 first draft: winter 1999

CHAPTER 5

TWO-COLORED BORDER PATTERNS

5.0 Color and colorings

5.0.1 Design, color, and background. Color is present everywhere, even on this page: it is the contrast between black ink and white **background** that makes this page visible! One could even talk about 'active' design colors and 'passive' background colors; in more complex situations it is not so clear what belongs to the design and what belongs to the background -- nor is such a differentiation always important or even possible, of course. But it is often crucial in discussions of '**colored patterns**'.

Consider the following **pma2** border pattern:

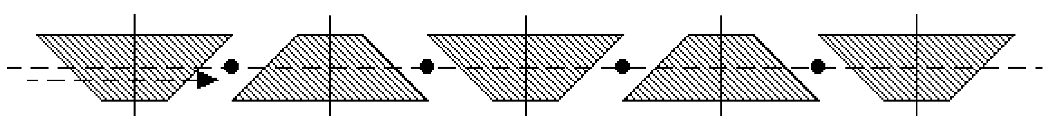


Fig. 5.1

One can certainly talk about three colors being present here: the trapezoids are 'grey', their boundaries are black, and the background is, of course, white; since we generally ignore the boundary lines' color, we can safely talk about two colors being present and deal with an **'one-colored'** (grey) **pattern** on white background.

5.0.2 Colorings. Let us now leave blank (white) the pattern of figure 5.1, planning to color it in two colors as indicated below:

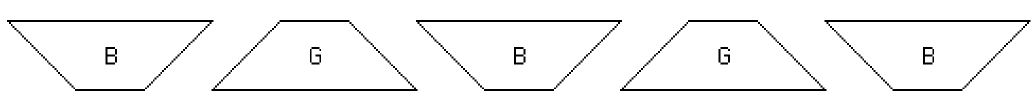


Fig. 5.2

In figure 5.2, **B** and **G** could stand for blue and green, or any other pair of distinct colors: if you feel that blue and green are too close to each other or that they do not get along that well, whatever, you are free to use, for example, red for **B** and yellow for **G**, etc. (The choice of the two colors is not an entirely trivial matter, as it could in fact affect perception of space and interaction between shapes.) For our purposes, and in an effort to keep printing costs low, we limit ourselves to black for **B** and grey for **G** (in this and the next chapters, where, with the exception of 5.0.4 below, only two colors are involved). In particular, our coloring of figure 5.2 yields the following **'two-colored'** border pattern:

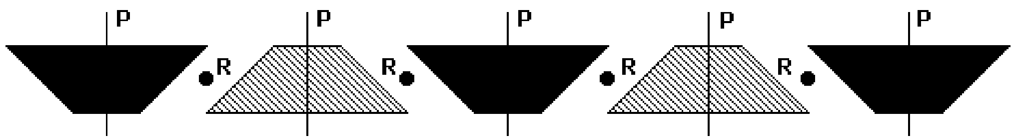


Fig. 5.3

5.0.3 <u>Color preservation and reversal.</u> Does the border pattern in figure 5.3 look the same if you flip this page (or if you trace it and flip the tracing paper)? Your first response might be "no", as the black trapezoids that were 'inverted' in figure 5.3 are now 'upright':

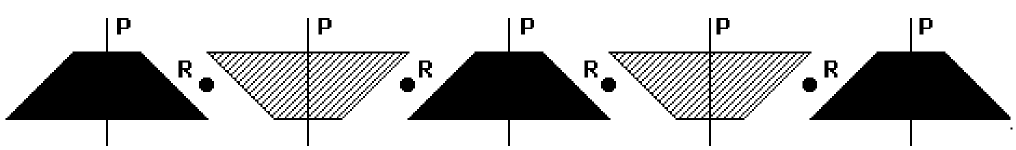


Fig. 5.4

In other words, it is tempting to say that the pattern in figure 5.3 (or 5.4) does not have half turn. But then the question arises: what happened to the half turn of the original pattern, and to the rotation centers indicated in figure 5.1, in particular? A closer look at figures 5.3 & 5.4 shows that a 180^0 rotation (half turn) about each of those rotation centers maps **every** black trapezoid to a grey one and vice versa! In the language that we will be using from here on, each half turn in figures 5.3 & 5.4 **reverses colors**. We end up saying that our pattern has **color-reversing half turn**, indicating this fact by placing a capital **R** right next to each rotation center: indeed those **R**s that you see right next to the rotation centers in figures 5.3 & 5.4 do not stand for “rotation” but for “reverses”!

How about the vertical reflections of our pattern? In figures 5.3 & 5.4 you see a **P** right next to each reflection axis, standing for “preserves”: indeed vertical reflection about each axis maps **every** black trapezoid to a black one (possibly even itself) and **every** grey trapezoid to a grey one! In our new language we say that the pattern in figures 5.3 & 5.4 has **color-preserving vertical reflection**.

So, every half turn in the border pattern of figures 5.3 & 5.4 reverses colors, while every vertical reflection preserves colors. Does that mean that, in such two-colored border patterns, half turns always reverse colors and vertical reflections always preserve colors? **Not at all**: as you are going to see in what follows, all combinations are possible; further, it is possible for a single border pattern to have **both** color-preserving and color-reversing half turns, or **both** color-preserving and color-reversing vertical reflections. On the other hand, a border pattern can have only one glide reflection or horizontal reflection: so these isometries, if there to begin with, must be either color-preserving or color-reversing; but wait until section 5.8, too!

5.0.4 <u>More than two colors?</u> Let’s have a look at a few colorings -- indicated by letters rather than real colors -- of another **pma2** border pattern:

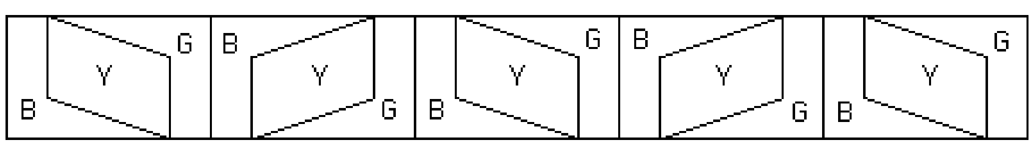


Fig. 5.5

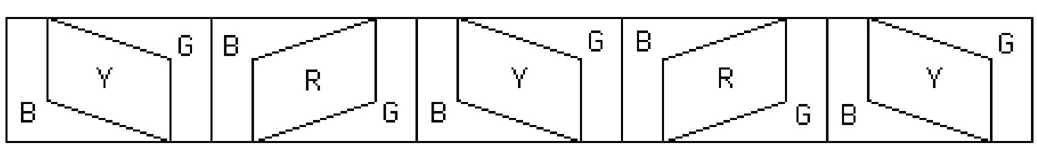


Fig. 5.6

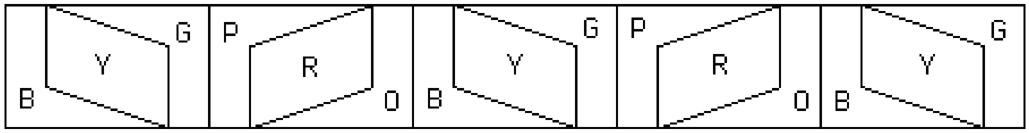


Fig. 5.7

In the pattern of figure 5.5, there are two colors ‘in balance’ with each other (blue and green) and a third color (yellow) not ‘in balance’ with either of the other two: it is more reasonable then to describe the pattern as two-colored (blue-green) **pma2** on yellow-white background; at the same time, one could ‘see’ an one-colored **pma2** border pattern (yellow) on three-colored (blue-green-white) background!

Much more so, it is not clear what the background is in figure 5.6, so one could easily talk of **two** two-colored **pma2** border patterns, a blue-green one **and** a yellow-red one, on white background: indeed yellow and red are as much ‘in balance’ with each other as blue and green are! Finally, it does make sense to talk of a **four-colored** (blue-green-purple-orange) **pma2** border pattern **and** a two-colored (yellow-red) **pma2** border pattern in figure 5.7!

Complicated, isn’t it? Well, in this and the next chapter all border and wallpaper patterns will be simple enough to be viewed as two-colored, with no room for confusion; one or more **background colors** might be there from time to time, but it will be clear that those are indeed background colors. The concept of background color is more important in the context of **‘real world’** patterns, found in textiles, mosaics, and other artifacts.

But what is, after all, and in our context of border or wallpaper patterns always, that “background (color)”? It is reasonable to say that, in the presence of more than one colors in a pattern, a color is viewed as background if and only if the pattern has **no isometry that swaps it with another color**. Under this definition, the situation is certainly clear in figure 5.5 (yellow is background) but not quite so clear in figures 5.6 & 5.7: it would be best to view those border patterns as four-colored and **six-colored** patterns, respectively. (On another note, this definition resolves the

‘Alhambra controversy’ of 4.17.4 by rendering the ‘entrapped white’ in figure 4.73 a **second color**!)

5.0.5 One-colored or two-colored? Motivated by the discussion in 5.0.4, we say that a border or wallpaper pattern is **two-colored** if and only if **precisely two** of its colors are **swapped by at least one isometry** that maps the pattern to itself. In particular, this definition implies that the two colors are **‘in balance’** with each other: for example there is as much grey as black in figures 5.3 & 5.4 (with grey and black swapped by half turn), and as much blue as green in figure 5.5 (with blue and green swapped by both vertical reflection and half turn).

For a change, let’s have a look at the following border pattern:

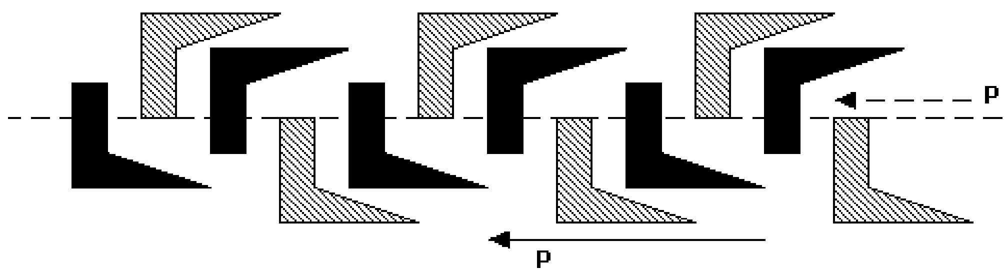


Fig. 5.8

How many isometries do you see that swap black and grey? None! Our pattern has color-preserving translation and color-preserving glide reflection, and that’s about it! On the other hand, it is clear that black and grey are in perfect balance with each other: is it a two-colored pattern, then? No, under our sound definition the border pattern in figure 5.8 is **one-colored**, classifiable in fact as a **p1a1**!

Confused? Well, don’t worry, the next seven sections will be relatively straightforward; patterns such as that of figure 5.8 are indeed rare... (Can you create any others, by the way? Having, for example, only color-preserving translation and color-preserving vertical reflection?) For the rest of the chapter we will be dealing mostly with ‘genuine’ two-colored patterns, colorings in fact of the seven types of one-colored patterns we studied in chapter 2. Now things can at times go a bit ‘wrong’ with those colorings, too, but you will have to wait until section 5.8 to see how that can happen!

