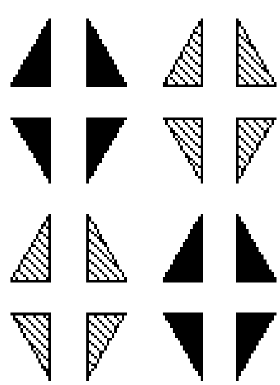
$p'_b mm$



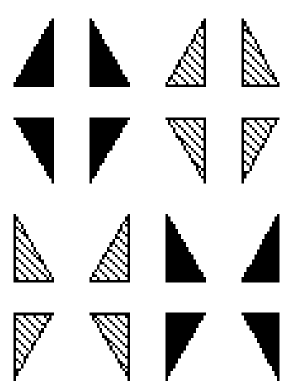
$c'm\ m$



$c'mm$



$c'mm$



$p'_b mm$

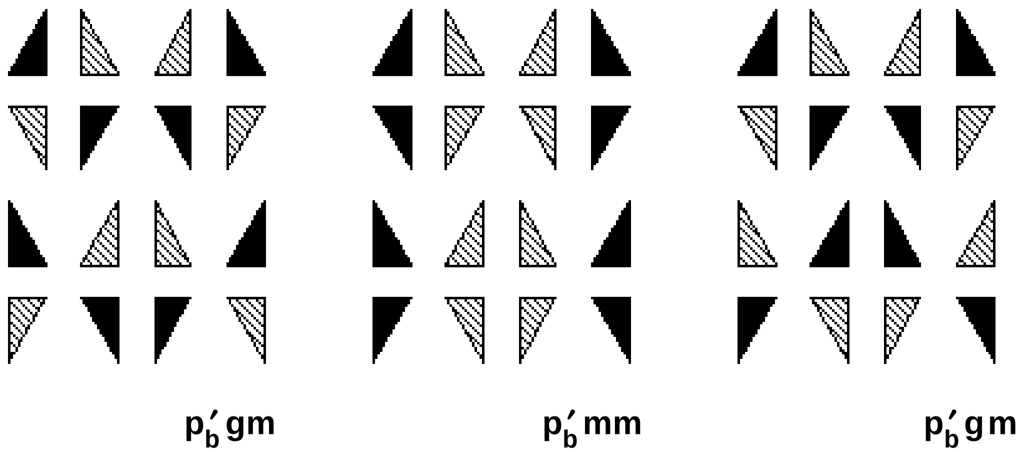


Fig. 6.78

6.8.4 Symmetry plans. No surprises here, just remember that all rotations are combinations of the two perpendicular reflection axes intersecting at their center (a **special case** of the fact illustrated in figure 6.54), hence their effect on color is determined by that of the reflections (and according to the 'multiplication' rules of 5.6.2).

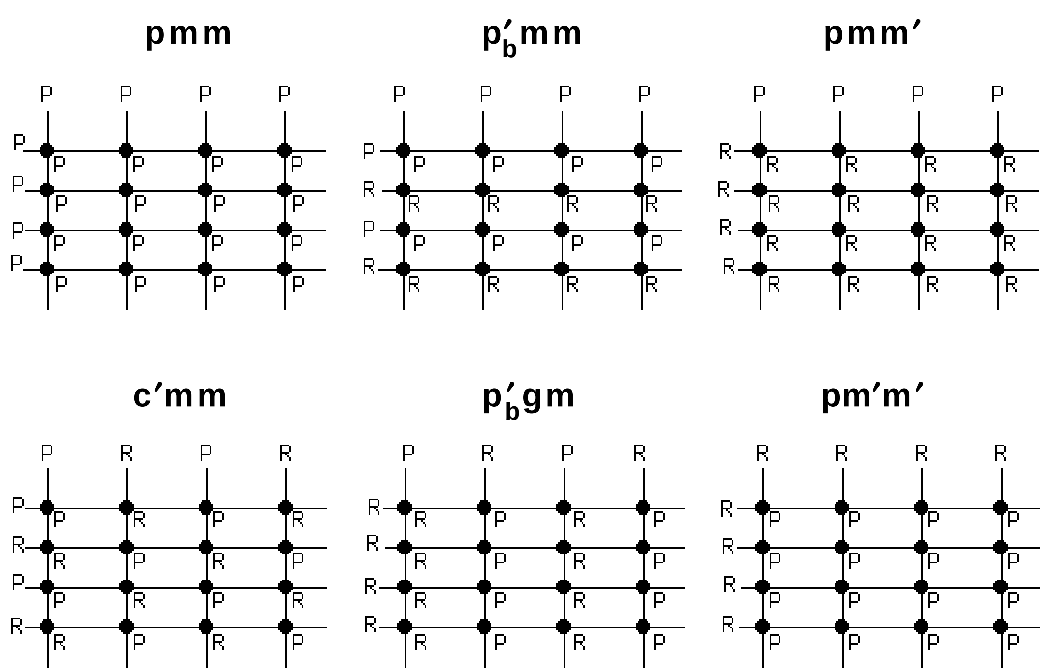


Fig. 6.79

We conclude by expressing each type as a 'product' of **pm** types:

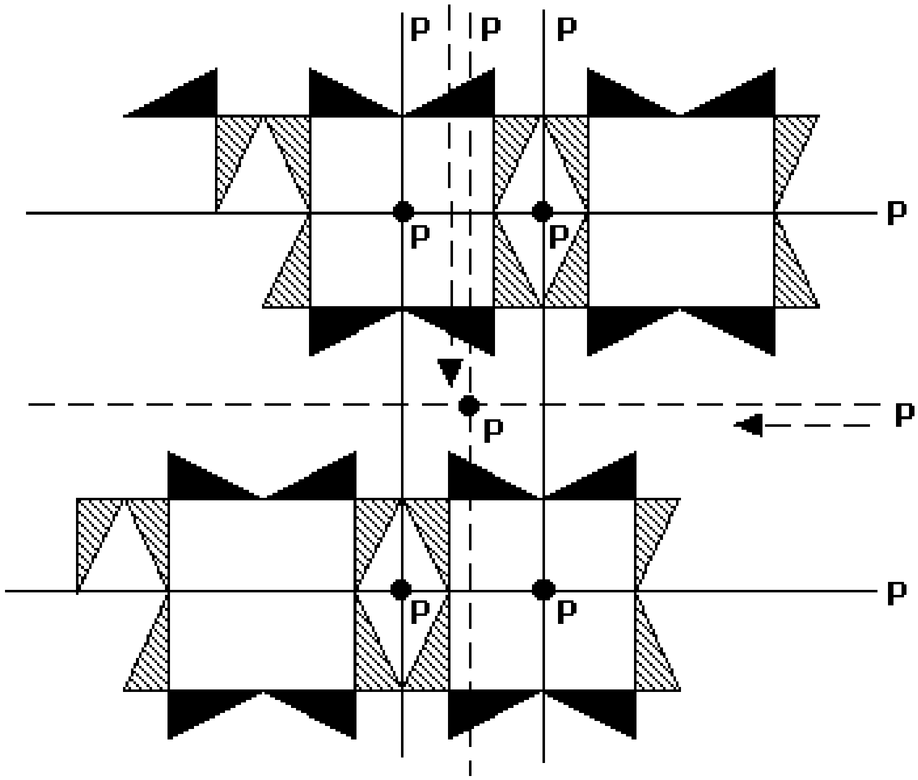
$\mathbf{pmm} = \mathbf{pm} \times \mathbf{pm}$, $\mathbf{p'_b mm} = \mathbf{p'm} \times \mathbf{p'_b 1m}$, $\mathbf{pmm'} = \mathbf{pm} \times \mathbf{pm'}$,
$\mathbf{c'mm} = \mathbf{c'm} \times \mathbf{c'm}$, $\mathbf{p'_b gm} = \mathbf{p'_b 1m} \times \mathbf{p'_b g}$, $\mathbf{pm'm'} = \mathbf{pm'} \times \mathbf{pm'}$

In connection to figure 6.79 (**pmm** symmetry plans) always, the 'first' factor corresponds to the 'vertical' direction and the 'second' factor corresponds to the 'horizontal' direction. Color-reversing translation is no longer crucial enough to be explicitly indicated; consistently with 5.5.1 and 6.5.3, it is to be found precisely in those directions in which there exist half turn centers of **opposite** effect on color. In particular the elusive **c′mm** is the only **pmm**-like type with color-reversing translation in **both** the vertical and horizontal directions, while **pmm** and **pm′m′** are the only ones with no color-reversing translation **at all**. More to the point, and arguing as in 6.7.1, we see that there exist vertical reflections of opposite color effect if and only if there exists horizontal color-reversing translation (and vice versa).

6.9 cmm types (cmm, cmm′, cm′m′, $\mathbf{p'_c mm}$, $\mathbf{p'_c mg}$, $\mathbf{p'_c gg}$)

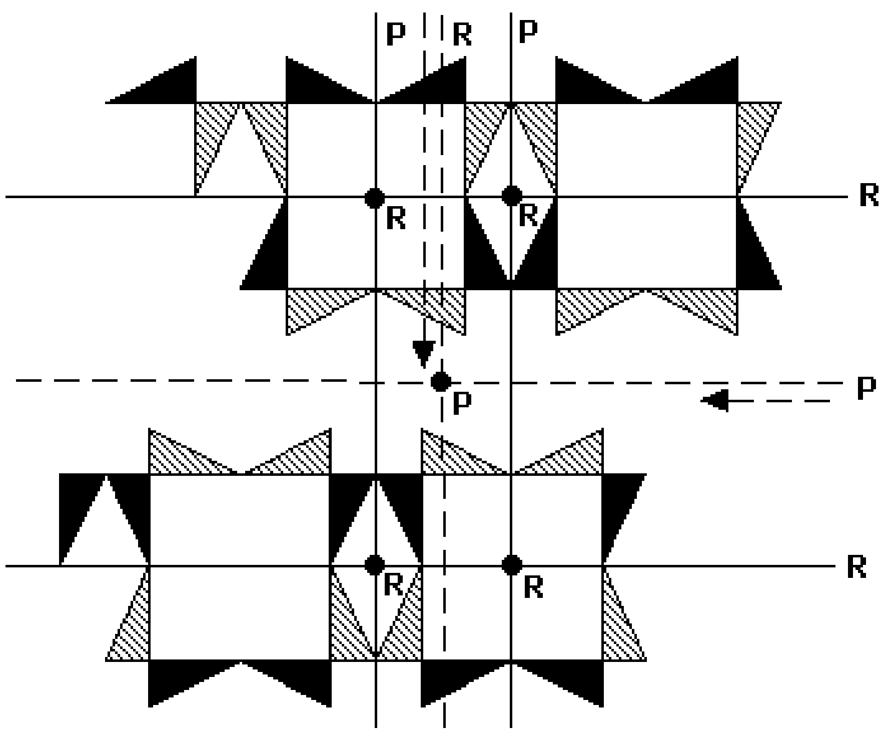
6.9.1 How many types? Following the approach in 6.6.2, 6.7.1, and 6.8.1, we view a **cmm**-like pattern as a 'product' of two **cm**-like patterns. Having four possibilities for each 'factor' (**PP**, **PR**, **RP**, **RR**, where the **first** letter now stands for reflection and the **second** letter for in-between glide reflection), and keeping in mind that 'multiplication' is **commutative** (again the 'horizontal' versus 'vertical' non-issue), we see that there can be **at most** ten possible **cmm** types, defined by the 'products' **PP** × **PP**, **PP** × **PR**, **PP** × **RP**, **PP** × **RR**, **PR** × **PR**, **PR** × **RP**, **PR** × **RR**, **RP** × **RP**, **RP** × **RR**, **RR** × **RR**. Let's first check how many types we can get 'experimentally' (6.9.2), and then check how many types are in fact impossible (6.9.3).

6.9.2 Perfectly shifting the **pmm**s. We trace each new (**cmm**) type back to a **pmg** type, showing also the 'intermediate' **pmm** type (the perfect shifting of which led to the **cmm** type):



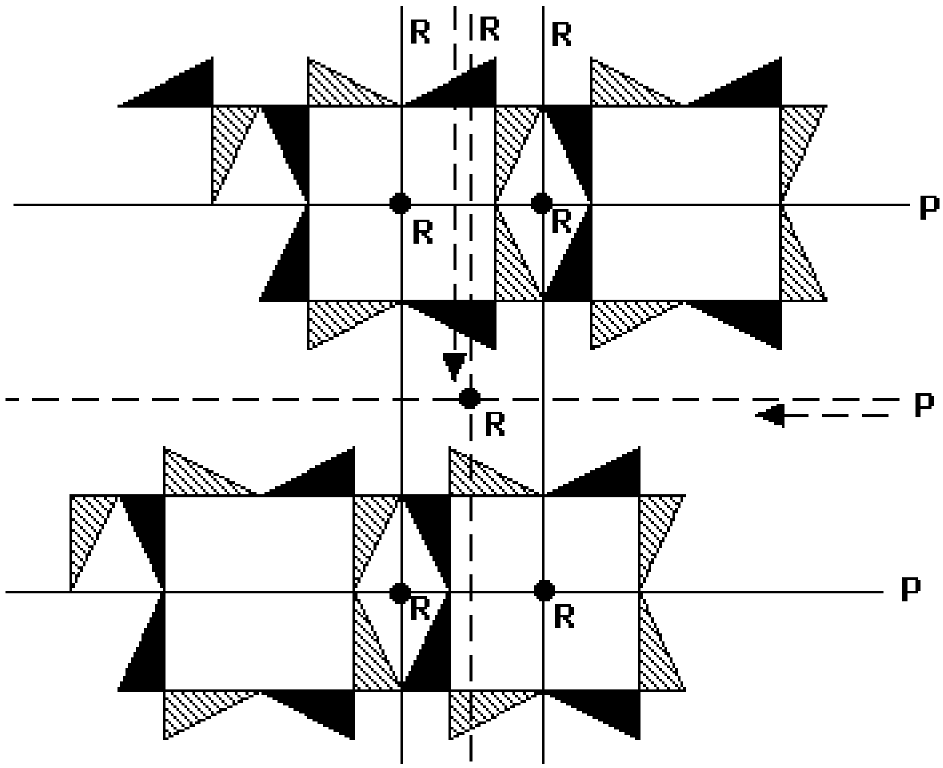
pmg → pmm → cmm

Fig. 6.80



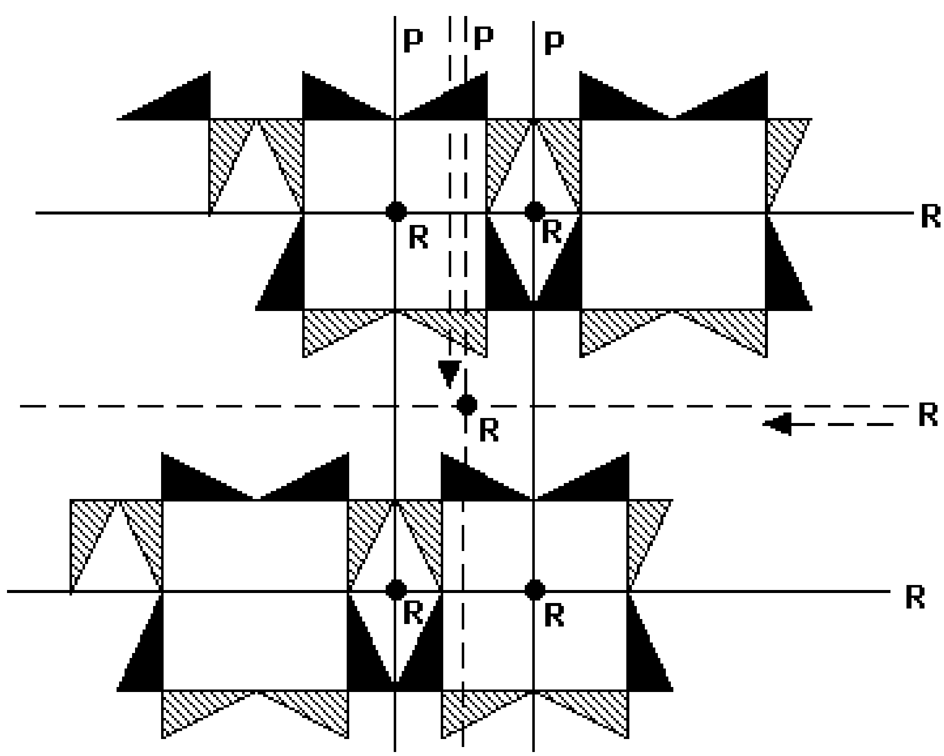
$p_b'mg \rightarrow p_b'mm \rightarrow p_c'mg$

Fig. 6.81



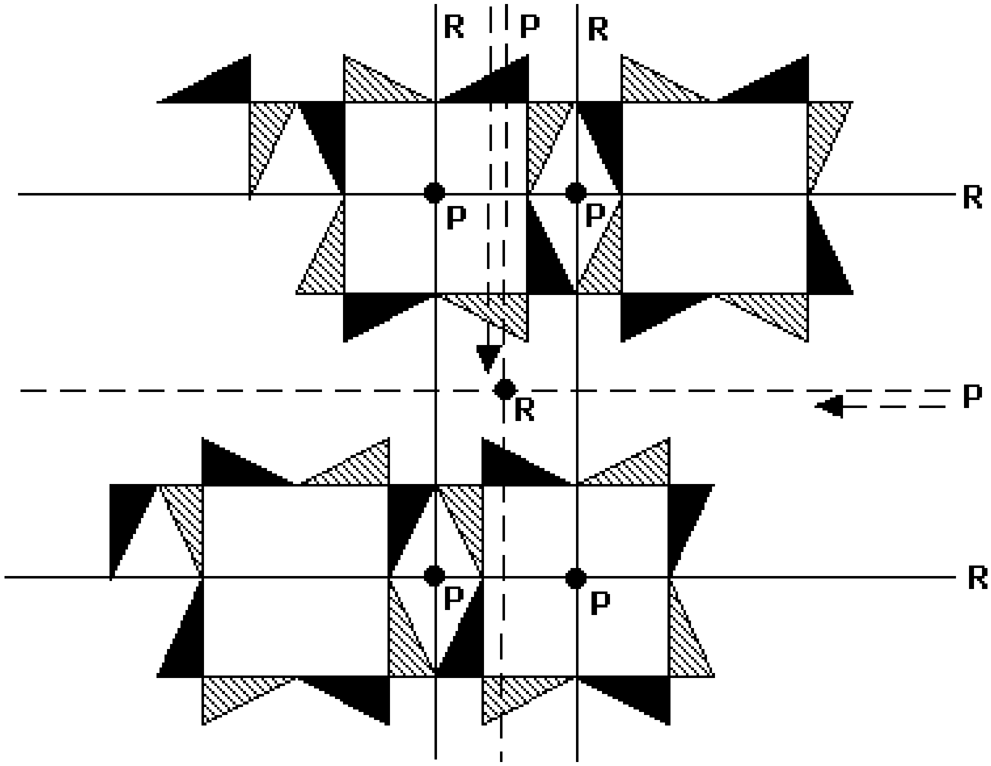
pm′g → pmm′ → cmm′

Fig. 6.82



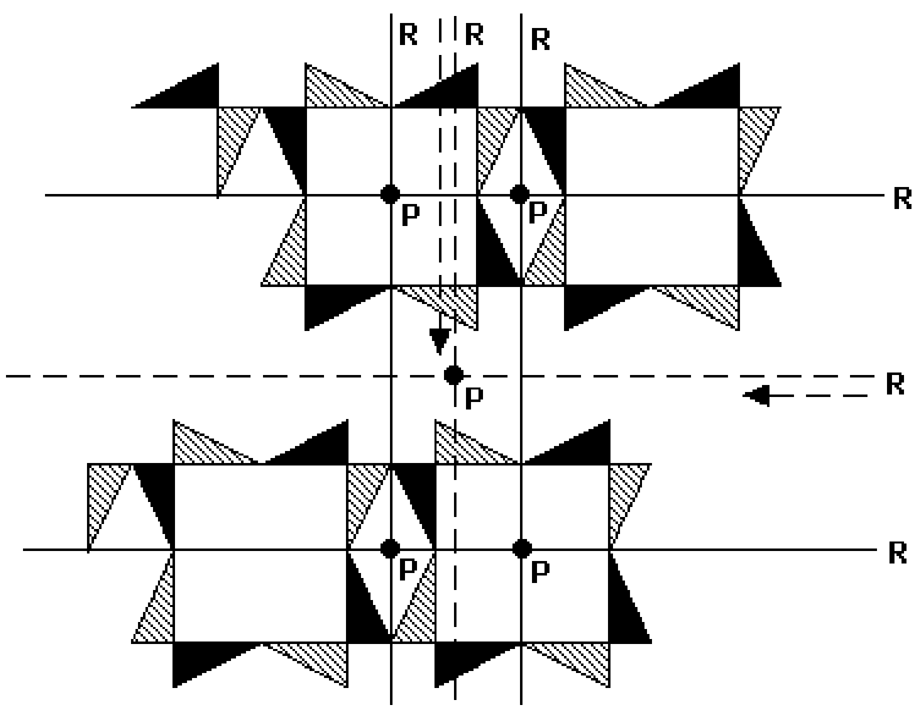
pmg′ → pmm′ → cmm′

Fig. 6.83



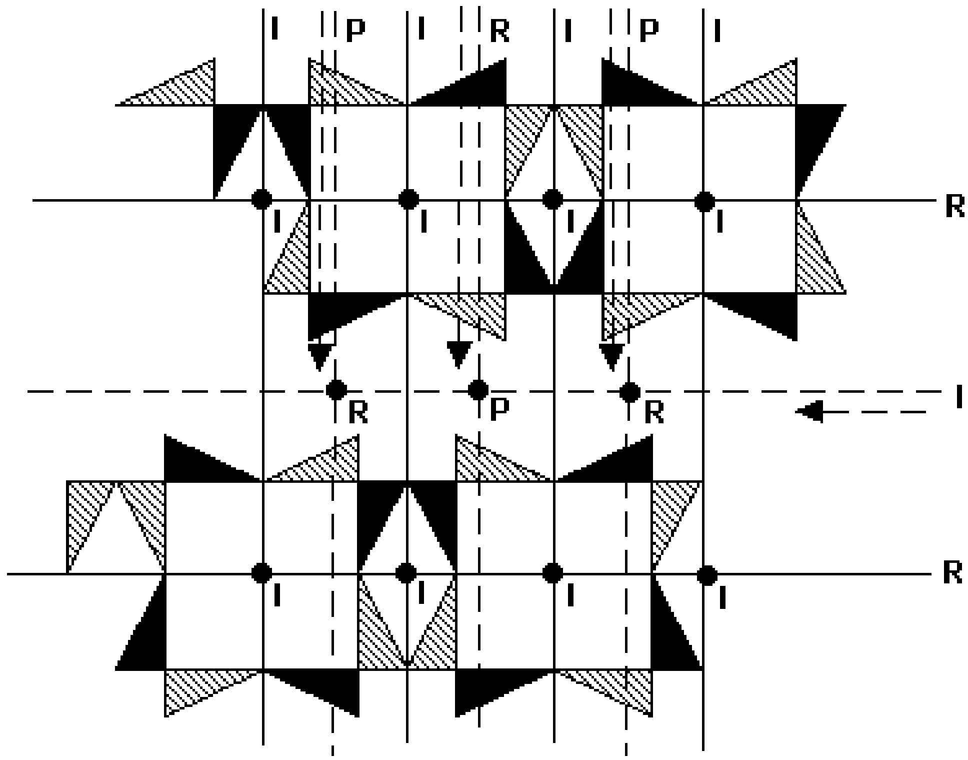
$p'_b gg \rightarrow p'_b gm \rightarrow p'_c gg$

Fig. 6.84



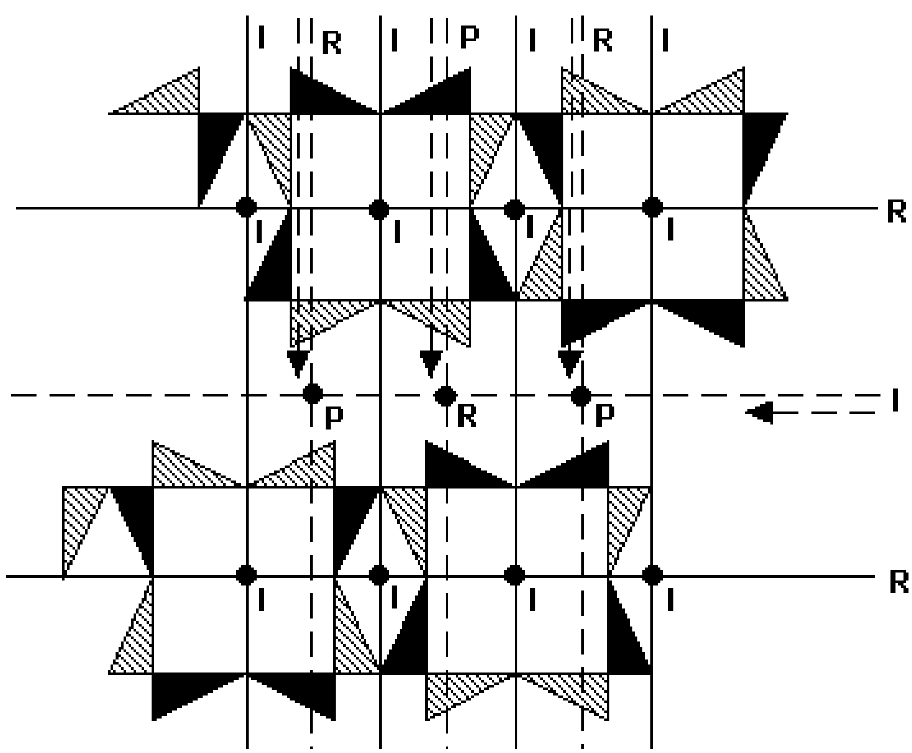
$pm'g' \rightarrow pm'm' \rightarrow cm'm'$

Fig. 6.85



$p'_b gm \rightarrow c'mm \rightarrow p'_b gg$

Fig. 6.86



$p'_b mm \rightarrow c'mm \rightarrow p'_b gg$

Fig. 6.87

Due to not-that-obvious color inconsistencies, the last two patterns are of the same **pmg**-like type! Our shifting process has produced **five** out of at most ten possible types corresponding to the ten combinations listed in 6.9.1: **PP** × **PP** (**cmm**), **PR** × **RP** (**p'_cmg**), **PP** × **RR** (**cmm′**), **RP** × **RP** (**p'_cgg**), **RR** × **RR** (**cm′m′**). But this 50% rate of success is a bit too low in view of our experience with the other types! Is it possible that some or all of the remaining five combinations are in fact **impossible**?

