A Remark on Strongly Nonlinear Elliptic Boundary Value Problems.

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This note is concerned with the existence of solutions for variational boundary value problems for quasilinear elliptic operators of the form

(1) $$\Sigma_{|\alpha| \leq m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x, u, \nabla u, \ldots, \nabla^m_u)$$

in the case where the coefficients A_α do not have polynomial growth in $u, \nabla u, \ldots, \nabla^m_u$. Existence theorems for problems of this type have been given in [4, 5]; they provide natural extensions to the case of rapidly or slowly increasing A_α's of the basic results of Browder [1] and Leray-Lions [9]. These existence theorems apply to the Dirichlet or the Neumann problems, but, as remarked at the end of [5], it is not immediate to use them to treat other kinds of boundary conditions. The main reason for this is explained in section 2 below.

It is our purpose here to indicate one way of treating the so-called "third problem". As will be seen, although our result allows the consideration of boundary conditions different from the Dirichlet's or the Neumann's, it does not go in the direction of the questions raised in section 2.

To avoid technicalities, we will concentrate on the simple equation

(2) $$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[\varphi\left(\frac{\partial u}{\partial x_i}\right)\right] + \varphi(u) = f,$$

but our result could easily be extended to operators of type (1), along the lines of [4, 5]. Here $\varphi\colon \mathbb{R} \to \mathbb{R}$ is assumed to be *continuous, increasing with* $\varphi(\pm\infty) = \pm\infty$, and *odd at infinity*, i.e.

$$r \leq \left|\frac{\varphi(\alpha t)}{\varphi(-t)}\right| \leq R$$

for some $r, R, \alpha > 0$ and all t sufficiently large; note that no growth assumption is imposed on φ which can behave at infinity for instance as an exponential,

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or as a logarithm. The third problem for (2) asks for a solution u of (2) in Ω satisfying

$$\sum_{i=1}^{n} \varphi\left(\frac{\partial u}{\partial x_i}\right) v_i + Su = 0 \tag{3}$$

on the boundary Γ or Ω, where v_i denotes the i^{th} component of the exterior normal to Γ and S in an operator acting on functions defined on Γ, for instance $Su = \varphi(u)$.

Section 2 introduces some notations and briefly recalls how to solve the Dirichlet and the Neumann problems for (2). In section 3, the results of Fougeres [3] about the trace of a function in $W^1 L_M(\Omega)$ are extended to the case where the N-function M has arbitrary growth. The third problem for (2) is studied in section 4.

2. Preliminares

Let Y and Z be two real Banach spaces in duality with respect to a continuous pairing $\langle,\rangle$ and let Y_0 and Z_0 be closed subspaces of Y and Z respectively; the quadruple $(Y, Y_0; Z, Z_0)$ is called a complementary system if, by means of $\langle,\rangle$, Y_0^* can be identified to Z and Z_0^* to Y. For instance, if $L_M(\Omega)$ denotes the Orlicz space on a bounded open subset Ω of $\mathbb{R}^n$ corresponding to a N-function M, if $E_M(\Omega)$ denotes the closure in $L_M(\Omega)$ of $L^\infty(\Omega)$ and if $\overline{M}$ denotes the N-function conjugate to M, the $(L_M(\Omega). E_M(\Omega); L_{\overline{M}}(\Omega), E_{\overline{M}}(\Omega))$ constitues a complementary system.

The Sobolev space of functions u such that u and its first distributional derivatives lie in $L_M(\Omega)\,[E_M(\Omega)]$ is denoted by $W^1 L_M(\Omega)\,[W^1 E_M(\Omega)]$. These spaces will always be identified to subspaces of the product $\prod L_M$ ($n+1$ factors); they are Banach spaces. The $\sigma(\Pi L_M; \Pi E_{\overline{M}})$ closure of $\mathfrak{D}(\Omega)$ in $W^1 L_M(\Omega)$ is denoted by $W_0^1 L_M(\Omega)$ and the norm closure of $\mathfrak{D}(\Omega)$ in $W^1 L_M(\Omega)$ by $W_0^1 E_M(\Omega)$.