5.1 Colorings of p111

5.1.1 Color-reversing translation. All border patterns presented in section 5.0 have color-preserving translation, common in fact **by definition** to **all** border patterns, but none of them has **color-reversing translation**. Does that mean that no translation can be color-reversing? Not at all, in fact sometimes a color-reversing translation is the **only** isometry that makes a border pattern two-colored! This will have to be the case in this section: if you start with a border pattern that has only translation (**p111**), coloring it in two colors can at most make it have **both** color-reversing and color-preserving translation instead of just color-preserving translation; **coloring may not increase symmetry!** Here is an example of two distinct colorings of the same **p111** pattern:

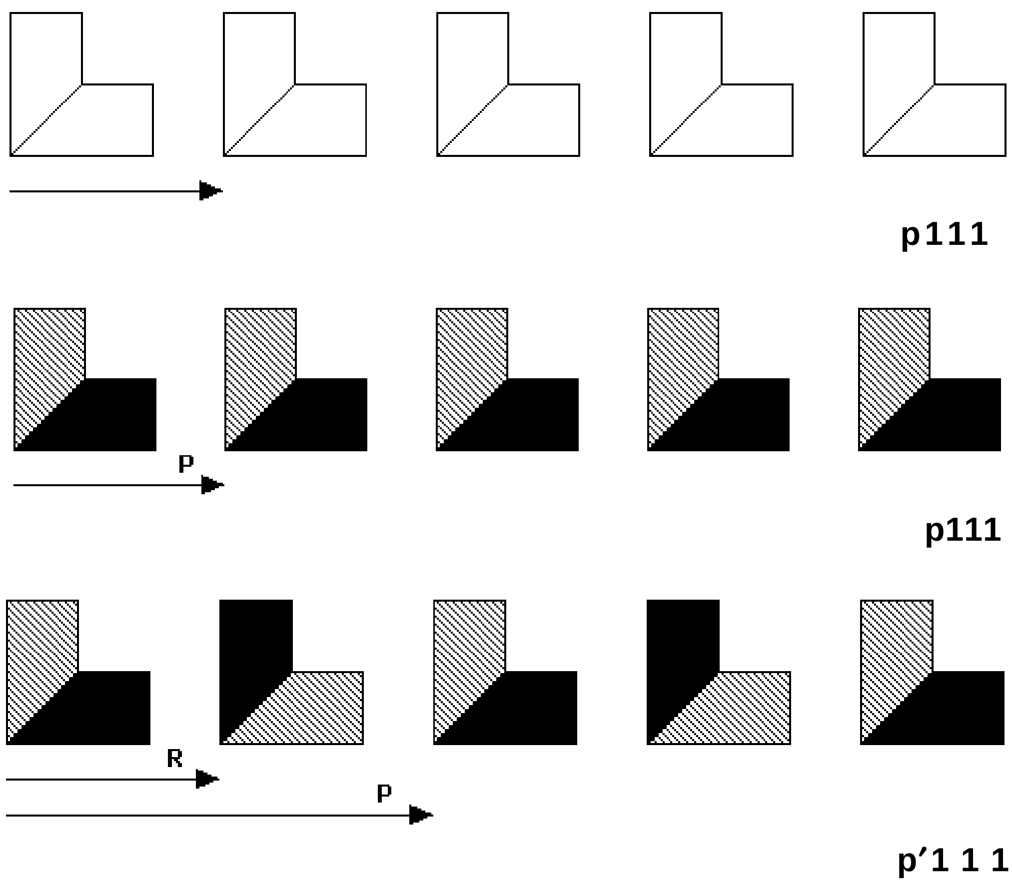
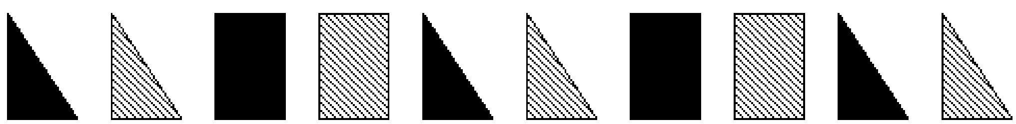


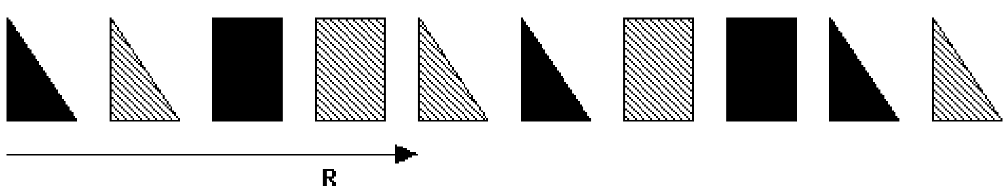
Fig. 5.9

Only the second coloring above allows for color-reversing translation (indicated by **R**–vector), in addition of course to color-preserving translation (indicated by **P**–vector, **twice as long** as the **R**–vector): this yields a **two-colored** pattern known as **p′111**. The first and second patterns in figure 5.9, despite looking like colorless and two-colored, respectively, are **both** classified as **p111**: they both have color-preserving translation and nothing else!

5.1.2 A word on notation. That little **'accent'** (like the one above **p** in **p′111**) will always indicate a **color-reversing isometry** in this and the next chapter; in particular, **p′** always stands for **color-reversing translation**. In figure 5.9 we indicated color-reversing translations with **R**–vectors and color-preserving translations with **P**– vectors. From here on we will no longer bother to indicate color-preserving translations: they are present in **all** border patterns, be them one-colored or two-colored; moreover, the **doubling** of any color-reversing translation vector produces a color-preserving one! Here is an example, again of two distinct colorings of the same border pattern, illustrating this approach:



p111



p′1 1 1

Fig. 5.10

5.2 Colorings of pm11

5.2.1 Color-reversing vertical reflection. Let us now start with a ‘colorless’ **pm11** border pattern and color it following the colorings employed in figure 5.9:

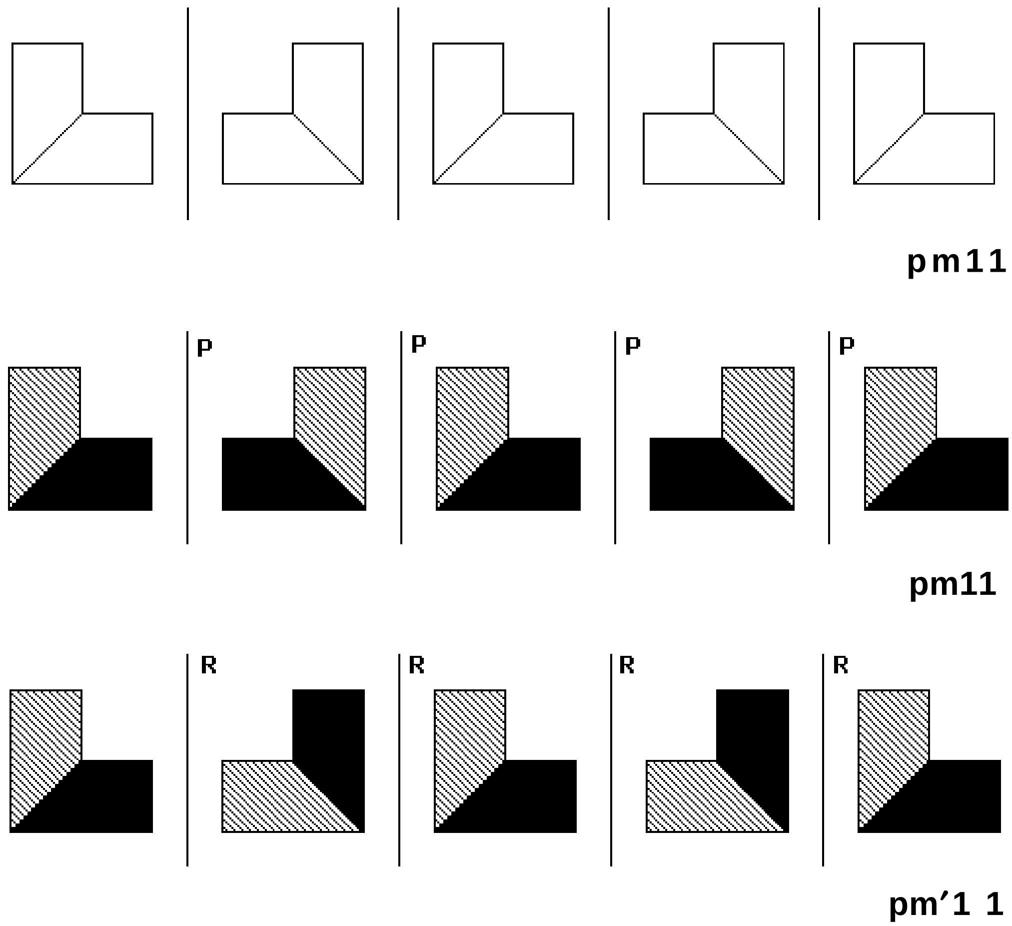
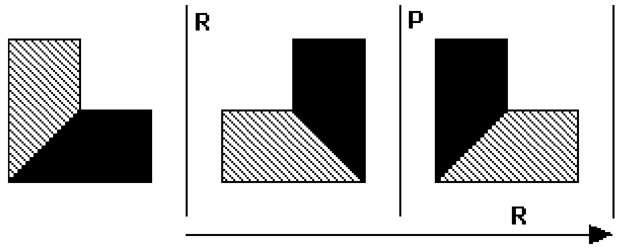


Fig. 5.11

Once again, only the third pattern is two-colored, not because of color-reversing translation (which it doesn’t have) but because of its **color-reversing vertical reflection**: such border patterns, having only color-reversing vertical reflection (in addition, of course, to that ubiquitous color-preserving translation) are denoted, rather predictably in view of 5.1.2, by **pm′11**.

5.2.2 How about color-reversing translation? Can we ‘force’ the third pattern in figure 5.11 to also have color-reversing translation? One thing to try is to reverse the colors in **every other pair** of motifs ... and see what happens:

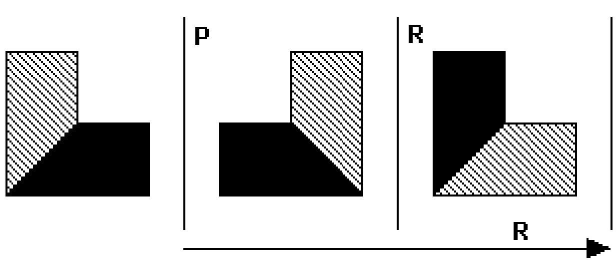


p′m11

Fig. 5.12

Clearly, color-reversing translation has been achieved. How about vertical reflection? Well, here we got a bonus: instead of color-preserving vertical reflections **only** or color-reversing vertical reflections **only**, as in the second (**pm11**) and third (**pm′11**) patterns in figure 5.11, respectively, our new pattern has **both** color-preserving (**P**) and color-reversing (**R**) vertical reflections; such border patterns are known as **p′m11**, in honor (**p′**) of the color-reversing translation that actually allows for the two kinds of vertical reflections ... and is in turn implied by them!

5.2.3 Two kinds of mirrors. As we pointed out in 2.2.3, every **pm11** border pattern has two kinds of vertical reflection axes (mirrors). This is nicely illustrated in the context of figure 5.12, where one kind of reflection axes preserve colors and the other kind of reflection axes reverse colors. Can we get the two kinds of axes to have the exact opposite effect on color? Surely we can, in fact the same process that led from the third pattern of figure 5.11 to the pattern of figure 5.12 leads from the second pattern in figure 5.11 to the following border pattern:



p′m11

Fig. 5.13

While the two-colored patterns in figures 5.12 & 5.13 are **distinct to the eye**, they are mathematically identical (**p′m11**): each of them has color-reversing translation and **both kinds** of vertical reflection (color-preserving and color-reversing). In more mathematical terms, we may say, never forgetting that all border patterns are infinite, that the patterns in figures 5.12 & 5.13 share the same **symmetry plan**:

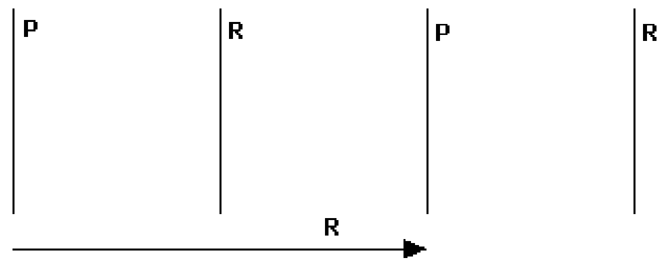


Fig. 5.14

Symmetry plans will be revisited in full in section 5.9.