6.9.3 Ruling out the non-obvious. It turns out that another four of the combinations in 6.9.1 are impossible (**PP** × **PR**, **PP** × **RP**, **PR** × **RR**, **RP** × **RR**), leaving thus only **one** question mark around **PR** × **PR**. Let's see for example why a situation such as **PP** × **PR** is impossible, using a version of the argument in 6.6.2 (figure 6.55):

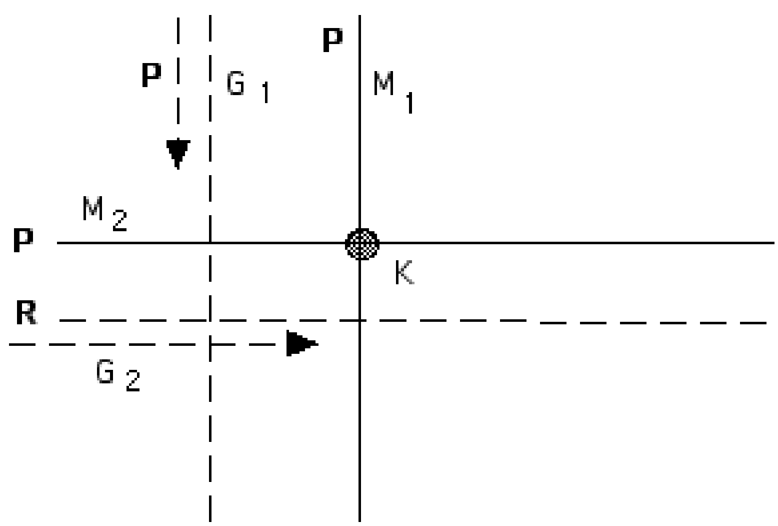


Fig. 6.88

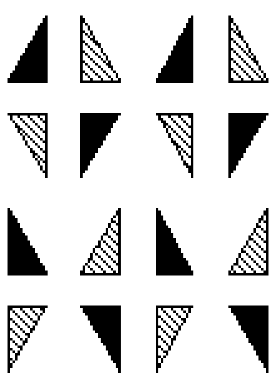
Is the half turn (at) K color-preserving or color-reversing? In view of **K = M_2∗M_1 = M_1∗M_2** (reflection **M_1** (**P**) followed by reflection **M_2** (**P**) **or** the other way around) and **K = G_2∗G_1** (glide reflection **G_1** (**P**) followed by glide reflection **G_2** (**R**)) we conclude that the half turn at K must be **both** color-preserving and color-reversing, which is certainly **impossible**.

So, the **cmm** does not allow a ‘mixed’ combination of reflection and glide reflection in one direction **and** a ‘pure’ combination in the other direction. Unlike in 6.6.3, this impossibility cannot be deduced from the Conjugacy Principle; it is solely a consequence of the pattern’s structure and the way its isometries are **‘weaved’** into each other.

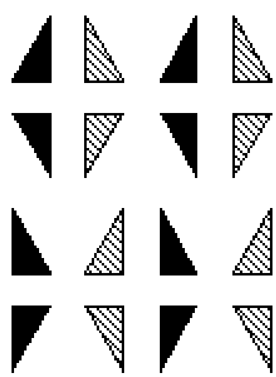
6.9.4 <u>One more type!</u> The question mark around **PR** $\times$ **PR** would not have been there at all were we blessed with photo memory: indeed the pattern in figure 6.2 has just what we were looking for, color-preserving reflections **and** in-between color-reversing glide reflections in **both** directions! Such patterns are known as **p'_cmm**.

But here is another question: how could we **possibly** get a **p'_cmm** out of those ‘root’ **pg** patterns through our usual operations? This is something for you to wonder about as we are bidding farewell to our ‘roots’: even though the **p4m** types in section 6.12 may be viewed as special (‘square’) versions of the **pmm**, and likewise for **p4g** (section 6.11) and **cmm** (as our last example on **p'_cmm** indicates), the **pg** excursion cannot go on for ever, as color inconsistencies and worse stand on our way...

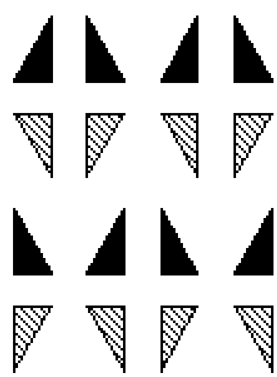
6.9.5 <u>Examples.</u> First a few ‘triangles’: **compare** with 6.8.3!



p'_cgg



p'_cmg



cmm′

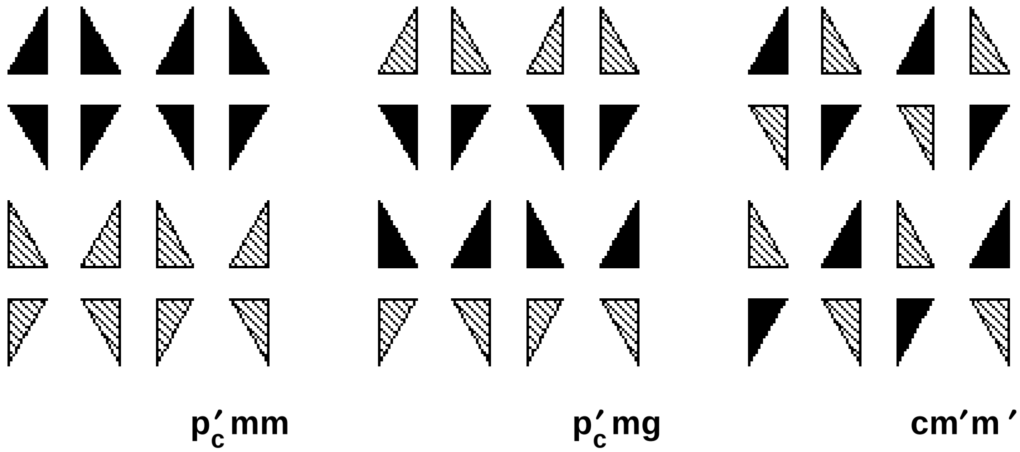


Fig. 6.89

And now a collection of examples in the spirit of 6.4.2, with or without **color inconsistencies** (and consequent **reductions** of symmetry from **cmm** to 'lower' types):

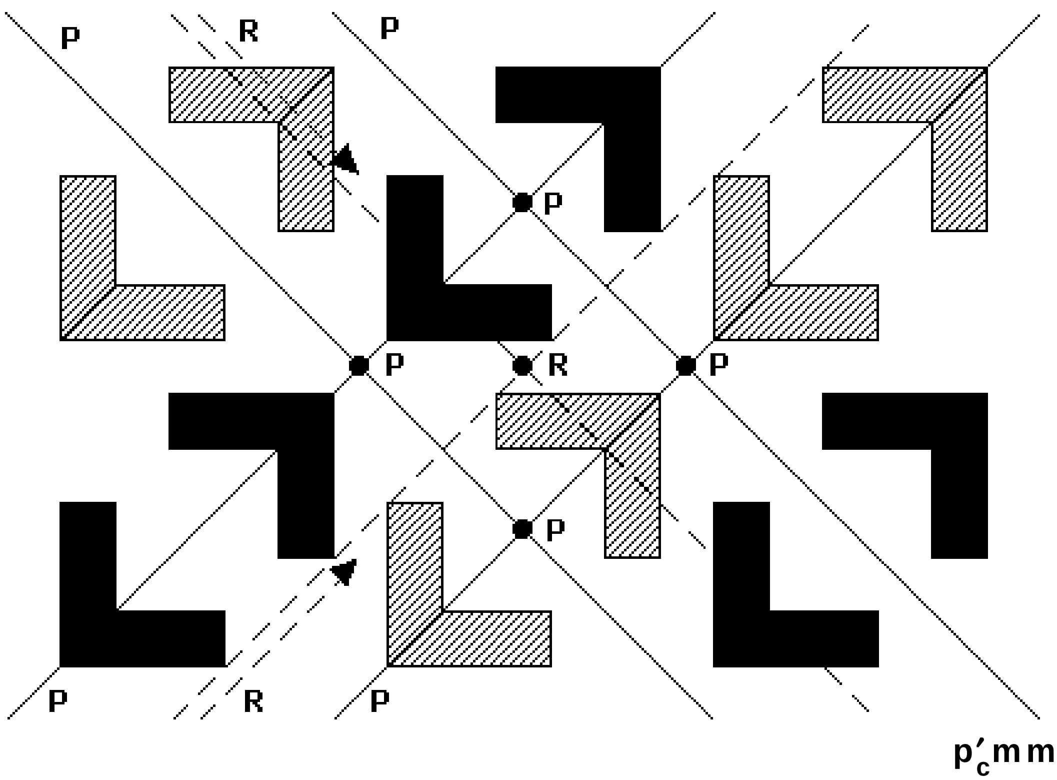


Fig. 6.90

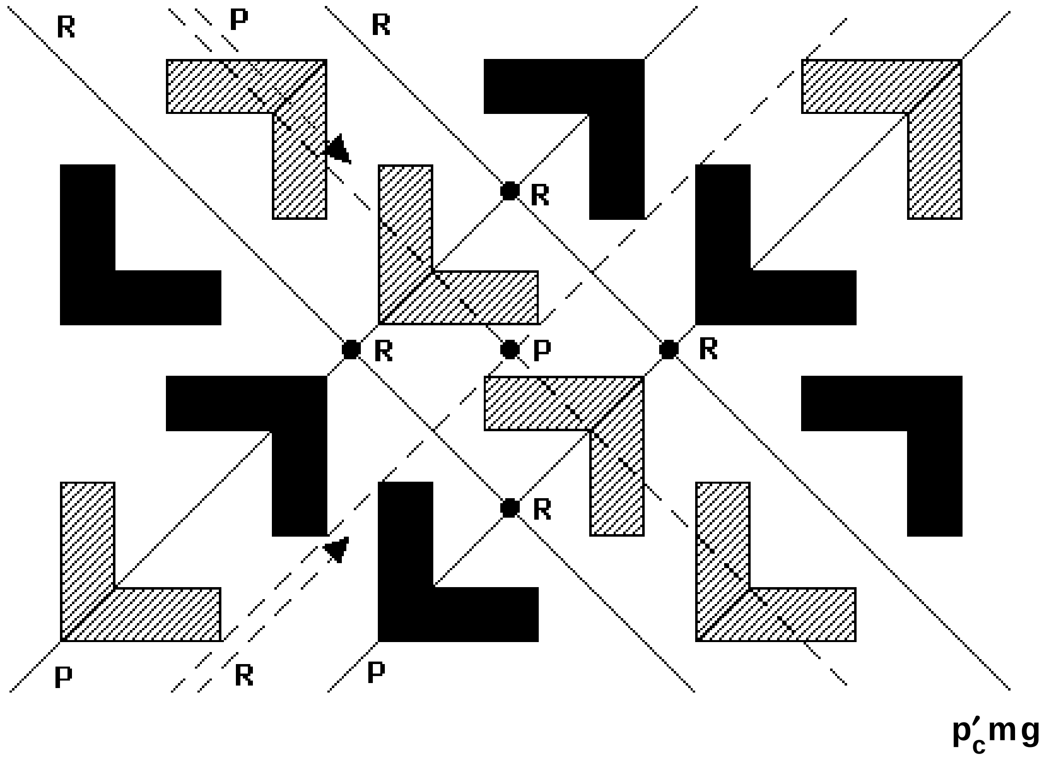


Fig. 6.91

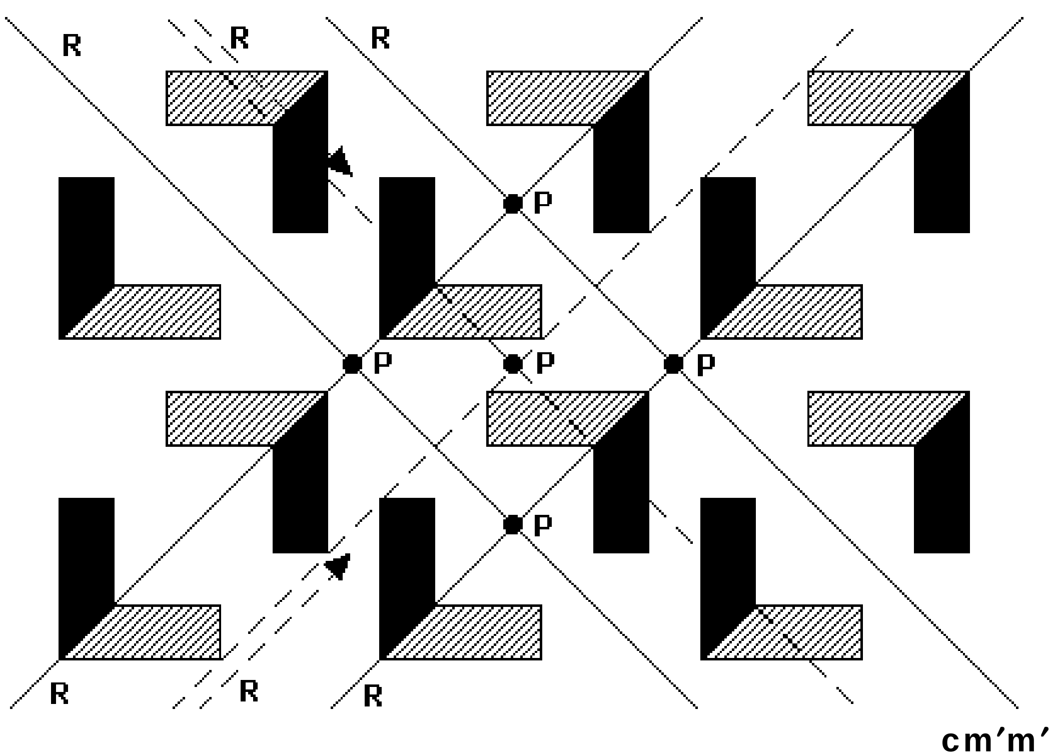


Fig. 6.92

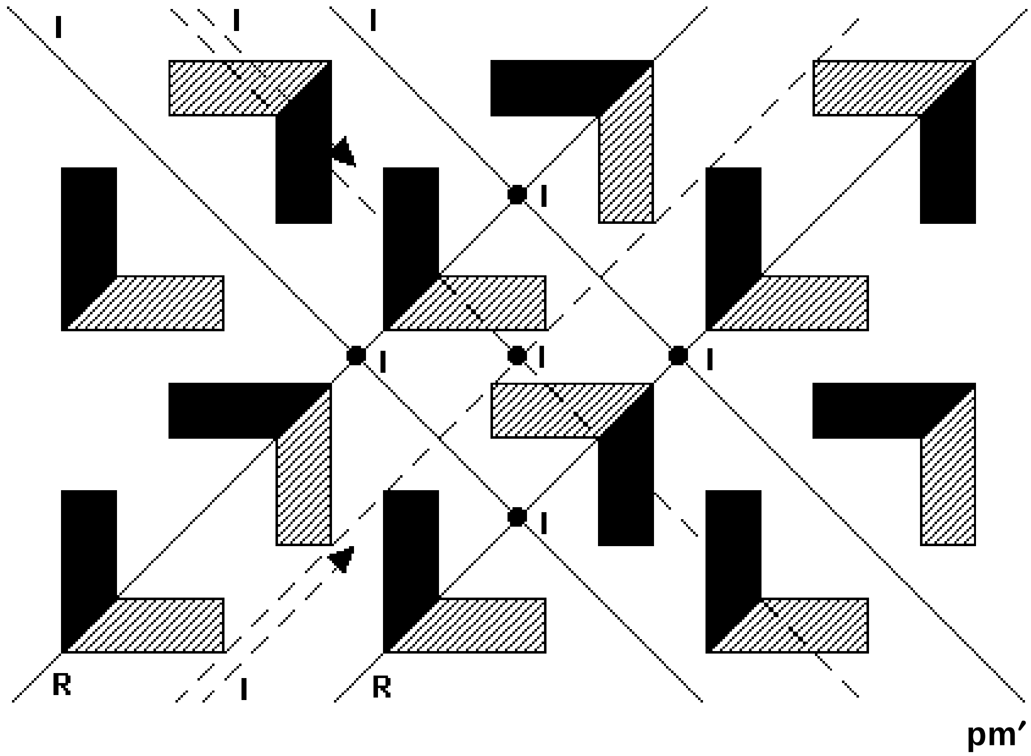


Fig. 6.93

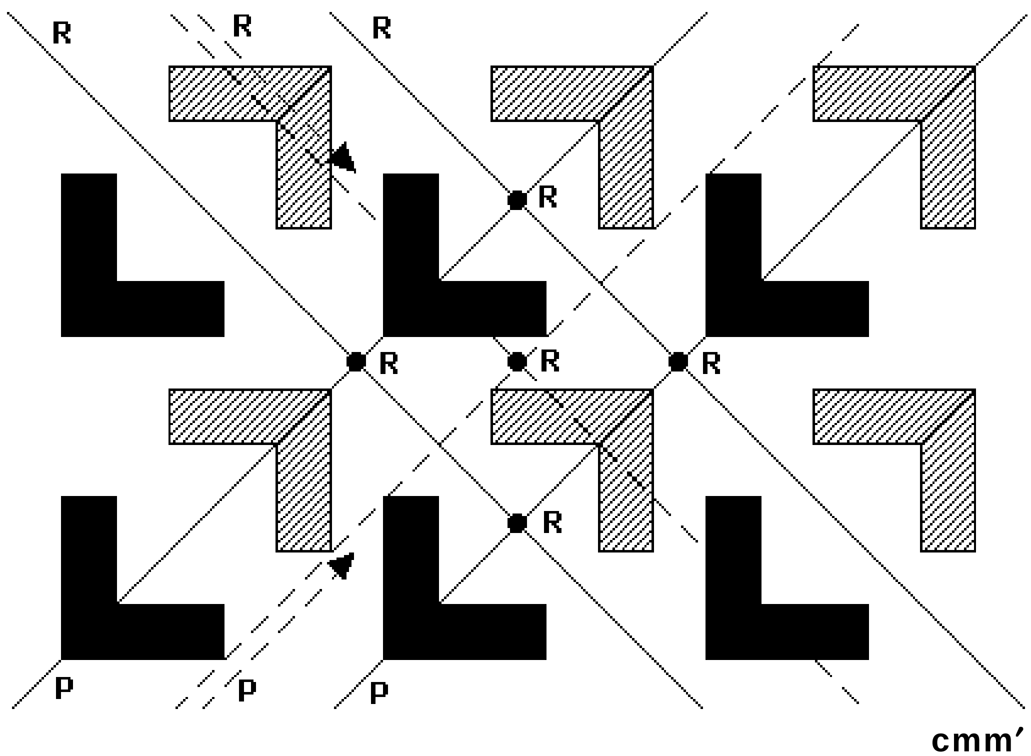


Fig. 6.94

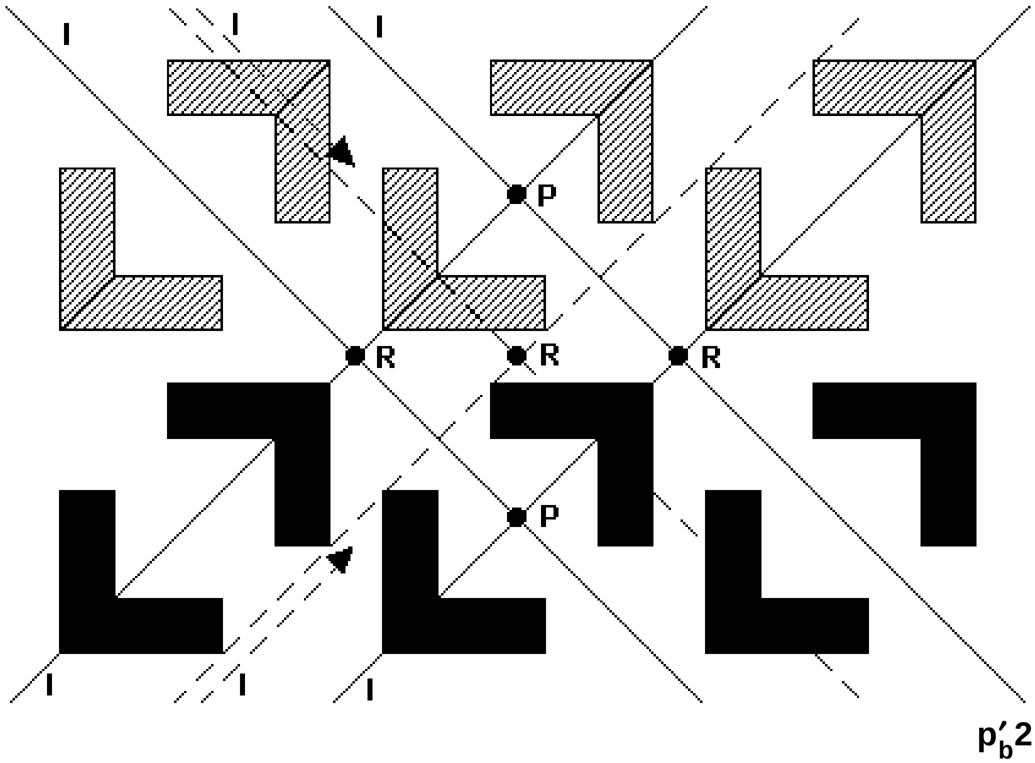


Fig. 6.95

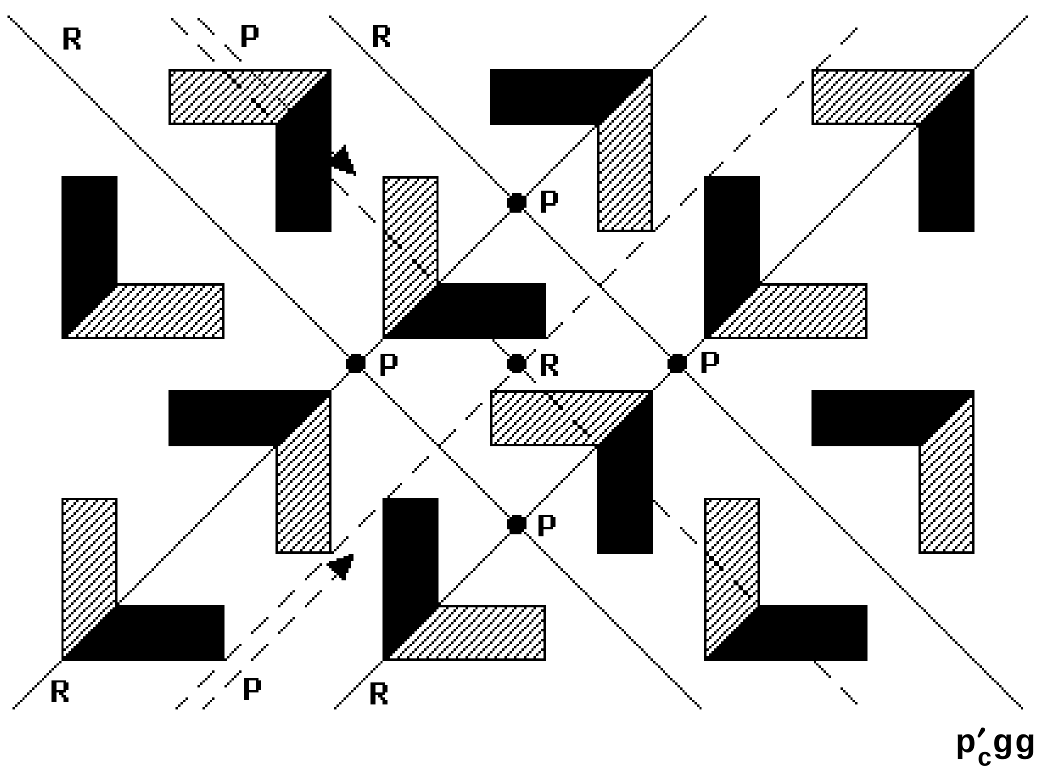
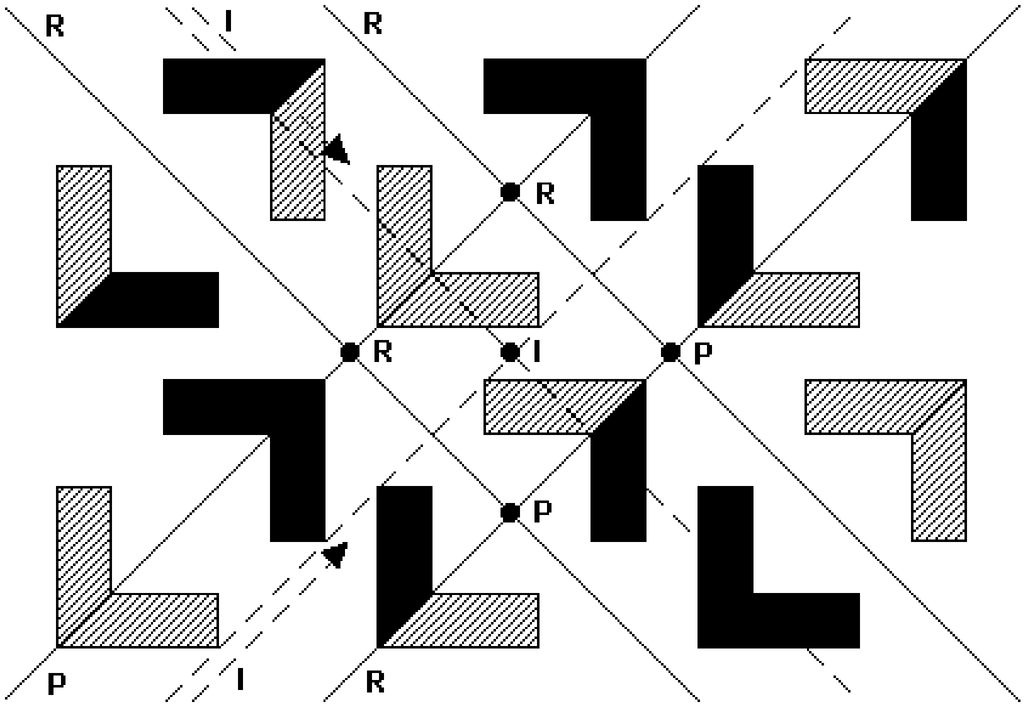


Fig. 6.96



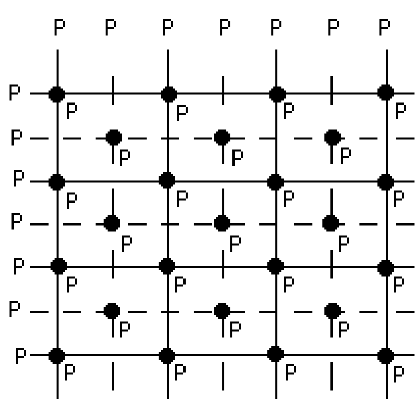
$p'_b gm$

Fig. 6.97

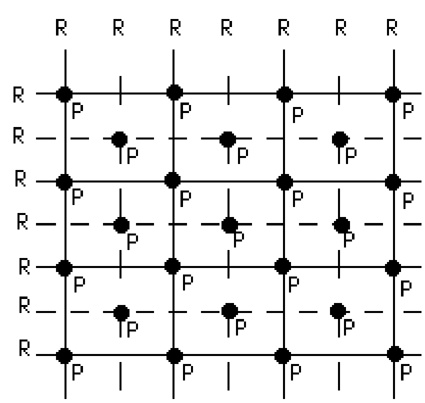
You should be able to derive more types out of the original **cmm** pattern of figures 6.90-6.97 using yet more imaginative colorings!

