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Lifting Positive Elements

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1. Introduction

In the recent papers D. Voiculescu [15] and M.D. Choi and E.G. Effros [8] have shown that the theory of extensions of C^*-algebras as developed by L. Brown, R. Douglas and P. Fillmore in [18] admits a useful generalization to large classes of not necessarily commutative C^*-algebras. The idea for how this generalization could by carried out was suggested by Arveson's paper [17]. This approach required a theorem stating that any *-morphism f from a separable C^*-algebra A to a quotient B/I lifts to a completely positive map $\tilde{f}$. Work of Andersen [3] and Vesterstrøm [14] showed that this was the case for commutative A; Davie also provided a direct proof which can be found in the paper [19]. The case for A nuclear (which covers most interesting cases) was proven by Choi and Effros; they rely heavily on tools from convexity theory: notably those dealing with split faces of compact convex sets and M-ideals [1], [2].

This paper provides a different approach which simplifies many of the technical details of [8]. More specifically we prove a lifting theorem (Proposition 1) which generalizes the extension theorem for continuous affine functions defined on a split face of a compact convex set and from which the completely positive lifting theorem follows almost immediately.

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2. Bidual Systems and approximate liftings.

Suppose M is a locally convex space, $F = M^*$ and $M \subseteq E \subseteq F'$ a vector space with a locally convex topology (referred to as the strong topology) such that

(i) E has a neighborhood base at 0 (for its strong topology) consisting of convex sets V such that $V \cap M$ is $\sigma(E, F)$ dense in V.

(ii) The restriction of the strong topology of E to M is the original topology of M. In other words, a convex set $A \subseteq M$ is closed in the relative strong topology iff it is closed in the relative $\sigma(E, F)$ topology, i.e. the $\sigma(M, F)$ topology.

The archetype of this situation is a Banach space M imbedded in its bidual $E = M^{**}$ with $F = M^*$; In this example M^{**} has the norm topology.

Lemma 1. *Conserving the notations and hypothesis of the preceding paragraph, let A, B be convex intersecting subsets of M. Let W be a neighborhood of 0 in M in the relative strong topology. Then $(A + W) \cap B$ is $\sigma(E, F)$ dense in $A^{-\sigma} \cap B^{-\sigma}$, the closures taken in the $\sigma(E, F)$ topology of E.*

Proof: Let $x_0 \in A \cap B$; As translation by $-x_0$ is a homeomorphism for the $\sigma(E, F)$ topology there is no loss of generality in supposing $0 \in A \cap B$. To avoid confusion, a single bar (e.g. A^-) denotes closure in E where as two bars, $A^=$ denotes closure in M. Closures in F are also denoted by single bars.

Let V be a convex strong neighborhood of 0, such that $V \cap M = V'$ is $\sigma(E, F)$ dense in V and $V' + V' - V' - V' \subseteq W$. By (*i*) such V form a neighborhood basis for 0 in E. We first prove $(A + V' + V') \cap (B + V' + V')$ is $\sigma(E, F)$ dense in $A^{-\sigma} \cap B^{-\sigma}$. Notice $A + V' + V' \supseteq (A + V')^=$, $B + V' + V' \supseteq (B + V')^=$ [12, *I*, 1.1]. Now $(A + V')^=$, $(B + V')^=$ are strongly closed convex sets (in M) and hence by (ii) they are $\sigma(M, F)$ closed. Thus by [12, IV, 1.5, Corollary 2]

$$Po((A + V')^= \cap (B + V')^=) = \langle Po(A + V')^= \cup Po(B + V')^= \rangle^{-\sigma(F, M)}$$

Now by the Banach Alaouglu theorem since $A + V'$, $B + V'$ are neighborhoods of zero in the strong topology, $Po\{(A + V')^=\}, Po\{(B + V')^=\}$ *are* $\sigma(F, M)$ compact, and hence $\langle Po\{(A + V')^=\} \cup Po\{(B + V')^=\}\rangle$ is $\sigma(F, M)$ compact [12, II, 10.1, Corollary]. Thus

$$Po((A + V')^= \cap (B + V')^=) = \langle Po\{(A + V')^=\} \cup Po\{(B + V')^=\}\rangle.$$

On the other hand by [12, IV, 1.5, Corollary 2]

$$Po(A^{-\sigma} \cap B^{-\sigma}) = \langle Po(A^{-\sigma}) \cup Po(B^{-\sigma})\rangle^{-\sigma(F,E)}$$
$$= \langle Po(A) \cap Po(B)\rangle^{-\sigma(F,E)} \supseteq \langle Po\{\{A + V'\}^=\} \cup Po\{\{B + V\}^=\}\rangle.$$

