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Replacing Tangencies by Saddle-nodes

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1. In this paper, M^n is a closed n-dimensional manifold ($n \geq 2$) of class C^{r+1} ($r \geq 2$), and all the flows and ares of flows on M are supposed of class C^r. We shall prove the following:

Theorem. *et X_λ, $\lambda \varepsilon [-1, 1]$ be an are of gradient-like flows on M^n, except for the value $\lambda = 0$, which is a bifurcation point of type tangency. Then, $X_{\lambda=-1}$, $X_{\lambda=1}$ can be joined by another are with only two bifurcations, which are of type saddle-node.*

That $\lambda = 0$ is a bifurcation of type tangency means that there exists a generic tangency between the invariant manifolds of two singularities of $X_{\lambda=0}$. That λ_0 is a bifurcation of type saddle-node means that X_{λ_0} has a singularity which is a saddle-node and the are is generic. These definitions will be precised in the next paragraph. From [5] and the theorem above, it results the following.

Corollary. *Any two Morse-Smale flows on M^n can be joined by an are with a finite number of bifurcation points, which are of type saddle-node.*

It is shown in [4] that this are is stable.

2. Recall of definitions.

A flow X on M^n is called *gradient-like* if:
(1) The α and ω limit of every orbit of X is contained in $\{p_1, \ldots, p_k\}$, where p_i is a hyperbolic singularity, i.e., no eigenvalue of $DX(p_i)$ has real part zero.
(2) The stable and unstable manifolds of the singularities $p_1, \ldots, p_k$ meet transversely.
$W^u(p)$, $W^s(p)$ denote the unstable and stable manifolds of a critical point p.

Let X_λ be a 1-parameter family of flows on M^n. We say that λ_0 is a *bifurcation point of type tangency* if the following conditions are satisfied:

(i) X_{λ_0} verifies all the conditions of gradient-like, except that it has a pair of singularities p_{λ_0}, q_{λ_0} such that $W^u(p_{\lambda_0})$ $W^s(q_{\lambda_0})$ have a common orbit 0 where their intersection is not transversal, and $0 < \dim W^u(p_{\lambda_0}) < n$, $0 < \dim W^s(q_{\lambda_0}) < n$.

(ii) Let r be a point on 0, and let N be a local normal section to X_λ on r, for λ in a neighborhood $U(\lambda_0)$. There exists a λ-family of coordinates $(z, u, w)_\lambda \in R \times R^{k1} \times R^{k2}$ on N such that $r = (0,0,0)_{\lambda_0}$ and $W^u(p_\lambda) \cap N$ is the plane $z = 0$, $u = 0$. There also exists a λ-family of coordinates $(x, y)_\lambda \in R^{k1} \times R^{k3}$ on $W^s(q_\lambda) \cap N$ with $r = (0,0)_{\lambda_0}$. And, if F is the projection of $W^s(q_\lambda) \cap N$ on the plane $w = 0$, expressed with these coordinates, it is verified:

$$\det\left[\frac{\partial p_{u'} F}{\partial x}(0, \lambda_0)\right] \neq 0, \quad \frac{\partial p_{u'} F}{\partial y}(0, \lambda_0) = 0,$$

$$\frac{\partial p_{z'} F}{\partial x}(0, \lambda_0) = \frac{\partial p_{z'} F}{\partial y}(0, \lambda_0) = 0, \quad \frac{\partial p_{z'} F}{\partial y}(0, \lambda_0) \neq 0 \text{ and } \frac{\partial^2 p_{z'} F}{\partial y^2}(0, \lambda_0) \text{ is}$$

not degenerated.

It may happens $k_1 = 0$, $k_2 = 0$ or both; the last when $n = 2$.

(iii) The eigenvalues $\mu_1, \ldots, \mu_\alpha$ of $DX/(p_{\lambda_0})$ with negative real part verifie:

$$Re\, \mu_\alpha \leq \ldots \leq Re\, \mu_2 < \mu_1 < 0$$

μ_1 is real and has multiplicity one. Let $K_{p\lambda}$ be the eigenspace corresponding to the eigenvalues $\mu_2, \ldots, \mu_\alpha$ of $DX_\lambda(p_\lambda)$. It is shown [2] that there exists a unique stable manifold $W^{ss}(p_\lambda)$ which is tangent to $K_{p\lambda}$. We suppose that $W^{sss}(p)$ is transversal to the unstable manifolds of all the singularities, and that the limit (when $t \to -\infty$) of the tangent space to $W^s(q_\lambda{}^o)$ in r: $\lim_{t \to -\infty}$ $(T_r \varphi^t_{X_{\lambda_0}})\,(T_r W^s(q_{\lambda_0}))$ is the sum of $K_{p\lambda_0}$ and a subspace which is tangent to $W^u(p_{\lambda_0})$. Symmetrical conditions are also required for q_{λ_0} and $W^u(p_{\lambda_0})$.