To get duality results for these spaces, we need a method by which, given a complementary system $(E, E_0; F, F_0)$ and a closed subspace Y of E, one can construct a new complementary system $(Y, Y_0; Z, Z_0)$. Define $Y_0 = Y \cap E_0$, $Z = F/Y_0^{\perp}$ and $Z_0 = \{u + Y_0^{\perp};\ u \in F_0\}$, where $Y_0^{\perp} = \{u \in F;\ \langle v, v\rangle = 0$ for all $v \in Y_0\}$.

Lemma 1. *cf.* [4, *p.* 166; 6, *section* 2.1]). *The pairing between E and F induces a pairing between Y and Z if and only if Y_0 is $\sigma(E, F)$ dense in Y. In this case, $(Y, Y_0; Z, Z_0)$ is a complementary system if and only if Y is $\sigma(E, F_0)$ closed in E.*

When Ω has the segment property, then $C^\infty(\Omega)$ is $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ dense in $W^1 L_M(\Omega)$ and $\mathfrak{D}(\Omega)$ is $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ dense in $W_0^1 L_M(\Omega)$ (cf. [4, p. 167]) so that the above lemma can be applied to both $W^1 L_M(\Omega)$ and $W_0^1 L_M(\Omega)$, starting with the complementary system $(\Pi L_M, \Pi E_M; \Pi L_{\overline{M}}, \Pi E_{\overline{M}})$. We get in this way two complementary systems $(W^1 L_M(\Omega), W^1 E_M(\Omega); *, *)$ and $(W_0^1 L_M(\Omega), W_0^1 E_M(\Omega); *, *)$, which, for brievety, will be denoted below by $(Y', Y_0'; Z', Z_0')$ and $(Y'', Y_0''; Z'', Z_0'')$ respectively.

More generally the above lemma can be applied to a space Y satisfying the following three conditions:

$$W_0^1 L_M(\Omega) \subset Y \subset W^1 L_M(\Omega),$$
$$Y\ \sigma(\Pi L_M, \Pi E_{\overline{M}}) \text{ closed in } W^1 L_M(\Omega),$$
$$Y \cap \Pi E_M\ \sigma(\Pi L_M, \Pi L_{\overline{M}}) \text{ dense in } Y.$$

This is in contrast with the L^P variational theory where one starts with an *arbitrary* closed space Y lying between $W_0^{1,P}(\Omega)$ and $W^{1,P}(\Omega)$. The understanding of the above rectrictions on Y in terms of boundary conditions is not clear yet. For instance, what about the space of functions u in $W^1 L_M(\Omega)$ which are zero on some given part Γ_1 of Γ? Of course, when $\overline{M}$ satisfies the Δ_2 condition (i.e. $\overline{M}(2t) \leq k\,M(t)$ for some $k > 0$ and all t sufficiently large), things are easier: $E_{\overline{M}} = L_{\overline{M}}$, and so $\sigma(\Pi L_M, \Pi E_{\overline{M}}) = \sigma(\Pi L_M, \Pi L_{\overline{M}})$.

We now briefly indicate how to solve the Neumann problem for (2). The treatment of the Dirichlet problem is similar.

Take a N-function $M(t)$ such that

$$|\varphi(t)| \leq C_1 \overline{M}^{-1} M(C_2 t) + C_3, \tag{4}$$

$$\varphi(t).\,t \geq C_4\, M(C_5 t) - C_6, \tag{5}$$

for all $t \in \mathbb{R}$, where the C_i's are positive constants with $C_4, C_5 > 0$ and where $\overline{M}^{-1}$ denotes the reciprocal function of $\overline{M}$ on $\mathbb{R}^+$. Such a N-function is easily constructed; if φ were odd and strictly increasing, one could simply write $M(t) = \int_0^t \varphi(\tau)\, d\tau$.

Consider now the Orlicz-Sobolev spaces corresponding to $M(t)$, as above. Let $F \in Z_0'$. The Neumann problem for (2) asks for an element $u \in W^1 L_M(\Omega)$ such that $\varphi(\partial u/\partial x_i) \in L_{\overline{M}}(\Omega)$ for $i = 1, \ldots, n$, $\varphi(u) \in L_{\overline{M}}(\Omega)$ and

$$\int_\Omega \left[\sum_{i=1}^{n} \varphi\left(\frac{\partial u}{\partial x_i}\right)\frac{\partial v}{\partial x_i} + \varphi(u)\, v\right] dx = \langle v, f\rangle$$

for all $v \in W^1 L_M(\Omega)$. The existence of a solution for this problem is a consequence of the following two propositions.