6.9.6 <u>Symmetry plans.</u> Notice the **location** and effect on color of rotation centers (**determined** by that of (glide) reflection axes).

cmm



cm′m′



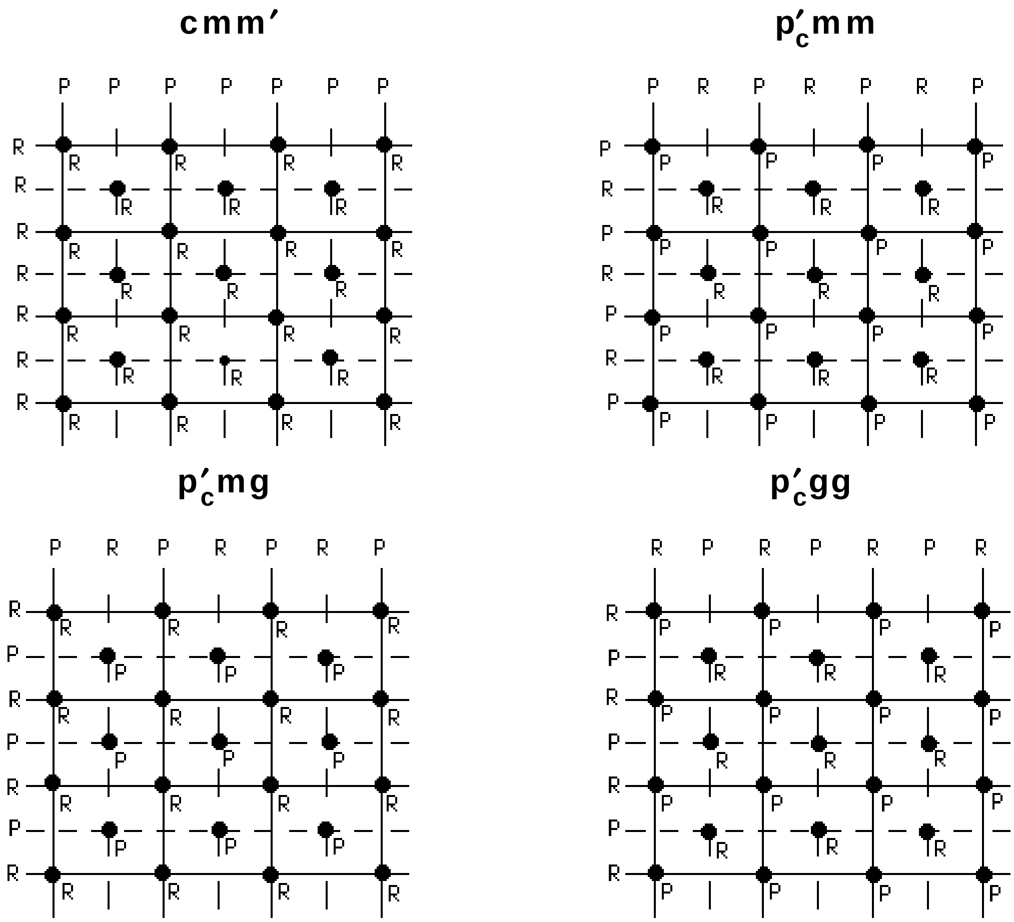


Fig. 6.98

A couple of remarks: the half turn centers found at the intersection of any two perpendicular **glide** reflection axes are ‘products’ of any one of the two glide reflections and a **reflection perpendicular** to it (as in the case of the **pmg**), **not** of the two glide reflections; and the length of the glide reflection vector is equal to the distance between two nearest half turn centers on a parallel to it (glide) reflection axis (as in the case of the **pmg**).

Finally, a ‘factorization’ of our **cmm** types into simpler ones:

cmm = **cm** × **cm**, **cm′m′** = **cm′** × **cm′**, **cmm′** = **cm** × **cm′**,
$\mathbf{p'_c mm} = \mathbf{p'_c m} \times \mathbf{p'_c m}$, $\mathbf{p'_c mg} = \mathbf{p'_c m} \times \mathbf{p'_c g}$, $\mathbf{p'_c gg} = \mathbf{p'_c g} \times \mathbf{p'_c g}$

You may also ‘factor’ the **cmm**s using either **pmm**s and **pgg**s or, in resonance with the remarks made above, **pmg**s in **both** directions!

6.10 p4 types (p4, p4′, p'_c4)

6.10.1 A look at fourfold rotations. We begin with a picture:

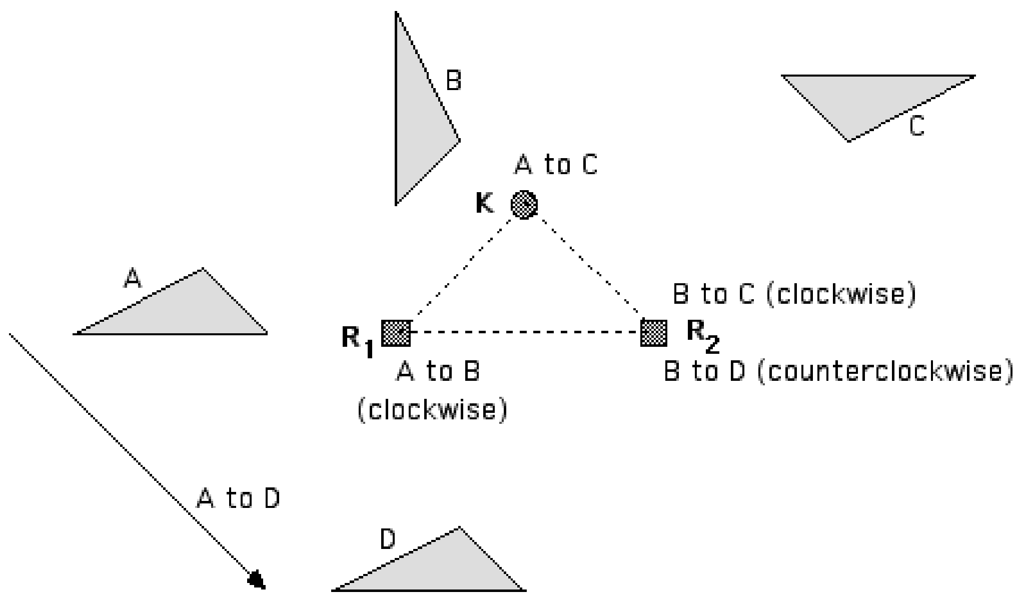


Fig. 6.99

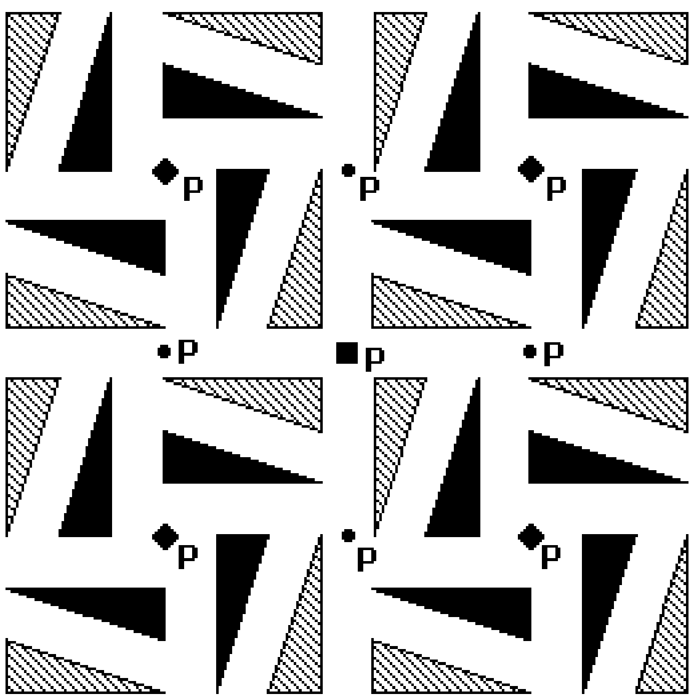
That is, a **clockwise** $\mathbf{90^0}$ rotation centered at $\mathbf{R_1}$ (mapping A to B) followed by a **clockwise** $\mathbf{90^0}$ rotation centered at $\mathbf{R_2}$ (mapping B to C) result into a $\mathbf{180^0}$ **rotation** centered at **K** (mapping A to C); and a **clockwise** $\mathbf{90^0}$ rotation centered at $\mathbf{R_1}$ (mapping A to B) followed by a **counterclockwise** $\mathbf{90^0}$ rotation centered at $\mathbf{R_2}$ (mapping B to D) result into a **translation** (mapping A to D). Figure 6.99 offers of course illustrations rather than proofs (which are special cases of 7.5.1 and 7.5.2, respectively).

You can also use figure 6.99 ‘backwards’ to illustrate how the combination of a translation (mapping D to A) and a 90^0 rotation (mapping A to B) is another 90^0 rotation (mapping D to B).

6.10.2 The lattice of rotation centers revisited. Figure 6.99

throws quite a bit of light into the lattice of rotation centers featured in figure 4.5. Indeed it is not a coincidence that we always get **two** fourfold centers and **one** twofold center in an **isosceles right triangle** (90^0-45^0-45^0) configuration: you can see this rather special triangle being formed by the composition of the two fourfold rotations in figure 6.99; and it is true that **every** wallpaper pattern with 90^0 rotation is **bound** to have 180^0 rotation as well, with **all** the twofold centers 'produced' by fourfold centers as in figure 6.99.

6.10.3 Precisely three types. With all 180^0 rotations fully determined by 90^0 rotations, and an interplay between fourfold rotations and translations (figure 6.99) fully reminiscent of the one between the **pg**'s glide reflections and translations (figure 6.13) or the one between the **p2**'s half turns and translations (figure 6.44), it is easy to follow the approach in sections 6.2 (**pg**) or 6.5 (**p2**) and conclude without much effort that there exist **at most** three **p4** types: **p4** (**all** 90^0 rotations **preserve** colors), **p4′** (**all** 90^0 rotations **reverse** colors), and $\mathbf{p'_c4}$ (90^0 rotations of **both** kinds). As usual, we need to show that such types do indeed exist:



p4

Fig. 6.100

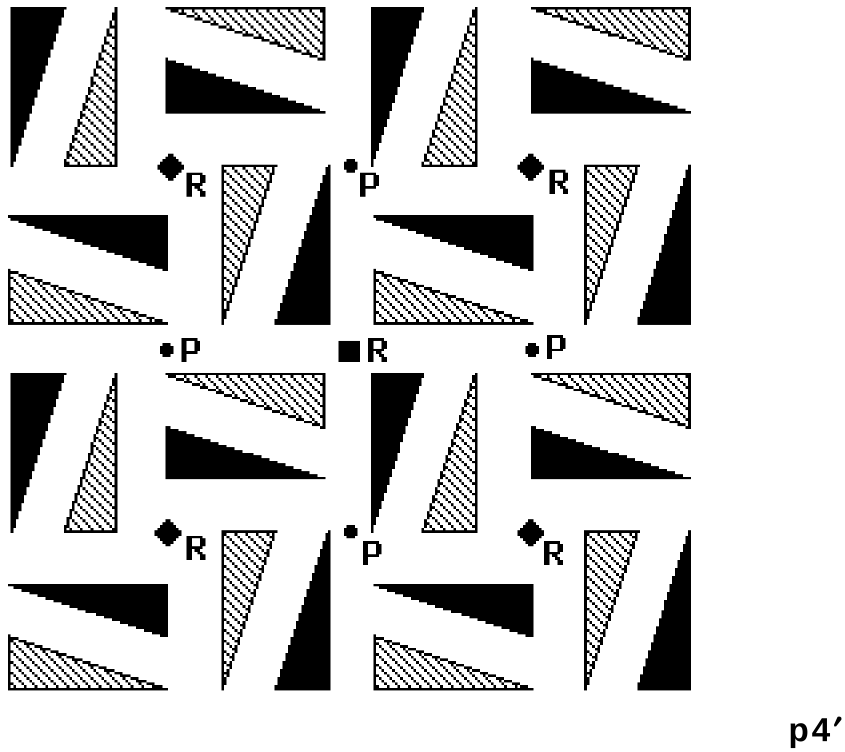


Fig. 6.101

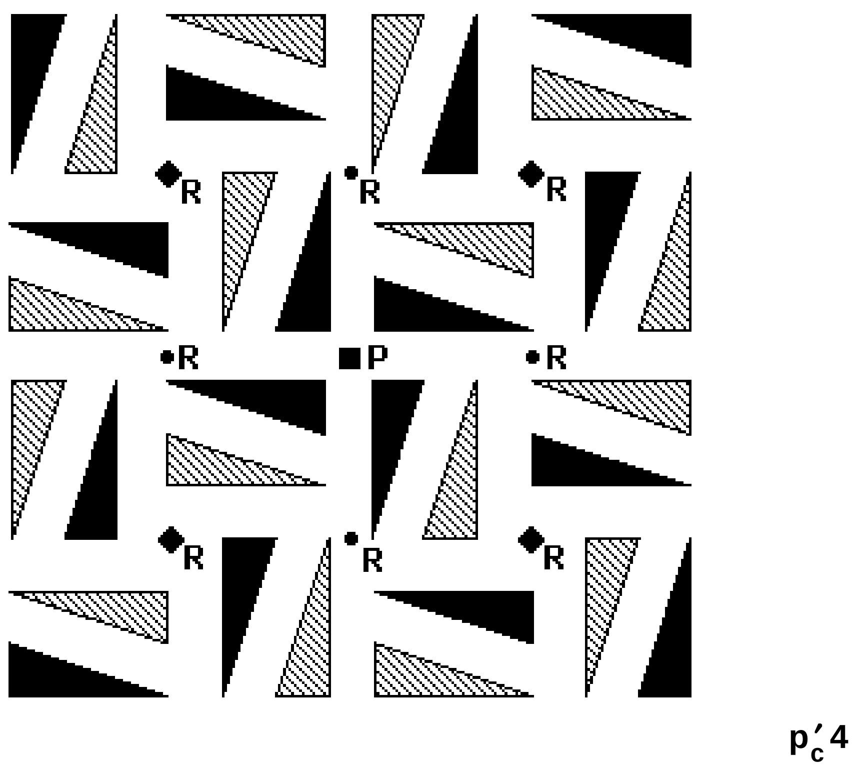
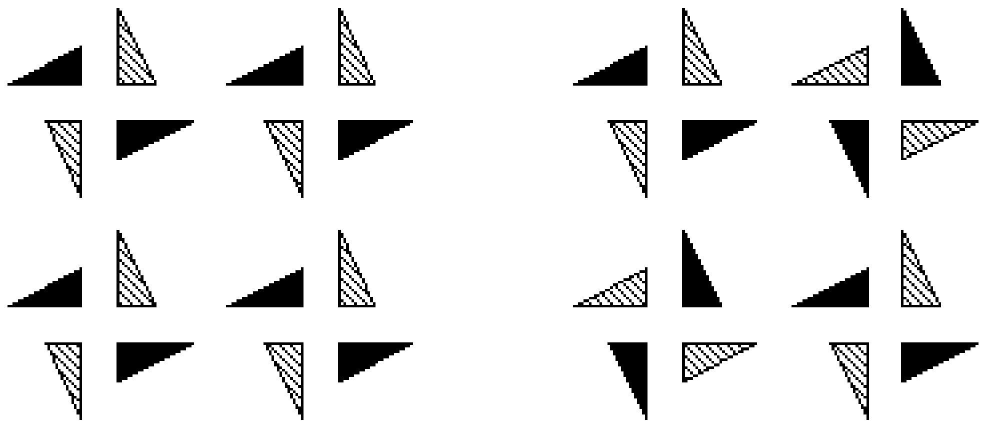


Fig. 6.102

We leave it to you to investigate the complex relationship between the **p4**-like patterns in figures 6.100-6.102 above and the **p2**-like patterns in figures 6.5, 6.39, and 6.40!

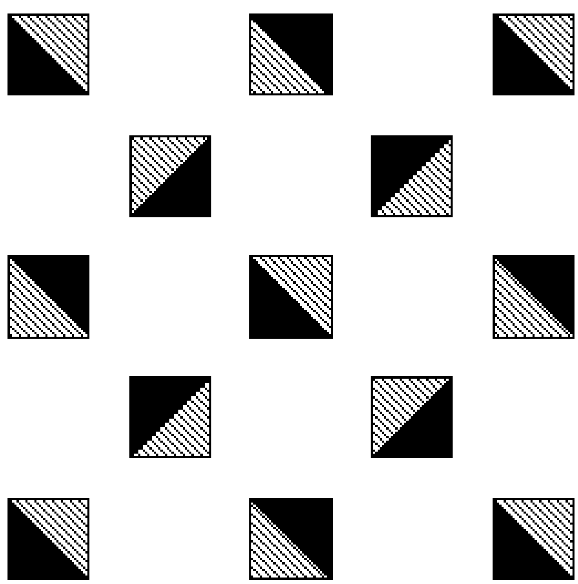
6.10.4 Examples. First a couple of ‘triangles’ and ‘windmills’:



$p4'$ **p'_c4**

Fig. 6.103

Next, a rather complicated **p'_c4**, offspring of a **p4g** of which **all** reflections and glide reflections have been destroyed by coloring:



p'_c4

Fig. 6.104

6.10.5 <u>Symmetry plans.</u> We use 'straight' and 'slanted' squares for the **two kinds** of fourfold centers (4.0.4), and dots for the twofold centers (included for reference only, as 90^0 patterns can be classified based **solely** on the effect on color of their fourfold rotations).

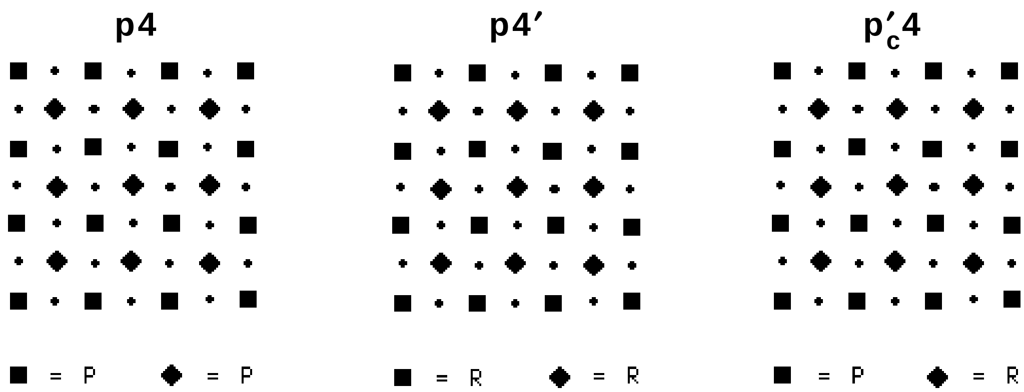


Fig. 6.105

Recall (4.0.3) that every fourfold center is **also** a twofold center by way of **double** application of the 90^0 rotation; this means that the resulting 180^0 rotation is color-preserving: $\mathbf{P} \times \mathbf{P} = \mathbf{R} \times \mathbf{R} = \mathbf{P}$. Observe that, by the same 'multiplication' rules, **all** 'genuine' twofold centers must be color-preserving in **p4** and **p4′**, but color-reversing ($\mathbf{P} \times \mathbf{R} = \mathbf{R}$) in $\mathbf{p'_c4}$: this follows from our remarks in 6.10.1 and 6.10.2.

6.11 p4g types (p4g, p4′g′m, p4′gm′, p4g′m′)

6.11.1 <u>Studying the symmetry plan.</u> Of course a **p4g** may be viewed as a 'merge' of a **cmm** ('vertical'-'horizontal' direction) and a **pgg** ('diagonal' direction). This leads to a rather complex interaction between the two structures, severely limiting the number of possible two-colored **p4g**-like patterns and best understood by having a close look at the **p4g**'s symmetry plan:

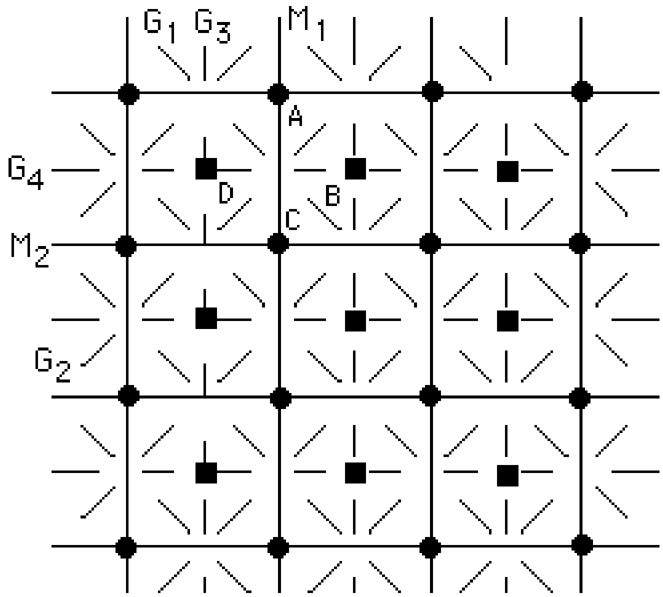


Fig. 6.106

Depending on their vector's direction, the diagonal glide reflections **G_1** and **G_2** produce **four** distinct **twofold** rotations, centered at A, B, C, D. (This relies on figure 6.54 and, primarily, on common sense: **where else** could the four centers be?) Of course B and D are centers for fourfold rotations, but, as pointed out in 6.10.5, such centers are also centers for **color-preserving** twofold rotations. A first consequence of this is that the **pgg**-like component of a **p4g** type can only be a **pgg** or a **pg′g′**: **G_1** and **G_2** must have the same effect on color, otherwise we get **color-reversing** twofold centers at B and D! Another consequence is that the **cmm**-like component of a **p4g** type can only (and **possibly**) be a **cmm** or a **cm′m′** or a **p'_cmg**: by figure 6.54 again, **M_1** and **G_4** combined produce a **color-preserving** twofold center at D, and so do **M_2** and **G_3**; this means that horizontal/vertical glide reflections (**G_4/G_3**) must have the **same** effect on color as vertical/horizontal reflections (**M_1/M_2**).

A further analysis of the symmetry plan rules out the **p'_cmg** as a possible **cmm** 'factor'. Indeed the **Conjugacy Principle** (and also a precursory remark in 4.11.2) tells us that the **fourfold** centers at B and D (**reflected** to each other by **M_1**) must have the same effect on color, hence the **twofold** center at C, produced by a combination of two fourfold rotations (figure 6.99), must be **color-preserving**. But then the two reflection axes **M_1** and **M_2**, which **also** produce the twofold rotation at C, must **both** be either color-preserving (**cmm**) or color-reversing (**cm′m′**). (One may also appeal directly to the **Conjugacy Principle**: $\mathbf{M_1}$ and $\mathbf{M_2}$ must have the same effect on color because they are rotated to each other by a 90^0 rotation at D!)

6.11.2 <u>Precisely four types.</u> We are already familiar with the **p4g** = **cmm** × **pgg** (but see 6.11.3 for a 'two-colored' version), and we had in fact produced a **p4′g′m** = **cmm** × **pg′g′** back in figure 6.10 (color-reversing 90^0 rotations, color-reversing 'diagonal' glide reflections (**pg′g′**), color-preserving 'vertical'-'horizontal' reflections and glide reflections (**cmm**)). One way to arrive at a **p4′gm′** = **cm′m′** × **pgg** is this: start with a **'p4′-unit'** like the one occupying the four central squares in figure 6.107, and then use color-reversing reflections to extend it to a full-fledged pattern:

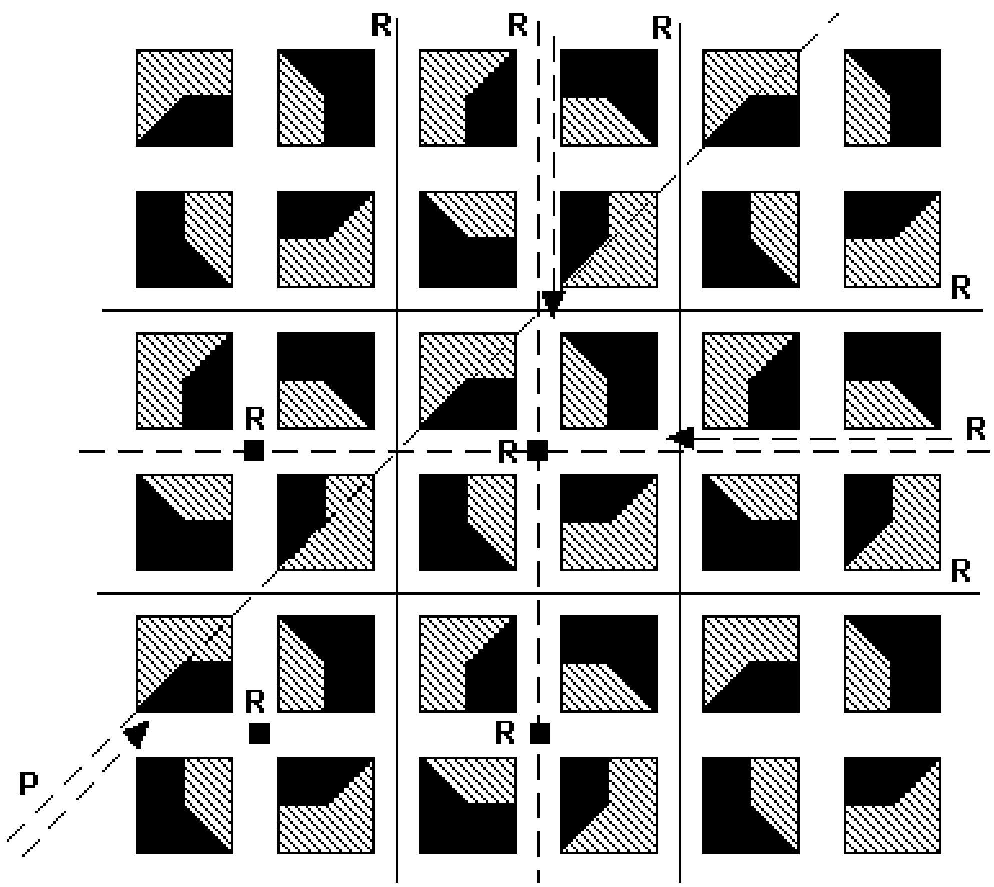
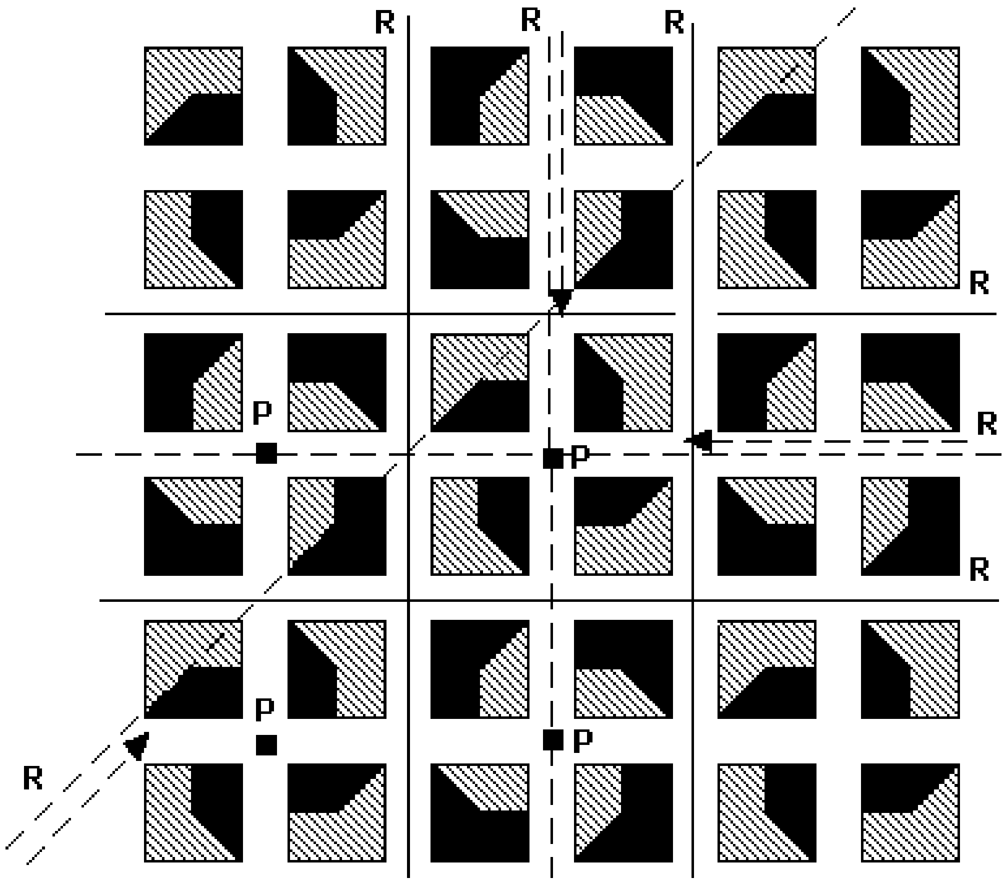


Fig. 6.107

A variation on this approach, starting now with a **‘p4-unit’**, yields the fourth type, **p4g′m′** = **cm′m′** × **pg′g′**:



p4g′m ′

Fig. 6.108

Of course this approach would lead to the other two types if we used **color-preserving** mirrors around our starting unit. Notice also that, while there **seem** to be two kinds of fourfold centers in figures 6.107 & 6.108, their effect on color is the same in each case: for the reflection that maps them to each other makes them to have the same effect on color (**Conjugacy Principle**), even though it makes them look different (**heterostrophic**) at the same time.