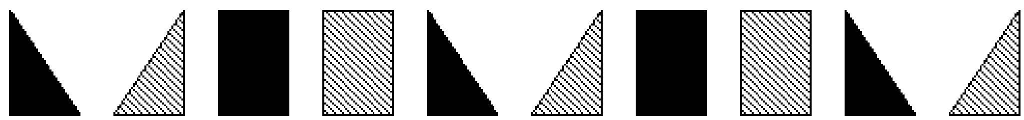
5.2.4 <u>How many colorings?</u> There are two kinds of reflection axes in every **pm11**-like pattern and two possibilities for each kind of reflection axis (color-preserving and color-reversing), hence there should be two $\times$ two = four **possible** types all together. In view of our remarks in 5.2.3, however, two of those four types are viewed as identical, hence there exist four minus one = three types of **pm11**-like border patterns: **pm11**, **pm′11**, and **p′m11** (each of them discussed and exhibited already).

Another way of arriving at this conclusion follows the approach employed in 2.8.3 for classifying all one-colored border patterns:

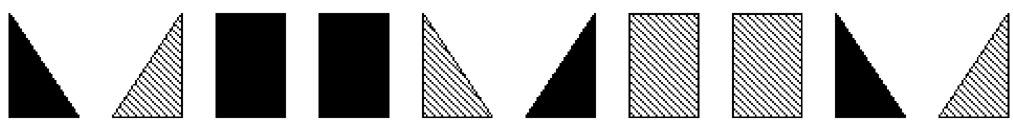
Color-reversing translation	Color-reversing vertical reflection	Border pattern type
Y	Y	**p′m11**
Y	N	impossible
N	Y	**pm′11**
N	N	**pm11**

The ruling out of the second possibility above relies on the following observation: the existence of color-reversing translation in a border pattern with vertical reflection **implies** the existence of color-reversing vertical reflection; check also 7.3.1 and 5.6.2!

5.2.5 Further examples. Here are a couple of suggested colorings further illustrating the role of **pm11**'s two kinds of vertical reflection:



pm′1 1



p′m 1 1

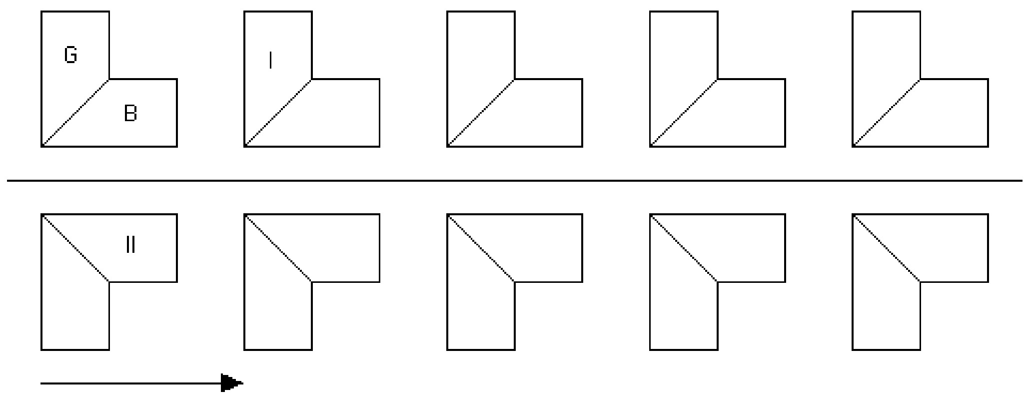
Fig. 5.15

5.3 Colorings of p1m1

5.3.1 Four possibilities. The **p1m1** border pattern may of course be viewed as a **pm11** pattern with vertical reflection **'replaced'** by horizontal reflection. Replacing "vertical" by "horizontal" in the table of 5.2.4 we obtain the following list of possibilities and border pattern types:

Color-reversing translation	Color-reversing horizontal reflection	Border pattern type
Y	Y	**p′1a1**
Y	N	**p′1m1**
N	Y	**p1m′1**
N	N	**p1m1**

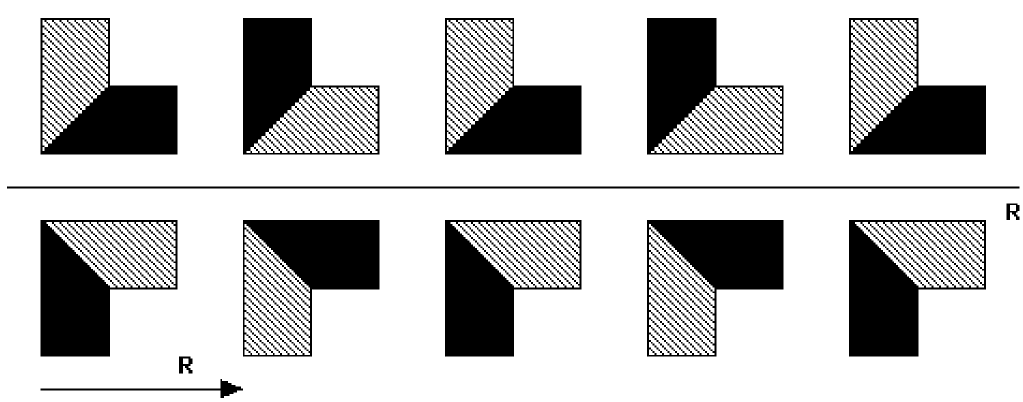
In other words, we claim that this time **all** two × two = four types are indeed possible. In particular there is no problem having color-reversing translation without color-reversing horizontal reflection in a **p1m1**-like border pattern. The best way to establish this claim is of course to provide examples for each one of the four possibilities: one picture is worth one thousand words! To do that, we start with a 'colorless' **p1m1** border pattern and then we color it in two colors and in every possible way, exactly as in the two preceding sections:



p1m1

Fig. 5.16

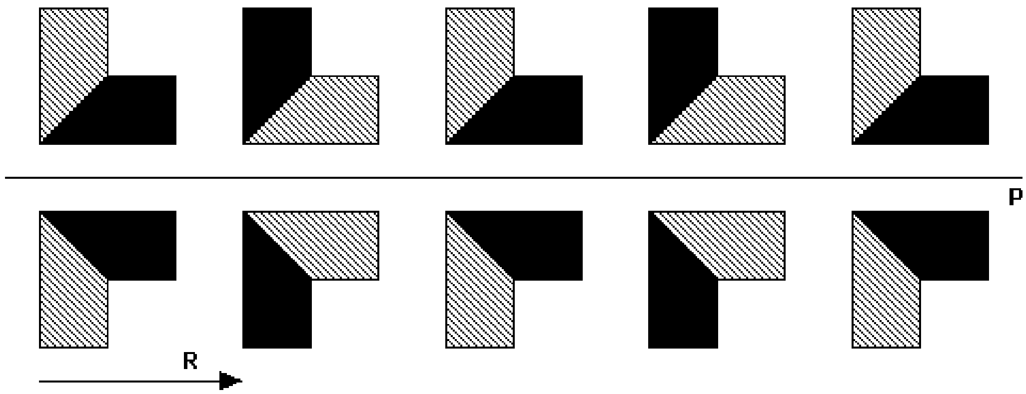
Now if we start by coloring the first two 'cells' grey (**G**) and black (**B**) as indicated in figure 5.16, then there exist **two** choices (**G** or **B**) for **each** of the two 'adjacent' cells I and II; so there exist indeed four possibilities altogether shown in figures 5.17-5.20.



(I = **B**, II = **G**) **p′1a1**

Fig. 5.17

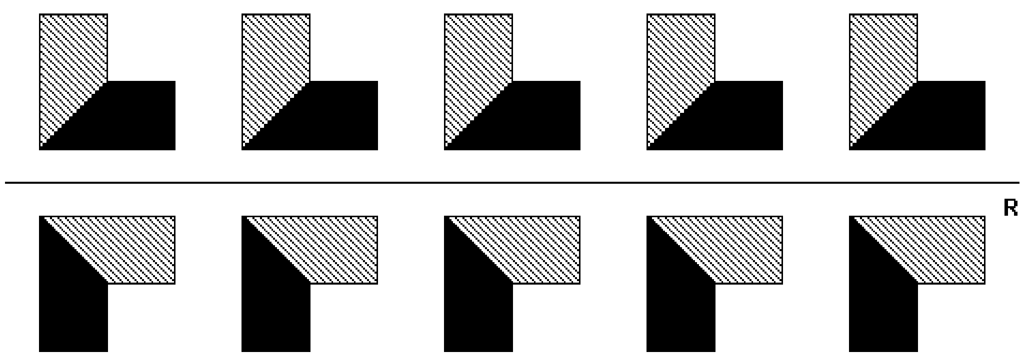
This type is known as **p′1a1**, ‘in honor’ of the **color-preserving** glide reflection (**a**) implied ‘automatically’ by the horizontal reflection (and the translation) in the spirit of 2.7.1. It should more appropriately be denoted by “p′1m′1”, perhaps, but the **crystallographic notation** has its own little secrets!



(I = **B**, II = **B**) **p′1m1**

Fig. 5.18

This type is known as **p′1m1** and may be viewed as a ‘doubled’ version of a **p′111**, with the bottom half being a mirror image of a **p′111** border pattern at the top.

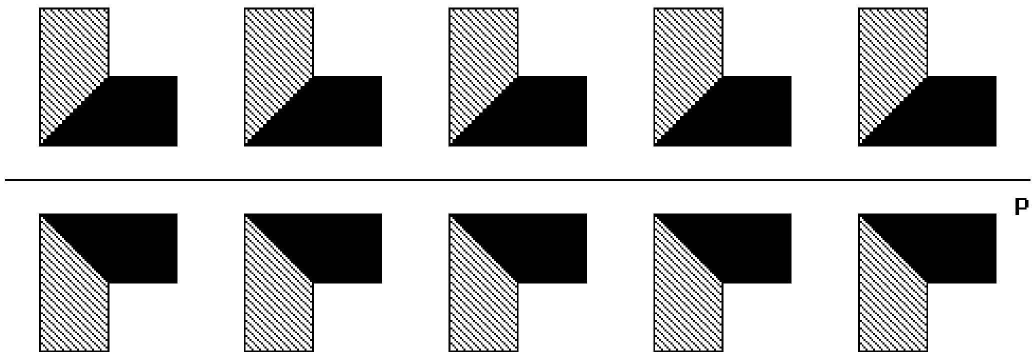


(I = **G**, II = **G**) **p1m′1**

Fig. 5.19

This type is known as **p1m′1**; it could also be called “p1a′1”, thanks to its ‘hidden’ color-reversing glide reflection and in conformity with the **p′1a1** type’s naming above, except that ... this ‘name’ is reserved for a type we will introduce in the next section!