Proposition 1. cf. [4, 5]). *Let $(Y, Y_0; Z, Z_0)$ be a complementary system with an admissible norm and let T be a mapping with domain $D(T)$ in Y and values in Z. Assume that (i) T is monotone, (ii) T is pseudo-monotone with respect to Y_0, (iii) each $z \in Z_0$ has a norm neighbourhood $\mathcal{N}$ in Z such that $\{u \in D(T); Tu \in \mathcal{N}\}$ is bounded in Y. Assume also that $T(Y_0)$ meets Z_0. Them $R(T) \supset Z_0$.*

Proposition 2. cf. [4, 5]). *Consider the complementary system $(Y', Y'_0; Z', Z'_0)$ and define T: $D(T) \subset Y' \to Z'$ by*

$$D(T) = \left\{u \in Y'; \varphi\left(\frac{\partial u}{\partial x_i}\right), \varphi(u) \in L_M(\Omega) \quad for \quad i = 1, \ldots, n\right\}$$

$$\langle v, \tau u\rangle = \int_\Omega \left[\sum_{i=1}^n \varphi\left(\frac{\partial u}{\partial x_i}\right)\frac{\partial v}{\partial x_i} + \varphi(u)\, v\right] dx \quad for \quad v \in Y'$$

Then T satisfies the assumptions (i), (ii), (iii) of proposition 1.

Some explanations are needed. The existence of an admissible norm is not a serious restriction; it is guaranteed in any complementary system built from $(\Pi L_M, \Pi E_M; \Pi L_M, \Pi E_M)$ by applying lemma 1 to a space Y satisfying the therее conditions mentionned above (cf. [4, p. 170]). Now a mapping $T: D(T) \subset Y \to Z$ is called pseudo-monotone with respect to Y_0 if (a) $D(T) \supset Y_0$ and T is continuous from each finite-dimensional subspace of Y_0 to the $\sigma(Z, Y_0)$ topology of Z, (b) for each bounded net $y_i \in Y_0$ with $y_i \to y \in Y$ for $\sigma(Y, Z_0)$, $Ty_i \to z \in Z$ for $\sigma(Z, Y_0)$ and $\lim\sup \langle y_i, Tu_i\rangle \le \langle y, z\rangle$, it follows that $y \in D(T)$, $Ty = z$ and $\langle y_i, Ty_i\rangle \to \langle y, z\rangle$.

3. Trace

In this section and in the following one, we assume that the boundary Γ of our open bounded set Ω is sufficiently good so that questions in Ω, near Γ, can be transformed, by using a partition of unity and local charts, into similar questions in $\mathbb{R}^n_+$, near $\mathbb{R}^{n-1}$. This will be certainly so, for our purposes below, if Γ is assumed to be C^1.

Consider the "restriction to Γ" mapping:

$$\tilde{\gamma}: C^\infty(\bar{\Omega}) \to C(\Gamma): u \to u_{|\Gamma}$$

We will show that it is continuous for the following topologies on $C^\infty(\bar{\Omega})$ and $C(\Gamma)$ respectively:

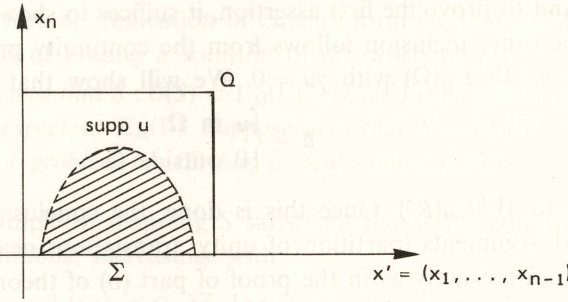
$$(6) \qquad \|\ \|\, W^1 L_M(\Omega) \to \|\ \|\, L_M(\Gamma),$$

$$(7) \qquad \sigma(\Pi L_M(\Omega), \Pi E_{\bar{M}}(\Omega)) \to \sigma(L_M(\Gamma), E_{\bar{M}}(\Gamma)),$$

$$(8) \qquad \sigma(\Pi L_M(\Omega), \Pi L_{\bar{M}}(\Omega)) \to \sigma(L_M(\Gamma), L_{\bar{M}}(\Gamma)).$$