Thus $A^{-\sigma} \cap B^{-\sigma} = PoPo(A^{-\sigma} \cap B^{-\sigma}) \subseteq PoPo((A + V')^= \cap (B + V')^=) = ((A + V')^= \cap (B + V')^=)^{-\sigma} \subseteq ((A + V' + V' \cap (B + V' + V'))^{-\sigma}$ proving our first assertion. Now for any $\sigma(E, F)$ neighborhood U of 0 in E $(AA + V' + V') \cap (B + V' + V'))^{-\sigma} \subseteq ((A + V' + V' - V' - V') \cap B + V' + V')^{-\sigma} \subseteq (A + W) \cap B + V' + V' + U$ [12, I, 1.1]. Now as V varies through a neighborhood basis $\mathcal{N}$ in the strong topology of 0 and U through a neighborhood basis in the $\sigma(E, F)$ topology, $V' + V' + U$ forms a neighborhood

basis of 0 in the $\sigma(E, F)$ topology: For $V' + V' = V \cap M + V \cap M$ is a basis for the strong topology of 0 in M; thus given $f_i \in F$ $i = 1, \ldots, n$ there is a $V \in \mathcal{N}$, and U a $\sigma(E, F)$ neighborhood of 0 such that Re $f_i \mid V' + V' \leq 1/2$, *Re* $f_i \mid U \leq 1/2$ $i = 1, \ldots, n$. Thus *Re* $f_i \mid U + V' + V' \leq 1$, and as sets $\{x : Re\, f_i(x) \leq 1\ i = 1, \ldots, n\}$ form a $\sigma(E, F)$ basis for 0 in E the assertion is established. Thus

$$(A + W) \cap B^{-\sigma} = \cap \{(A + W) \cap B + V' + V' + U\} \supseteq A^{-\sigma} \cap B^{-\sigma}.$$

Definition 1. A triple (E, F, M) where $M \subseteq E$, $F = M^*$ and $E \subseteq F'$ which satisfy conditions (i), (ii) above is called a *bidual system*. If M is given then (E, F, M) is a bidual system for M if the strong topology of E restricted to M is the given topology.

Example. Let (E, F, M) be a bidual system, X a locally convex space. Clearly the restriction of the topology of simple strong convergence on $\mathcal{B}(X, E)$ to $\mathcal{B}(X, M)$ is also the topology of simple strong convergence. On the other hand if $T \in \mathcal{BF}(M, X)$, then T has a unique extension to a $\tilde{T} \in \mathcal{BF}(E, X)$ which is continuous for the $\sigma(E, F)$ topology on E, since M is $\sigma(E, F)$ dense in E and T is continuous of finite rank. Thus there is a pairing $\mathcal{B}(X, E) \times \mathcal{BF}(M, X) \to K$ given by $(T, S) \to$ trace $(\tilde{S}T)$ which extends the canonical pairing $\mathcal{B}(X, M) \times \mathcal{BF}(M, X) \to K$. Now the simple weak (i.e. simple $\sigma(E, F)$) topology on $\mathcal{B}(X, E)$ is the same as the $\sigma(\mathcal{B}(X, E), \mathcal{BF}(M, X))$ topology: In fact any $S \in \mathcal{BF}(M, X)$ is of the form $S(x) = \sum_{i=1}^{n} \langle x, f_i\rangle \cdot y_i$ with $f_i \in F$, $y_i \in X$ and $\tilde{S}(x) = \sum_{i=1}^{n} \langle x, f_i\rangle \cdot y_i$. Thus if $T \in \mathcal{B}(X, E)$, trace $(\tilde{S}T) = \sum_{i=1}^{n} \langle Ty_i, f_i\rangle$ which shows the equivalence of the topologies.