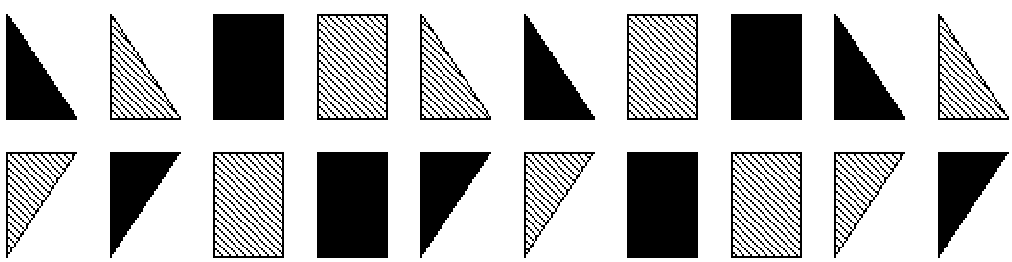
Finally, we get the standard two-colored looking, one-colored classifiable border pattern that is found in every group, a **p1m1**:



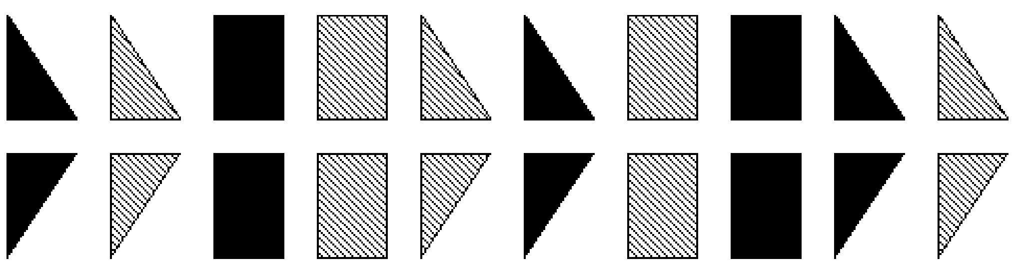
(I = **G**, II = **B**) **p1m1**

Fig. 5.20

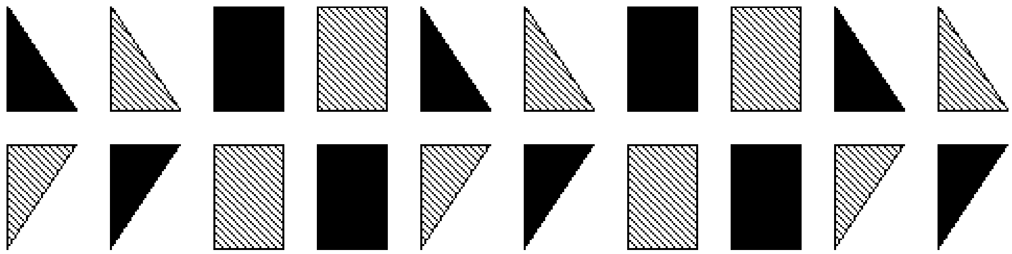
5.3.2 Further examples. Here are three colorings illustrating the three new members of the **p1m1** group:



p′1a1



p′1 m 1

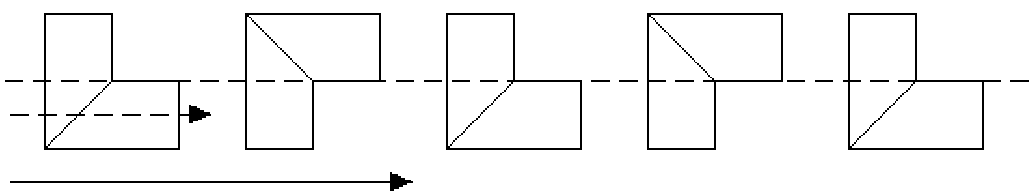


p1m′1

Fig. 5.21

5.4 Colorings of p1a1

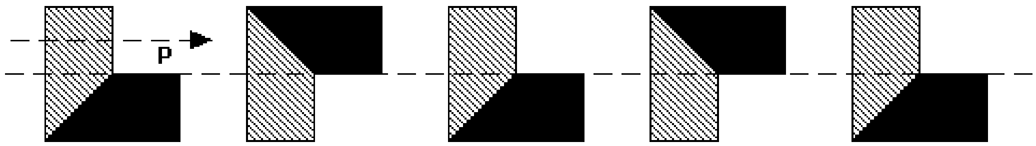
5.4.1 Only two possibilities. In 2.4.2 we observed that the glide reflection vector in every **p1a1** border pattern is equal to **half** the pattern's minimal translation vector. Pushing that observation one step further we see that a **double** application of the **p1a1**'s minimal glide reflection results in the **p1a1**'s minimal translation. You may verify that yourself for the following 'colorless' **p1a1** pattern:



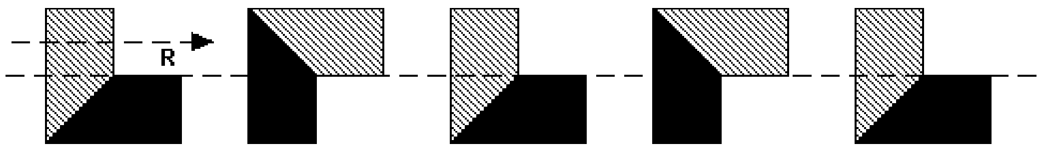
p1a1

Fig. 5.22

Our observation has a crucial implication: no matter how one colors a **p1a1** border pattern, the resulting two-colored pattern cannot possibly have color-reversing translation! Indeed, the **p1a1**'s translation is the **'square'** of either a color-preserving glide reflection or a color-reversing glide reflection: in either case, that 'square' **must** be **color-preserving**, in the same way that the square of every non-zero number must be positive. But this means that there is **only one question** to ask ("does the pattern have color-reversing glide reflection?”) and as many types of **p1a1**-like two-colored patterns as possible answers to that question:



p1a1

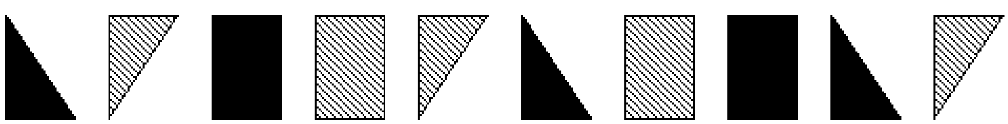


p1a′1

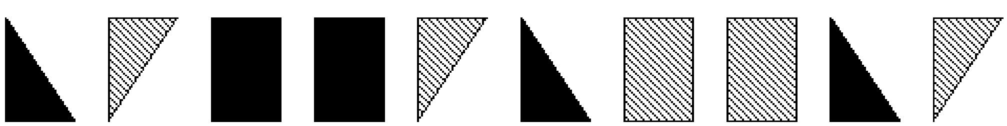
Fig. 5.23

So we have only one genuinely two-colored pattern in the **p1a1** group, characterized by color-reversing glide reflection (**p1a′1**).

5.4.2 Example. Our usual coloring example follows, involving two distinct but closely related **p1a′1**s:



p1a′1



p1a′1

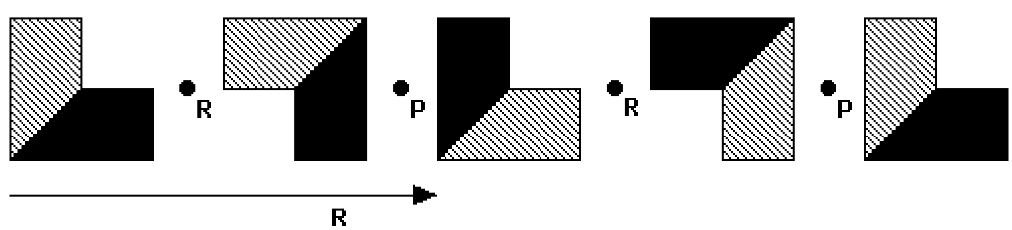
Fig. 5.24

5.5 Colorings of p112

5.5.1 A familiar story! Back in 2.5.4 we alluded to a certain similarity between vertical mirrors (in the **pm11** type) and half turn centers (in the **p112** type). This similarity is in fact so strong that virtually all our observations in section 5.2 remain valid when "vertical reflection" is replaced by "half turn". In particular, color-reversing translation in a **p112**-like border pattern **implies** color-reversing half turn -- check also 7.6.4 -- and that table from 5.2.4 migrates here as follows:

Color-reversing translation	Color-reversing half turn	Border pattern type
Y	Y	**p′112**
Y	N	impossible
N	Y	**p112′**
N	N	**p112**

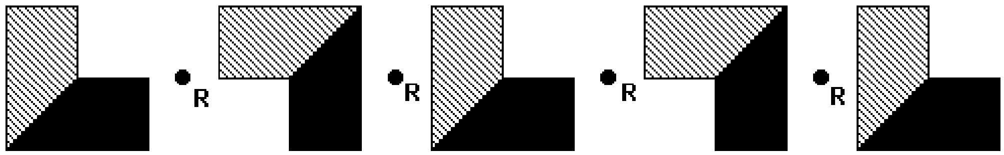
That is, there are precisely three types of patterns in the **p112** group, only two of them genuinely two-colored, shown right below:



p′1 1 2

Fig. 5.25

This type, known as **p′112**, has **both** color-reversing and color-preserving half turn centers, thanks to color-reversing translation. But why is color-reversing translation associated with two adjacent half turn centers (or vertical reflection axes) of **opposite** effect on color? Well, the easiest way to see this right now is to argue as in 5.4.1, observing in particular that the successive application (**'product'**) of two adjacent half turns (or vertical reflections) yields the pattern’s minimal translation; check also 7.5.3 (and 7.2.1)!

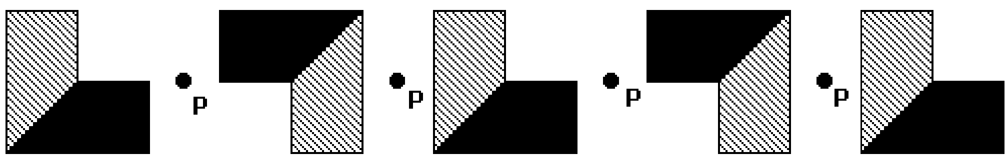


p112′

Fig. 5.26

This type, known as **p112′**, has only color-reversing half turn: now **both** types of centers correspond to color-reversing half turns.

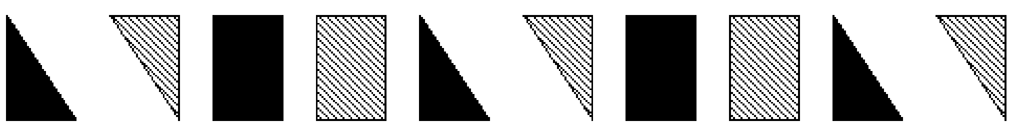
As usual, we are ‘tolerant’ enough to include a ‘two-colored looking’ one-colored pattern in our collection:



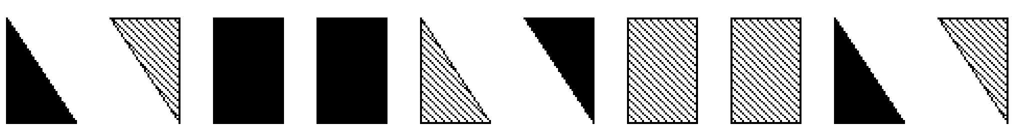
p112

Fig. 5.27

5.5.2 Examples. Here are two colorings of a **p112** pattern that should be compared to the colorings of **pm11** in 5.2.5 (as well as the colorings of **p1a1** in 5.4.2):



p112′



p′1 1 2

Fig. 5.28

5.6 Colorings of pma2

5.6.1 Half turns and mirrors of one kind only! Combining the discussions in 5.4.1 and 5.5.1 we are forced to conclude that in every **pma2**-like two-colored pattern **all** half turns must be either color-preserving or color-reversing, and likewise for vertical reflections. Indeed, the 'combination' of two adjacent half turn centers or vertical reflections of opposite effect on color would produce a color-reversing translation, which is ruled out by the presence of glide reflection!