From (8) and from the fact that $C^\infty(\bar{\Omega})$ is $\sigma(\Pi L_M, \Pi L_{\bar{M}})$ dense in $W^1 L_M(\Omega)$, it follows that $\tilde{\gamma}$ can be extended into a continuous mapping γ from $W^1 L_M(\Omega)$, $\sigma(\Pi L_M, \Pi L_{\bar{M}})$ to $L_M(\Gamma)$, $\sigma(L_M, L_{\bar{M}})$. Condition (7) implies that γ is continuous from $W^1 L_M(\Omega)$, $\sigma(\Pi L_M, \Pi E_{\bar{M}})$ to $L_M(\Gamma)$, $\sigma(L_M, E_{\bar{M}})$. Since $C^\infty(\bar{\Omega})$ is norm dense into $W^1 E_M(\Omega)$ (an easy consequence of the $\sigma(\Pi L_M, \Pi L_{\bar{M}})$ density of $C^\infty(\bar{\Omega})$ in $W^1 L_M(\Omega)$), condition (6) implies that γ is continuous from $W^1 E_M(\Omega)$, $\|\ \|$ to $E_M(\Gamma)$, $\|\ \|$. For u in $W^1 L_M(\Omega)$, γu is called the trace of u on Γ.

Proof of (6), (7), (8). By using a partition of unity and local charts, we are reduced to the following situation: $u \in C^1(Q)$, with support intersecting only the part Σ of ∂Q, where Q is, say, a cube in $\mathbb{R}^n_+$ and $\Sigma = \partial Q \cap \mathbb{R}^{n-1}$:



We have

$$u(x', 0) = -\int_0^\infty \frac{\partial u}{\partial x_n}(x', x_n)\, dx_n$$

and so, for $v(x') \in L_{\bar{M}}(\Sigma)$,

$$(9) \qquad \int_\Sigma u(x', 0)\, v(x')\, dx' = -\int_Q \frac{\partial u}{\partial x_n}(x', x_n)\, \tilde{v}(x', x_n)\, dx'\, dx_n,$$

where $\tilde{v}(x', x_n)$ is defined by $\tilde{v}(x', x_n) = v(x')$. Since $v(x')$. Since $v(x') \in L_{\bar{M}}(\Sigma)$ $[E_{\bar{M}}(\Sigma)]$ implies $\tilde{v}(x', x_n) \in L_{\bar{M}}(Q)$ $[E_{\bar{M}}(Q)]$, we immediately deduce (7) and (8) from (9). By going to the supremum in (9) when $v(x')$ varies in a bounded set in $E_{\bar{M}}(\Sigma)$ and after noting that the mapping $v(x) \in E_{\bar{M}}(\Sigma) \to \tilde{v}(x', x_n) \in E_{\bar{M}}(Q)$ is bounded, we deduce (6). Q.E.D.

Green's formula holds: if $u \in W^1 L_M(\Omega)$ and $v \in W^1 L_{\bar{M}}(\Omega)$, then

$$(10) \qquad \int_\Omega u \frac{\partial v}{\partial x_i}\, dx + \int_\Omega \frac{\partial u}{\partial x_i}\, v\, dx = \int_\Gamma uv\, v_i\, d\Gamma.$$

Indeed first note that the three integrals in (10) are well defined. Now (10) is true for u and v in $C^\infty(\overline{\Omega})$. Since $C^\infty(\overline{\Omega})$ is $\sigma(\Pi L_{\overline{M}}, \Pi E_{\overline{M}})$ dense in $W^1 L_M(\Omega)$ and since γ is continuous for (7), we derive (10) for $u \in W^1 L_M(\Omega)$ and $v \in C^\infty(\overline{\Omega})$. Since $C^\infty(\overline{\Omega})$ is $\sigma(\Pi L_{\overline{M}}, \Pi L_M)$ dense in $W^1 L_{\overline{M}}(\Omega)$ and since γ is continuous for (8) (with M and $\overline{M}$ interchanged), we derive (10) for $u \in W^1 L_M(\Omega)$ and $v \in W^1 L_{\overline{M}}(\Omega)$.