6.11.3 Examples. Another way of getting **p4g**-like two-colored patterns is to start with a ‘**p4**-unit’ or a ‘**p4′**-unit’ and then extend it to a full pattern using **color-preserving** (by necessity) vertical-horizontal **translations** instead of reflections:

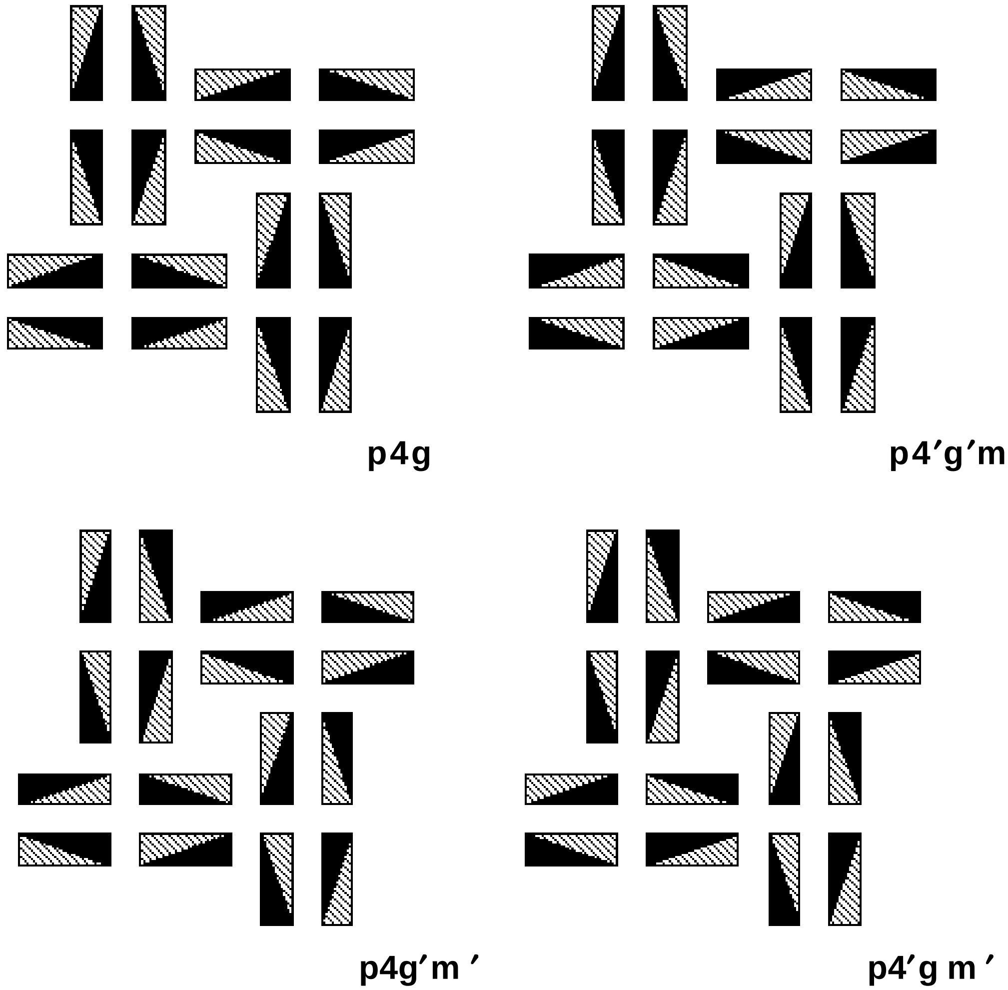


Fig. 6.109

6.11.4 Symmetry plans. Notice that a **p4g**-like pattern may be classified using **only** the underlying ‘vertical’-‘horizontal’ **cmm** (and, more specifically, its reflections) together with the effect on color of the fourfold centers (all of which are of **one kind** and therefore represented by the same type of square dot): this remark has some practical significance, as it is often **difficult** to ‘see’ a **p4g**-like pattern’s ‘diagonal’ **pgg** glide reflection. Notice by the way that the **g** or **g′** in the ‘names’ listed below stands for the diagonal (**pgg**) glide reflection, **not** for the vertical-horizontal (**cmm**) glide reflection. And do not forget that “diagonal”, “vertical”, and “horizontal” have always a lot to do ... with the way we ‘hold’ the pattern in question!

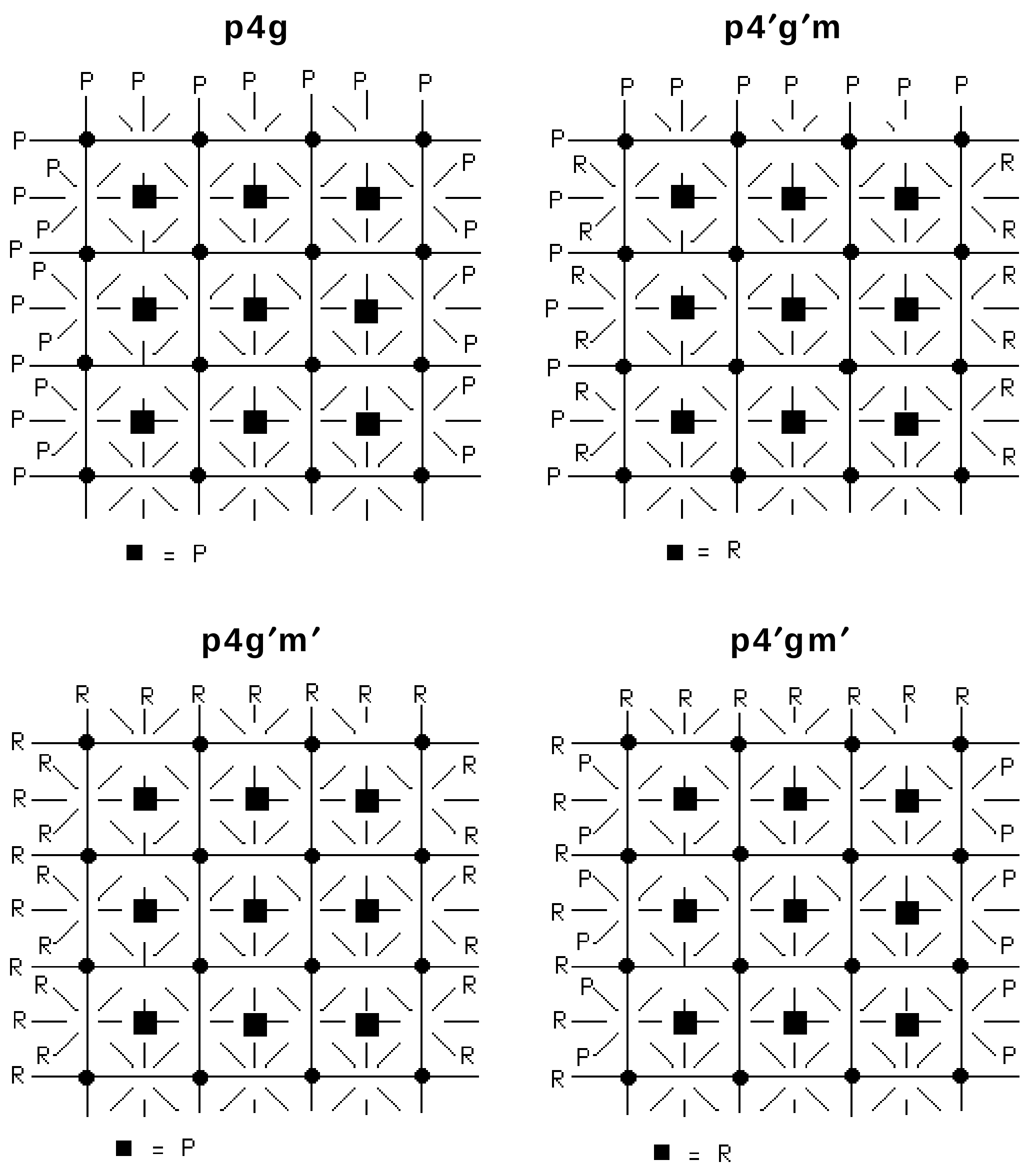


Fig. 6.110

6.12 p4m types (p4m, p4′mm′, p_c'4mm, p_c'4gm, p4′m′m, p4m′m′)

6.12.1 Studying the symmetry plan. Fortunately (a lot of fun) or unfortunately (a lot of work), we need to repeat the ‘break down’ process we applied to the **p4g** and its symmetry plan: that’s the only way to prove that there can only be six **p4m**-like two-colored patterns! So we start with a fresh look at the **p4m**’s symmetry plan:

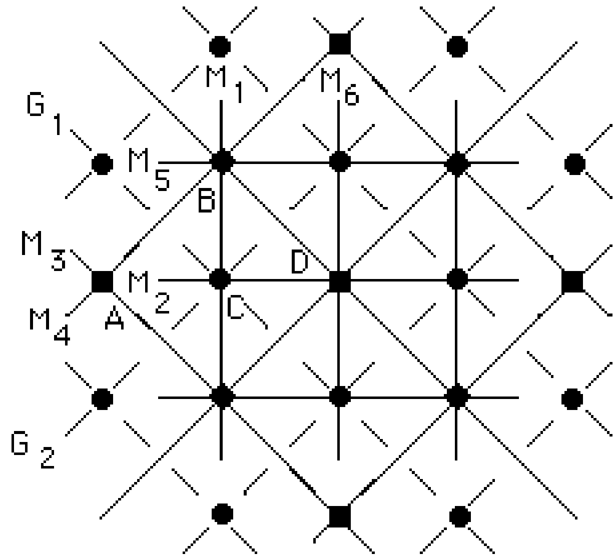


Fig. 6.111

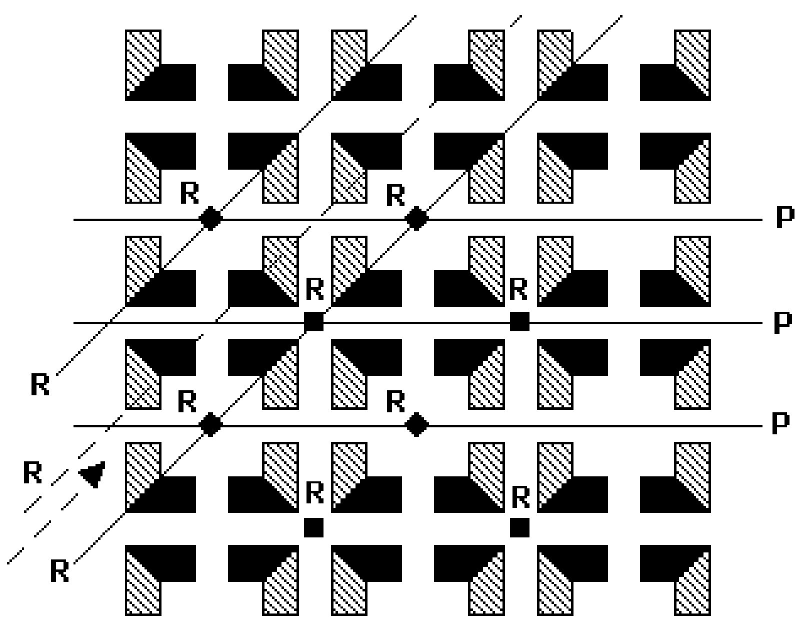
We see that the **p4m** may be viewed as a 'product' of a 'vertical'-'horizontal' **pmm** and a 'diagonal' **cmm**. Every two adjacent horizontal or vertical axes **may** have opposite effect on color, but this is not possible for either any two **parallel** diagonal reflection axes or any two **parallel** diagonal glide reflection axes (by the very structure of **cm**(**m**)-like patterns, see 6.4.4); moreover, every two **perpendicular** diagonal axes (such as $\mathbf{G_1}$, $\mathbf{G_2}$ or $\mathbf{M_3}$, $\mathbf{M_4}$, for example) must have the same effect on color by the **Conjugacy Principle**: indeed they are **rotated** to each other by **fourfold** centers (such as A). We conclude, by revisiting 6.9.6 if needed, that the **cmm** 'factor' could only (and **possibly**) be one of **cmm**, **cm′m′**, $\mathbf{p'_c}$**mm**, or $\mathbf{p'_c}$**gg**. Moreover, every horizontal and every vertical reflection axis intersecting each other at a **fourfold** center (**twofold** center within the **pmm**), such as $\mathbf{M_1}$, $\mathbf{M_5}$ at B or $\mathbf{M_2}$, $\mathbf{M_6}$ at D, must have the same effect on color (**Conjugacy Principle** again). Therefore the **pmm** 'factor' could only (and **possibly**) be one of **pmm**, **pm′m′**, or **c′mm** (6.8.4).

Let's now have a closer look at how the two types 'merge' into the **p4m**. The 'genuine' **twofold** center C is produced by the combination of a glide reflection and a reflection perpendicular to it within the **cmm** 'factor' ($\mathbf{G_1}$, $\mathbf{M_4}$ or $\mathbf{G_2}$, $\mathbf{M_3}$), **as well as** by the combination of two perpendicular reflections within the **pmm** ‘factor’ $(\mathbf{M_1}, \mathbf{M_2})$. The implication of this **‘weaving’** is that the **two perpendicular pairs** of diagonal (**cmm**) and vertical-horizontal (**pmm**) axes producing the same genuine twofold center **must** be of **same combined effect** on color: in the context of figure 6.111, $(\mathbf{M_4}, \mathbf{G_1})$ and $(\mathbf{M_1}, \mathbf{M_2})$ could **possibly** be nothing but **PP**/**PP**, **PP**/**RR**, **RR**/**PP**, **RR**/**RR** (if C preserves colors) or **PR**/**PR**, **RP**/**PR** (if C reverses colors). (Recall (6.9.1, 6.8.1) that the letter order in the latter case **is** crucial for **cmm** types (reflection, glide reflection) but **not** for **pmm** types (reflection, reflection).)

Converting our findings into ‘type multiplication’, we arrive at **six** possible combinations:

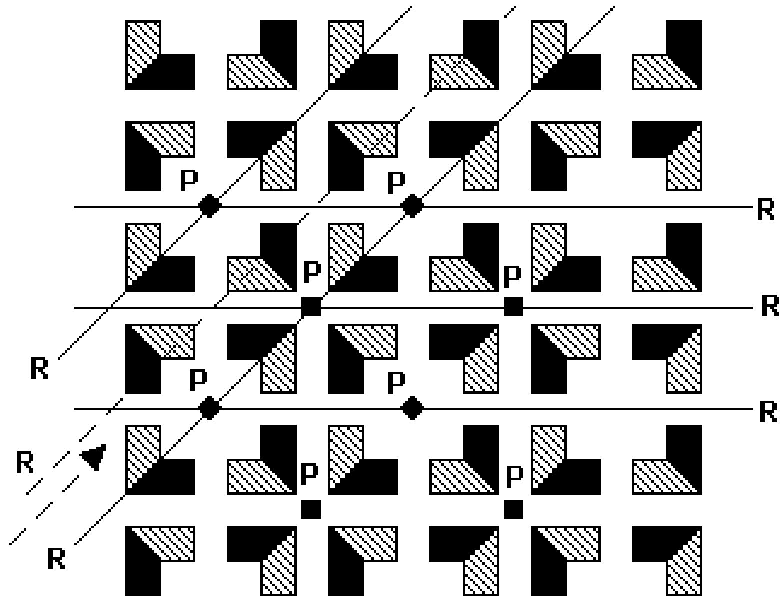
p4m = **cmm** × **pmm**, **p4′m′m** = **cmm** × **pm′m′**,
p4′mm′ = **cm′m′** × **pmm**, **p4m′m′** = **cm′m′** × **pm′m′**,
$\mathbf{p'_c4mm}$ = $\mathbf{p'_cmm}$ × **c′mm**, $\mathbf{p'_c4gm}$ = $\mathbf{p'_cgg}$ × **c′mm**

6.12.2 <u>Six types indeed.</u> A good source of patterns verifying the five ‘new’ types above is chapter 5: employing the stacking process of chapter 4, we can often stack copies of a two-colored **pmm2**-like **border** pattern into a **p4m**-like two-colored wallpaper pattern:



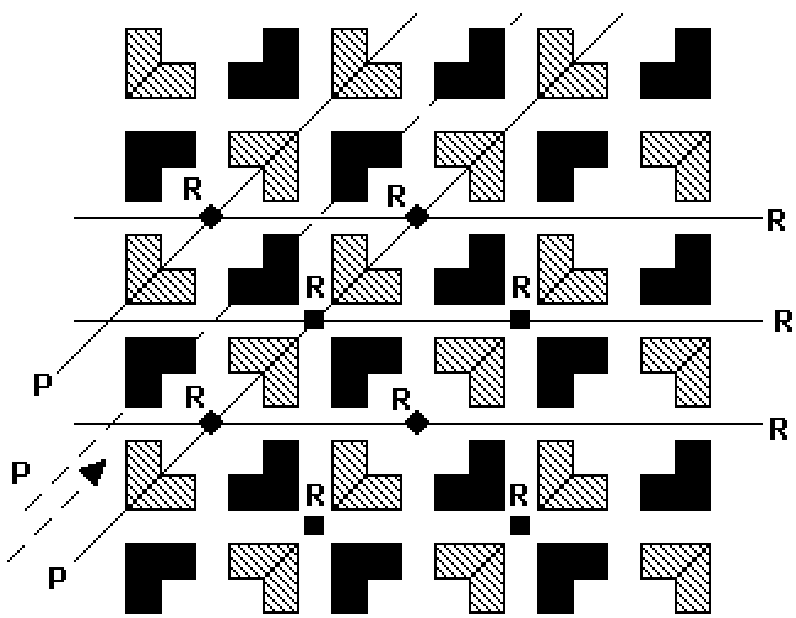
pmm2 → **p4′m m′**

Fig. 6.112



pm′m′2 → p4m′m′

Fig. 6.113

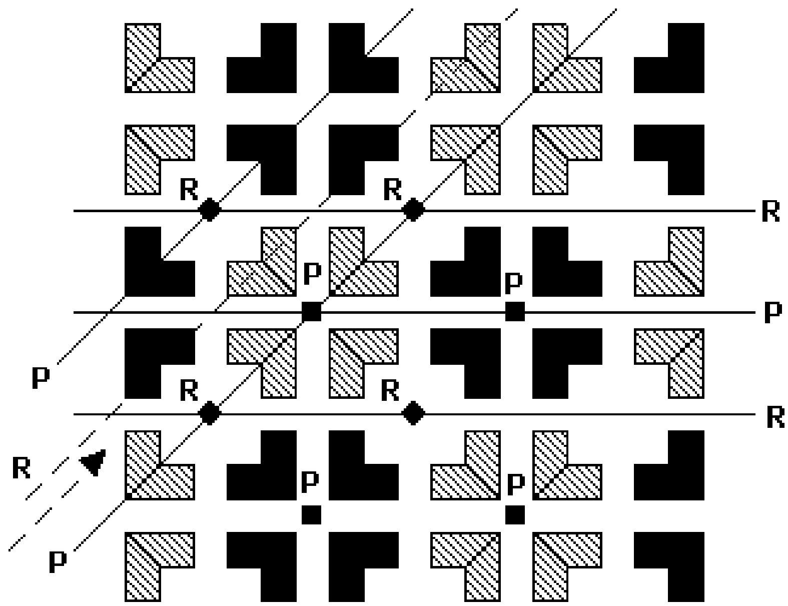


pm′m′2 → p4′m′m

Fig. 6.114

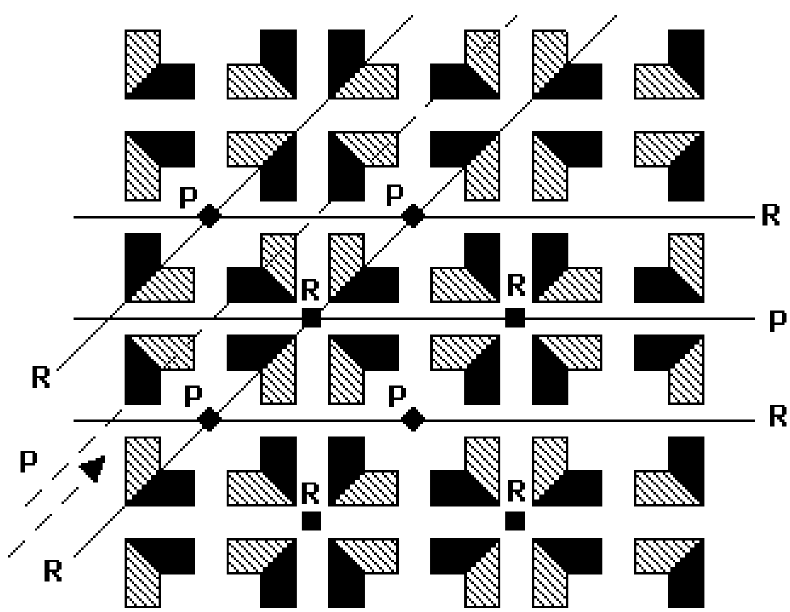
By now you can probably tell that the process is somewhat ‘unpredictable’: an **‘one-colored’** type (**pmm2**, figure 5.31) led to a genuinely two-colored type (**p4′mm′**, figure 6.112), while distinct representatives of the same type (**pm′m′2**), one of them also from figure 5.31, led to two distinct **p4m** types (figures 6.113 & 6.114).

We move on to get the remaining two **p4m** types; the last one (figure 6.116) ‘requires’ a **perfectly shifted** stacking of yet another border pattern -- distinct from the one employed in figure 6.115 despite being of the same type (**p′mm2**) -- from figure 5.31:



p′mm2 → p_c'4mm

Fig. 6.115



p′mm2 → p_c'4gm

Fig. 6.116

6.12.3 <u>Further examples.</u> Our ‘triangles’ have now been superficially merged into ‘squares’; notice the ‘two-colored’ **p4m**:

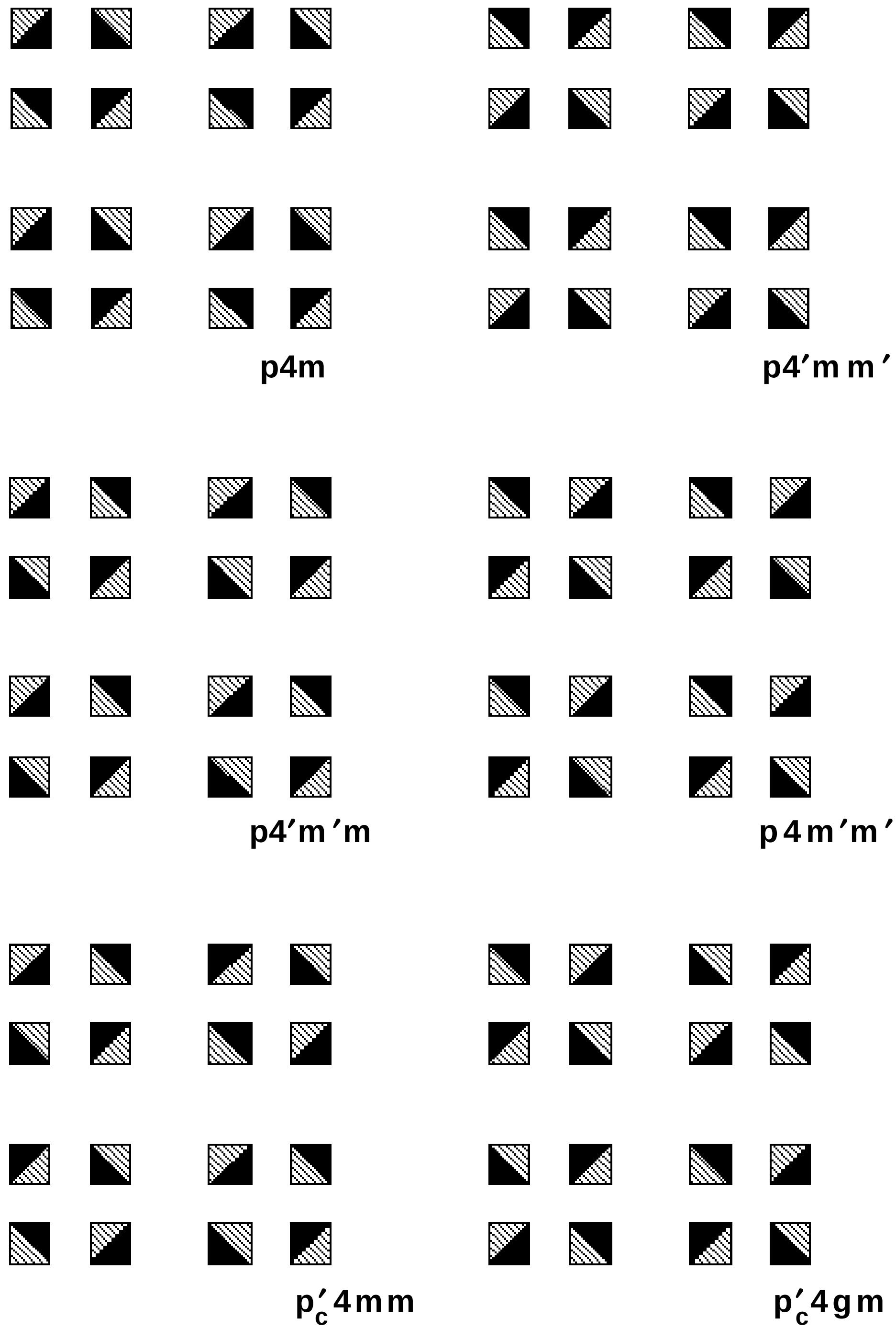
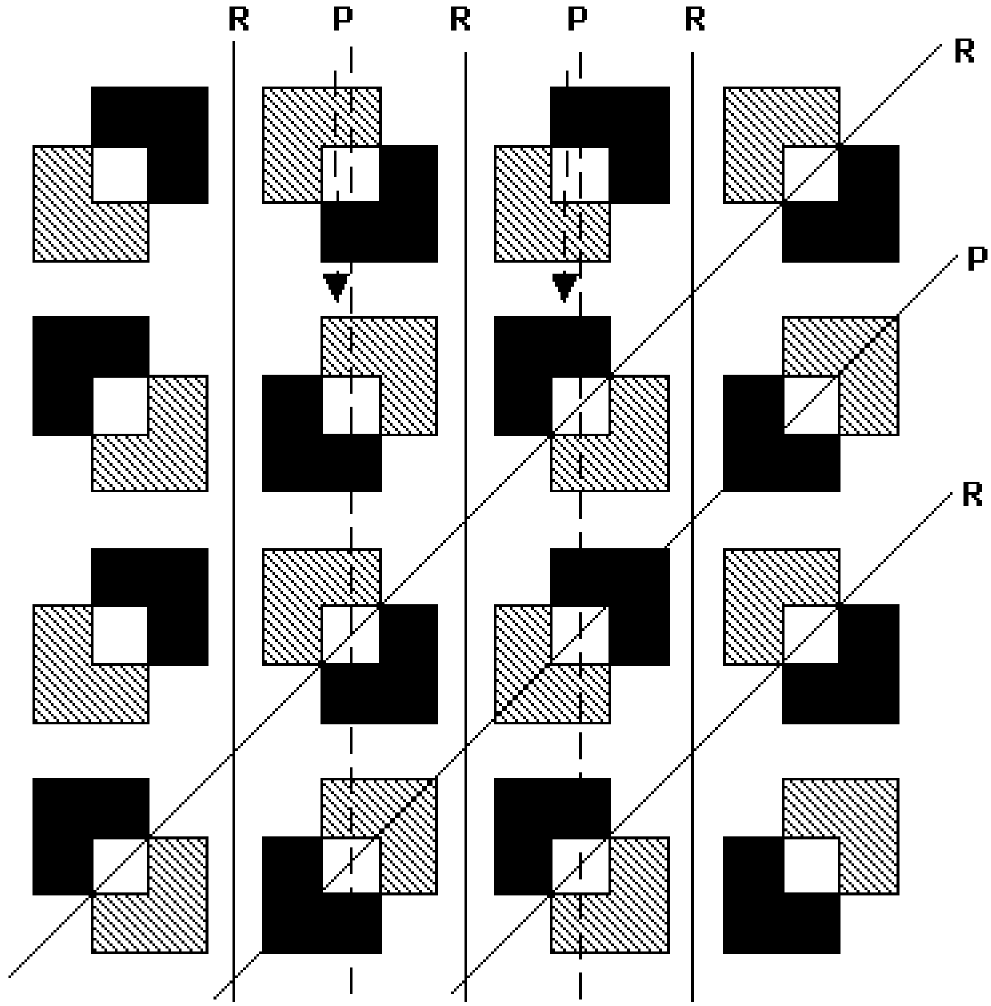


