A note on the Koszul homology of ordinary singularities

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Introduction

A great deal has by now been said about syzygies of an ideal in a noetherian ring. The theory is particularly enhanced in the case of a homogeneous ideal in a polynomial ring (over a field), where there is a pervasive invariance not to mention the concrete geometry waiting in the corner.

However, it is often the case when one needs precise information on the length of syzygies and the degrees of the generating cycles. I have personally found it very instructive, if not terribly important, to juggle with such computations. Recently, D. Buchsbaum and D. Eisenbud have together with some students engaged in developing a computer program to quickly recognize syzygies. Not seldom does happen such a computation to shed some light back into the theory.

It is my purpose in this note to give one instance of this phenomenon. Explicitly, I am concerned with classifying projective plane curves by means of invariants of the corresponding "gradient ideals". A complete classification is obtained for cubic curves.

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1. Gradient Ideals and Syzygies

Some general remarks.

Let R be a commutative ring with identity. Let M be an R-submodule of $R^n = R \oplus \ldots \oplus R$ (the elements of M will often be referred to as *vectors* of dimension n). Given a ring-automorphism $T: R \to R$ we let M^T denote the subset of R^n whose elements are the vectors $(T(F_1), \ldots, T(F_n))$ such that $(F_1, \ldots, F_n) \in M$. Since T admits an inverse, it readily follows that M^T is also an R-submodule of R^n. Clearly, the canonical mapping $\tau: M \to M^T$, given by $(F_1, \ldots, F_n) \to (T(F_1), \ldots, T(F_n))$ is a T-isomorphism, that is to say, it is an isomorphism of additive groups and, moreover, $\tau(G(F_1,\ldots,F_n)) = T(G)\,\tau(F_1,\ldots,F_n)$ for every $G\in R$ and for every $(F_1,\ldots,F_n)\in M$.

We now specialize to the case where $R = k[X_1,\ldots,X_n]$, a polynomial ring over a field, and T is a homogeneous linear change of coordinates. If $f\in R$, we have the *gradient ideal* of f, namely, the ideal generated by the partial derivatives $\partial f/\partial X_1,\ldots,\partial f/\partial X_n$. Let $T(X_i) = \sum_j a_{ij}X_j$, $i = 1,\ldots,n$. Then the matrix $A = (a_{ij})$ is invertible. A direct computation with derivatives shows that

$$\left(T\left(\frac{\partial f}{\partial X_1}\right),\ldots,T\left(\frac{\partial f}{\partial X_n}\right)\right)\cdot {}^tA = \left(\frac{\partial T(f)}{\partial X_1},\ldots,\frac{\partial T(f)}{\partial X_n}\right),$$

where ${}^tA$ denotes the transpose of A. It follows, in particular, that the gradient ideal of the transform $T(f)$ is also generated by the transforms $T(\partial f/\partial X_1),\ldots,T(\partial f/\partial X_n)$.

We next prove a basic lemma about the above transforms and their modules of "relations".

Lemma 1.1. *Notation as above. Let $\{e_1,\ldots,e_n\}$ stand for the canonical basis of R^n and let $\varphi: R^n\to R$ (respectively, $\psi: R^n\to R$) be the R-homomorphism defined by $\varphi(e_i) = \partial f/\partial X_i$ (respectively, $\psi(e_i) = \partial T(f)/\partial X_i$), $i = 1,\ldots,n$. Then $\ker\psi = B((\ker\varphi)^T)$, where $B = ({}^tA)^{-1}$.*

Proof. One has

$$\left(\frac{\partial T(f)}{\partial X_1},\ldots,\frac{\partial T(f)}{\partial X_n}\right)\cdot B = \left(T\left(\frac{\partial f}{\partial X_1}\right),\ldots,T\left(\frac{\partial f}{\partial X_n}\right)\right).$$