We will now show that, as in the usual L^p situation, $W_0^1 L_M(\Omega)$ $[W_0^1 E_M(\Omega)]$ can be interpreted as the space of functions in $W^1 L_M(\Omega)$ $[W^1 E_M(\Omega)]$ will zero trace on Γ.

Proposition 3. *The kernel of the trace mapping* $\gamma : W^1 L_M(\Omega) \to L_M(\Gamma)$ *is* $W_0^1 L_M(\Omega)$. *The kernel of the trace mapping* $\gamma : W^1 E_M(\Omega) \to E_M(\Gamma)$ *is* $W_0^1 E_M(\Omega)$.

Proof. Since $W_0^1 E_M = W_0^1 L_M \cap W^1 E_M$ (an easy consequence of the $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ density of $\mathfrak{D}(\Omega)$ in $W_0^1 L_M$), the first assertion implies the second. And to prove the first assertion, it suffices to show that $\ker \gamma \subset W_0^1 L_M$ since the other inclusion follows from the continuity properties of γ. So let us take $u \in W^1 L_M(\Omega)$ with $\gamma u = 0$. We will show that

$$\tilde{u} = \begin{cases} u & \text{in } \Omega \\ 0 & \text{outside } \Omega \end{cases}$$

belongs to $W^1 L_M(R^n)$. Once this is done, the conclusion follows by using standard arguments (partition of unity, translations near the boundary, regularization), exactly as in the proof of part (b) of theorem 1.3 of [4].

It is clear that $u \in L_M(\mathbb{R}^n)$. Write

$$v_i = \begin{cases} \dfrac{\partial u}{\partial x_i} & \text{in } \Omega \\ 0 & \text{outside } \Omega. \end{cases}$$

Of course $v_i \in L_M(\mathbb{R}^n)$, and we have to show that $\partial u / \partial v_i = v_i$ in the distributional sense on all $\mathbb{R}^n$. Let $\psi \in \mathfrak{D}(\mathbb{R}^n)$. We have, using Green's formula and the fact that $\gamma u = 0$,

$$\begin{aligned} \int_{\mathbb{R}_n} v_i \psi \, dx &= \int_\Omega \frac{\partial u}{\partial x_i} \psi \, dx \\ &= -\int_\Omega u \frac{\partial \psi}{\partial x_i} dx + \int_\Gamma u \psi v_i \, d\Gamma \\ &= -\int_\Omega u \frac{\partial \psi}{\partial x_i} dx = -\int_{\mathbb{R}_n} \tilde{u} \frac{\partial \psi}{\partial x_i} dx, \end{aligned}$$

which concludes the proof. Q.E.D.

In the particular case where $\overline{M}$ satisfies the Δ_2 condition, the above results have been obtained by Fougères. [3]. A basic question in the study of the trace of a function in $W^1 L_M(\Omega)$ is of course that of characterizing the range of the trace operator γ. Some results in this direction have been obtained by Lacroix [7] and Lami Dozo [8].

4. Third problem for (2)

Let S be a mapping with domain $D(S)$ in $L_M(\Gamma)$ and values in $L_{\overline{M}}(\Gamma)$. Here $M(t)$ is, as before, a N-function associated with φ in such a way that (4) and (5) hold. Let $f \in Z_0'$. The third problem for (2) asks for an element $u \in W^1 L_M(\Omega)$ such that $\varphi(\partial u/\partial x_i) \in L_{\overline{M}}(\Omega)$ for $i = 1, \ldots, n$, $\varphi(u) \in L_{\overline{M}}(\Omega)$, $\gamma u \in D(S)$ and $b(u, v) = \langle v, f \rangle$ for all $v \in W^1 L_M(\Omega)$, where

$$b(u, v) = \int_\Omega \left[\sum_{i=1}^n \varphi\left(\frac{\partial u}{\partial x_i}\right) \frac{\partial v}{\partial x_i} + \varphi(u) v \right] dx + \int_\Gamma S(\gamma u) \,.\, \gamma v \, d\Gamma.$$

It is easily seen, by formal application of Green's formula, that this problem is equivalent to that of finding a solution of (2) satisfying (3).