Fig. 6.117

6.12.4 <u>Hidden glide reflections revisited.</u> While color-reversing translations along reflection axes no longer matter (6.7.1), hidden glide reflections are now upgraded thanks to the second of the next two **p′$_c$4gm** examples provided by **Amber Sheldon** (Spring 1998):



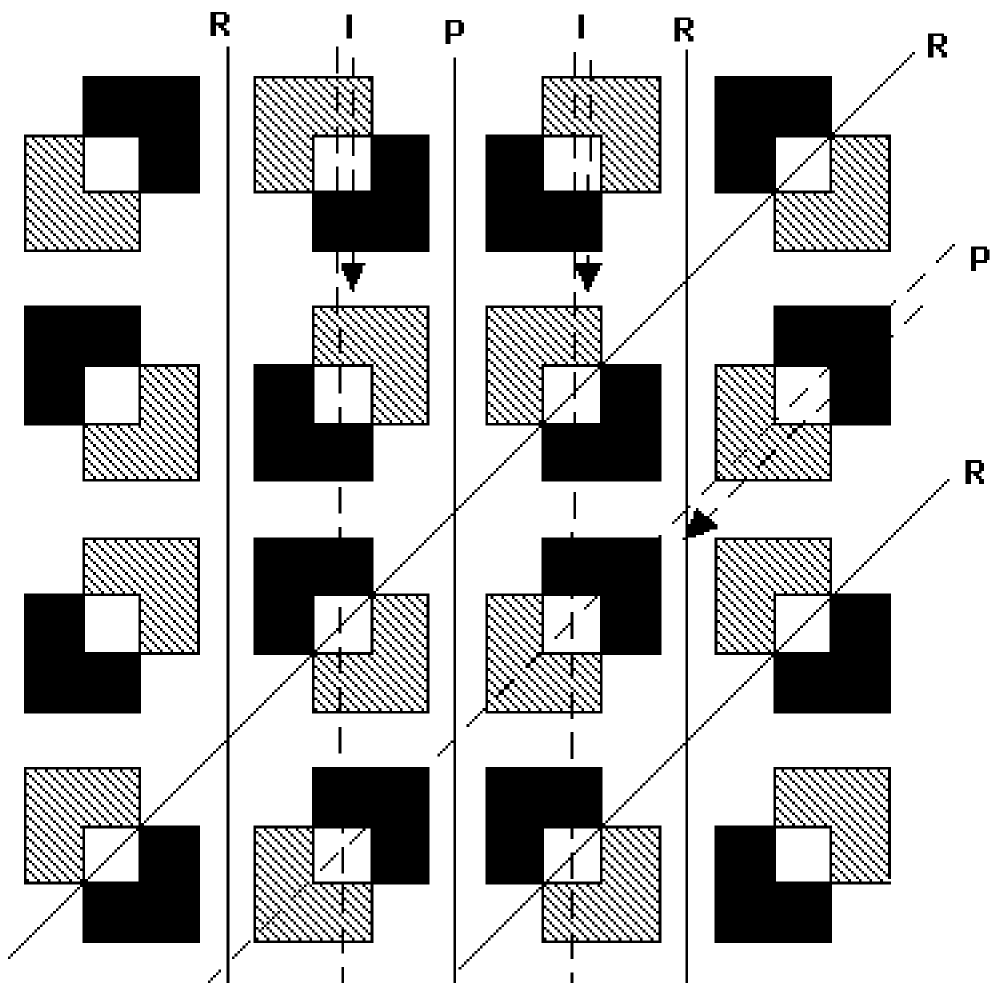
p′$_c$4gm

Fig. 6.118

This example is useful because it tells you once again that some patterns must be viewed **'diagonally'**: that is, the **cmm** direction (of in-between glide reflection) is vertical-horizontal rather than diagonal (as it has so far been the case with all our examples).

The next example is a clever variation on the previous one: **all** vertical-horizontal glide reflections are now **inconsistent** with color; and so is **every other** diagonal reflection, **but** the axes of **all** those diagonal reflections that are inconsistent with color **do** work for diagonal **glide reflections** (of vector shown below) that **are**

consistent with color! As a consequence of all this, we are now **back** to vertical-horizontal viewing and diagonal **cmm** subpattern (built, remarkably, on underlying structure of **pmm** type):

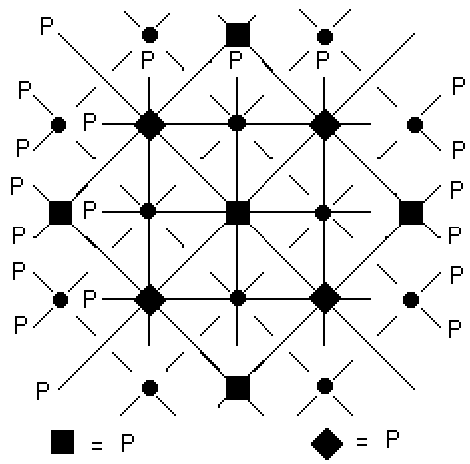


$\mathbf{p'_c4gm}$

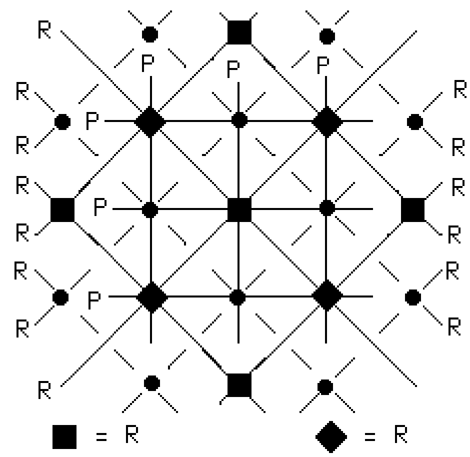
Fig. 6.119

6.12.5 <u>Symmetry plans.</u> The two $\mathbf{p'_c4gm}$ patterns of 6.12.4 are fully indicative of the **pitfalls** associated with the classification of **p4m**-like patterns. We suggest the following way of 'reading' the symmetry plans listed below, particularly helpful in distinguishing between **p4′m′m** and **p4′mm′**: **first** decide what the **cmm** direction (of **in-between glide reflection**) is and determine what the **cmm** type is, **then** work on the **pmm** 'factor', and **finally** 'merge' the two factors following the **p4m**'s 'factorizations' at the end of 6.12.1.

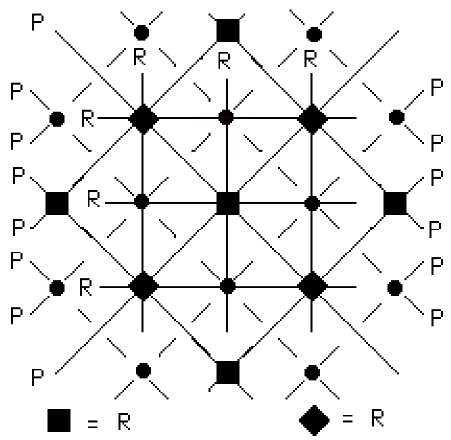
p4m



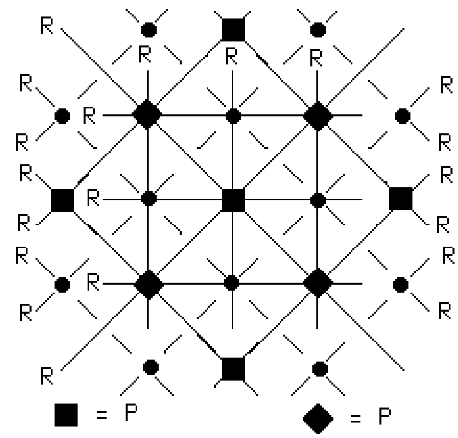
p4′mm′



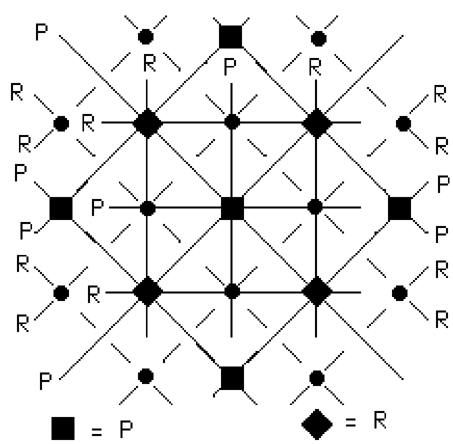
p4′m′m



p4m′m′



p'_c4mm



p'_c4gm

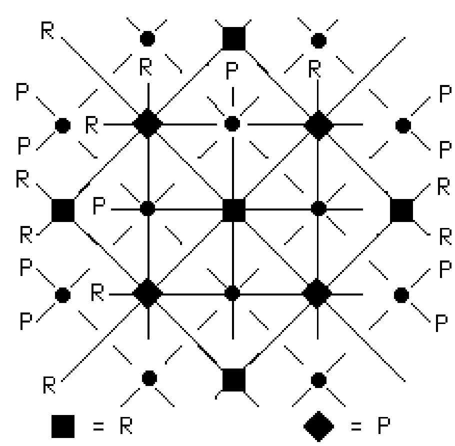


Fig. 6.120

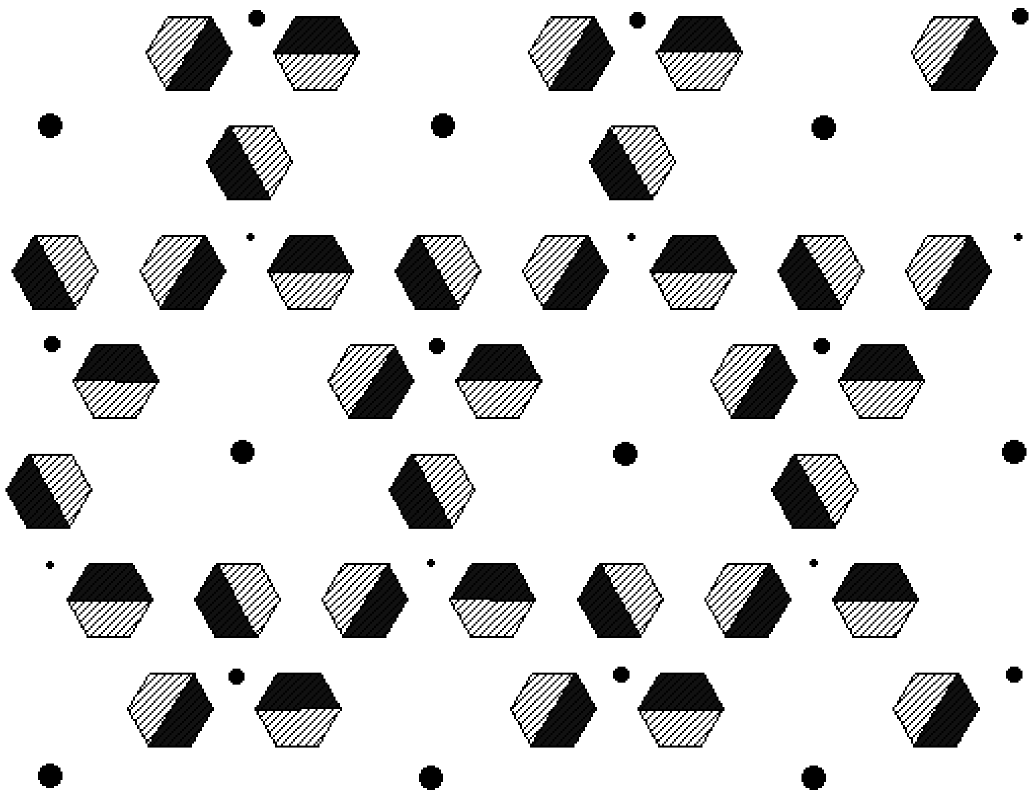
Predictably, fourfold centers **preserve** colors precisely when they lie at the intersection of **four** reflection axes of **same** effect on color. Of course it is only in the last two types that different kinds of fourfold centers (**not** mapped to each other by any of the pattern's isometries) have **opposite** effect on color. (Again we have not marked the effect on color of the twofold centers, which are not essential for classification purposes; and that effect is in any case easily determined (within the **pmm** 'factor'), as twofold centers lie at the intersection of **two** reflection axes **only**.)

6.13 p3 types (p3)

6.13.1 <u>No threefold color-reversing rotations.</u> The assumption that there exists a 120^0 color-reversing rotation leads to an immediate **contradiction**: starting with a black point A, its image must be grey, then the image of the image must be black, and finally the third image, which is no other than the departing point A, must be grey! More generally, the same argument shows that no 'oddfold' rotation (in a **finite** pattern) can be color-reversing.

6.13.2 <u>Farewell to color-reversing translations.</u> As we show in section 7.6, and have indicated in 6.10.1 for the special case of 90^0, the combination of a rotation and a translation leads to a rotation of **same angle** but different center. It follows from 6.13.1 and the $\mathbf{P} \times \mathbf{R} = \mathbf{R}$ rule that **no** wallpaper pattern with 120^0 rotation (such as the **p3** or actually **every** type we are going to study from here on) can have **color-reversing** translation.

6.13.3 <u>Only one type</u>. In the absence of (glide) reflection, 6.13.1 and 6.13.2 imply that the **only** possible type in the **p3** group is the **p3** itself. Below we offer an example of a 'two-colored' **p3** pattern. Notice the **three** different kinds of 120^0 rotation centers (denoted by dots of different sizes): no two centers of different kind are mapped to each other by either a rotation or a translation. Notice also the **rhombuses** formed by rotation centers of the same kind: their importance will be made clear in later sections and chapters!



p3

Fig. 6.121

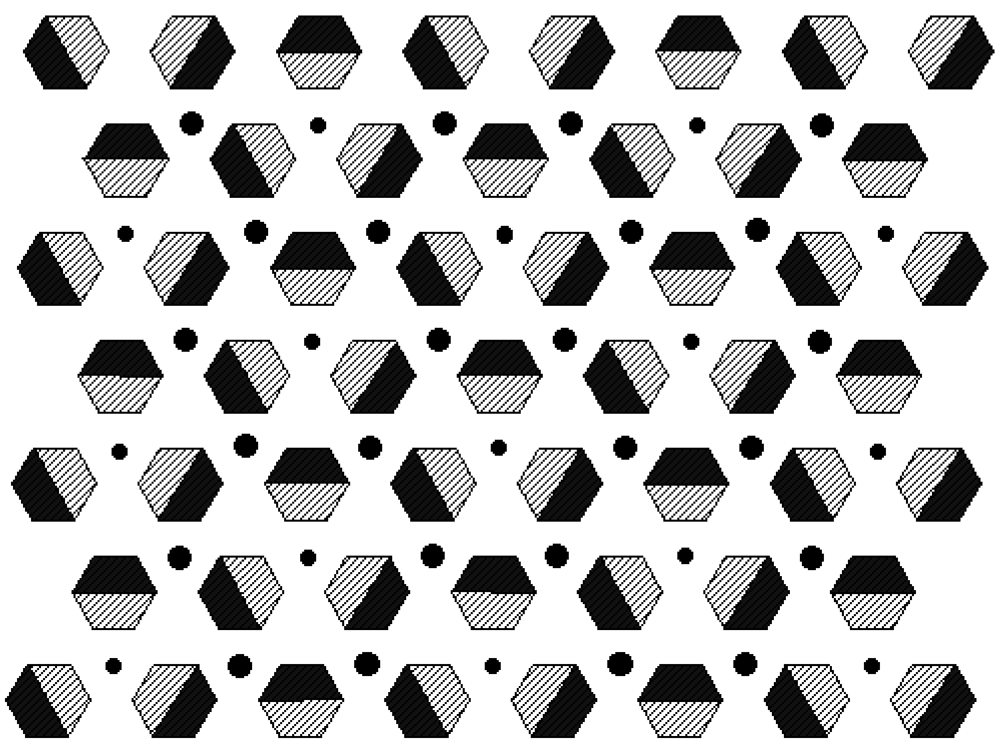
The pattern in figure 6.121 may not be the simplest two-colored **p3** in the world, but it should be compared to the **p31m** pattern in figure 6.122 below in order to illustrate a basic symmetry principle: **less** symmetry is often **harder** to achieve than **more** symmetry!

6.14 p31m types (p31m, p31m′)

6.14.1 <u>'Products' of three **cm**s.</u> The **p31m** has reflection and in-between glide reflection in three directions, hence it may be viewed as the 'product' of three **cm**s. Therefore all reflections within each one of the three directions must have the same effect on color and likewise all glide reflections within each of the three directions must have the same effect on color (6.4.4). **Moreover**, every two axes (be them glide reflection axes **or** reflection axes) of **non-parallel** direction must have the same effect on color: indeed any two such axes (intersecting each other at **60^0**) produce a **120^0 rotation** (see sections 7.2, 7.9, and 7.10); but this rotation **must** leave the pattern invariant, therefore it **must** be color-preserving (6.13.1), so that the two axes must have the **same** effect on color.

What all this means is that, in every **p31m**-like wallpaper pattern, **all axes** -- be them reflection or glide reflection axes -- must have the same effect on color. That is, either all axes **preserve** colors (**cm** × **cm** × **cm** = **p31m**) or all axes **reverse** colors (**cm′** × **cm′** × **cm′** = **p31m′**): no other possibilities!

6.14.2 Examples. First a ‘two-colored’ **p31m**:

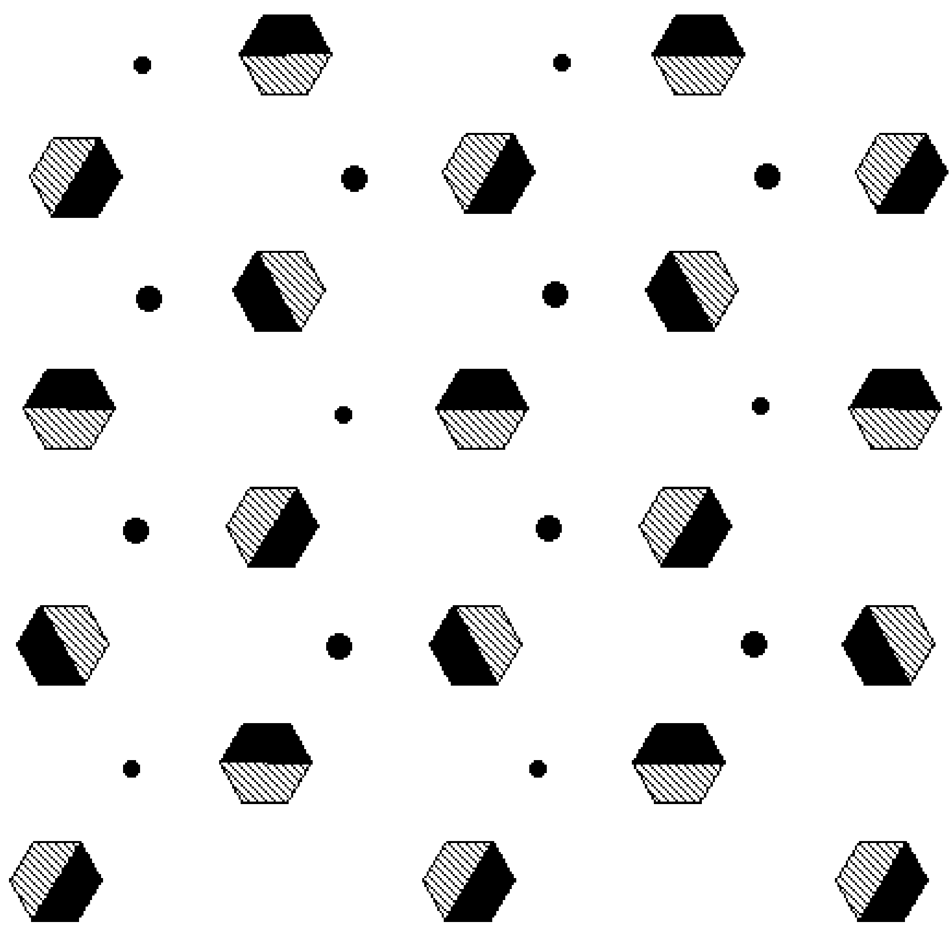


p31m

Fig. 6.122

We have marked the **two** different kinds of rotation centers by dots of different sizes. If you are at a loss trying to determine the **color-preserving** reflection axes, simply connect the **smaller** dots! As for glide reflection axes (**not** crucial for classification purposes), those pass not only half way between every two adjacent rows of **on-axis** centers (smaller dots), but also half way between every two adjacent rows of **off-axis** centers (larger dots).

Next comes an example of a **p31m′**:

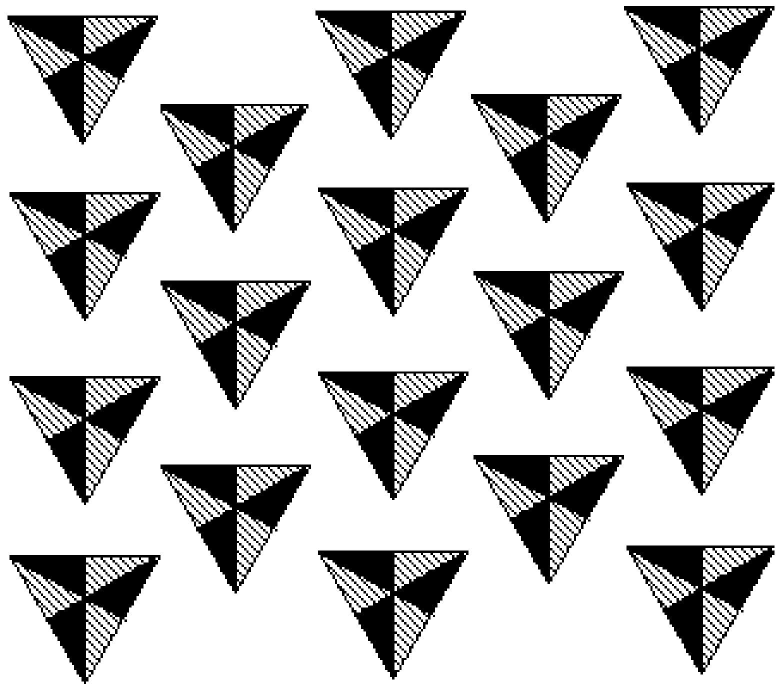


p31m′

Fig. 6.123

As in the case of figure 6.122, **color-reversing** reflection axes lie on lines formed by the smaller dots. And once again each on-axis center (smaller dot) is 'surrounded' by **six** off-axis centers (larger dots) symmetrically placed on the vertices of an invisible **hexagon**.

Finally, an example of a **p31m′**, 'offspring' of figure 4.69:



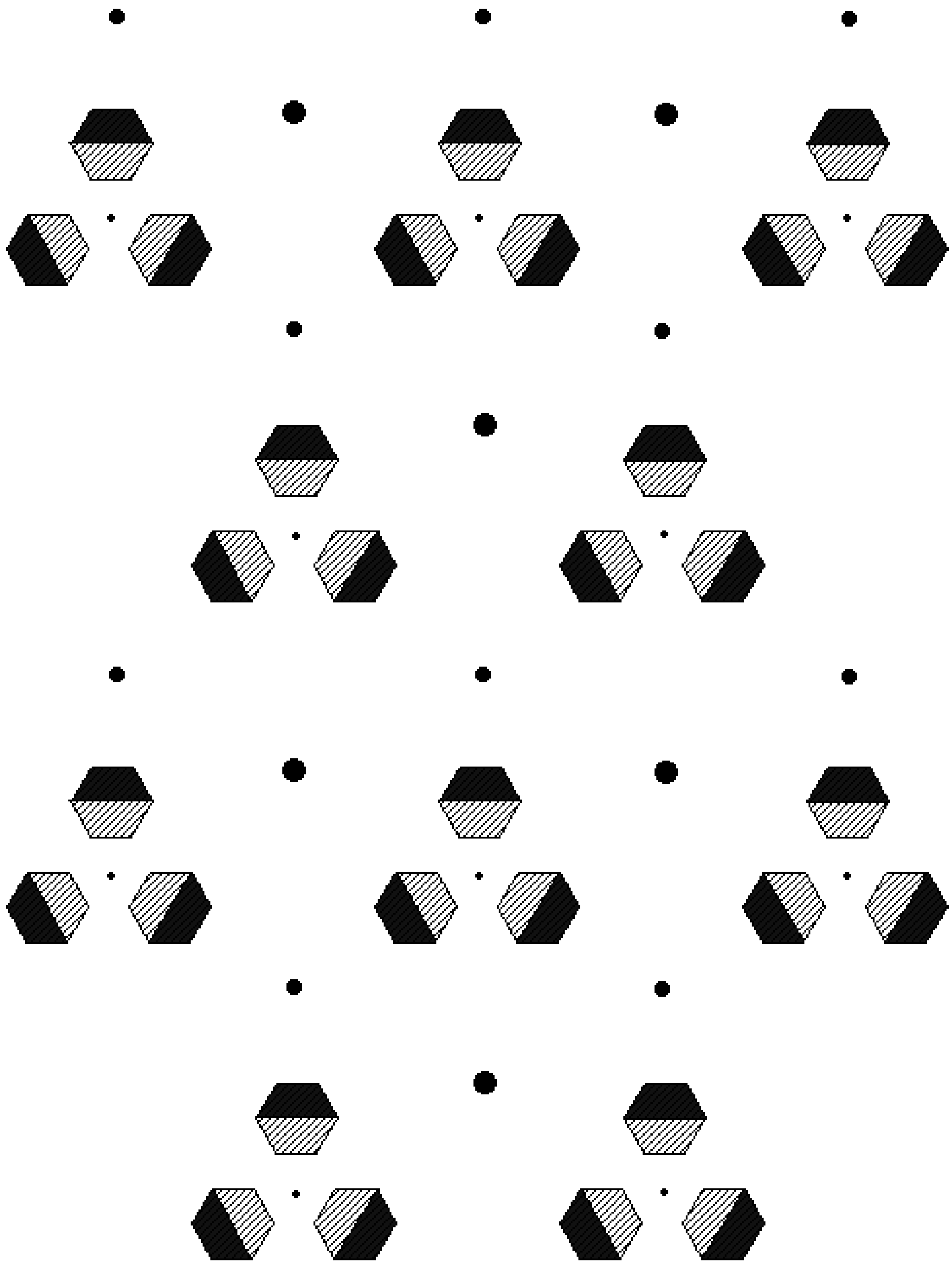
p31m′

Fig. 6.124

6.15 p3m1 types (p3m1, p3m′)

6.15.1 Three **cm**s again. As we indicated in 4.17.4, and will analyse further in chapters 7 and 8, the difference between the **p31m** and **p3m1** types is rather **subtle**, having in fact more to do with the glide reflection vector's length and the distances between glide reflection axes and rotation centers than with 'symmetry plan' structure (and the off-axis centers of the **p31m** specifically). It is clear in particular that **both** the **p3m1** and the **p31m** are 'products' of three **cm** patterns, and the entire **p31m** analysis of 6.14.1 is also applicable to the **p3m1** word by word. We conclude again that, depending on the (glide) reflections' **uniform** effect on color, there can only be two **p3m1**-like patterns, **p3m1** = **cm** × **cm** × **cm** (all (glide) reflections **preserve** colors) and **p3m′** = **cm′** × **cm′** × **cm′** (all (glide) reflections **reverse** colors).