Set $B = (b_{ij})$. Then, for $(F_1,\ldots,F_n)\in R^n$, one has

$$\begin{aligned}T(\varphi(F_1,\ldots,F_n)) &= T\left(F_1\frac{\partial f}{\partial X_1}+\ldots+F_n\frac{\partial f}{\partial X_n}\right) = T(F_1)\,T\left(\frac{\partial f}{\partial X_1}\right)+\ldots+\\ &+T(F_n)\,T\left(\frac{\partial f}{\partial X_n}\right) = T(F_1)\sum_i b_{i1}\,T\left(\frac{\partial f}{\partial X_i}\right)+\ldots+\\ &+T(F_n)\sum_i b_{in}T\left(\frac{\partial f}{\partial X_i}\right) = \left(\sum_j b_{1j}T(F_j)\right)\frac{\partial T(f)}{\partial X_1}+\ldots+\\ &+\left(\sum_j b_{nj}T(F_j)\right)\frac{\partial T(f)}{\partial X_n} = (\psi(T(F_1),\ldots,T(F_n))).\,B.\end{aligned}$$

It follows from this equality that $(\ker\varphi)^T$ is mapped into $\ker\psi$ by means of B. To see the inclusion in the other direction, note that the first inclusion yields $(\ker\varphi)^T\subset B^{-1}(\ker\psi)$. By a symmetric argument, one obtains $(\ker\psi)^{T^{-1}}\subset C^{-1}(\ker\varphi)$, where $C = ({}^t(A^{-1}))^{-1}$. Consequently, $\ker\psi = (\ker\psi)^{T^{-1}T}\subset (C^{-1}(\ker\varphi))^T = ({}^t(A^{-1})(\ker\varphi))^T = (B(\ker\varphi))^T\subset B((\ker\varphi)^T)$, as required.

Remark. The elements of $\ker\varphi$ as in Lemma 1.1 are identified with the 1-cycles of the Koszul complex attached to $\partial f/\partial X_1,\ldots,\partial f/\partial X_n$. Henceforth, they will be referred to simply as cycles. In general, for a given sequence of elements $f_1,\ldots,f_n$, $Z_1(f_1,\ldots,f_n)$ will denote the module of cycles of the sequence.

Now, let $I\underset{\neq}{\subset} R = k[X_1,\ldots,X_n]$ be an arbitrary homogeneous ideal. By Hilbert's theorem on syzygies $[H]$, one can find a resolution

$$0\longrightarrow F_r\xrightarrow{\varphi_r}\ldots\xrightarrow{\varphi_2}F_1\xrightarrow{\varphi_1}R\longrightarrow R/I\longrightarrow 0$$

where:

(i) F_t is a graded free module and φ_t is a homogeneous homomorphism of degree 0, for $t = 1,\ldots,r$. Besides, $F_r\neq(0)$.

(ii) $Im(\varphi_t)\subset(X_1,\ldots,X_n)\,F_{t-1}$, $t = 1,\ldots,r$.

Such a resolution is called a *minimal free (homogeneous) resolution* of the R-module R/I. Any two such resolutions of R/I are R-isomorphic [Se, Remarque, p. IV-48], so, in particular, the rank of F_r is well defined. Call the type of the ideal I the rank of F_r.

Let me briefly recall how to construct a minimal free resolution of I. Thus, let $\{f_1,\ldots,f_m\}$ be a minimal set of homogeneous generators of I and let $\deg(f_i) = d_i$, $i = 1,\ldots,m$. Set $R = \bigoplus_{d\geq 0}R_d$ for the standard gradation of $R = k[X_1,\ldots,X_n]$. For any integer p, let $R(-p)$ denote the graded R-module defined by $R(-p)_d = R_{d-p}$, $d\geq 0$. Clearly, $R(-p)\simeq R$ as R-modules, but the gradation has undergone a shift.

Consider the graded direct sum $\bigoplus_{i=1}^m R(-d_i)$, where $(\bigoplus_{i=1}^m R(-d_i))_d = \bigoplus_{i=1}^m (R(-d_i)_d)$, $d\geq 0$. One may identify $\bigoplus_{i=1}^m R(-d_i)$ with R^m as non-graded R-modules, although the gradation of $\bigoplus_{i=1}^m R(-d_i)$ is not the standard gradation of R^m. Rather, the basis vector $(0,\ldots,0,1,0,\ldots,0)$, with 1 in the i^{th}-place, is a homogeneous vector of degree d_i in $\bigoplus_{i=1}^m R(-d_i)$. This allows one to define a homogeneous homomorphism of degree 0,

$$\varphi_1:\bigoplus_{i=1}^m R(-d_i)\to R,$$

by sending $(0,\ldots,0,\underset{i}{1},\ldots,0)$ to f_i, $i = 1,\ldots,m$.