Theorem 1. *Assume that* $S : D(S) \subset L_M(\Gamma) \to L_{\overline{M}}(\Gamma)$ *is monotone and pseudo-monotone with respect to* $E_M(\Gamma)$. *Suppose also that* $S o \in E_M(\Gamma)$. *Then, for any* $f \in Z_0'$, *the third problem for* (2) *has at least one solution.*

Here is an example of mapping S satisfying the above conditions. Let $\psi : \mathbb{R} \to \mathbb{R}$ be continuous, increasing, with

$$|\psi(t)| \leq C_7 \, \overline{M}^{-1} M(C_8 t) + C_9$$

for all $t \in \mathbb{R}$ and some positive constants C_7, C_8, C_9. Define

$$\begin{aligned} D(S) &= \{ w \in L_{\overline{M}}(\Gamma);\ \psi(w(x)) \in L_{\overline{M}}(\Gamma) \}, \\ Sw &= \psi(w). \end{aligned}$$

Clearly S is monotone and $S o \in E_{\overline{M}}(\Gamma)$; the pseudo-monotonity of S follows from [4; example 2.3 and theorem 4.1].

To prove theorem 1, denote by $T_1 : D(T_1) \subset Y' \to Z'$ the operator introduced in proposition 2 and by $T_2 : D(T_2) \subset Y' \to Z'$ the operator defined by

$$\begin{aligned} D(T_2) &= \{ u \in Y';\ \gamma u \in D(S) \}, \\ \langle v, T_2 u \rangle &= \textstyle\int_\Gamma S(\gamma u) \,.\, \gamma v \, d\Gamma \quad \text{for} \quad v \in Y'. \end{aligned}$$

Then theorem 1 will be proved if we show that $f \in R(T_1 + T_2)$, where $T_1 + T_2$ is defined as usual by

$$\begin{aligned} D(T_1 + T_2) &= D(T_1) \cap D(T_2), \\ (T_1 + T_2)u &= T_1 u + T_2 u. \end{aligned}$$

The following two lemmas will be needed.

Lemma 2. *Assume that $S : D(S) \subset L_M(\Gamma) \to L_{\overline{M}}(\Gamma)$ is pseudo-monotone with respect to $E_M(\Gamma)$ and strongly quasibounded with respect to some γu with $\bar{u} \in W E_M(\Omega)$. Then the corresponding mapping $T_2 : D(T_2) \subset Y' \to Z'$ is pseudo-monotone with respect to Y'_0 and strongly quasibounded with respect to $\bar{u}$.*

Here a mapping $T : D(T) \subset Y \to Z$ in a complementary system $(Y, Y_0; Z, Z_0)$ is called strongly quadibounded with respect to $\bar{y} \in Y_0$ if for each $a_1, a_2 > 0$ there exists $k(a_1, a_2)$ such that for $y \in D(T) \cap Y_0$ with $\|y\| \le a_1$ and $\langle y - \bar{y}, Ty \rangle \le a_2$, one has $\|Ty\| \le k(a_1, a_2)$. Of course "bounded" implies "strongly quaseibounded", but the converse is not true (cf. [4, p. 172]). It is known that if T is monotone and if $D(T)$ contains some ball $B_\varepsilon(\bar{y}, Y_0)$, $\varepsilon > 0$, then T is strongly quasibounded with respect to $\bar{y}$ (cf. [2, proposition 14]). Consequently the mapping S of theorem 1 satisfies the assumptions of lemma 2.

Proof of Lemma 2. We first show that T_2 is pseudo-monotone with respect to Y'_0. Condition (a) about continuity on finite dimensional subspaces of Y'_0 is immediate. To verify condition (b), let u_i be a bounded net in Y'_0 such that $u_i \to u \in Y'$ for $\sigma(Y', Z'_0)$, $T_2 u_i \to g \in Z'$ for $\sigma(Z', Y'_0)$ and lim sup $\langle u_i, T_2 u_i \rangle \le \langle u, g \rangle$; we must show that $u \in D(T_2)$, $T_2 u = g$ and $\langle u_i, T_2 u_i \rangle \to$ $\to \langle u, g \rangle$; as usual, it suffices to prove the latter for a subnet.