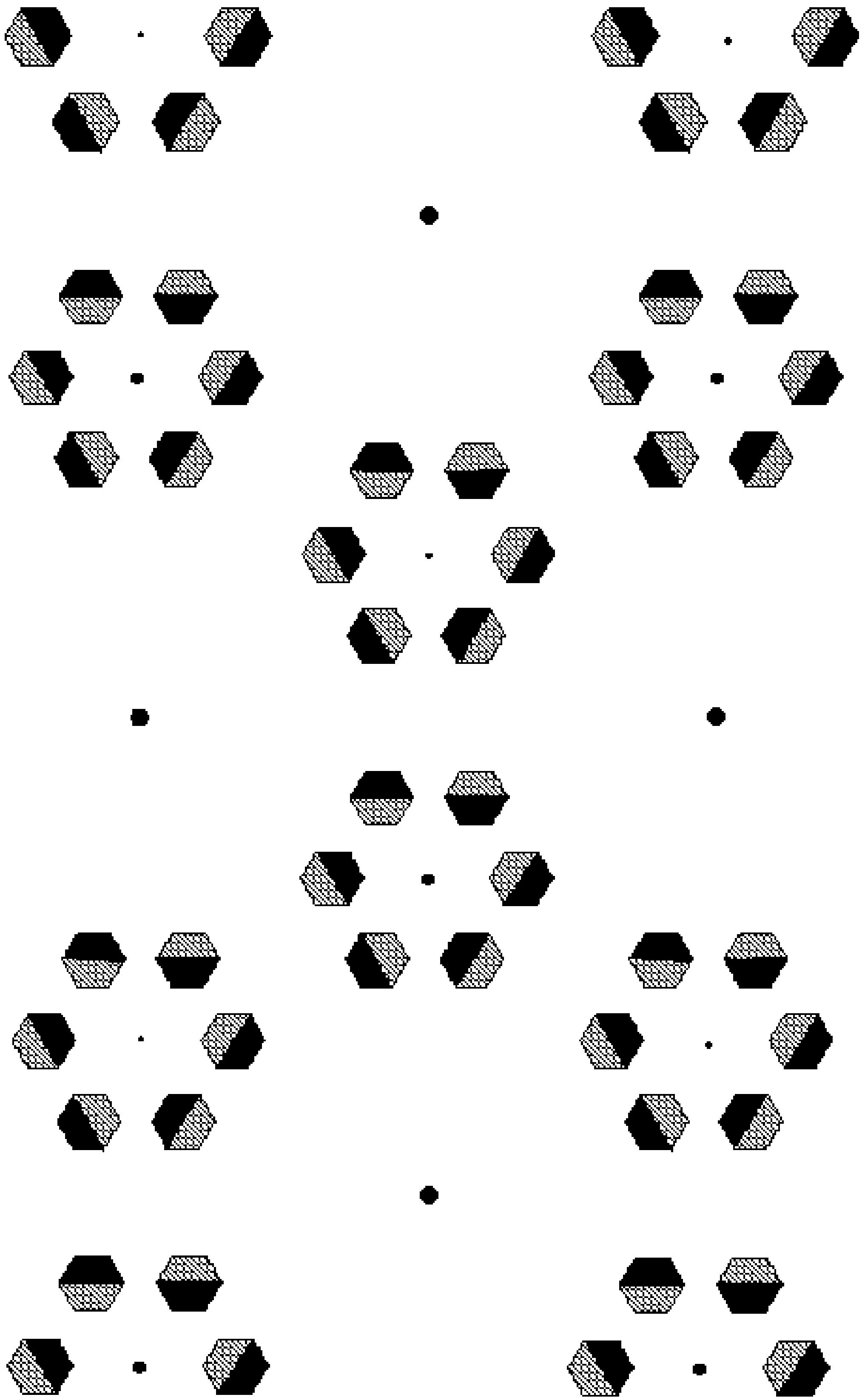
6.15.2 Examples. We begin with a 'two-colored' **p3m1**:



p3m1

Fig. 6.125

Next, a rather ‘exotic’ **p3m′** that probably celebrates the sacred concept of hexagon more than any other figure in this book:

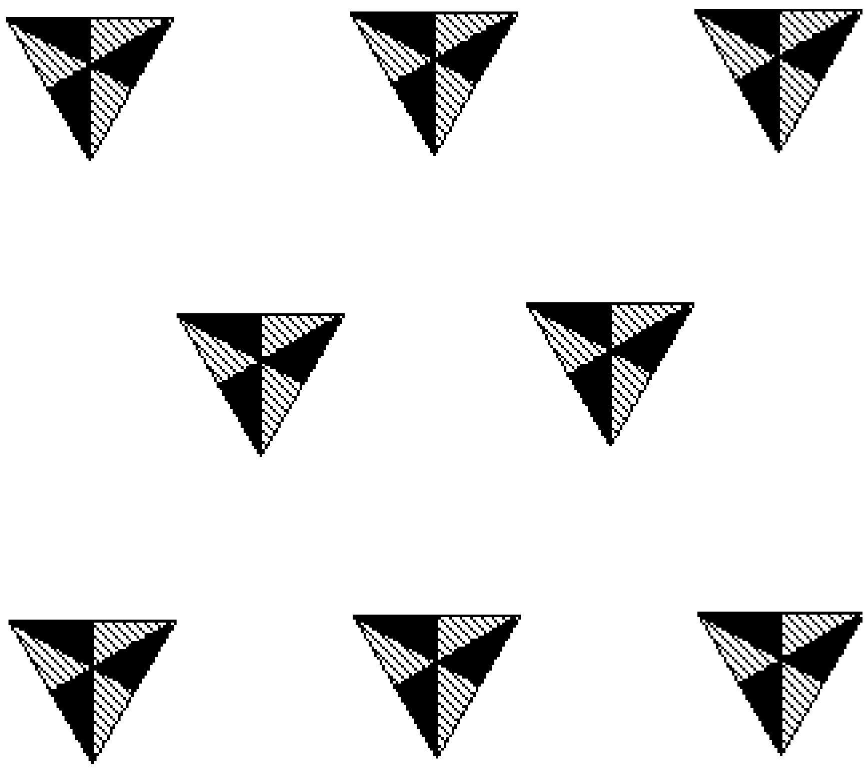


p3m′

Fig. 6.126

We are back to **three** kinds of rotation centers (**p3 structure**). (This is not obvious for the **p3m′** in figure 6.126, which at first glance **seems** to have only two kinds of threefold centers: do you see why the 'hexagon middle' centers are of two kinds?) As in figures 4.71 & 4.73, reflection axes are defined by any three **collinear** centers of **different** kind.

Finally, let's **'dilute'** (4.17.1) the **p31m′** pattern in figure 6.124 in order to get a 'triangular' **p3m′**:



p3m′

Fig. 6.127

Observe here that rotating **all** triangles above about their center by **any** angle other than 60^0 or 120^0 or 180^0 turns the **p3m′** into a **p3**; exactly the same observation holds for the triangular **p31m′** of figure 6.124 (and in fact for all **p3m1**-like or **p31m**-like patterns)!

6.15.3 How about symmetry plans? You have probably noticed by now that we have not provided symmetry plans for **p3**, **p31m**, and

p3m1 types. They are not that crucial, because the classification is very easy within each type (two cases at most). Anyway, you will find symmetry plans for all sixty three two-colored types at the end of the chapter (section 6.18); before that, look also for the **p31m** and the **p3m1** 'symmetry plans' (inside the **p6m**) in section 6.17!

6.16 p6 types (p6, p6′)

6.16.1 The lattice of rotation centers. Let us first explain the arrangement of rotation centers (twofold, threefold, and sixfold) in **60^0** wallpaper patterns (both **p6**-like and **p6m**-like), shown already in figure 4.5 (**'beehive'**), by way of the following demonstration:

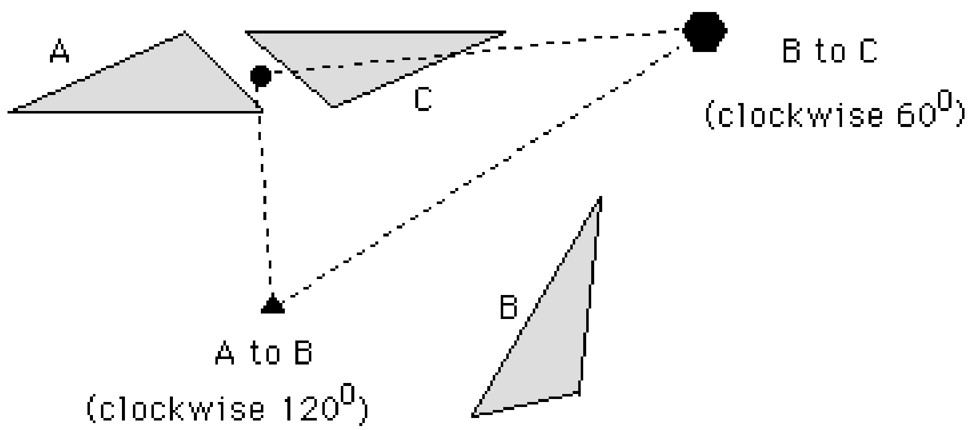


Fig. 6.128

We see that a **clockwise 120^0** rotation (mapping A to B) followed by a **clockwise 60^0** rotation (mapping B to C) results into a **180^0** rotation (mapping A to C). While a complete proof of this will be offered only in 7.5.4, figure 6.128 is rather convincing; especially in view of the fact that the three rotation centers are located at the three vertices of a **90^0-60^0-30^0** triangle, **exactly** as in figure 4.5!

You should use a demonstration similar to figure 6.128 in order to verify that the combination of two clockwise 60^0 rotations is a clockwise 120^0 rotation, that the combination of a 180^0 rotation and a clockwise 120^0 rotation is a counterclockwise 60^0 rotation, etc.

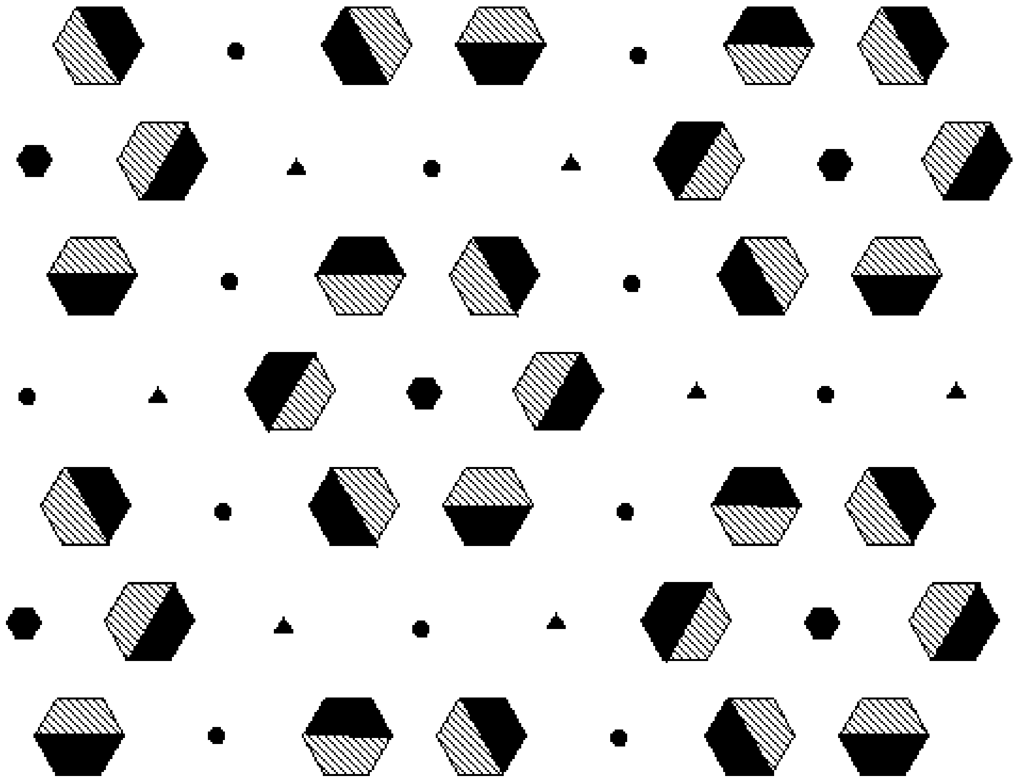
6.16.2 <u>One kind of sixfold rotations at a time!</u> Another fact valid for both **p6**-like and **p6m**-like types is that **no** 60^0 wallpaper pattern can possibly have **both** color-preserving and color-reversing sixfold centers. Indeed, should two 60^0 rotations of **opposite** effect on color coexist in a pattern, their combination would generate a **color-reversing** 120^0 rotation (see section 7.5 or proceed as in figure 6.99), which is impossible (6.13.1).

6.16.3 <u>Only two types.</u> In the **absence** of (glide) reflection and color-reversing translation (6.13.2), and in view of 6.16.2 above, we conclude at once that only two **p6**-like types are possible: one with **only color-preserving** 60^0 rotations (**p6**) and one with **only color-reversing** 60^0 rotations (**p6′**); no 'mixed' type like $\mathbf{p'_b2}$ (180^0) or $\mathbf{p'_c4}$ (90^0) is possible in the 60^0 case!

Of course all 120^0 rotations in either the **p6** or the **p6′** are color-preserving as usual. Then figure 6.128, together with the $\mathbf{P} \times \mathbf{P} = \mathbf{P}$ and $\mathbf{P} \times \mathbf{R} = \mathbf{R}$ rules, leads to an observation that may at times help you distinguish between **p6** and **p6′**: **all** 180^0 rotations in a **p6** pattern are color-preserving, and **all** 180^0 rotations in a **p6′** pattern are color-reversing. (Sometimes you may even **miss** the sixfold rotation and see only the twofold one, thus misclassifying a **p6′** or a **p6** as a **p2′** or a **p2**, respectively; more likely, you may only see the threefold rotation and misclassify a **p6** or a **p6′** as a **p3**!)

These remarks make it clear that the classification process within the **p6** type is rather simple, and the need for symmetry plans drastically reduced: therefore we follow the example set by the three previous sections, simply exiling the **p6** symmetry plans to the 'review' section 6.18.

6.16.4 <u>Examples.</u> First a 'two-colored' **p6** (with sixfold, threefold, and twofold centers represented by hexagons, triangles, and dots, respectively):

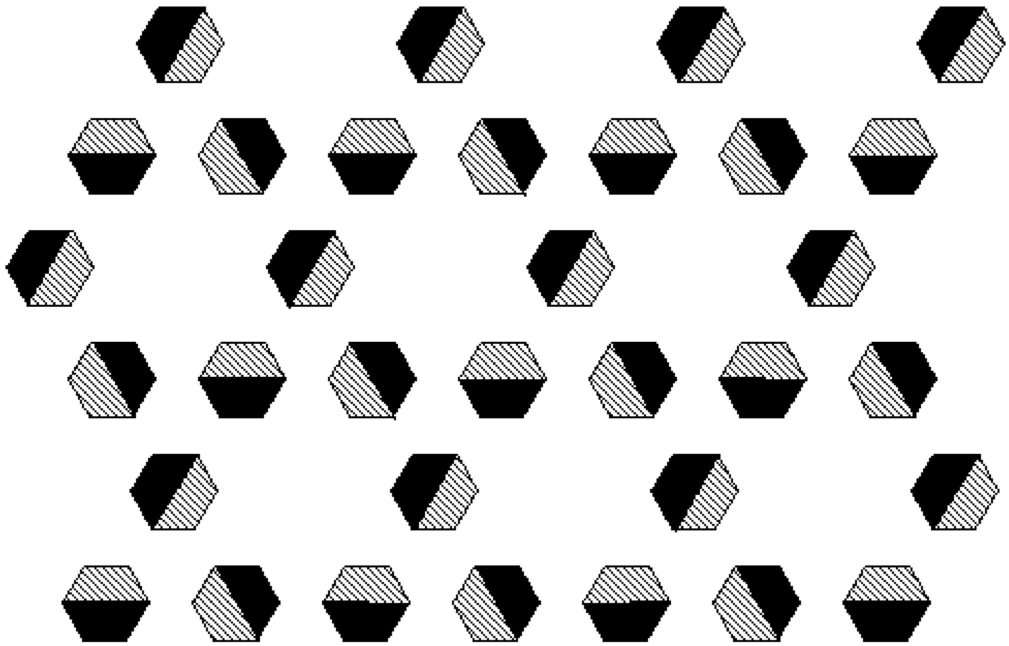


p6

Fig. 6.129

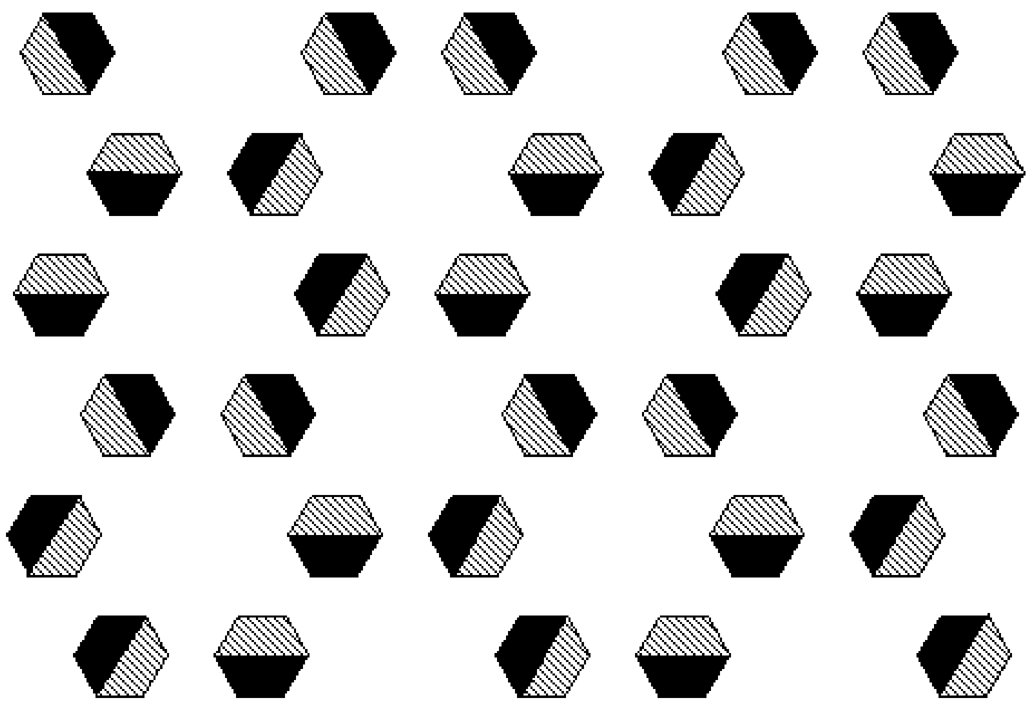
The 90^0-60^0-30^0 triangles of figure 6.128 are certainly ubiquitous. There are many other remarks one could make with regard to the positioning of the three kinds of rotation centers. One non-trivial remark is that every two rotation centers corresponding to equal rotation angles are **conjugate** in the sense of 6.4.4: at least one of the pattern's isometries (in fact a **rotation**) maps one to the other; in particular, this fact provides another explanation for the uniform effect on color within each kind of rotation. A related remark concerns the perfectly hexagonal arrangement of both twofold and threefold centers around every sixfold center. And so on.

We leave it to you to check that the **p6** pattern in figure 6.129 may be split into two identical **p6′** patterns the threefold centers of which are sixfold centers of the original **p6** pattern and vice versa! Further, here are two similar yet distinct **p6′** patterns (the second of which is sum of two copies of the **p31m′** pattern in figure 6.123):



p6′

Fig. 6.130



p6′

Fig. 6.131

6.17 p6m types (p6m, p6′mm′, p6′m′m, p6m′m′)

6.17.1 A symmetry plan in two parts. We begin with a very **visual** introduction to the most complex of wallpaper patterns:

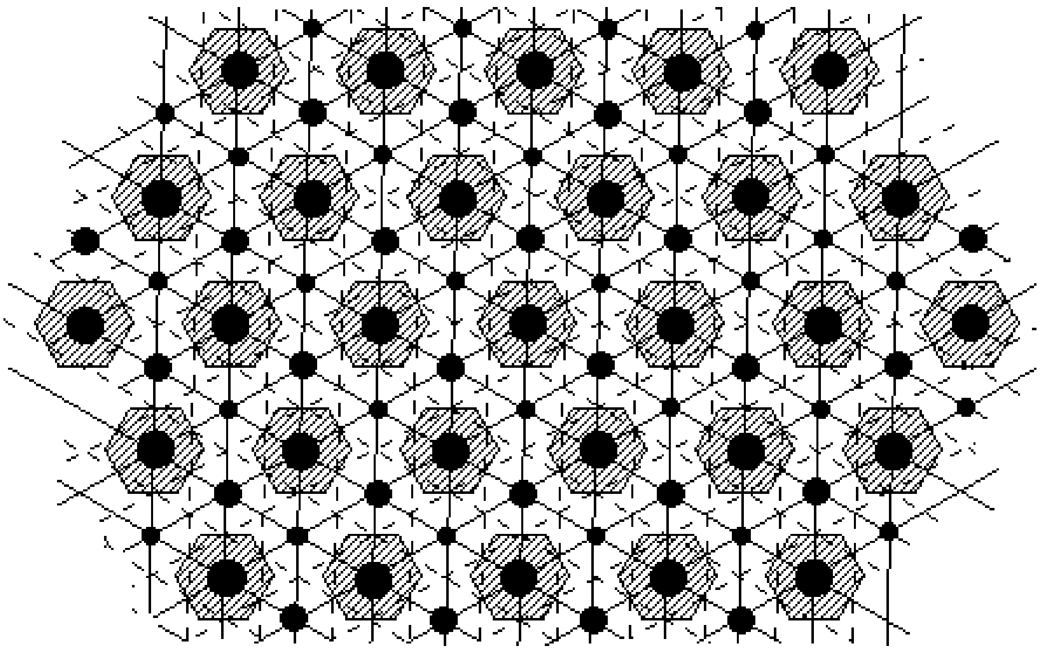


Fig. 6.132

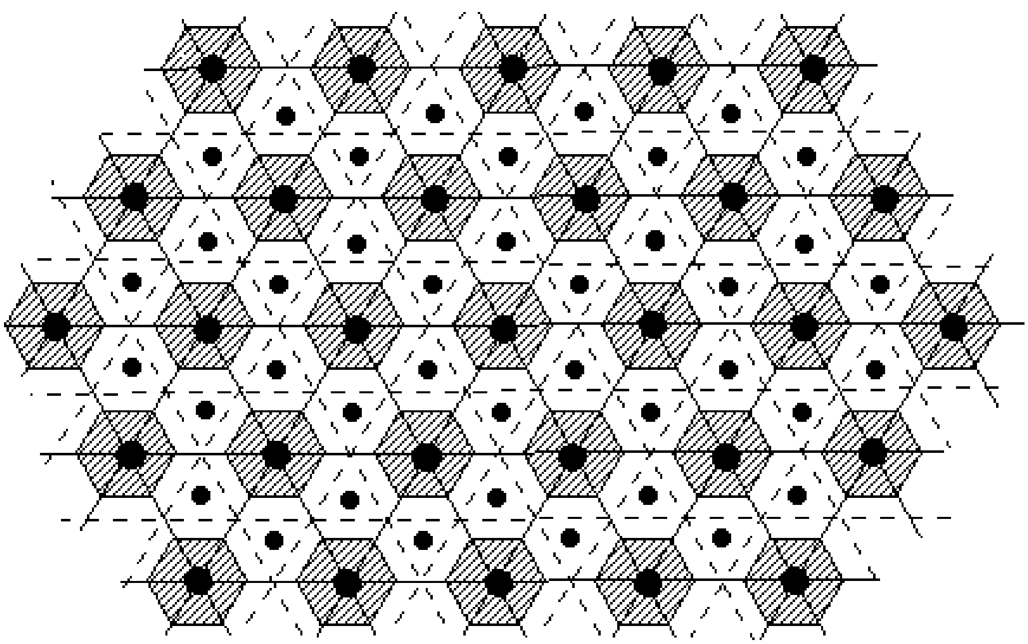


Fig. 6.133

Two pictures are worth two thousand words, one may say! All we did was to **'analyse'** a typical, hexagon-based **p6m** pattern into one **'p3m1'** pattern (with reflection axes passing through the **p6m**'s threefold centers and **hexagons' edges**, figure 6.132) and one **'p31m'** pattern (with reflection axes avoiding the **p6m**'s threefold centers and passing through the **hexagons' vertices**, figure 6.133). This is in fact the **'game'** we have been playing throughout most of this chapter, and some **'cheating'** was always involved! In the present case, for example, we do know that the 'off-axis' centers in figure 6.133 (**p31m** pattern) **do** in fact **lie** on the reflection axes of the **p3m1** pattern (figure 6.132); and that the 'distinct' on-axis centers of the **p3m1** pattern in figure 6.132 (small and medium sized dots) **are** in fact **mapped** to each other by the **p31m**'s reflection axes (figure 6.133). Moreover, the largest dots in both figures represent rotation centers for 60^0 rather than just 120^0, and so on. At the same time, the 'lower' patterns (**p3m1**, **p31m**) included in the 'higher' pattern (**p6m**) do **determine** its structure: for example, wherever two reflection axes (one from each 120^0 subpattern) cross each other at a **30^0** angle -- see figures 6.132 **and** 6.133 -- they do 'produce' a 60^0 rotation center (7.2.2). Conversely, the **p6m**'s properties are **inherited** by the **p3m1** and the **p31m** contained in it: for example, we may always 'use' a sixfold center as a threefold one; after all, a double application of a 60^0 rotation produces a 120^0 rotation (4.0.3). **In brief**, our **reduction** of the study of complex structures to that of simpler ones employed so far is **sound**, and we will appeal to it for one last time in 6.17.3.