Let $S_1 = \ker \varphi_1$, a homogeneous R-submodule of $\bigoplus_{i=1}^{m} R(-d_i)$. Let $(\phi_{1j}, \ldots, \phi_{mj})$, $j = 1, \ldots, q$, be a minimal set of homogeneous generators of S_1. Then each ϕ_{ij} is homogeneous in the standard gradation of R and, moreover, if $\deg(\phi_{ij}) = d_{ij}$ in this gradation, one must have the equalities $d_1 + d_{1j} = d_2 + d_{2j} = \ldots = d_m + d_{mj}$ and this common integer is the degree of the vector $(\phi_{1j}, \ldots, \phi_{mj})$ as element of the graded module $\bigoplus_{i=1}^{m} R(-d_i)$, for $j = 1, \ldots, q$.

Next look at the graded free module $\bigoplus_{j=1}^{q} R(-d_1 - d_{1j})$. As before, one gets a homogeneous homomorphism of degree 0:

$$\varphi_2 : \bigoplus_{j=1}^{q} R(-d_1 - d_{1j}) \to \bigoplus_{i=1}^{m} R(-d_i)$$

$$(0, \ldots, 0, 1, 0, \ldots, 0) \to (\phi_{1j}, \ldots, \phi_{mj})$$

Let $S_2 = \ker \varphi_2$, and so on. As a final remark, the length of the so constructed resolution (i.e., the number of non-zero free modules apart from R itself) must equal $\mathrm{h.d.}_R(R/I)$ – the homological dimension of the R-module R/I. Also, the alternate sum of the ranks of the nonzero free modules F_t is 0; this is seen, e.g., by passing to the corresponding long exact sequence of vector spaces over the field of rational functions $k(X_1, \ldots, X_n)$.

2. Non-degenerate Ideals

The following concept will play a central role in this and next paragraph.

Definition 2.1. Let $I \subset R = k[X_1, \ldots, X_n]$ be a homogeneous ideal and assume that I is minimally generated by homogeneous polynomials $f_1, \ldots, f_m$ of the same degree; let d denote this common degree. I will call the ideal I *non-degenerate* if the R-module $Z_1(f_1, \ldots, f_m)$ can be generated by homogeneous vectors of degree $\leq 2d$.

Here, the degree of a homogeneous vector of $Z_1(f_1, \ldots, f_m)$ is to be understood in the sense of the graded free module $R(-d) \oplus \ldots \oplus R(-d)$. Equivalently, the definition is to the effect that the coordinates of the generators of $Z_1(f_1, \ldots, f_m)$ have (usual) degree $\leq d$.

The definition apparently depends on the given set of generators, but actually it does not. Indeed, if $\{g_1, \ldots, g_r\}$ is any other minimal set of homogeneous generators of I then, in the first place, $r = m$ ($= \dim(I/(X_1, \ldots, X_n)I$ as k-vector space) and next, $\deg(g_i) \geq d$, for every i. Finally, if $\deg(g_i) > d$, for some i, then the original f's are k-linear combinations of at most $m - 1$ of the g's; but this is impossible as the f's are k-linearly independent. Consequently, all the g's have degree d as well. A fortiori, the passage from the f's to the g's is accomplished by means of a matrix in $GL(m; k)$ and one can then use the same technique as the one employed in Lemma 1.1 to deduce that, degreewise, $Z_1(g_1, \ldots, g_r)$ does not differ from $Z_1(f_1, \ldots, f_m)$.

It is as yet not clearly understood what ingredients intervence in the cooking-up of a non-degenerate ideal I. At any rate, there is the following result, which will be fully used here.

Proposition 2.2. (*Lazard*) *If I is generated by at most quadratic polynomials (i.e., $d \leq 2$ in Definition 2.1), then I is non-degenerate.*

The proposition is a very special case of a general result that determines upper bounds for the degrees of the generators of the module $Z_1(f_1, \ldots, f_m)$. This result is to be found in [L, Prop. 6].