5.6.2 Another kind of multiplication. As we are going to see in 6.6.2 and 7.7.4, and have already hinted on in 2.6.3, the **pma2**'s glide reflection may be viewed as the 'product' (successive application) of the pattern's half turn and vertical reflection. This means that the glide reflection's effect on color is **completely determined** by those of the half turn and the vertical reflection: if both are either color-preserving (**P**) or color-reversing (**R**), then the glide reflection has to be color-preserving (**P × P = P**, **R × R = P**); and if one is color-preserving (**P**) and the other one is color-reversing (**R**), then the glide reflection must be color-reversing (**P × R = R**, **R × P = R**). This **'multiplication rule'**, partially introduced in 5.4.1 and 5.5.1, is something you should be able to verify on your own: for **P** × **R** (color-reversing isometry **followed** by color-preserving isometry), for example, **B** → **G** → **G** and **G** → **B** → **B**, hence **B** → **G** and **G** → **B**. You may also draw an analogy with ordinary multiplication, thinking of **P** as **'positive'** and **R** as **'negative'**!

5.6.3 Precisely four possibilities. The discussion in 5.6.1 and 5.6.2 allows for a quick determination of all the **pma2** colorings. Indeed it suffices to look only at the pattern's half turn and vertical reflection, each of which has a well defined effect on color (either **P** or **R**), and the following table captures all two × two = four types:

Half turn	Vertical reflection	Pattern type	Glide reflection
P	**P**	**pma2**	**P** × **P** = **P**
P	**R**	**pm′a′2**	**P** × **R** = **R**
R	**P**	**pma′2′**	**R** × **P** = **R**
R	**R**	**pm′a2′**	**R** × **R** = **P**

Once again, we better provide one example per type in order to show that each type is indeed possible:

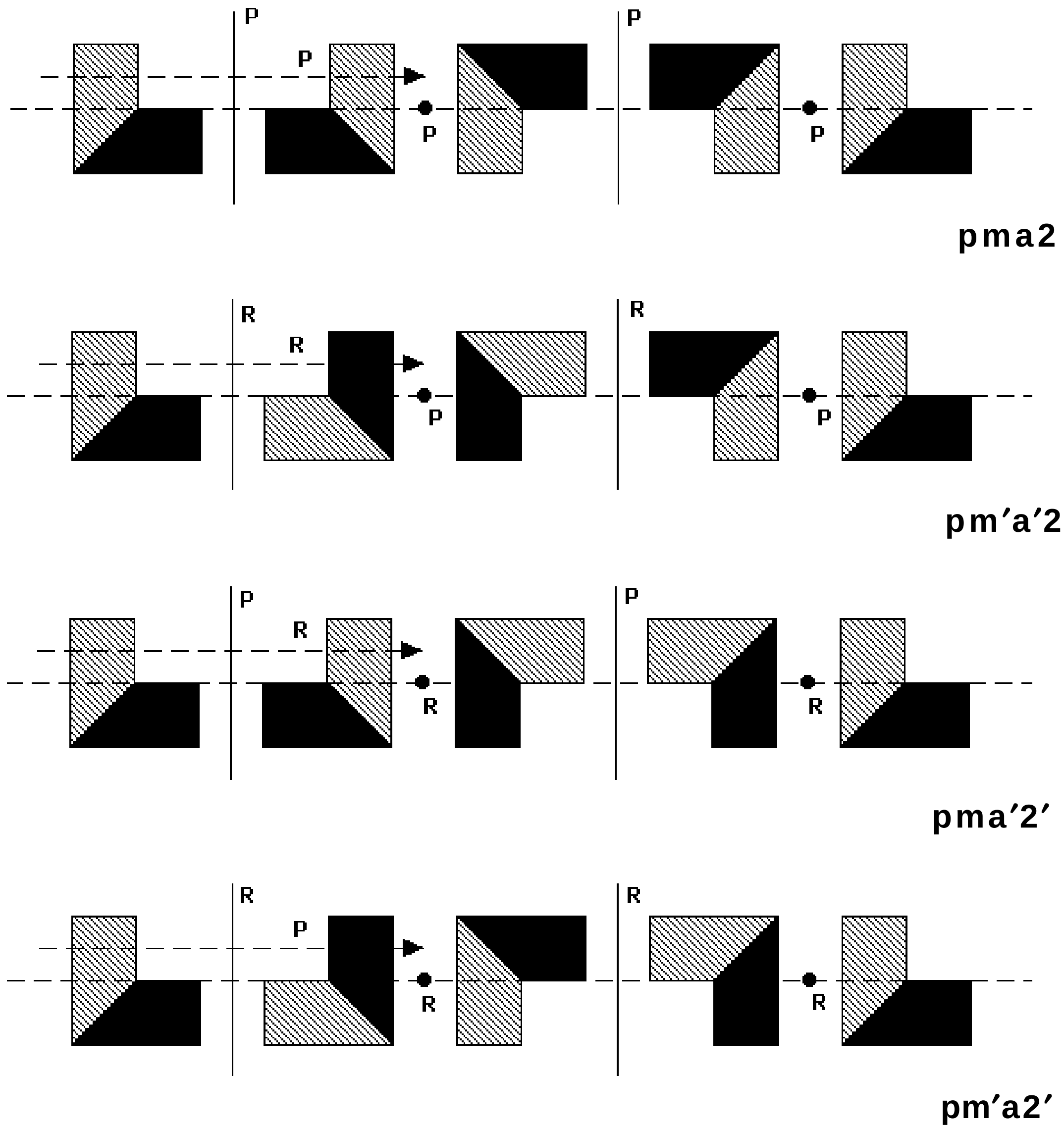
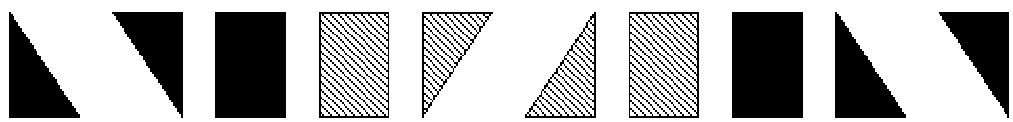


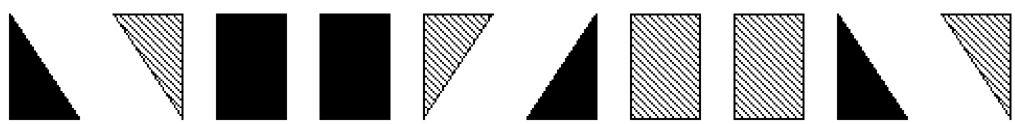
Fig. 5.29

Now you can go back to section 5.0, practice what you just learned and confirm that the two-colored patterns presented there indeed belong to the **pma2** family as follows: **pma′2′** (figures 5.3 & 5.4 (black-grey)), **pm′a2′** (figures 5.5 & 5.6 (blue-green)), and **pm′a′2** (figures 5.6 & 5.7 (yellow-red)).

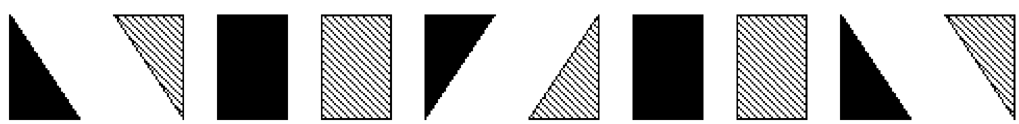
5.6.4 <u>Further examples.</u> Three more **pma2**-like border patterns:



pm′a′2



pma′2′



pm′a2′

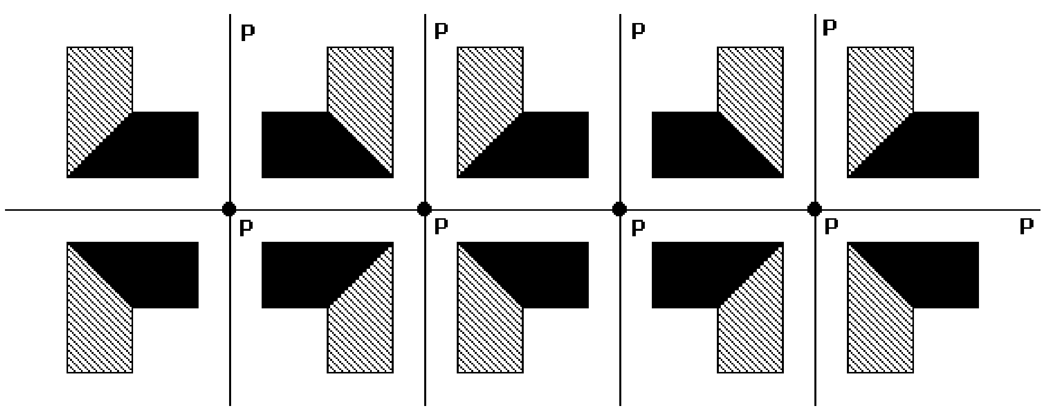
Fig. 5.30

5.7 Colorings of pmm2

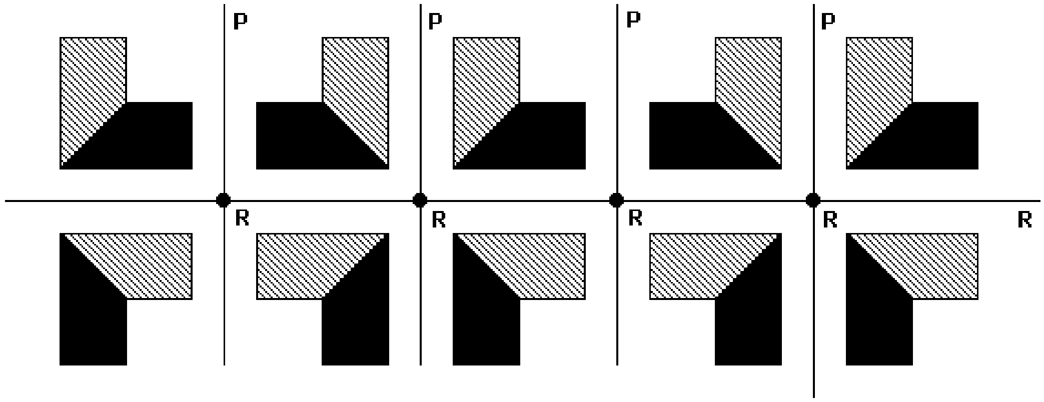
5.7.1 <u>‘Multiplying’ two types now!</u> As we noticed in 4.6.1, the half turn of the **pmm2** border pattern may be seen as the ‘product’ of that pattern’s vertical and horizontal reflections, with the half turn centers found at the **intersection points** of the horizontal reflection axis with the vertical reflection axes. That is, the effect of a two-colored **pmm2**'s half turns on color is determined by the effect on color of its horizontal reflection and vertical reflections, following again the 'multiplication' rules of 5.6.2. It follows that we only need to focus on the effect on color of the horizontal reflection (viewing for a moment our **pmm2** pattern as merely a **p1m1** one), the vertical reflection (now treating the **pmm2** pattern as merely a **pm11** one), and the translation (present of course in both **'factor types'**). Focusing on whether or not **both** 'factors' have color-reversing translation or not, as well as on the color effect of their reflections, we build a **'multiplication table'** as follows:

color-reversing translation	**p1m1** 'factor'	**pm11** 'factor'	**pmm2** type	half turn (H.R.×V.R.)
N	**p1m1**	**pm11**	**pmm2**	**P** only
N	**p1m′1**	**pm11**	**pmm′2′**	**R** only
N	**p1m1**	**pm′11**	**pm′m2′**	**R** only
N	**p1m′1**	**pm′11**	**pm′m′2**	**P** only
Y	**p′1m1**	**p′m11**	**p′mm2**	**P** and **R**
Y	**p′1a1**	**p′m11**	**p′ma2**	**P** and **R**