6.17.2 <u>A complex structure indeed.</u> Figures 6.132 and 6.133 **together** make it clear that the **p6m**'s **sixfold** centers lie at the intersection of **six** reflection axes and that its **threefold** centers lie at the intersection of **three** reflection axes. Missing from both figures (and from 120^0 patterns!) are the **p6m**'s **twofold** centers, which nonetheless exist, located **half way** between every two adjacent hexagons; they are in fact located at the intersection of **one** reflection axis and **two** glide reflection axes **perpendicular** to hexagons' edges (figure 6.132) **and** at the intersection of **one** reflection axis and **two** glide reflection axes **parallel** to hexagons' edges (figure 6.133). We conclude that the **p6m**'s twofold centers lie at the intersection of **two** reflection axes and **four** glide reflection axes, still adhering to the **'p6 rule'** set by figure 6.128, and in full agreement with figure 4.5 as well. See also figure 8.42!

6.17.3 Only four types! Given the **p6m**'s complexity and what has happened in the case of other complex types (such as the **pmg** or the **p4m**), you would probably expect a long story here, too, right? Well, sometimes we get a break, rather predictable in this case: since the **p6m** is the 'product' of two **simple** (in terms of two-color possibilities) types, its study may not be that complicated after all. Indeed there can be **at most** two × two = four types, all of which **do** in fact exist (6.17.4): **p6m** = **p3m1** × **p31m** (both the **p3m1**'s and the **p31m**'s (glide) reflections preserve colors), **p6′mm′** = **p3m1** × **p31m′** (the **p3m1**'s (glide) reflection preserves colors and the **p31m**'s (glide) reflection reverses colors), **p6′m′m** = **p3m′** × **p31m** (the **p3m1**'s (glide) reflection reverses colors and the **p31m**'s (glide) reflection preserves colors), **p6m′m′** = **p3m′** × **p31m′** (both the **p3m1**'s and the **p31m**'s (glide) reflections reverse colors).

Our analysis so far is rather effective yet not terribly user-friendly. Taking advantage of the fact that one of the **p31m**'s (glide) reflection's directions is **'horizontal'** (figure 6.133) and that one of the **p3m1**'s (glide) reflection's directions is **'vertical'** (figure 6.132), we capture the preceding paragraph's findings in a simple diagram (and effective **substitute** for symmetry plan) as follows:

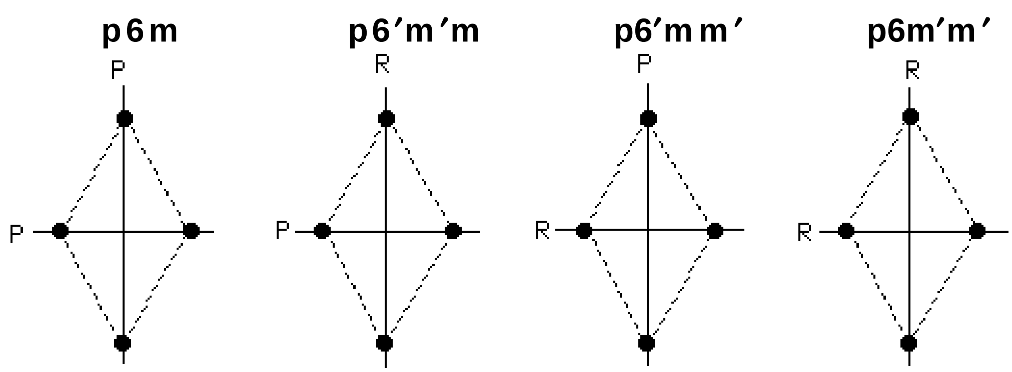
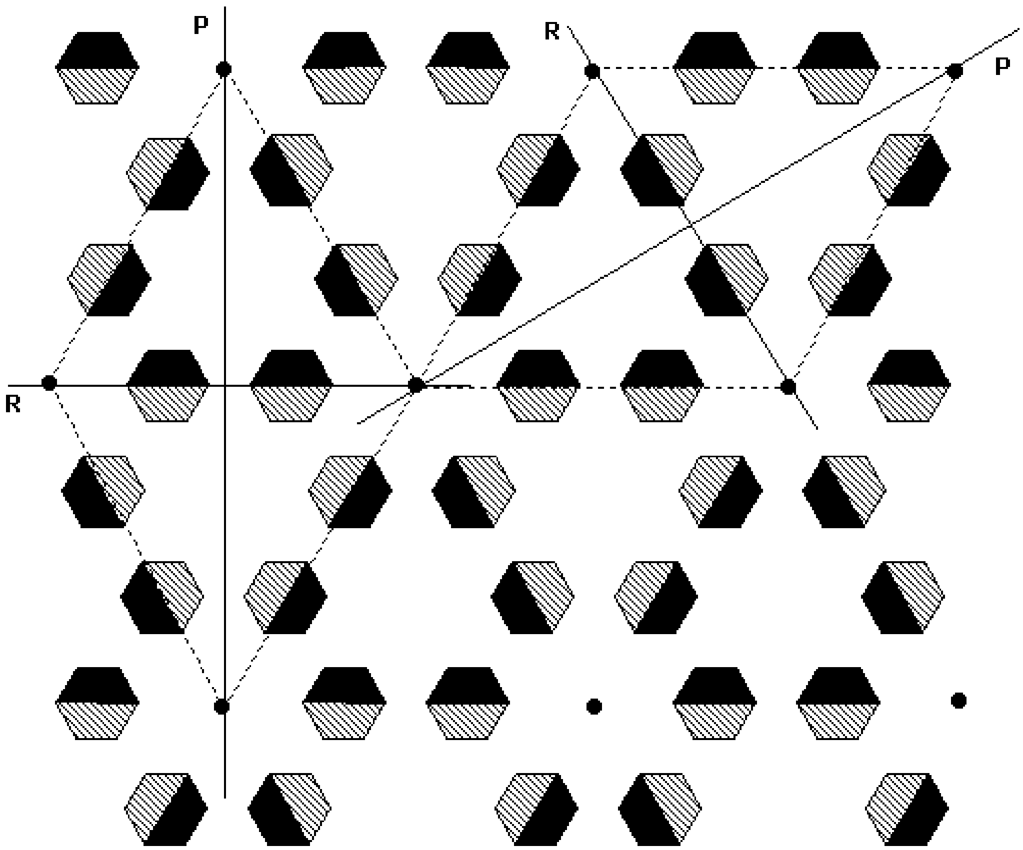


Fig. 6.134

So, once you decide that a two-colored pattern belongs to the **p6m** family (60^0 rotation plus 'some' reflection), locate all sixfold centers, pick four of them arranged in a (not necessarily 'vertical')

rhombus configuration as above, and then simply determine the effect on color of that rhombus' **short diagonal** (**p31m**) and **long diagonal** (**p3m1**). Notice that you do not need at all the effect on color of the **p6m**'s sixfold centers (represented by dots in figure 6.134), but you may still use them to **check** your classification: they of course have to **preserve** colors (**6**) in the cases of **p6m** and **p6m′m′**, and **reverse** colors (**6′**) in the cases of **p6′mm′** and **p6′m′m**.

6.17.4 Examples. First a **p6′mm′** and a **p6m′m′**:



p6′mm′

Fig. 6.135

We indicate **two** rhombuses of sixfold centers, one vertical and one non-vertical: the process is the same for both cases, the only thing that matters is the correct identification of the **long** and **short** diagonals. Make sure you can locate all the other isometries: threefold and twofold centers, in-between glide reflections, etc.

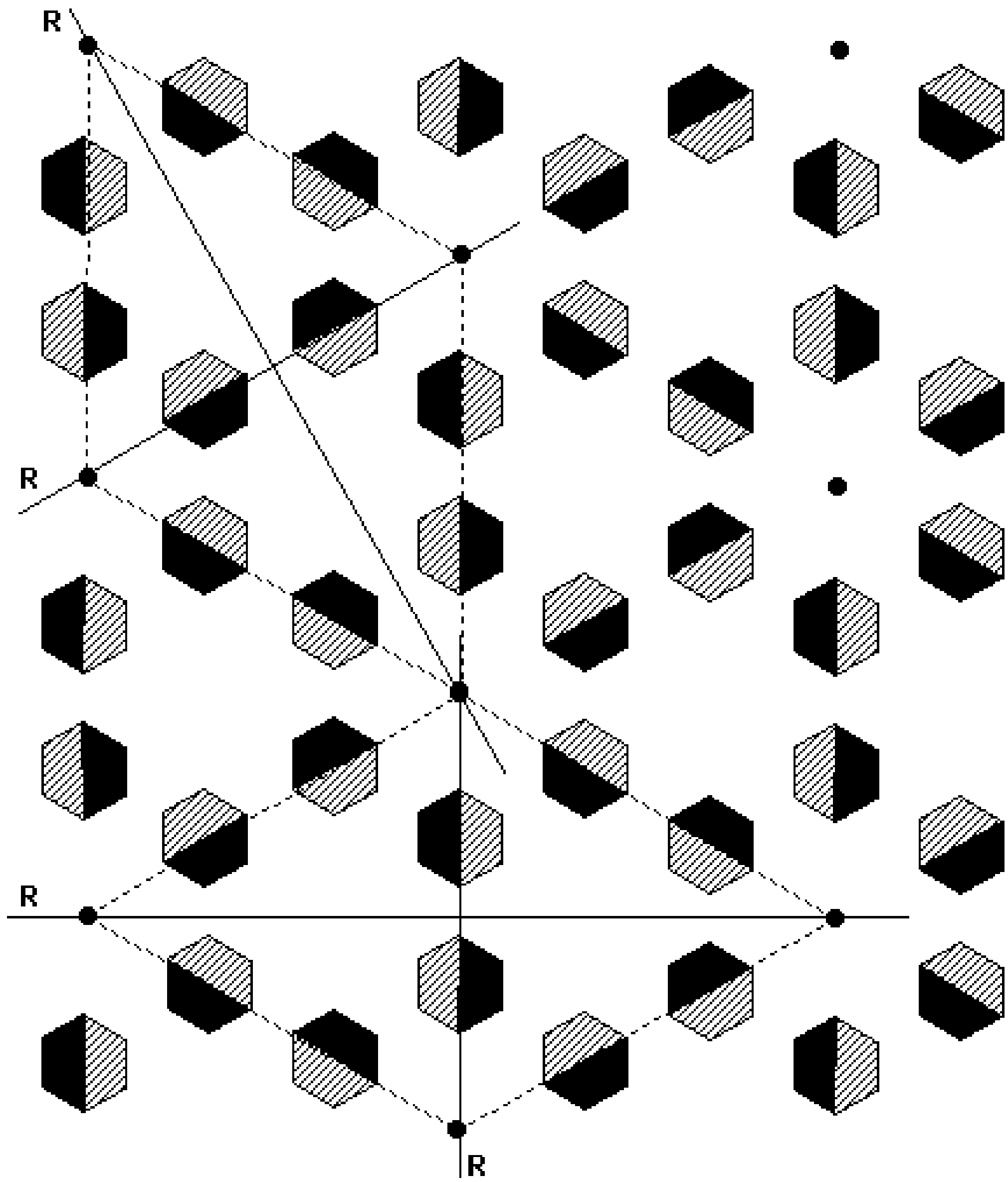


Fig. 6.136

In this example, closer than you might think to the previous one, we see that there is no **vertical** rhombus of sixfold centers. But we can still classify the pattern, using for example the **horizontal** rhombus at the bottom: **both** its diagonals **reverse** colors.

A slight yet necessary modification of the two-colored hexagons (and not only!) leads to examples of the remaining two **p6m** types, **p6′m′m** and ('two-colored') **p6m**. This time there is only one rhombus of sixfold centers shown per example:

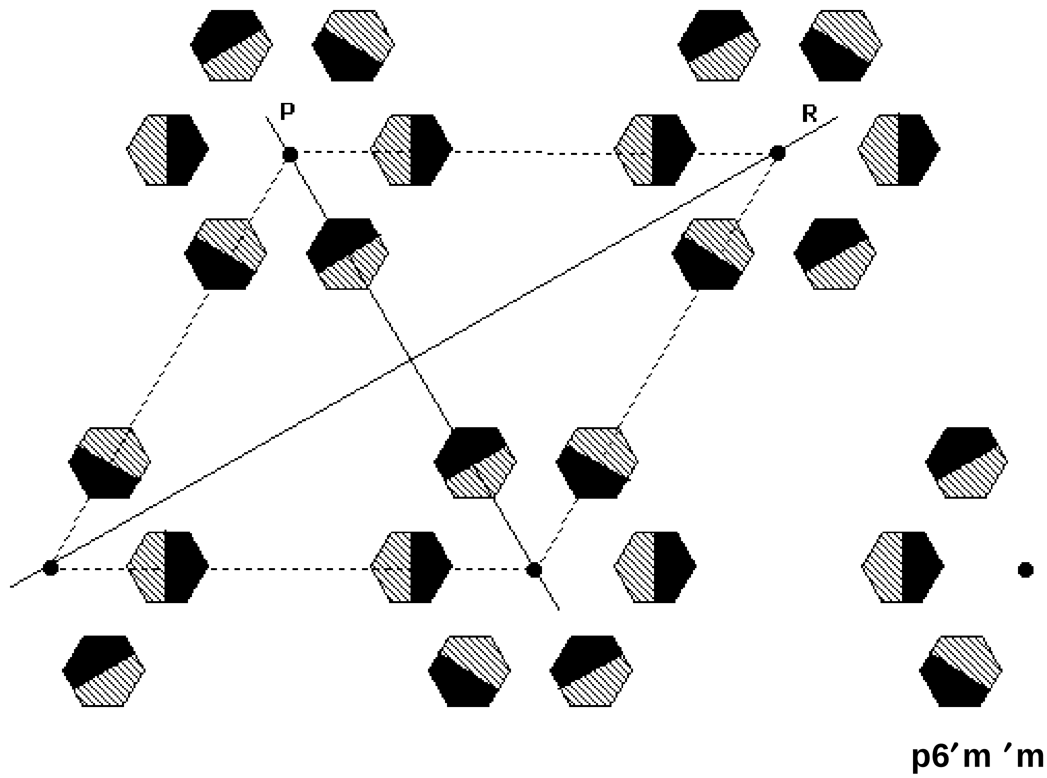


Fig. 6.137

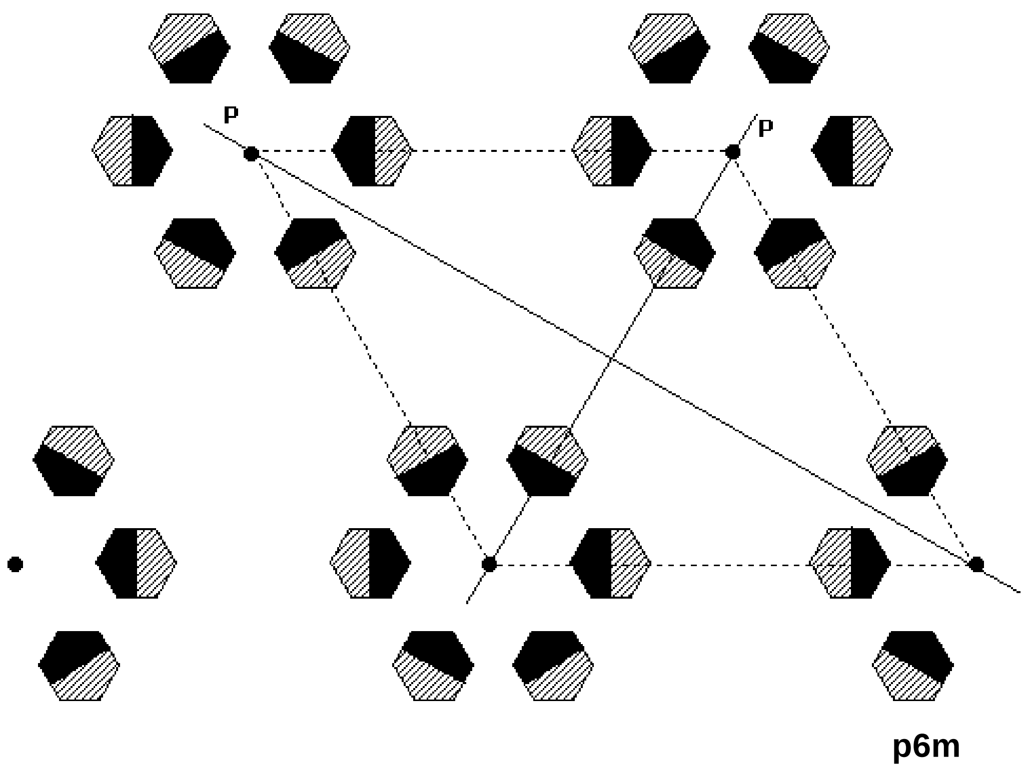
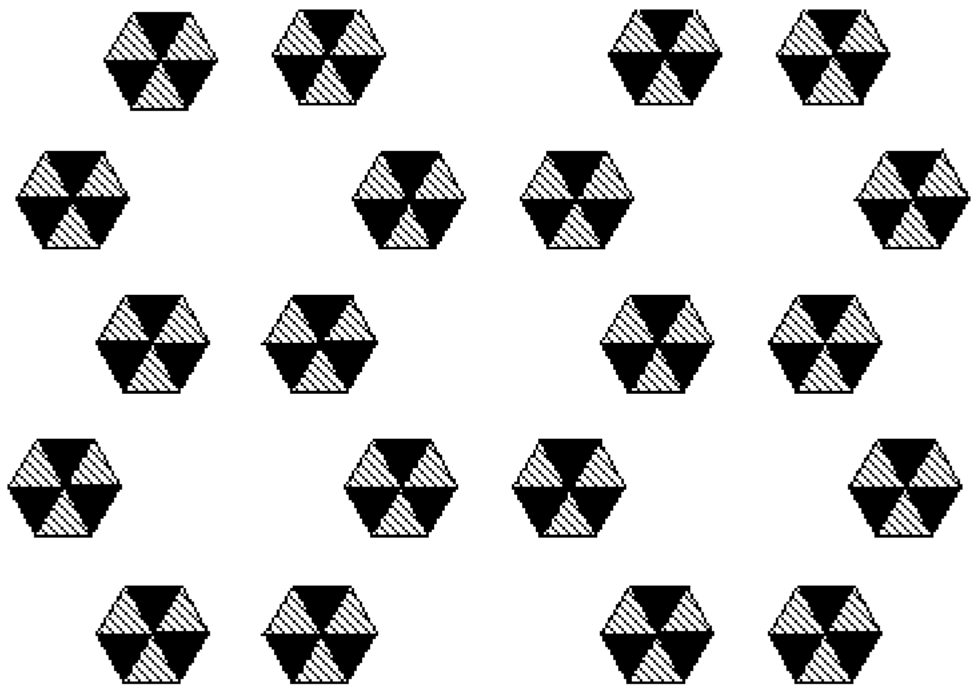
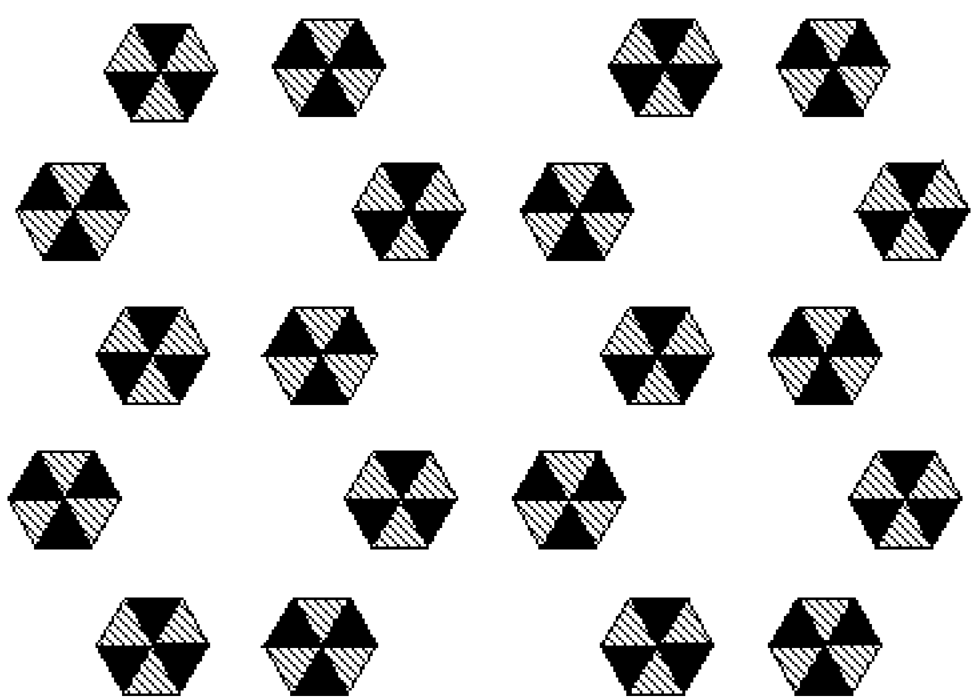


Fig. 6.138

Finally, some triangles inside the hexagons:



p6′m′m



p6m′m′

Fig. 6.139

6.17.5 Reduction of symmetry revisited. You must have noticed that we provided no 'triangular' example of a **p6′mm′** in figure 6.139. While we leave it to you to decide whether or not such a particular example is possible, we compensate with the following variation:

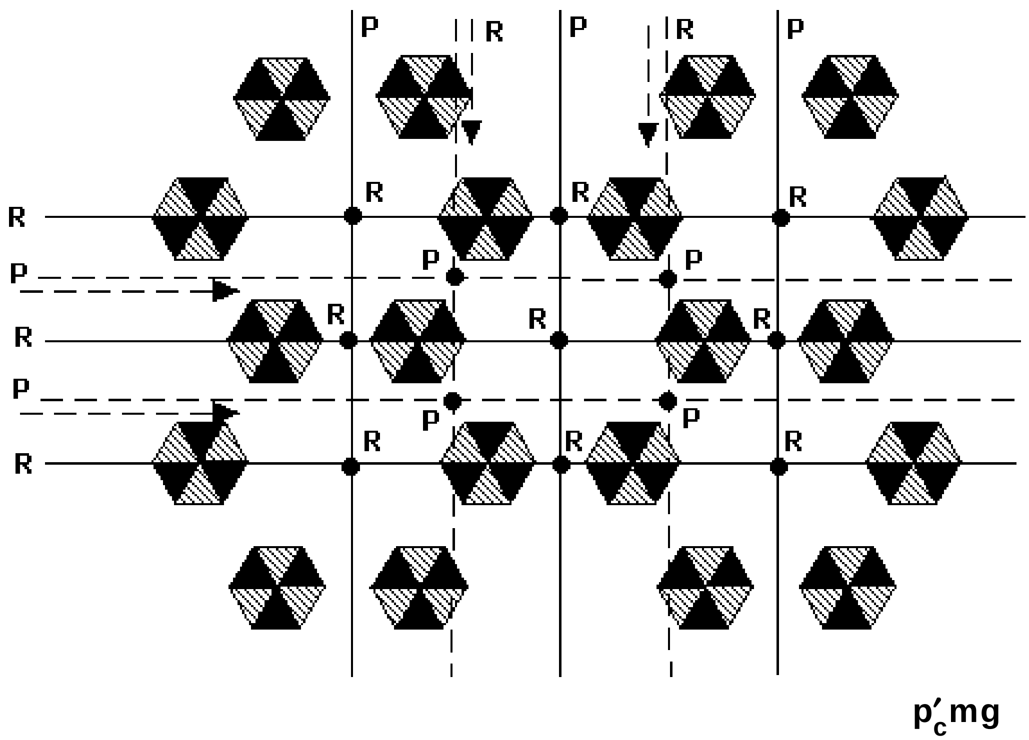


Fig. 6.140

What happened? Quite simply, our coloring has rendered **all** the **p6m** isometries **but** the ones shown in figure 6.140 inconsistent with color. More specifically, only **one** direction of (glide) reflection has survived within each of the two 120^0 patterns hidden behind the **p6m**. As a result, all sixfold and threefold rotations are gone, but the twofold ones are left intact; to be more precise, all the sixfold centers have been **'downgraded'** to twofold ones. It is certainly not difficult now to classify this 'fallen' pattern as a **cmm**-like type.

The **p6m** is a type that can generate many two-colored types (in all groups save for 90^0) by way of color inconsistency (just like the **p4g** of figure 4.57 has produced several 180^0 and 360^0 types). You should **experiment** on your own and **explore** its rich underground.

6.18 All sixty three types together (symmetry plans)

6.18.1 Reading and using the symmetry plans. As in section 4.18, we split the seventeen families of two-colored wallpaper patterns into **five groups** based on the angle of smallest rotation, indicating the **'parent types'** within each group in parenthesis; but the descriptions of the various patterns and types here are going to be visual rather than verbal, based on the symmetry plans developed throughout this chapter. As in section 5.9, isometries inconsistent with color are discarded: there are **no I**s in the symmetry plans!

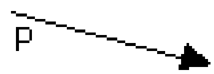
In what follows, and as in the rest of the book, **solid** lines stand for **reflection** axes and **dotted** lines stand for **glide reflection** axes; once again, however, space limitations dictate the omission of all the glide reflection vectors and most translation vectors. Twofold, threefold, fourfold, and sixfold rotations are represented by dots, triangles, squares, and hexagons, respectively.

Recall at this point that all 120^0 rotations preserve colors (6.13.1) and that **all** reflections and glide reflections in two-colored 120^0 patterns must have the same effect on color (6.14.1): therefore in the respective symmetry plans a **P** or **R** to the symmetry plan's lower left or right indicates that **all** axes preserve colors or that **all** axes reverse colors, respectively. Along the same lines, rather than marking with a **P** or **R**, or even showing, every single (glide) reflection axis in each of the four **p6m** types, we limit ourselves to a single rhombus formed by sixfold centers (in the spirit of 6.17.3); a single unmarked **p6m** symmetry plan is shown **in full**.

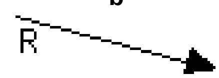
Keep in mind that a symmetry plan not only provides the answer (as to what type a given two-colored wallpaper pattern belongs to), but it also may well **lead** to the answer; it shows, for example, how rotation centers are positioned with respect to (glide) reflection axes **and** vice versa: in some cases you may first locate the rotation centers, in other cases you may first find the (glide) reflection axes.