Also the sets $V_A = \{T \in \mathcal{B}(X, E) : T(A) \subseteq V\}$ for $A \subseteq X$ a finite set linearly independent over K and V a convex neighborhood of 0 in E such that $V \subseteq (V \cap M)^{-\sigma}$, form a base at 0 in the topology of simple strong convergence for $\mathcal{B}(X, E)$. These sets form a base at 0 for the simple strong topology of $\mathcal{B}(X, E)$ because for any finite set $Q \subseteq X$ and neighborhood W of 0 in the strong topology of E there is a finite independent set A whose convex circled hull contains Q and a convex circled neighborhood $V \subseteq W$. Thus if $T(A) \subseteq V$ then $T(Q) \subseteq V \subseteq W$. It is clear that for such V, V_A is convex and is such that $V_A \cap \mathcal{B}(X, M)$ is dense in V_A in the simple $\sigma(E, F)$ topology. To see that $V_A \cap \mathcal{B}(X, M)$ is dense in V_A, let U be a $\sigma(E, F)$ neighborhood of 0 in E, P a finite independent set which constains A, and $T \in V_A$. If $x \in A$ there is an $S_y \in V \cap M$ such that $S_y - T(x) \in U$; if $x \in P - A$ then as M is dense E in the $\sigma(E, F)$ topology, there is an $S_y \in M$ such that $S_y - T(x) \in U$. Thus there is an operator S on the space Y spanned by P such that $S(x) = S_y$ for $x \in P$. Clearly $S \in \mathscr{B}(Y, M)$. As S is of finite rank there is a continuous extension of S to an operator in $\mathscr{B}(X, M)$. $T_{P,U} \to T$ in the point weak topology. It follows that $T_{P,U} \to T$ in the $\sigma(\mathscr{B}(X, E), \mathscr{B}\mathscr{F}(M, X))$ topology. Consequently $(\mathscr{B}(X, E), \mathscr{B}\mathscr{F}(M, X), \mathscr{B}(X, M))$ is a bidual system.

We add that if X has a countable basis then if E is metrizable $\mathscr{B}(X, E)$ with the simple strong topology will also be metrizable. Of course X cannot be an infinite dimensional Banach space for this to hold, but we will apply this observation to a dense subspace of X in case X is separable. Unfortunately for no infinite dimensional X will $\mathscr{B}(X, E)$ be complete in the simple strong topology.

The following definition is inspired by the notion of M ideal [2, § 5]. The similarities should be evident.

Definition 2. Let M, N be ordered locally convex spaces, $\pi \in \mathscr{B}(M, N)$ a positive linear map. An *aproximate lifting* for π consists of the following data:

a) Bidual systems (E, F, M), (G, H, N) for M, N respectively.

b) Maps given in the following diagram

$$\begin{array}{ccc} M & \xrightarrow{id_{ME}} & E \\ \pi \downarrow & & \pi' \downarrow \uparrow L \\ N & \xrightarrow[id_{*G}]{} & G \end{array}$$

satisfying:

(i) $\pi' \; id_{ME} = id_{NG} \; \pi$, $\pi' L = id_G$

(ii) $L \; id_{NG}(N^+) \subseteq (id_{ME}(M^+))^{-\sigma}$

$(id_{ME} - L \, id_{NG}\pi')M^+ \subseteq (id_{ME}(M^+))^{-\sigma}$

(iii) π' is continuous for the strong topologies on E, G and for the $\sigma(E, F)$, $\sigma(G, H)$ topologies; L is continuous for the strong topologies on G, E.

Remarks. 1) An approximate lifting is defined for π and particular orderings on M, N. It is clear however that if we have the data for an approximate lifting for π and the ordering given by the cones M^+, N^+ then the same data is an approximate lifting for π with the orderings given by the cones $(M^+)^=$, $(N^+)^=$ (closures taken in M, N resp., in accordance with usage in Lemma 1). This is clear because of the strong continuity of L.

2) It is an important consequence of (ii) that $\pi(M^+)^= \supseteq N^+$. For

$$N^+ = \pi' L(N^+) \subseteq \pi'((M^+)^{-\sigma}) \subseteq \pi'(M^+)^{-\sigma}$$

so

$$N^+ \subseteq \pi'(M^+)^{-\sigma} \cap N = \pi(M^+)^{=\sigma} = \pi(M^+)^=$$

In proposition 1 we will prove something much stronger.

Example Let A, B be C^*-algebras and $\pi : A \to B$ a surjective *-homomorphism. Then π has as approximate lifting; namely consider the diagram

$$\begin{array}{ccc} A & \longrightarrow & A^{**} \\ \pi \downarrow & & \pi^{**} \downarrow \uparrow L \\ B & \longrightarrow & B^{**} \end{array}$$

where $\pi^{**} : A^{**} \to B^{**}$ is the canonical morphism. There is a unique central projection $e \in A^{**}$ such that $\pi^{**}(xe) = \pi^{**}(x)$ and is s.t. $\pi^{**} | A^{**}e$ is bijective. L is then the inverse of $\pi^{**} | A^{**} e$. L is a morphism of C^*-algebras and is thus norm continuous; the commutativity conditions (i) are clear. Property (ii) is a consequence of the fact that any $x \in (A^{**})^+$ is a limit in the ultraweak topology of a net of positive elements in A, and the ultraweak topology is the $\sigma(A^{**}, A^*)$ topology.