Let us agree to call a homogeneous ideal $I \subset k[X_1, \ldots, X_n]$ *strongly non-degenerate* if it is non-degenerate and, moreover, the minimel generators of $Z_1(f_1, \ldots, f_m)$ have degree $2d$ exactly. For example, if the elements $f_1, \ldots, f_m$ form a regular sequence then $I = (f_1, \ldots, f_m)$ is a strongly non-degenerate ideal.

Let me proced to the main result.

Theorem 2.3. *Let $f \in k[X, Y, Z]$ be an irreducible homogeneous polynomial and let I denote its gradient ideal. The following conditions are equivalent:*
(i) f is the equation of a nodal cubic in $\mathbb{P}_k^2$.
(ii) I is strongly non-degenerate and has arithmetic genus 0.

Proof. (i) ⇒ (ii) Both assertions in (ii) can be checked once known explicitly the ranks of the free modules in a minimal free (homogeneous) resolution of I as well as the degrees of the cycles along such a resolution. So, proceed to compute these ingredients.

Firstly, in order to compute minimal generators for $Z_1(\partial f/\partial X, \partial f/\partial Y, \partial f/\partial Z)$ it suffices, according to Lemma 1.1 and the classical theory of cubics, to work with the standard nodal cubic $f = Y^2Z - X^2(X + Z)$. By Lemma 2.2, I is non-degenerate, so it is enough to look for cycles whose coordinates have degree ≤ 2.

Thus, let (G_1, G_2, G_3) be a 1-cycle such that $\deg(G_1) \leq 2$. I claim that $G_1 \in (XZ, YZ, Y^2 - X^2)$. In fact, the gradient ideal of f is $(X(3X + 2Z), YZ, Y^2 - X^2)$, so $G_1 X(3X + 2Z) \in (YZ, Y^2 - X^2)$. Since each of the polynomials YZ and $Y^2 - X^2$ vanishes at the points $(1:1:0)$ and $(1:-1:0)$ of $\mathbb{P}^2$ and since $X(3X+2Z)$ does not vanish at either of these points, it follows that G_1 vanishes on both points.

Now, if $\deg(G_1) = 1$ then G_1 must be of the form αZ, for some $\alpha \in k$. Therefore, in this case, one gets the relation

$$\alpha ZX(3X + 2Z) + G_2 YZ + G_3(Y^2 - X^2) = 0,$$

from which it readily follows that $X(3X + 2Z) \in (Y, Y^2 - X^2) = (X^2, Y)$. But $X \notin (X^2, Y)$ and no power of $3X + 2Z$ can possibly belong to (X^2, Y). As (X^2, Y) is a primary ideal, we have a contradiction.

Thus, one ought to have $\deg(G_1) = 2$ and G_1 is then a member of the the linear system of conics through $(1:1:0)$ and $(1:-1:0)$. Therefore, $G_1 \in (XZ, YZ, Z^2, Y^2 - X^2)$. By an elementary argument, entirely similar to the one above, one sees that Z^2 "cannot appear", i.e. actually G_1 belongs to $(XZ, YZ, Y^2 - X^2)$, proving the claim.

To continue, observe that $(XZ, -Y(3X+2Z), Z(3X+2Z))$ is a cycle. Thus, using a simple induction device – going back to Hilbert – one has only to produce cycles whose first coordinate is 0 in order to describe a set of generators for $Z_1(\partial f/\partial X, \partial f/\partial Y, \partial f/\partial Z)$. But YZ and $Y^2 - X^2$ are relatively prime, so the only such cycle is the boundary $(0, Y^2 - X^2, -YZ)$ (and its multiples). Thus, altogether $Z_1(\partial f/\partial X, \partial f/\partial Y, \partial f/\partial Z)$ is generated by the cycles $(XZ, \; Y(3X+2Z), \; Z(3X+2Z))$, $(YZ, -X(3X+2Z), 0)$, $(Y^2 - X^2, 0, -X(3X+2Z))$ and $(0, Y^2 - X^2, -YZ)$.