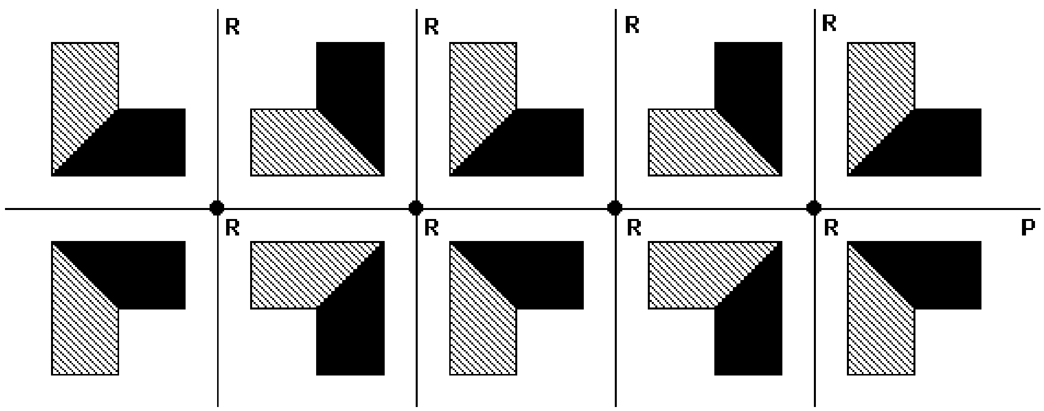
There are many things one could say about this complicated 'multiplication', but we would rather let you discover those on your own and verify our table with the help of the following examples:



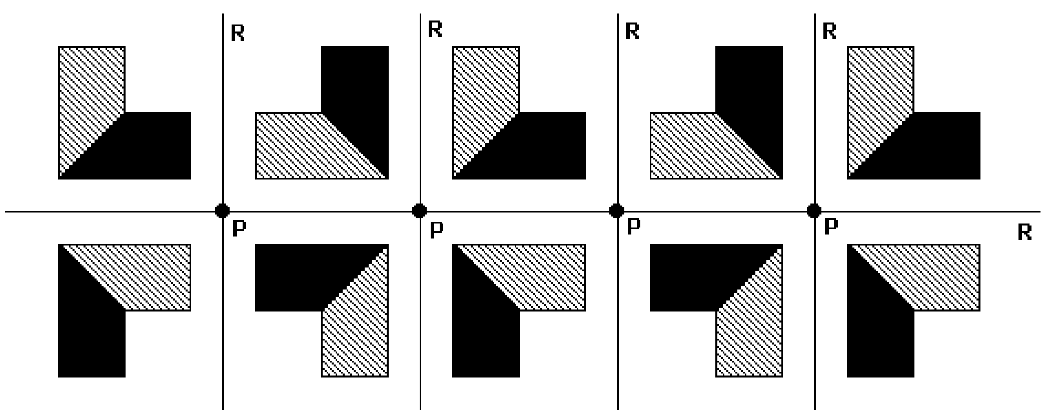
pmm2



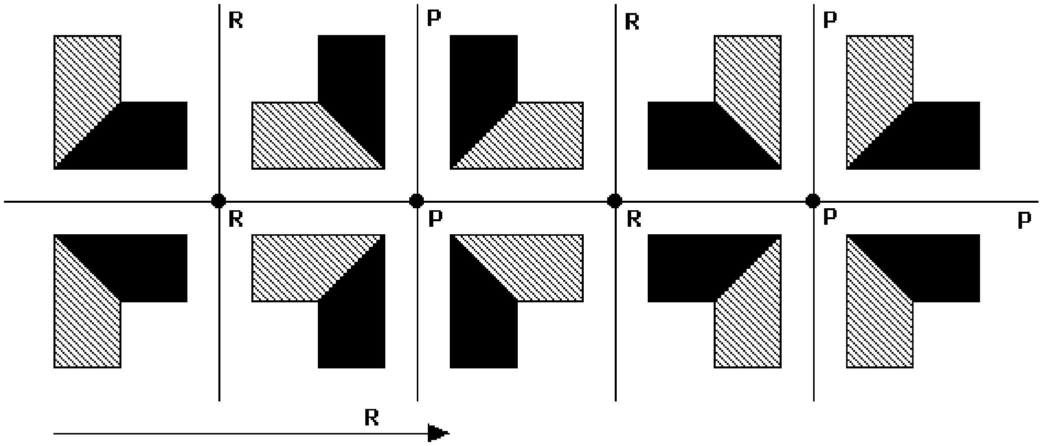
pmm′2′



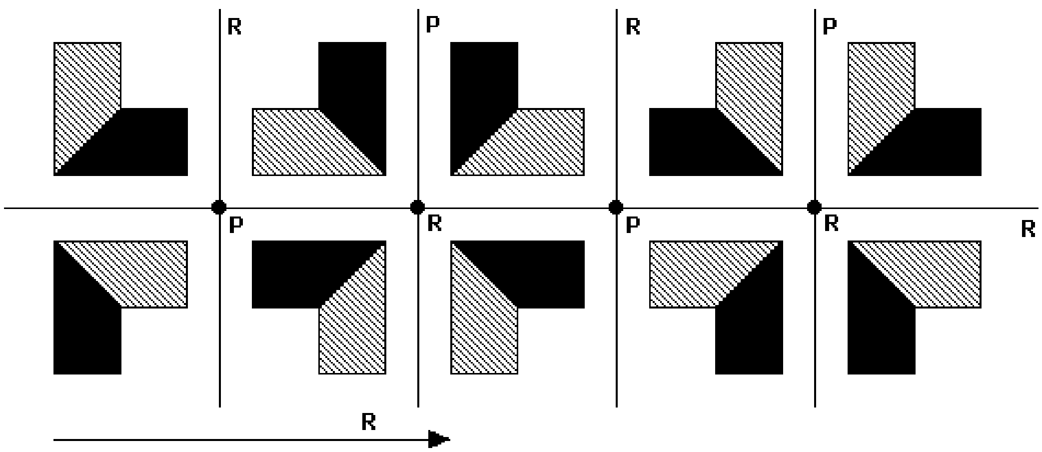
pm′m 2′



pm′m′2



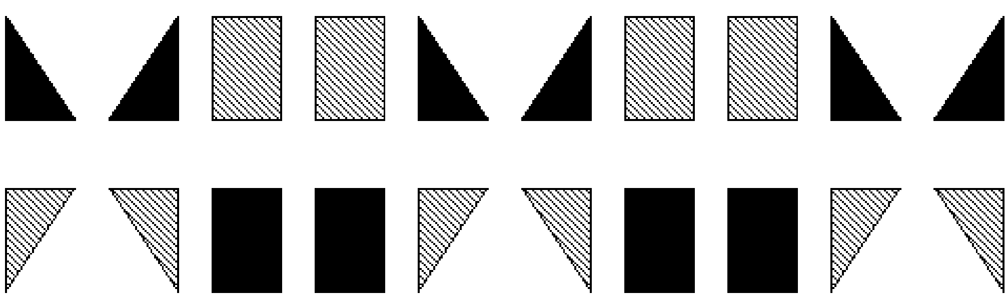
p′m m 2



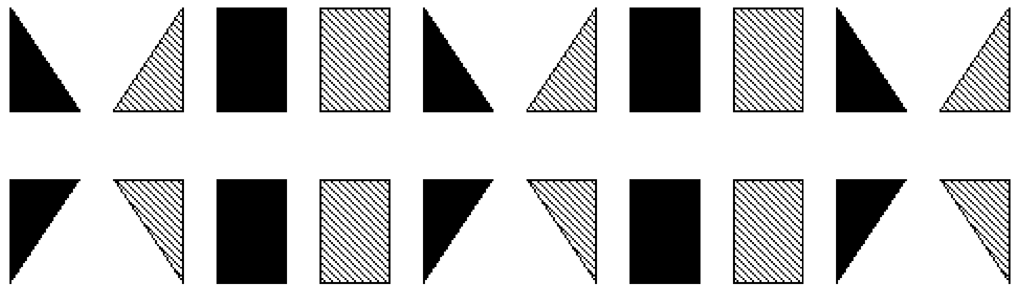
p′ma2

Fig. 5.31

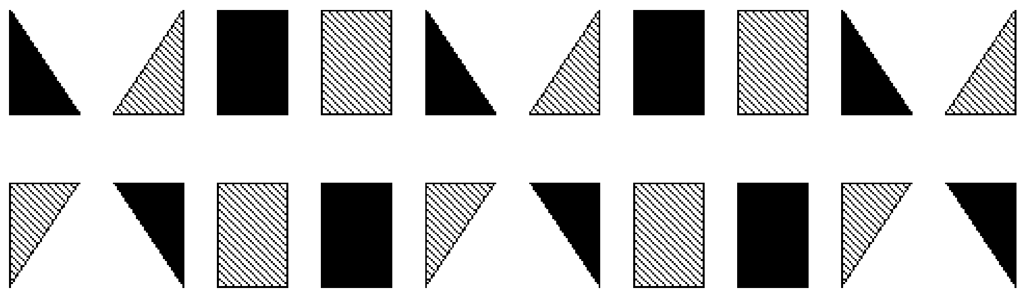
5.7.2 Further examples. Once again, here are five colorings, corresponding to each one of the five new **pmm2**-like patterns:



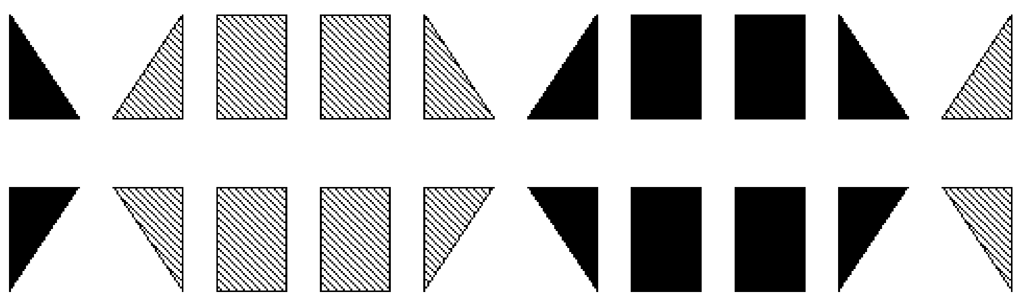
pmm′2′



pm′m 2 ′



pm′m ′2



p′m m 2

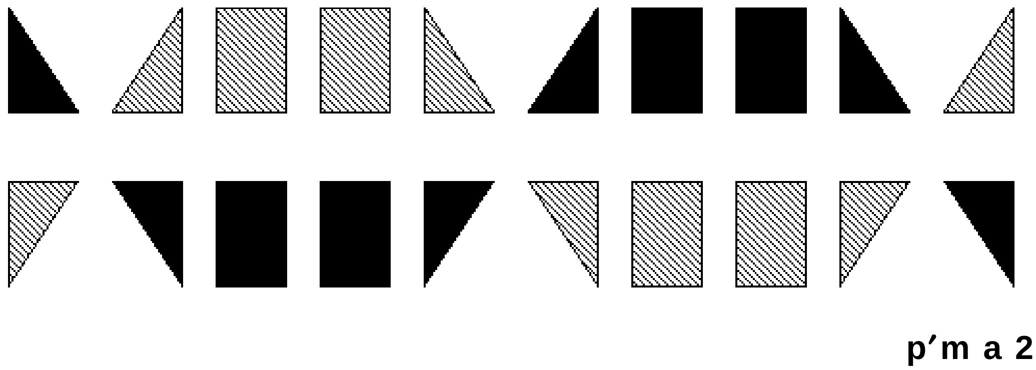


Fig. 5.32

5.8 Consistency with color

5.8.1 What happened to that reflection? Let's try a coloring for the **pma2** pattern of figures 5.1 & 5.2 a bit different from the one featured in figures 5.3 or 5.4:

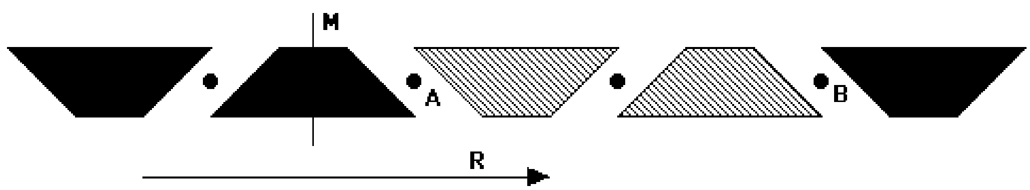


Fig. 5.33

Do the vertical reflection axes carried over from figure 5.1 preserve or reverse colors? The answer is neither "yes" nor "no": looking at axis **M**, for example, we see that it leaves the black trapezoid it bisects inevitably unchanged, but it maps the black trapezoid to its left to the grey trapezoid to its right. One might say that our reflection axis acts **inconsistently** with respect to color, preserving it in some instances and reversing it in others. And it doesn't take long to notice that all reflection axes in figure 5.33 'behave' the same way, being **inconsistent with color**.