We say that λ_0 is a *bifurcation point of type saddle-node* if:

(i) $X_\lambda{}^o$ verifies all the conditions of gradient-like, except that it has a singularity p_{λ_0} which is a saddle-node, i.e. zero is an eigenvalue of $Dp_{\lambda_0}(X_{\lambda_0})$ with multiplicity one, and all the other eigenvalues have real part different from zero. Then, it is shown [2] that $p_\lambda{}^o$ has invariant stable and unstable manifolds with boundary $(W^s(p_{\lambda_0})$ $W^u(p_{\lambda_0}))$. All the invariant manifolds are supposed transversal.

(ii) There exist local coordinates $(x_1, \ldots, x_n)$ such that:

$$p_{\lambda_0} = 0, \frac{\partial X^1}{\partial x_i}(0, \lambda_0) = \frac{\partial X^i}{\partial x_1}(0, \lambda_0) = 0 \; i = 1, \ldots, n.$$

$$\frac{\partial(X^2, \ldots, X^n)}{\partial(x_2, \ldots, x_n)}(0, \lambda_0) \neq 0$$

$$\frac{\partial^2 X^1}{\partial x_1^2}(0, \lambda_0) \neq 0, \frac{\partial X^1}{\partial \lambda}(0, \lambda_0) \neq 0.$$

If λ_0 is a bifurcation of type saddle-node or tangency, it can be proved [7] is gradient-like for $\lambda \in V(\lambda_0)$, $\lambda \neq \lambda_0$.

3. Proof of the theorem.

Suppose an are of flows X_λ, $-1. \geq \lambda \geq 1$, with a bifurcation point of type tangency at $\lambda_0 = 0$, in such a way that $W^u(p_0)$ has a tangency with $W^s(q_0)$, as above. We will construct an are Y_λ, $-1 \geq \lambda \geq 1$, such that $Y_\lambda = X_\lambda$ for all λ near $\lambda = 1$, $\lambda = -1$, and Y_λ has only two bifurcation points, which are of type saddle-node. All the constructions will be made in a neighborhood of $p_0 \cdot p_\lambda$ shall denote the critical point near to p_0 of X_λ.

A) We shall transform X_λ locally into a product $X_\lambda = X^u_\lambda \times X^w_\lambda \times X^z_\lambda$. Consider a λ-family of local coordinates $(w, u, z)_\lambda$ in M, such that, for $\lambda \in V(\lambda_0)$:

$$\begin{cases} p_\lambda = 0 \\ W^u(p_\lambda) = \text{the plane } z = 0, \quad u = 0 \\ W^s(p_\lambda) = \text{the plane } w = 0 \\ W^{ss}(p_\lambda) = \text{the plane } z = 0, \quad w = 0 \end{cases}$$

and $\dfrac{\partial X^h}{\partial k}(0, \lambda) = 0$; $h, k = w, u, z \quad h \neq k$

Consider

$$\bar{X}_{\lambda,}(x) = \mu\varphi(x)\, DX_\lambda(0)x + [1 - \mu\varphi(x)]\, X_\lambda(x) x \in V(0), 0 \geq \mu \geq 1$$

where φ is a bump-function.

$\bar{X}_{\lambda,\mu}$ has 0 as an hyperbolic critical point, whose invariant manifolds coincide with those of p_λ. Transversalities are conserved, by the λ-lemma [6]. So, for μ fixed, the are $\bar{X}_{\lambda,\mu}$ is similar to the are X_λ. Consider a C^∞-function $\mu(\lambda)$: $\mathbb{R} \to [0,1]$ s.t. $\mu(\lambda) = 0$ when $|\lambda| \geq \varepsilon_1 > 0$ and $\mu(\lambda) = 1$ when $|\lambda| \leq \varepsilon_2 < \varepsilon_1$, $\varepsilon_2 > 0$. Then, $\bar{X}_\lambda = \bar{X}_{\lambda,\mu(\lambda)}$ is similar to X_λ, $\bar{X}_\lambda = X_\lambda$ for all λ near to $\lambda = 1$ or $\lambda = -1$, and $\bar{X}_\lambda(x) = DX_\lambda(0)x$ for $x \in V(0)$, $\lambda \in V(\lambda_0)$.