Going back to the original nodal cubic f and its gradient ideal I, the minimal free (homogeneous) resolution of R/I has the form

$$0 \to F_3 \to R(-4) \oplus R(-4) \oplus R(-4) \oplus R(-4) \to$$
$$\to R(-2) \oplus R(-2) \oplus R(-2) \to R \to R/I \to 0,$$

where $F_3 = R(-4-e_{11}) \oplus R(-4-e_{12})$, for certain e_{11}, e_{12} which will soon be evaluated (cf. final remarks at the end of Section 1).

In particular, I is strongly non-degenerate. Furthermore, one can actually compute the arithmetic genus of I by using Hilbert's formula [Z. S, Ch. VII, 13]. I will isolate this computation in the form of a general lemma as it will be needed, in this general formulation, for the implication (ii) ⇒ (i).

Here is how it reads.

Lemma 2.4. *Let $I = (f_1, f_2, f_3) \subset k[X, Y, Z]$ be a homogeneous ideal such that $\deg(f_1) = \deg(f_2) = \deg(f_3) = d$, the f's forming a minimal set of generators. If I has height ≥ 2 then R/I admits a minimal free resolution of the form*

$$0 \to \bigoplus_{k=1}^{r-2} R(-d-d_{11}-e_{1k}) \to \bigoplus_{j=1}^{r} R(-d-d_{1j}) \to$$
$$\to R(-d) \oplus R(-d) \oplus (-d) \to R \to R/I \to 0,$$

where $r \geq 2$, Moreover,

$$d = \sum_{j=1}^{r} d_{1j} - \sum_{k=1}^{r-2} (d_{11} + e_{1k})$$

$$\text{(1)} \qquad p_a(I) = (-1)^s \left[\frac{1}{2}\left(d^2 + \sum_{j=1}^{r} d_{1j}^2 - \sum_{k=1}^{r-2} (d_{11} + e_{1k})^2\right) - 1 \right],$$

where $p_a(I)$ denotes the arithmetic genus of I and $s = $ Krull-dim $(R/I) - 1$.

N.B. *It is understood that for $r = 2$, $\bigoplus_{k=1}^{r-2} R(-d-d_{11}-e_{1k}) = 0$ (this will happen if and only if $h.d._R(R/I) = 2$) and also $\sum_{k=1}^{r-2} (d_{11} + e_{1k}) = \sum_{k=1}^{r-2} (d_{11} + e_{1k})^2 = 0$.*

Proof. Since I has height ≥ 2, then $2 \leq h.d._R(R/I) \leq 3$, so R/I admits indeed a minimal free resolution of the indicated form. Now use Hilbert's formula. Letting $H(R/I, q)$ denote the Hilbert function of R/I at q, one has (for $q >> 0$):

$$H(R/I.q) = \binom{q+2}{2} - 3\binom{q+2-d}{2} + \sum_{j=1}^{r} \binom{q+2-d-d_{1j}}{2} - \sum_{k=1}^{r-2} \binom{q+2-d-d_{11}-e_{1k}}{2}$$

$$= \frac{1}{2}\Big[(q+2)(q+1) - 3(q-(d-2))(q-(d-1)) + \sum_{j=1}^{r} (q-(d+d_{1j}-2))(q-(d+d_{1j}-1)) - \sum_{k=1}^{r-2} (q-(d+d_{11}+e_{1k}-2))(q-(d+d_{11}+e_{1k}-1))\Big].$$

This expression is a polynomial $h_1 q + h_0$ of degree $\leq$ in q. As R/I has Krull-dimension ≤ 1, necessarily $h_1 = 0$. Therefore, computing h_1 explicitly, one straighforwardly obtains the first claimed relation:

$$d = \sum_{j=1}^{r} d_{1j} - \sum_{k=1}^{r-2} (d_{11} + e_{1k}).$$

Likewise, a similar computation for h_0 yields:

$$2h_0 = 2 - 3(d-2)(d-1) + \sum_{j=1}^{r} (d+d_{1j}-2)(d+d_{1j}-1) -$$
$$- \sum_{k=1}^{r-2} (d+d_{11}+e_{1k}-2)(d+d_{11}+e_{1k}-1)$$
$$= 2 - 3(d-2)(d-1) + r(d-2)(d-1) - (r-2)(d-2)(d-1) + (2d-3)\sum_{j=1}^{r} d_{1j}$$
$$- (2d-3)\sum_{k=1}^{r-2} (d_{11}+e_{1k}) + \sum_{j=1}^{r} d_{1j}^2 - \sum_{k=1}^{r-2} (d_{11}+e_{1k})^2$$
$$= 2 - (d-2)(d-1) + d(2d-3) + \sum_{j=1}^{r} d_{1j}^2 - \sum_{k=1}^{r-2} (d_{11}+e_{1k})^2,$$

by substituting d according to the first relation;

$$= d^2 + \sum_{j=1}^{r} d_{1j}^2 - \sum_{k=1}^{r-2} (d_{11}+e_{1k})^2.$$

The required expression for the arithmetic genus will then follow suit by recalling that $p_a(I) = (-1)$, where $s = \text{Krull-dim}(R/I) - 1$.

Now, back to the proof of the theorem.

Since the gradient ideal of the nodal cubic is strongly non-degenerate (hence, automatically, $r \geq 3$), expressions (1) reduce, respectively, to the following ones:

$$d = \sum_{k=1}^{r-2} e_{1k}$$

(2)

$$p_a(I) = (-1)^s \left[\frac{1}{2}(d^2 - \sum_{k=1}^{r-2} e_{1k}^2) - 1\right], \quad s = \text{Krull-dim}(R/I) - 1.$$

Moreover, as has been shown before, $d = 2$ and I is of type 2 (i.e., $r = 4$). Therefore, necessarily, $e_{11} = e_{12} = 1$ and $p_a(I) = \frac{1}{2}(4-2) - 1 = 0$.

This proves the implication (i) $\Rightarrow$ (ii).

As to the reverse implication, one is given, first of all, that I is strongly non-degenerate. Therefore, formulae (2) hold. On the other hand, the asssumption $p_a(I) = 0$ implies $d^2 - \sum_{k=1}^{r-2} e_{1k}^2 = 2$. As $d^2 = (\sum_{k=1}^{r-2} e_{1k})^2 = \sum_{k=1}^{r-2} e_{1k}^2 + 2\sum_{k<k'} e_{1k}e_{1k'}$, it follows that $\sum_{k<k'} e_{1k}e_{1k'} = 1$. But this is possible only provided $r-2=2$ and $e_{11} = e_{12} = 1$. Therefore, $d = e_{11} + e_{12} = 2$, so f is a cubic polynomial with strongly nondegenerate gradient ideal of type 2.

However, by the classical theory of irreducible cubic curves, there are only the following possibilities (up to projective equivalence): a smooth cubic, the cusp $Y^2Z - X^3$, the nodal cubic. For a smooth curve, the partial derivatives form a regular sequence (cf. Proposition 3.1 of next section), hence the associated Koszul complex is a minimal free resolution. Therefore, the gradient ideal is of type 1. As to the cusp $Y^2Z - X^3$, the gradient ideal is easily seen to be of type 1 (furthermore, it is *not* strongly non-degenerate); cf. [Si, Applications].

This completes the proof of the implication (ii) $\Rightarrow$ (i), closing the proof of the theorem.

3. Strong non-degeneracy & type 1

Using the material of the preceding section, one can prove without much ado (char. 0):

Proposition 3.1. *Let $f \in k[X, Y, Z]$ be a square-free homogeneous polynomial of degree ≥ 2 and $\neq 4$, and let I denote its gradient ideal. The following conditions are equivalent:*

(i) *The projective curve $f = 0$ is smooth*
(ii) *I is a primary ideal for (X, Y, Z)*
(iii) *I is generated by a regular sequence*
(iv) *I is strongly non-degenerate and of type 1*
(v) *I is strongly non-degenerate and $p_a(I) = 1$.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) This is well known and quite general, holding for a hypersurface in $\mathbb{P}^{n-1}$.

(iii) $\Rightarrow$ (iv) A minimal free resolution of R/I is the Koszul complex attached to the partial derivatives of f.