How would you classify the pattern in question? First, you would certainly not think of vertical reflection anymore (our pattern **looks** now like a row of **homostrophic** black and grey **parallelograms**, leaving no room for reflection or even glide reflection), but you would probably notice the **color-reversing half turns** centered at **A** and **B** and nothing else: a **p112′**, then? Well, if you remember or review section 5.5 and also notice the pattern's **color-reversing translation**, you will disagree: **p112′** does not have color-reversing translation, hence our pattern must be a **p′112.** But every **p′112** has both color-reversing and color-preserving centers: where are the **color-preserving half turn centers**, in that case? Well, a bit of experience would have led you to look right at the center of each **parallelogram-like pair of trapezoids** of same color, the 'internal' half turn of which naturally extends to the entire border pattern. Alternatively, both color-preserving and color-reversing half turn centers have been 'inherited' from the original **pma2** pattern, hence they should not be missed. One way or another, we have reached a conclusion: the pattern in figure 5.33 is a **p′112**.

5.8.2 Coloring, symmetry, and perception. Back in 5.1.1 we made an 'innocent' remark to the effect that coloring cannot increase symmetry. We are now in a position to complete that remark by stating that **coloring may only decrease symmetry**. Indeed the example discussed in 5.8.1 is an appropriate illustration of this principle, in both **visual** and **conceptual** terms: the two-coloring of a **pma2** border pattern **'eliminated'** its vertical reflection -- and glide reflection, as you should verify on your own -- by rendering it inconsistent with color and reduced it to a **p′112** pattern; and instead of trapezoids, the viewer is now more likely to **'see'** parallelograms!

More generally, the rule born out of the discussion in 5.8.1 is: when classifying a two-colored border pattern, **discard every isometry that is inconsistent with color**.

5.8.3 Inconsistent half turns? Now you might ask: isn't the half turn in the 'trapezoidal' pattern discussed in 5.8.1 inconsistent with color? Do not half of the 180^0 centers reverse colors while the other half preserve them? **Attention!** When you examine consistency with color, you should focus on **one isometry at a time!** Indeed if you think of any individual half turn center in the pattern of figure 5.33, you will confirm for yourself that **it either preserves colors** (mapping every black trapezoid to a black one and every grey trapezoid to a grey one) **or it reverses colors** (mapping every black trapezoid to a grey one and vice versa); that is, every half turn in figure 5.33 is **consistent with color**, one way or another.

Does this mean that half turns are always consistent with color? Not at all! Here is another coloring of the original **pma2** pattern -- to be precise, of a **'partitioned'** version of it (in which every trapezoid is cut into two equal halves) -- that produces a **p′m11** pattern:

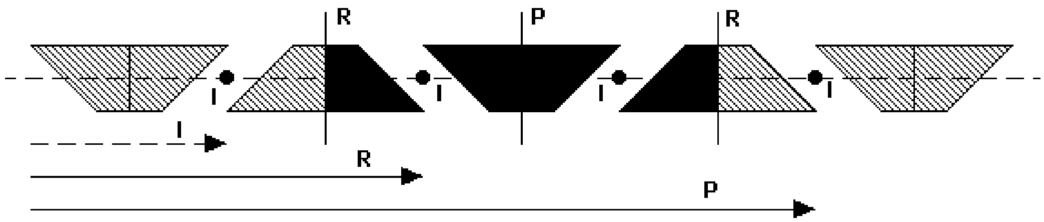


Fig. 5.34

Indeed all half turns are inconsistent with color (a fact denoted by an **I** right next to every half turn center), hence **discarded** (likewise for glide reflection); at the same time, vertical reflection axes alternate between color-preserving (**P**) and color-reversing (**R**).

5.8.4 Both kinds together. Here is yet another coloring of the original **pma2** pattern, producing a **pm′11** this time:

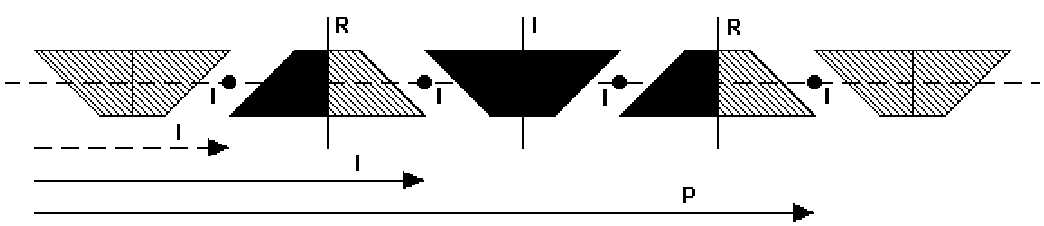


Fig. 5.35

Now **only half** of the vertical reflections in figure 5.35 are consistent, that is color-reversing (**R**); the other half are inconsistent (**I**), hence **discarded** together with the half turns, glide reflection, and even inconsistent translation! (There is still a color-preserving translation, of course, but the color-reversing one in figure 5.34 has been rendered inconsistent by our modification of coloring.)

5.8.5 Consistent glide reflection? Both colorings of the **pma2** pattern presented in 5.8.3 and 5.8.4 eliminated its glide reflection by rendering it inconsistent with color: are there any colorings that eliminate the pattern's half turn **and** vertical reflection but preserve its glide reflection? The answer is "yes", but such colorings are a bit harder to come up with; here is a coloring that reduces the **pma2** to a **p1a′1**, exhibited on a 'compressed' version of the original pattern (with the length of each trapezoid cut in half):

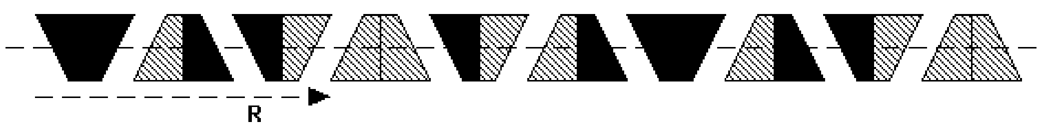
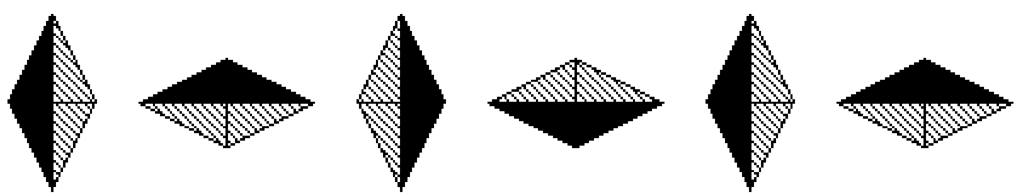
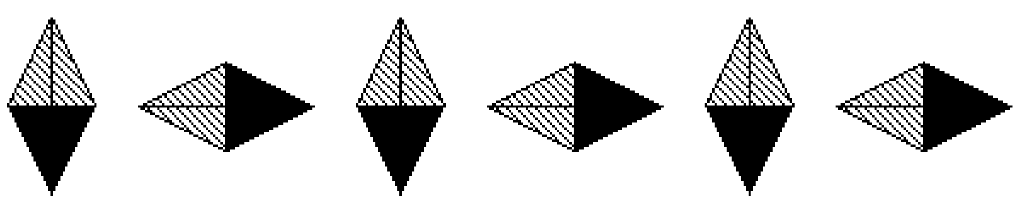


Fig. 5.36

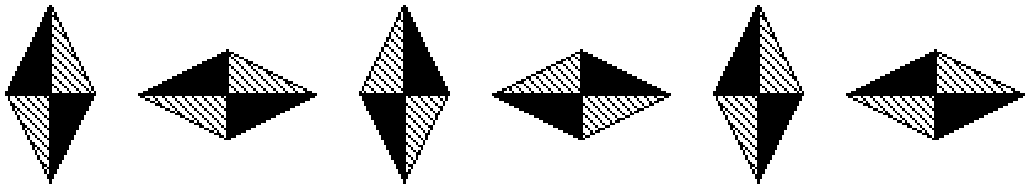
5.8.6 Further examples. Here are some inconsistent colorings reducing a 'partitioned diamond' **pmm2** pattern to 'lower' types:



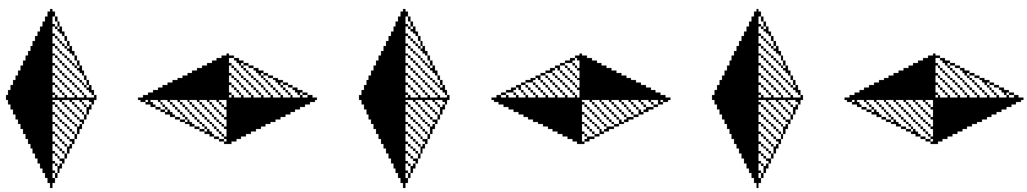
p′m11



p112′



p′1 a 1



pm′a 2′

Fig. 5.37

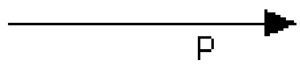
5.9 Symmetry plans

5.9.1 <u>The abstract way of looking at it.</u> As we indicated in 5.2.3, from the purely mathematical point of view, and for mere classification purposes, all that matters in a border pattern is its symmetry elements (isometries) and their effect on color, captured in what we called **symmetry plan**. Here we present symmetry plans for all **twenty four** types of two-colored border patterns: seventeen genuinely two-colored ones (introduced in this chapter) and seven one-colored ones (introduced in chapter 2); it is of course the various colorings of these seven **'parent types'** (presented below in bold face print) that generate the other seventeen types, hence the latter are appropriately grouped under the former.

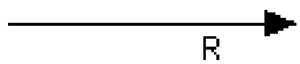
When trying to classify a border pattern, you should first locate its parent type and then match it with one of the parent type's 'offspring'. Do not forget that **isometries inconsistent with color --** which should **still** be marked with an **I -- are not taken into account at all**: there are no **I**s in the symmetry plans below!

Symmetry Plan notation: **solid** lines represent **translation** vectors (indicated even when color-preserving) and **reflection** axes; **dotted** lines represent **glide reflection** vectors and axes.