Note first that γu_i remains bounded in $E_M(\Gamma)$ and that, passing to a subnet if necessary,

$$\int_\Gamma S(\gamma u_i) \,.\, (\gamma u_i - \gamma \bar{u}) \, d\Gamma = \langle u_i - \bar{u}, T_2 u_i \rangle \le Cst.,$$

which implies, by the strong quasiboundedness of S, that $S(\gamma u_i)$ remains bounded in $L_{\overline{M}}(\Gamma)$; we can thus assume that $S(\gamma u_i) \to w \in L_{\overline{M}}(\Gamma)$ for $\sigma(L_{\overline{M}}(\Gamma), E_M(\Gamma))$. Now consider the two linear forms on Y':

$$v \to \langle v, g \rangle$$
$$v \to \int_\Gamma w \,.\, \gamma v \, d\Gamma;$$

both are $\sigma(Y', Z')$ continuous, and by going to the limit in

$$\langle v, T_2 u_i \rangle = \int_\Gamma S(\gamma u_i) \,.\, \gamma v \, d\Gamma$$

for $v \in Y'_0$, we see that they coincide on Y'_0; since Y'_0 is $\sigma(Y', Z')$ dense in Y', the above linear forms coincide on all Y'. We now apply the pseudo-monotonicity of S : γu_i remains bounded in $L_M(\Gamma)$, $\gamma u_i \to \gamma u$ for $\sigma(L_M, E_{\overline{M}})$, $S(\gamma u_i) \to w$ for $\sigma(L_{\overline{M}}, E_{\overline{M}})$, and

$$\limsup \int_\Gamma S(\gamma u) \,.\, \gamma u_i \, d\Gamma = \limsup \langle u_i, T_2 u_i \rangle \le \langle u, g \rangle = \int_\Gamma w \,.\, \gamma u \, d\Gamma;$$

consequently $\gamma u \in D(S)$, $S(\gamma u) = w$ and $\int_\Gamma S(\gamma u_i) \,.\, \gamma u_i \, d\Gamma \to \int_\Gamma w \,.\, \gamma u \, d\Gamma$. The corresponding conditions for T_2 follow then easily.

The verification that T_2 is strongly quasibounded with respect to $\bar{u}$ is immediate. Q.E.D.

Lemma 3. *cf.* [4, *proposition* 2.4]). *Let $(Y, Y_0; Z, Z_0)$ be a complementary system and let $T_1 : D(T_1) \subset Y \to Z$ and $T_2 : D(T_2) \subset Y \to Z$ be two pseudo-monotone mappings with respect to Y_0. Suppose that T_2 is strongly quasibounded with respect to some $\bar{y} \in Y_0$ and that there exists $l : \mathbb{R}^+ \to \mathbb{R}^+$ continuous such that $\langle y - \bar{y}, T_1 y \rangle \ge - l(\|y\|)$ for $y \in Y_0$. Then $T_1 + T_2$ is pseudo-monotone with respect to Y_0.*

Note that the existence of such a function l is automatically guaranteed when T_1 is monotone. This is the case in the application of lemma 3 below. Note also, in this application, that none of the mappings T_1 and T_2 is bounded, in general, which excludes the use of the more classial proposition 2.3 of [4].

Proof of theorem 1. We wish to apply proposition 1 to the mapping $T_1 + T_2$ in the complementary system $(Y', Y'_0; Z', Z'_0)$. Clearly $T_1 + T_2$ is monotone, and by lemmas 2 and 3, it is pseudo-monotone with respect to Y_0. Moreover $(T_1 + T_2)(Y'_0)$ meets Z'_0; in fact $(T_1 + T_2)0 \in Z'_0$.

Thus, to prove theorem 1, it suffices to verify the local a priori bound condition (iii) of proposition 1.

Since $S(0) \in E_{\overline{M}}(\Gamma)$, the linear form

$$v \to \int_\Gamma S(0) \,.\, v \, d\Gamma$$

is continuous on Y' for $\sigma(Y', Z'_0)$; consequently there exist functions a_i, a_0 in $E_M(\Omega)$ such that

$$\int_\Gamma S(0) \,.\, v \, d\Gamma = \int_\Omega \left[\sum_{i=1}^n a_i \frac{\partial v}{\partial x_i} + a_0 v \right] dx$$

for all $v \in Y'$. Now let $g \in Z'_0$ be given:

$$\langle v, g \rangle = \int_\Omega \left[\sum_{i=1}^n g_i \frac{\partial v}{\partial x_i} + g_0 v \right] dx$$