(iv) $\Rightarrow$ (v) Firstly, one notes that I has height ≥ 2. For, otherwise, $\partial f/\partial X$, $\partial f/\partial Y, \partial f/\partial Z$ have a proper common factor, which can only happen when $f = 0$ is a multiple curve (i.e., when f is not square-free). Thus, one can apply Lemma 2.4 of the preceding section and, together with the assumption that I is strongly non-degenerate, arrive at formulae (2) of that section. Bringing in the hypothesis of type 1 then yields $d = e_{11}$, hence $p_a(I) = 1$ (a close inspection of the signs in the expression of $p_a(I)$ indicates why -1 is not possible). (v) $\Rightarrow$ (i) Since $p_a(I) = 1$, the Hilbert function $H(R/I, q)$ is either constantly 2 or 0, for $q \gg 0$. Suppose $H(R/I, q) = 2$ $(q \gg 0)$. Since I is assumed to be strongly non-degenerated, $d^2 - \sum_{k=1}^{r-2} e_{1k}^2 = 4$ from formula (2) for $p_a(I)$. Therefore, one ought to have $\sum_{k<k'} e_{1k}e_{1k'} = 2$. The only possibility is then that $r-2=2$ and $e_{11}=2$, $e_{12}=1$. Hence, $d = e_{11} + e_{12} = 3$, thus implying that f is a quartic, which has been excluded in the assumptions. The remaining alternative if for $H(R/I, q)$ to vanish for every $q \gg 0$. This in turn implies that I contains all forms of every sufficiently high degree q. In other words, I is squeezed between, say, $(X, Y, Z)^t$ and (X, Y, Z). Therefore, I is primary for (X, Y, Z), consequently, it admits no zeroes in $\mathbb{P}^2$.

This completes the cycle of implications in Proposition 3.1.

I will close with some remarks and questions.

1. Is the condition that $\deg(f) \neq 4$ in Prop. 3.1 essential? To the preent, I have no method to decide it.

2. The question to the effect of what values the arithmetic genus of a gradient ideal can assume poses rather intriguing diophantine problems. To see what is meant, let, in the notation of Lemma 2.4, δ_k stand for $d_{11} + e_{1k}$, $k = 1, \ldots, r-2$. Rewriting expression (1) for $p_a(I)$ in terms of $h_0 = 1 + (-1)^s p_a(I)$, one gets $d^2 + d_{11}^2 + \ldots + d_{1r}^2 - (\delta_1^2 + \ldots + \delta_{r-2}^2) = 2h_0$. Of course, condition $\delta_k \geq 2$, $k = 1, \ldots, r-2$, and the relation $d = \sum_{j=1}^{r} d_{1j} - \sum_{k=1}^{r-2} \delta_k$ ought to be fulfilled too.

Thus, altogether, one is led to solve, if possible, the diophantine equation

$$x_0^2 + x_1^2 + \ldots + x_r^2 - (y_1^2 + \ldots + y_{r-2}^2) = 2h_0$$

subject to further conditions

$$x_j \geq 1,\ j = 1, \ldots, r$$
$$y_k \geq 2,\ k = 1, \ldots, r-2$$
$$x_0 = x_1 + \ldots + x_r - (y_1 + \ldots + y_{r-2}).$$

3. Knowing that the ideal I is non-degenerate adds further constraints to the quadratic diophantine problem mentioned above. However, except for Lazard's results and various loose cases, one knows no criterion for an ideal

I to be non-degenerate. On the other hand, the known examples of homogeneous ideals that are degenerate [L, VI-Un contre exemple] do not seem to be gradient ideals. One can thus wonder, in this connection, whether Lazard's bounds can be sharpened in the case of gradient ideals.

4. The difficulty in dealing with higher ordinary singularities lies not only on the lack of sharper bounds for the degrees of the cycles generating Z_1, but also on our ignorance in handling linear systems of curves of large degree. To the present moment, I couldn't get but cryptic calculations.

To the same effect, it would be important to know how the minimal free resolutions of the gradient ideals modify as one moves along a linear system of curves in $\mathbb{P}^2$. In this conection, see [Si, Applications].

5. Since one can practically dessingularize a curve $f = 0$ by means of the so-called quadratic Cremona transformations, the question of the behavior of the gradient ideal of f under such transformations needs some clarification.

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