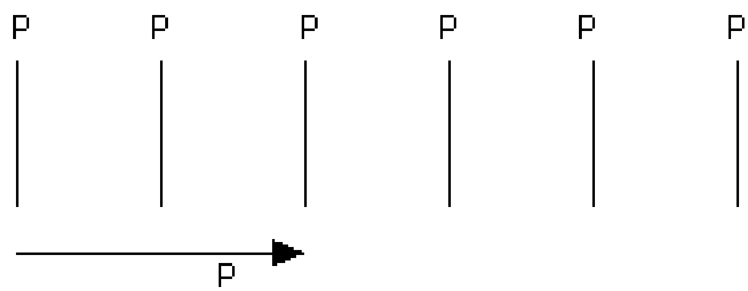
p111



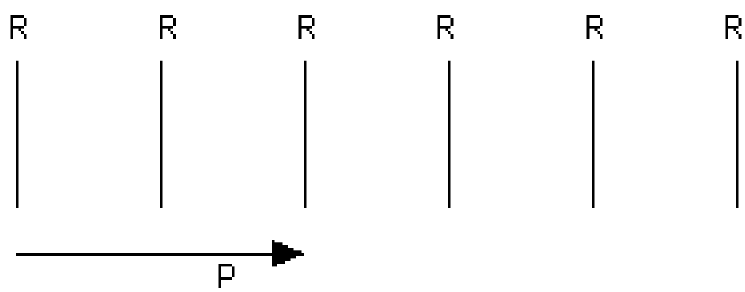
p'111



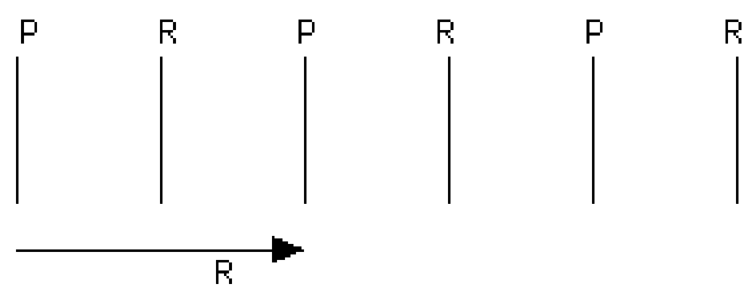
pm11

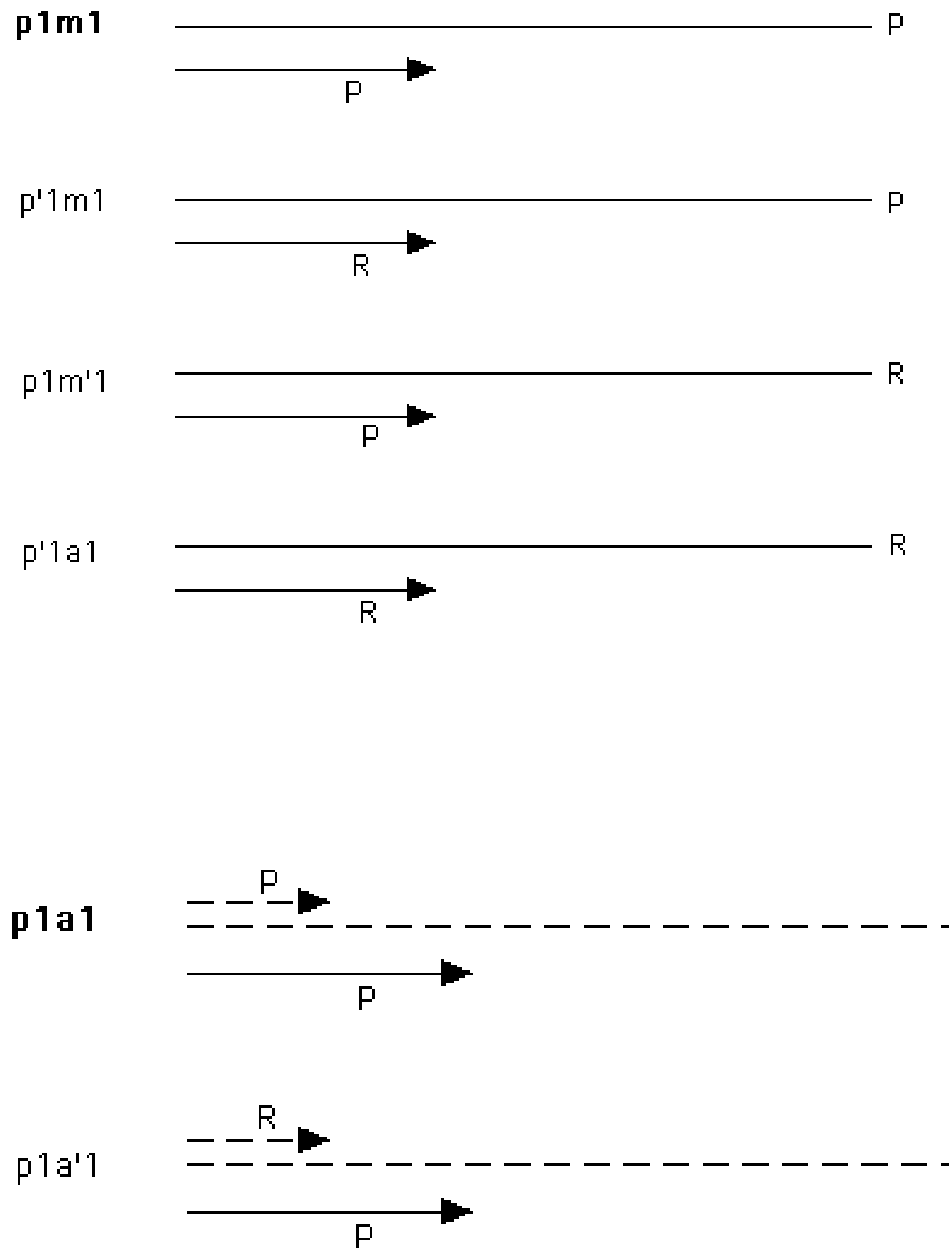


pm'11

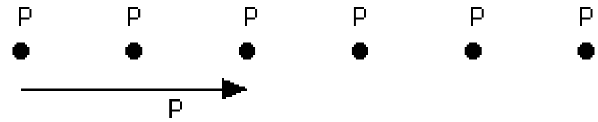


p'm11

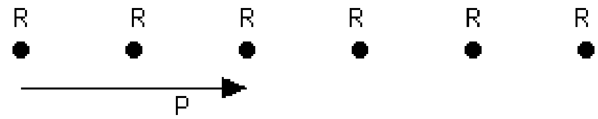



p1m1
P
P
p'1m1
P
R
p1m'1
R
P
p'1a1
R
R
p1a1
P
P
p1a'1
R
P

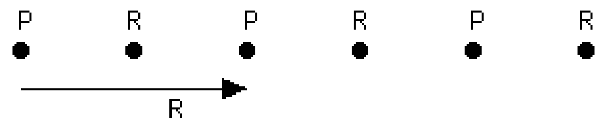
p112



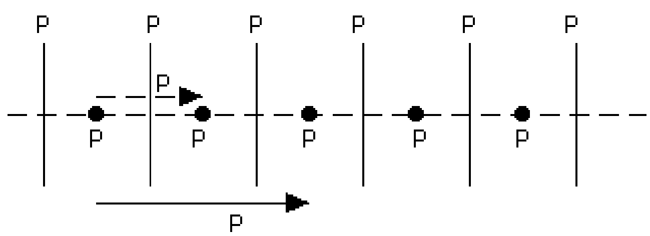
p112'



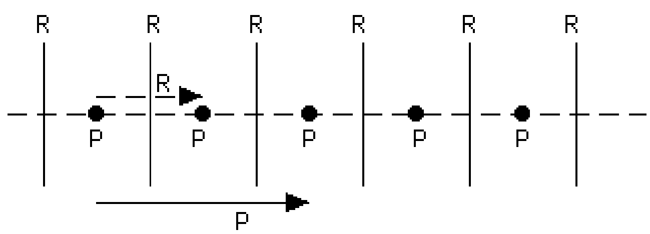
p'112



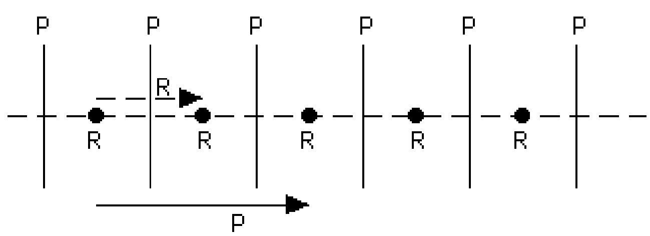
pma2



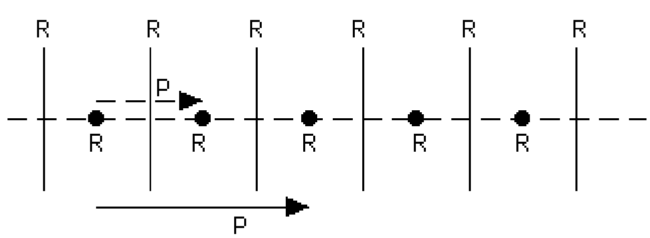
pm'a'2



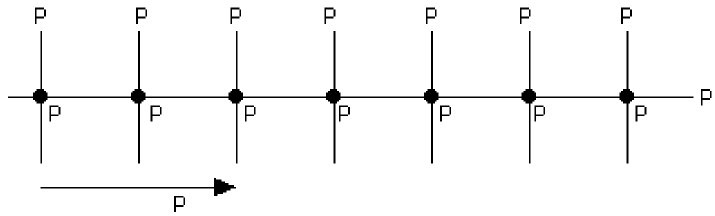
pma'2'



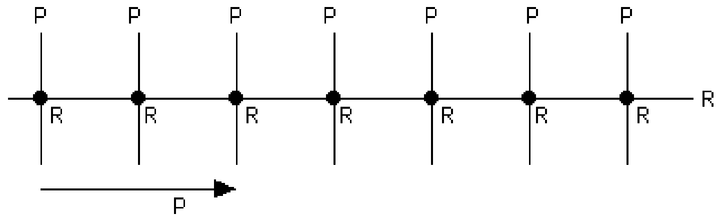
pm'a2'



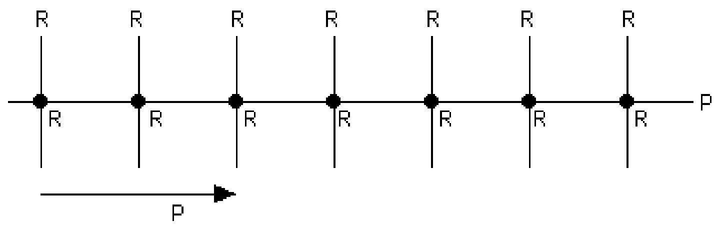
pmm2



pmm'2'



pm'm2'



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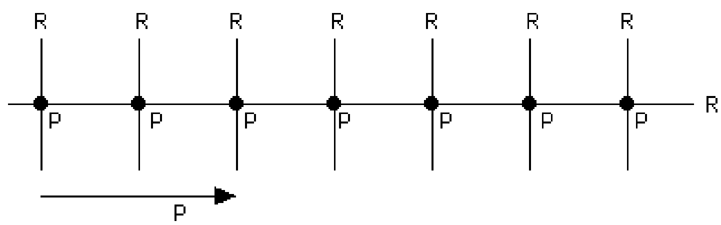

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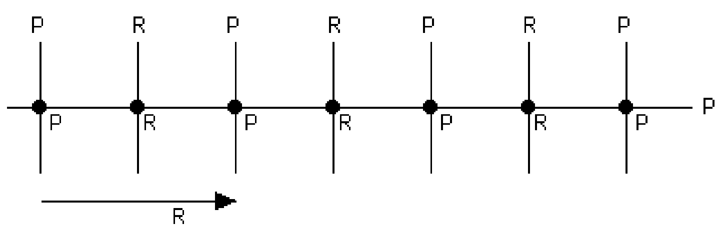

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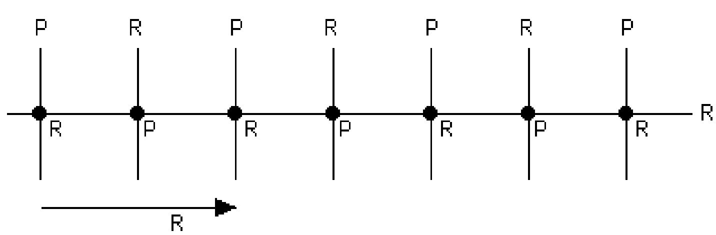

first draft: winter 1999