(I) 360^0 types (p1, pg, pm, cm)

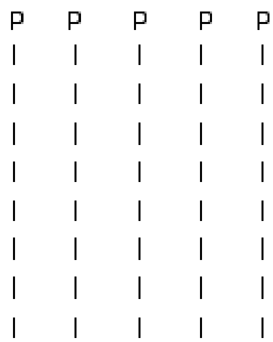
p1



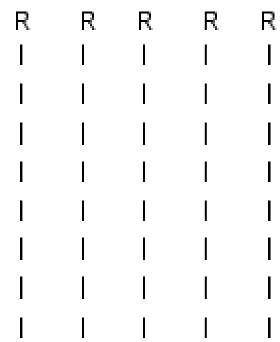
$p_b'1$



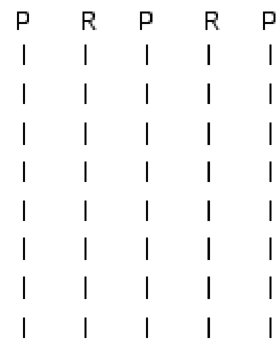
pg



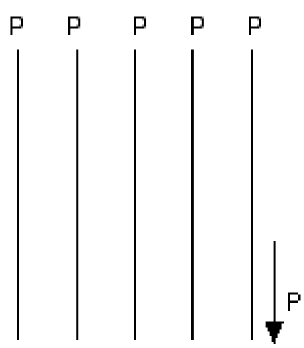
pg′



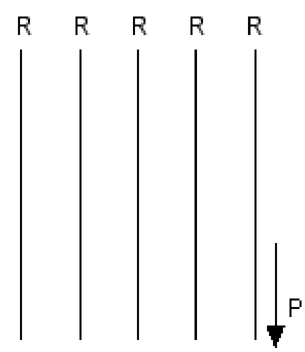
$p_b'1g$



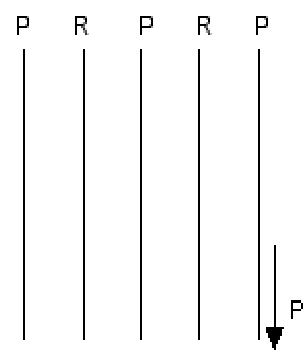
pm



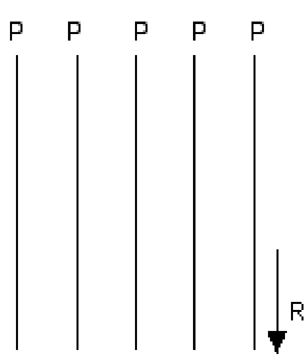
pm′



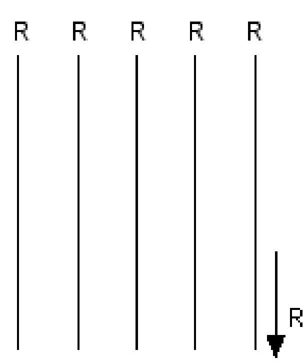
$p_b'1$ m



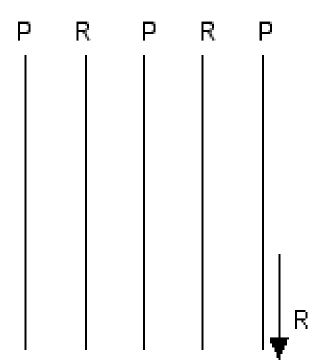
p′m



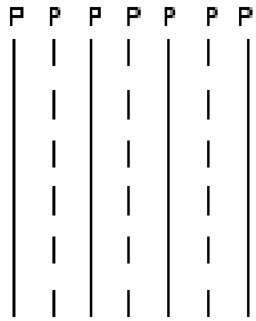
$p_b'g$



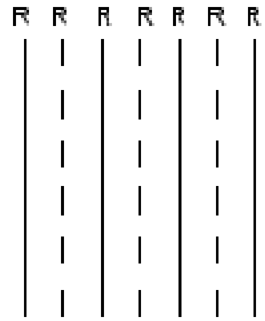
c′m



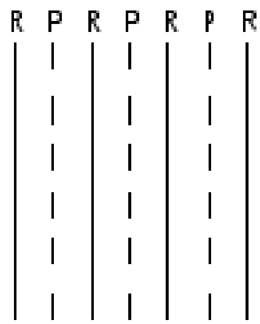
cm



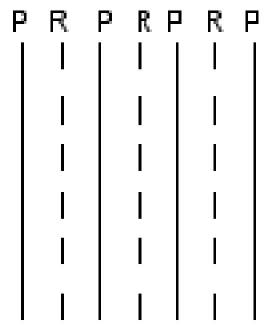
cm′



$p'_c g$

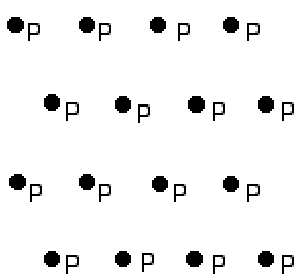


$p'_c m$

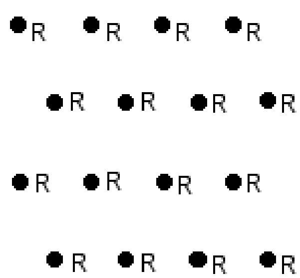


(II) 180^0 types (p2, pgg, pmg, pmm, cmm)

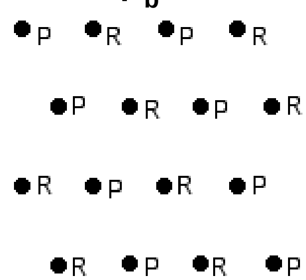
p2



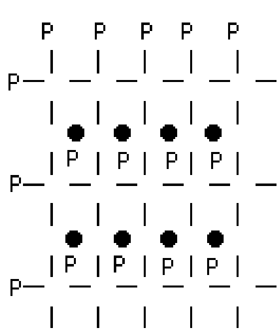
p2′



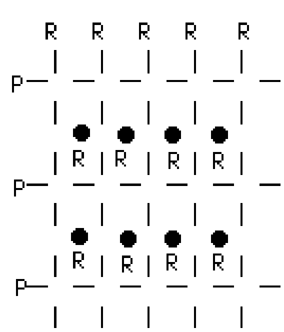
$p'_b 2$



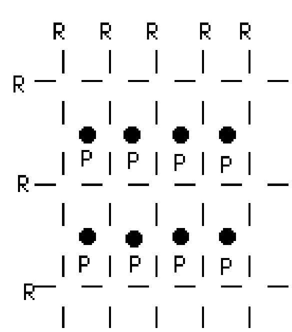
pgg



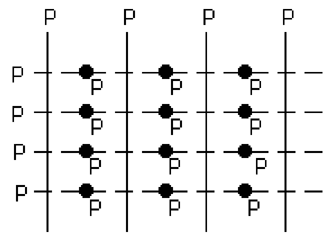
pgg′



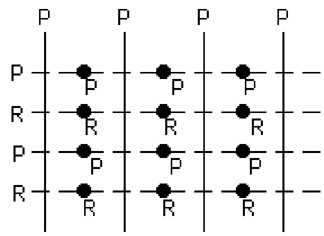
pg′g′



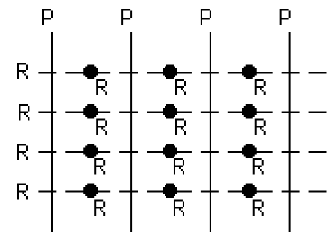
pmg



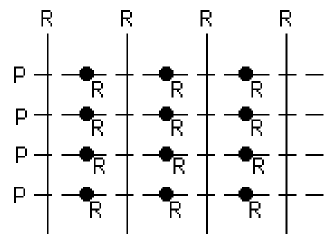
$p_b'mg$



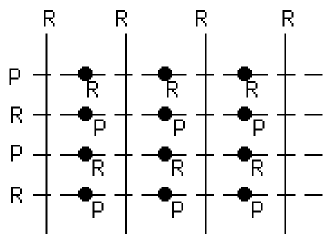
pmg'



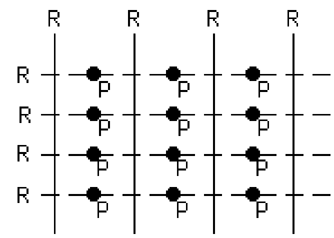
$pm'g$



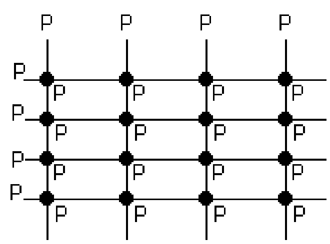
$p_b'gg$



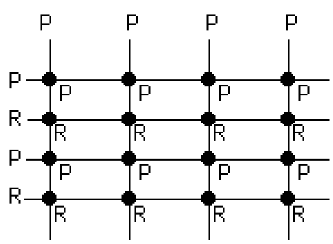
$pm'g'$



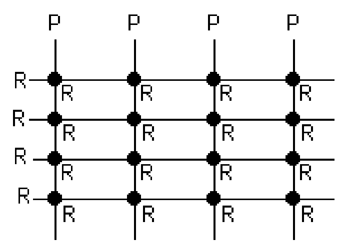
pmm



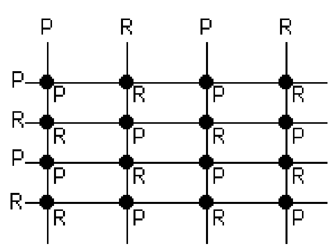
$p_b'mm$



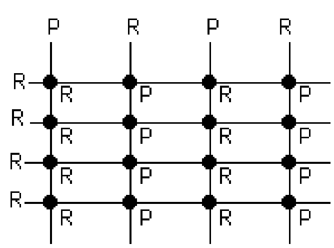
pmm'



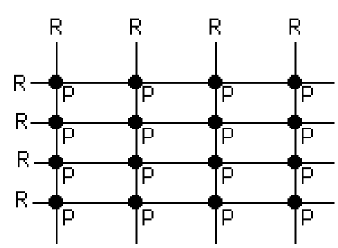
$c'mm$



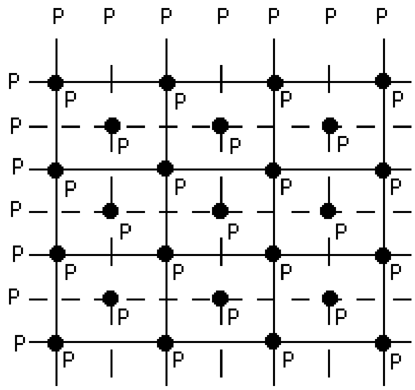
$p_b'gm$



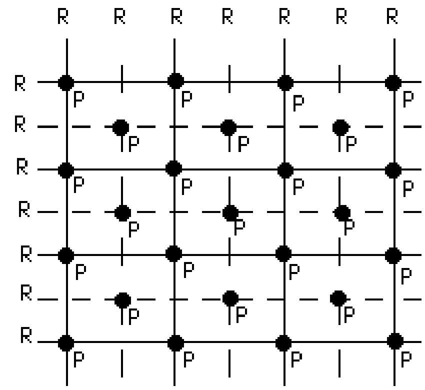
$pm'm'$



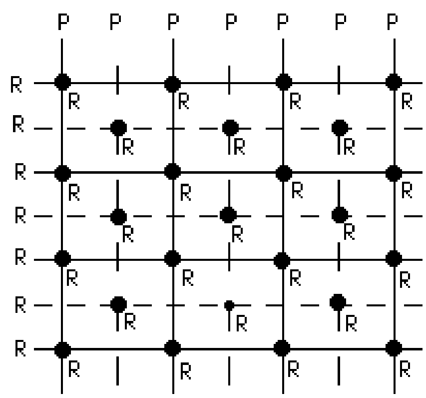
cmm



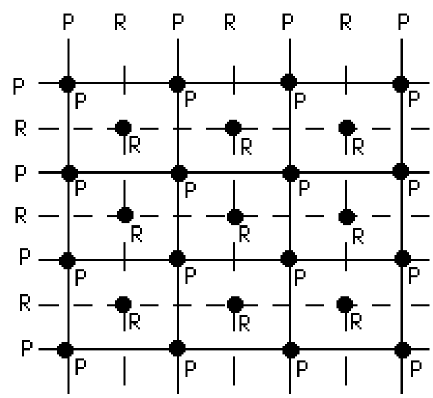
cm′m′



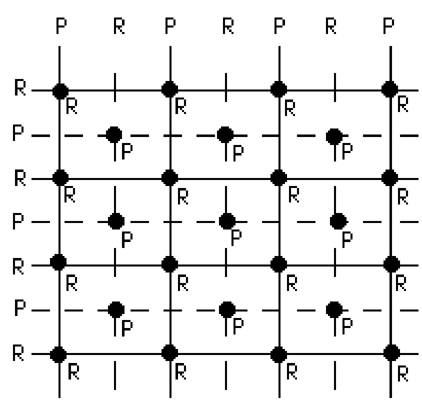
cmm′



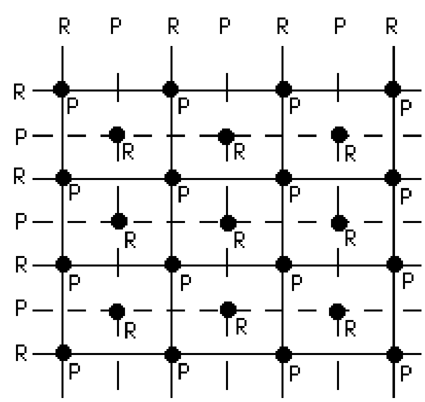
$p'_c mm$



$p'_c mg$

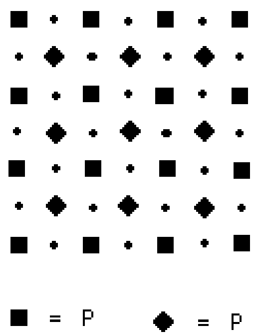


$p'_c gg$

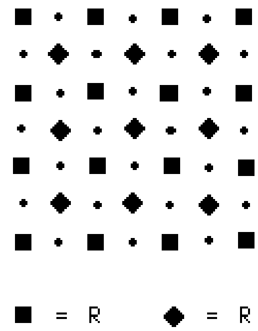


(III) 90^0 types (p4, p4g, p4m)

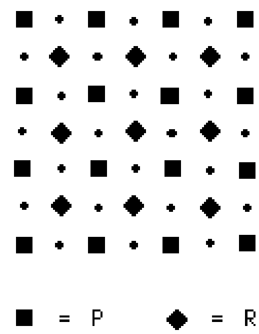
p4



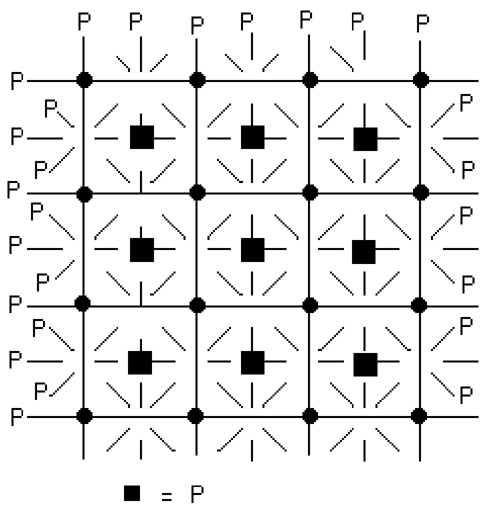
p4′



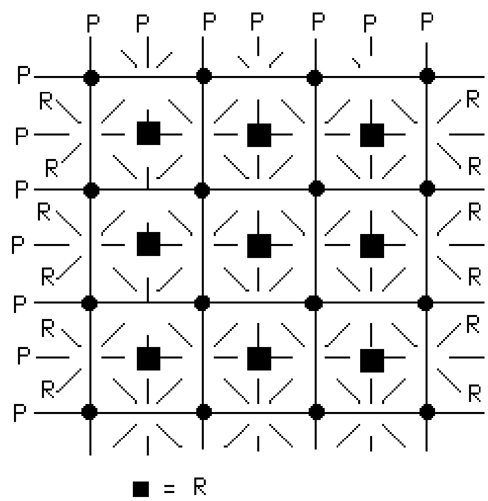
$p_c'4$



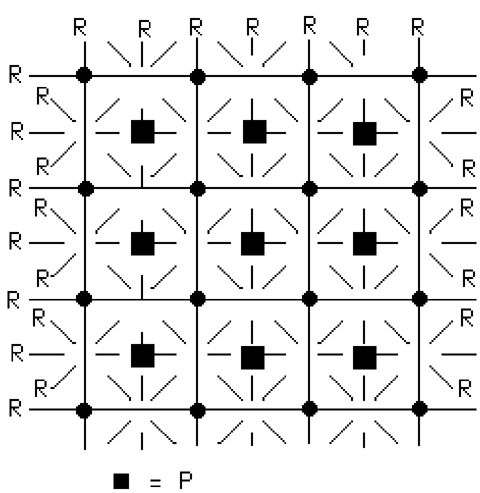
p4g



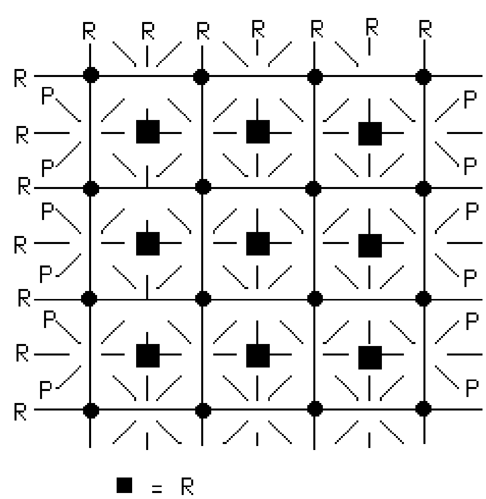
p4′g′m



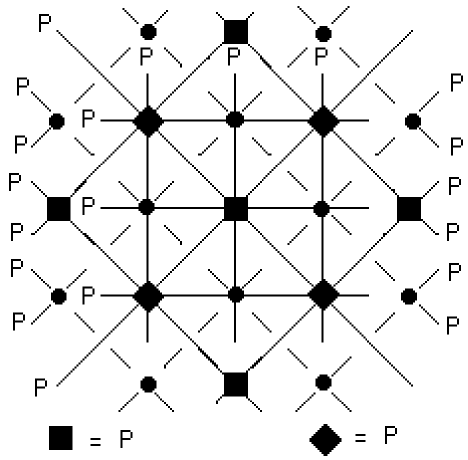
p4g′m′



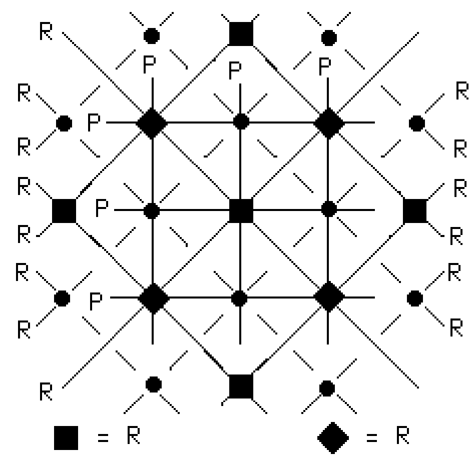
p4′gm′



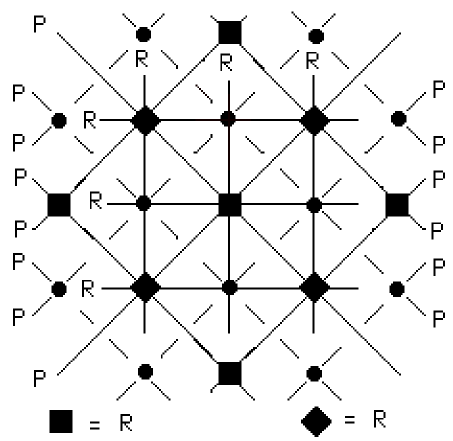
p4m



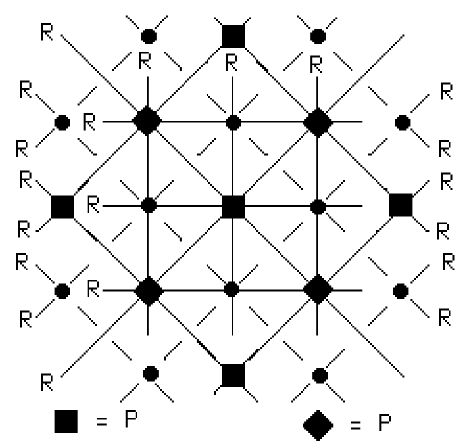
p4′m m′



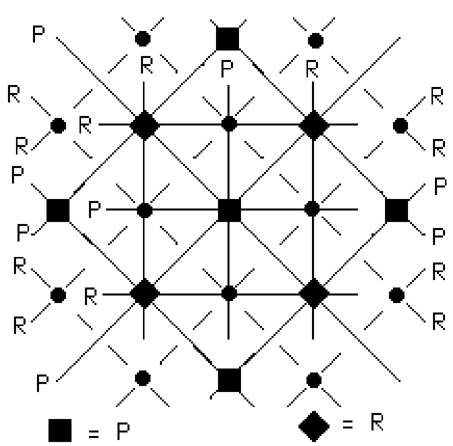
p4′m′m



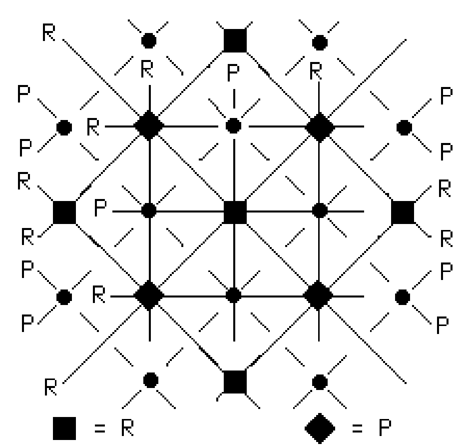
p4m′m′



p'_c4mm

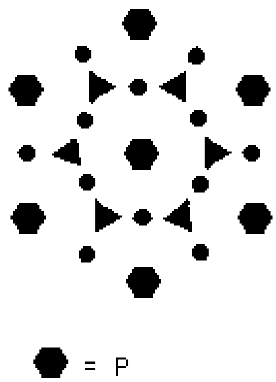


p'_c4gm

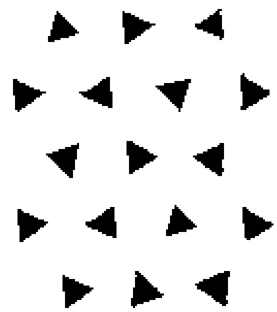


(IV-V) 120^0 (p3, p31m, p3m1) & 60^0 (p6, p6m) types

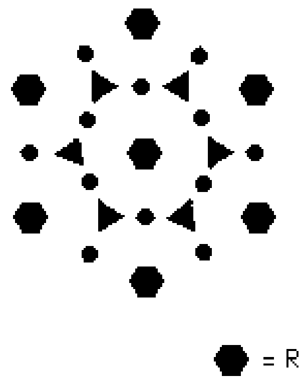
p6



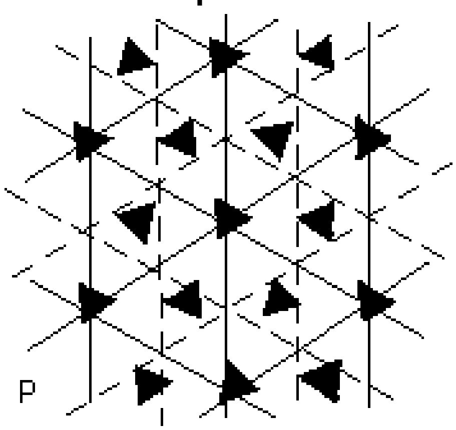
p3



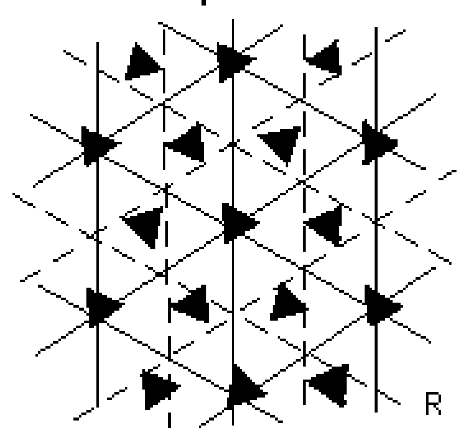
p6′



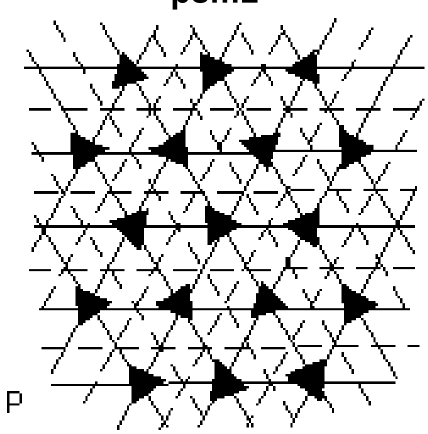
p31m



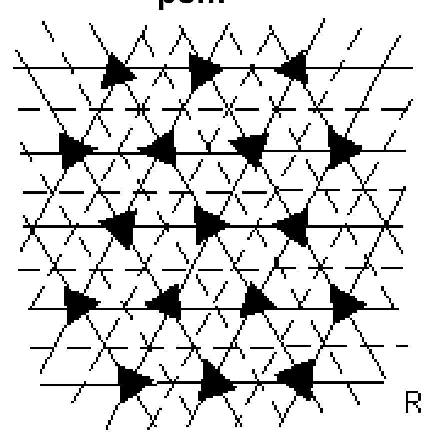
p31m′



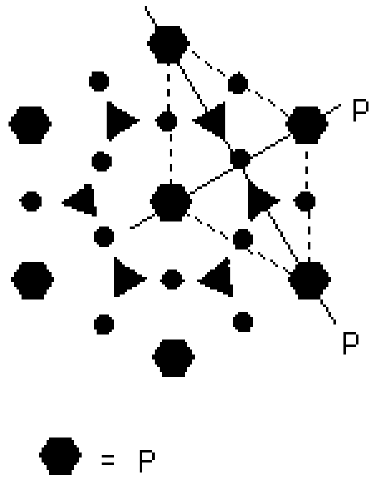
p3m1



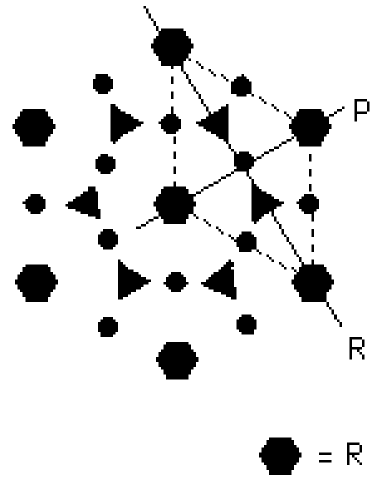
p3m′

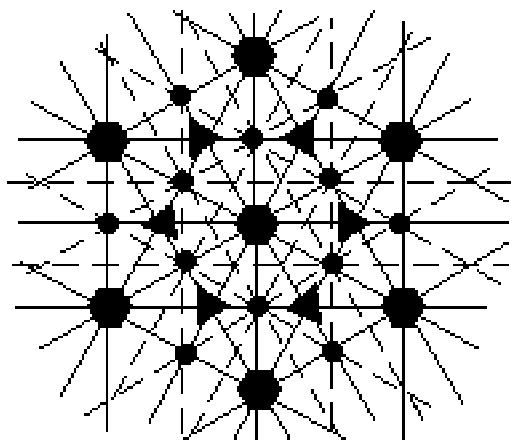


p6m

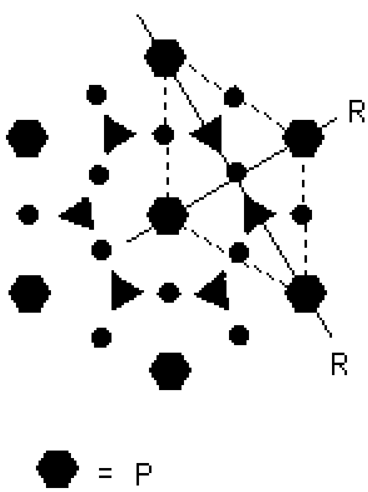


p6′m ′m

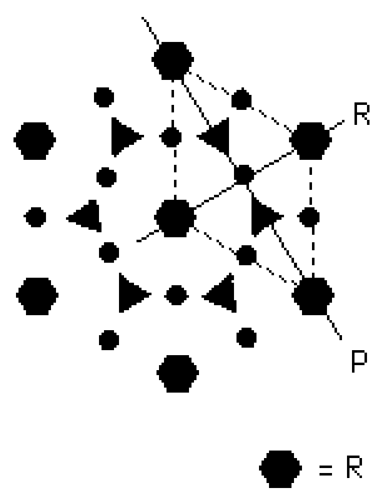




p6m′m ′



p6′m m ′



first draft: winter 2000