for $v \in Y'$, where g_i and $g_0 \in E_{\overline{M}}(\Omega)$. Take $r > \max\{1, 2/c_4, 1/c_5\}$ where C_4, C_5 are the constants appearing in (5), choose a number s such that

$$\int_\Omega \left[\sum_{i=1}^n \overline{M}(r^2 g_i) + \overline{M}(r^2 g_0) \right] dx \leq s$$

and define

$$\mathcal{N} = \{h \in Z' ; \exists\, h_i,\, h_0 \in L_M(\Omega) \text{ with}$$

$$\int_\Omega \left[\sum_{i=1}^n \overline{M}(r^2 h_i) + \overline{M}(r^2 h_0) \right] dx \leq s + 1 \text{ and}$$

$$\langle v, h \rangle = \int_\Omega \left[\sum_{i=1}^n h_i \frac{\partial v}{\partial x_i} + h_0 v \right] dx \text{ for all } v \in Y\}.$$

$\mathcal{N}$ is a (norm) neighbourhood of g in Z'; this follow from the construction of Z' as a quotient space (cf. section 2) and from the fact that the convex functional

$$w \to \int_\Omega \overline{M}(w)\, dx$$

is (norm) continuous on a (norm) neighbourhood of $E_{\overline{M}}(\Omega)$ in $L_{\overline{M}}(\Omega)$, cf. [4, example 4.10]. We claim that

$$\{u \in D(T_1 + T_2);\, (T_1 + T_2)\, u \in \mathcal{N}\}$$

is bounded in Y'. Indeed if $u \in D(T_1 + T_2)$ with $(T_1 + T_2)\, u \in \mathcal{N}$, then

$$b(u, v) = \int_\Omega \left[\sum_{i=1}^n h_i \frac{\partial v}{\partial x_i} + h_0 v \right] dx$$

for all $v \in Y$, and in particular

$$b(u, u) = \int_\Omega \left[\sum_{i=1}^n h_i \frac{\partial u}{\partial x_i} + h_0 u \right] dx.$$

Using inequality (5) in the definition of $b(u, u)$ together with the fact that

$$\int_\Gamma S(\gamma u) \,.\, \gamma u\, d\Gamma = \int_\Gamma [S(\gamma u) - S(0)\, \gamma u\, d\Gamma + \int_\Omega \left[\sum_{i=1}^n a_i \frac{\partial u}{\partial x_i} + a_0 u \right] dx$$

where the second integral on Γ is ≥ 0 by monotonicity of S, we obtain

$$C_4 \int_\Omega \left[\sum_{i=1}^n M\left(C \frac{\partial u}{\partial x_i} \right) + M(C_5 u) \right] dx \leq 2 C_6\, meas(\Omega) + \int_\Omega \left[\sum_{i=1}^n h_i \frac{\partial u}{\partial x_i} + h_0 u \right] dx$$

$$- \int_\Omega \left[\sum_{i=1}^n a_i \frac{\partial u}{\partial x_i} + a_0 u \right] dx.$$

By Young's inequality, the right-hand side above is less than

$$2 C_6\, meas(\Omega) + 2 \int_\Omega \left[\sum_{i=1}^n M\left(\frac{1}{r^2} \frac{\partial u}{\partial x_i} \right) + M\left(\frac{1}{r^2} u \right) \right] dx +$$

$$\int_\Omega \left[\sum_{i=1}^n \overline{M}(r^2 h_i) + \overline{M}(r^2 h_0) \right] dx +$$

$$+ \int_\Omega \left[\sum_{i=1}^n \overline{M}(r^2 a_i) + \overline{M}(r^2 a_0) \right] dx \leq Cst. +$$

$$+ \frac{2}{r} \int_\Omega \left[\sum_{i=1}^n M \left(\frac{1}{r} \frac{\partial u}{\partial x_i} \right) + M\left(\frac{1}{r} u \right) \right] dx,$$

where we have used facts that $r > 1$ and that M is convex. By the choice of r, a bound on $\int_\Omega \left[\sum_{i=1}^n M\left(C_5 \frac{\partial u}{\partial x_i} \right) + M(C_5 u) \right] dx$ can thus be derived and it follows that u remains bounded in Y. Q.E.D.

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