B) Let $r \in V(0) \cap W^u(p_\lambda)$ be a point of tangency of $W(q_0)$ with $W^u(p_0)$. Let $N = W \times U \times Z$. be a normal section to X_λ, where W, U, Z are contained in the planes w, u, z. Let F be the projection of $W^s(q_\lambda) \cap N$ on $0 \times U \times \times Z$. The image of $DF_{\lambda_0}(r)$ is close to the plane u. Let u' be coordinates on Im $DF_{\lambda_0}(r)$. F can be expressed in such a way that

$$x = P_{u'} \cdot F(x, y, \lambda); r = (0, 0)$$

$$\frac{\partial p_{z'} F}{\partial x}(r, \lambda_0) = \frac{\partial p_{z'} F}{\partial y}(r, \lambda_0) = 0.$$

Let C_λ be the set of points of $W^s(q_\lambda) \cap N$ where DF_λ is not onto; C_λ is the graph of a map $y = \varphi(x, \lambda)$. Then $F(C_\lambda)$ is the graph of $G_\lambda = p_z \cdot F(u', \varphi(u', \lambda))$ and $\frac{\partial G_\lambda}{\partial \lambda}(0, \lambda_0) = \frac{\partial p_z \cdot F}{\partial \lambda}(0, \lambda_0)$ which we suppose negative.

In the following, replace the coordinates $(w, u, z)_\lambda$ by $(w, u, z - k\lambda)$, where $k > 0$ is fixed and verifies:

$$\frac{\partial G_\lambda}{\partial \lambda}(0, \lambda_0) + k > 0, \quad \frac{\partial G_\lambda}{\partial \lambda}(0, \lambda_0) + \frac{1}{2}k < 0.$$

Then:

$$0 < G_\lambda(0) < \tfrac{1}{2}k\lambda < k\lambda = p_\lambda \quad \text{for } \lambda > 0$$

$$p_\lambda = k\lambda < \tfrac{1}{2}k\lambda < G_\lambda(0) < 0 \text{ for } \lambda < 0.$$

C) Consider the family of local flows $Y_{l,\, \varepsilon,\, \rho,\, \alpha,\, \beta,\, \lambda}$ where:

$$Y^z_{l,\, \varepsilon,\, \rho,\, \alpha,\, \beta,\, \lambda}(w, u, z) = lK(z, \varepsilon, \rho, \alpha, \beta)\,\varphi(w, u, z) + [1 - l\varphi(w, u, z)]\,\bar{X}^z_\lambda(z).$$

$$Y^w_\lambda = \bar{X}^w_\lambda \quad Y^u_\lambda = \bar{X}^u_\lambda \quad (w, u, z)\,\varepsilon\, V(0).$$

$$K(z, \varepsilon, \rho, \alpha, \beta) = -z^3 + (\varepsilon - \rho)z^2 + (\varepsilon\rho - \alpha - \beta)z - \alpha\rho + \beta\varepsilon.$$

In particular

$$K(z, \varepsilon, \rho, 0, \beta) = (\varepsilon - z)(z^2 + \rho z + \beta)$$

$$K(z, \varepsilon, \rho, \alpha, 0) = (-z^2 + \varepsilon z - \alpha)(\rho + z)$$

φ is a bump-function, $\operatorname{supp}(\varphi) \subset V(0)$ and $\varphi = 1$ in $V_1(0)$.
Consider the C^∞-real functions:

$$f(t) = e^{-1t},\ t > 0$$
$$= 0 \quad ,\ t \leq 0$$

$$g(t) = \frac{f(t)}{f(t) + f(1-t)} \qquad h(t) = g(2t+2)\,g(-2t+2)$$

$$l(\lambda) = h\left(\frac{\lambda}{a_0}\right) \qquad a_0 > 0 \text{ small and } 0 < a_1 < a < \tfrac{1}{2}a_0.$$

$$\varepsilon(\lambda) = k(\lambda - a_1)\,g\left(\frac{\lambda - a_1}{a - a_1}\right) + ka_1 \qquad \rho(\lambda) = \varepsilon(-\lambda)$$

$$\beta(\lambda) = \frac{k^2a^2}{4}\,g\left(\frac{\lambda}{a}\right) \qquad \alpha(\lambda) = \beta(-\lambda).$$

Then, for $\lambda \in [-1, 1]$, define

$$Y_\lambda(z, u, w) = Y_{l(\lambda),\, \varepsilon(\lambda),\, \rho(\lambda),\, \alpha(\lambda),\, \beta(\lambda),\, \lambda}(z, u, w) \text{ for } (z, u, w) \in V(0)$$

and $Y_\lambda = \bar{X}_\lambda$ otherwise.

Y_λ verifies the properties stated in the theorem.

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