In the following we use double bars to denote closures in M and N.

Proposition 1. *Let M, N be metrizable ordered locally convex spaces with M complete. Let $\pi \in \mathscr{B}(M, N)$ be a positive map which has an approximate lifting. Then $\pi(M^{+=}) = N^{+=}$.*

Proof. Considering the closures of M^+, N^+ there is no loss of generality in supposing M^+, N^+ are closed cones. We will also use without further clarification the notations and properties specified in the definition of an approximate lifting.

We first show that for any neighborhood V of 0 in M there is a neighborhood W' of 0 in N such that for any $x \in M^+$

$$\pi(M^+ \cap (x + V))^= \cap (\pi(x) + W').$$

Since (E, F, M) is a bidual system for M, there is a strong neighborhood U of 0 in E which is convex, $U \subseteq (U \cap M)^{-\sigma}$ and $U \cap M + U \cap M \subseteq V$. Let $W = L^{-1}(U)$. W is a strong neighborhood of 0 in G and $\pi'(M^{+-\sigma} \cap$

$\cap (x + U)) \supseteq N^+ \cap (\pi'(x) + W)$. For if $y \in N^+ \cap (\pi'(x) + W)$ then $My = = Ly + (x - L\pi' x)$ satisfies $My \in M^{+-\sigma}$ in virtue of (ii), $\pi' My = y$ and $My - x = Ly + (x - L\pi' x) - x = L(y - \pi' x) \subseteq LW \subseteq U$. Now $x + U \subseteq \subseteq (x + U \cap M)^{-\sigma}$ so by lemma 1, $M^+ \cap (x + U \cap M)$ is $\sigma(E, F)$ dense in $M^{+-\sigma} \cap (x + U \cap M)^{-\sigma} \supseteq M^{+-\sigma} \cap (x + U)$.
Thus

$$\begin{aligned} \pi'(M^+ \cap (x + V))^{-\sigma} &\supseteq \pi'((M^+ \cap (x + V))^{-\sigma}) \\ &\supseteq \pi'(M^{+-\sigma} \cap (x + U)) \supseteq N^+ \cap (\pi'(x) + W) \end{aligned}$$

Setting $W' = W \cap N$, clearly

$$\begin{aligned} \pi'(M^+ \cap (x + V))^{=} &= \pi'(M^+ \cap (x + V))^{=\sigma} \\ &= \pi'(M^+ \cap (x + V))^{-\sigma} \cap N \supseteq N^+ \cap (\pi'(x) + W'). \end{aligned}$$

Proving the assertion.

There exist translation invariant metrics d_M, d_N which define the topologies on M, N resp. [12, I, 6.1]. In terms of these metrics, the previous assertion states that for any $r > 0$ there is a $\rho(r) > 0$ such that

$$\pi(M^+ \cap \overline{B}_M(x, r))^{=} \supseteq N^+ \cap \overline{B}_*(\pi(x), \rho(r))$$

where $\overline{B}_M$, $\overline{B}_*$ denote closed balls in M, N resp. By [10, p. 202 or 5, §3, Lemma 2], and the completeness of M^+

$$\pi(M^+ \cap \overline{B}_M(x, s)) \supseteq N^+ \cap \overline{B}_*(\pi(x), \rho(r))$$

for $s > r$. By remark 2 after definition 6, we have $\pi(M^+)^{=} \supseteq N^+$ so the above inclusion implies $\pi(M^+) = N^+$.

Remarks. 1)It is obvious we have proven much more than we have stated in the proposition. Particularly, if $y \in N^+$ is such that $d_N(\pi(x), y) \leq \rho(r)$ then there is for any $s > r$ on $x' \in N^+$ such that $\pi(x') = y$ and $d_M(x, x') < s$. 2) It is also evident from the proof of [5, §3, Lemma 2] and the preceding that we can still say something in the absence of completeness: namely that there is a Cauchy sequence $x_n \in M^+$ such that $\pi(x_n) \to y$. This detail is important for the completely positive lifting problem.

It is worth pointing out that the proof of proposition 1 has elements similar to the proof of [1, Theorem II.6.15] and of [3, Lemma 1]. The condition of metrizability of M requires in applications, conditions of separability. We know of no significant extension of [5, §3, Lemma 2] or [10, p. 202] to non metrizable spaces.

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