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Stable Maps: An Introduction With Low Dimensional Examples

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Introduction

In this survey, I will discuss the following three topics about C^∞ stable maps from an n-manifold N to a p-manifold P.
1) The relationship between the topology of N and P and the stable maps from N to P.
2) The "simplest" stable maps (Removal of Singularities)
3) Classification of stable maps or germs under equivalence.

Much of what follows will be motivated by results in case the target P is 1 or 2-dimensional. We therefore begin with a general description of normal forms for a stable map germ and derive all of the local normal forms for maps with 1 or 2-dimensional targets.

Except where another description is given, in all that follows, *everything is* C^∞ and smooth always means C^∞.

1. Definitions and the Relation Between Stable and Finitely Determined Germs

A C^∞-map $f: N \to P$ is called stable if there is a neighborhood U of f in $C(N, P)$ such that if $g \in U$, there are diffeos $h : N ⥀$ and $h : P ⥀$ such that $k \circ f \circ h = g$. Here, $C(N, P)$ is the space of smooth maps from N to P with the Whitney C^∞-topology. For f a proper (pre-images of compacts are compact) map in $C(N, P)$, John Mather [15] proved that stability is equivalent to *infinitesimal stability* defined as follows: For each smooth mapping $\zeta : N \to TP$ over f (i.e $\pi_P \circ \zeta = f$), there are smooth sections ξ and η of the tangent bundles TN and TP respectively such that

$$\zeta = Tf \circ \xi + \eta \circ f$$

We introduce some notation: For any map $g: X \to Y$, let $\theta(g) = \{\sigma: X \to \to TY | \pi_Y \circ \sigma = g\}$ and let tg: $\theta(1_x) \to \theta(g)$: $\xi \to T_g \circ \xi$ and ωg: $\theta(1_y) \to \theta(g)$: $\eta \to \eta \circ g$. (Here 1_x and 1_y are the identity maps in X and Y resp.) Thus the condition of infinitesimal stability of f can be written:

$$\theta(f) = tf\theta(1_N) + \omega f\theta(1_P).$$

A germ $f: (N, x) - \to (P, y)$ is called *stable* if (in the obvious notation)

$$\theta_x(f) = tf\theta_x(1_N) + \omega f\theta_y(1_P).$$

More generally f is a finitely *$\mathscr{A}$-determined germ* if $\theta_x(f)/tf\theta_x(1_N) + wf\theta_y(1_P)$ is a finite dimensional real vector space. The dimension of this quotient is called the *$\mathscr{A}$-codimension of f*, $\mathscr{A}$-codim f. By a theorem of Mather [15-III], *f finitely $\mathscr{A}$-determined is equivalent to the following: There is an integer r* such *that if $g: (N, x) \longrightarrow (P, y)$ is any germ whose r-jet at $x, j^r g(x)$ agrees with $j^r f(x)$, then f and g are $\mathscr{A}$-equivalent*. (i.e. there are diffeo germs $h: (N, x) \supset$ and k: $(P, y) \supset$ such that $k \circ \circ h = g$). In the special case of f stable, f is $\mathscr{A}$-equivalent to any germ with the same $(p + 1)$-jet, where p is the dimension of the target.

The group $\mathscr{A}$ of germs of diffeos of the source and the target is contained in a larger group K, which also operates on the germs of maps and which is defined as the group of all diffeo germs $(h \times k)$: $(N \times P, (x, y)) \supset$ where $(\bar{x}, \bar{y}) \to (h(\bar{x}), k(\bar{x}, \bar{y}))$ and k is constant on $N \times y$.

If $f: (N, x) \longrightarrow (P, y)$ and $(h, k) \in K$, then the image of f by (h, k) is $k \circ (h^{-1} \times f \circ h^{-1})$. Two germs are K-*equivalent* if they are in the same K-orbit. A germ f is *finetely* K*-determined* if $\theta_x(f)/tf\theta_x(1_N) + f^* m_y \theta_x(f)$ is a finite dimensional real vector space. (m_y is the maximal ideal in the ring ε_y of germs of smooth real valued functions at y). The dimension of the quotient is called the K-*codimension* of f, K-codim f. Mather's theorem also applies to finitely K-determined germs, namely: *The* K*-equivalence class of a finitely* K*-determined germ f is determined by some finite jet of f.*

The connection between K-finitely determined germs and stable germs was established by Mather [15-IV], we give the result essentially as J. Martinet does in Springer Lect. Notes 535. Let $S_r(s, t)$ be the set of stable map germs of $(R^{r+s}, 0) \to (R^{r+t}, 0)$ of rank r, and let $K_r(s, t)$ be the set of germs of maps of $(R^s, 0) \to (R^t, 0)$ of rank 0 and K-codimension $\leq r + t$.

Then there is a one-to-one correspondence between the set of *$\mathscr{A}$-orbits* in *$S_r(s, t)$ and the set of* K*-orbits in $K_r(s, t)$.*

Let $\mathscr{A}F$ denote the $\mathscr{A}$-orbit of F and Kf denote the K-orbit of f. If $F \in S_r(s, t)$ it is no restriction to assume F his the form:

$$F: \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^r \times \mathbb{R}^t: (u, x) \to (u, f(u, x)),$$

since there is always an element in $\mathscr{A}F$ of that form.

The one-to-one correspondence is given by:

$$S_r(s, t)/_{\mathscr{A}} \in \mathscr{A}F \longrightarrow Kf(0, .) \in K_r(s, t)/_K.$$

Thus the classification of stable germs reduces to the classification of finitely K-determined germs. Knowing the inverse of this map allows us to give normal forms for stable maps. In fact let $\varphi \in K_r(s, t)$, then let $K\varphi = t\varphi\, \theta_0(1_s) + + \varphi^*(m_0)\, \theta_0(\varphi)$, then

$$\dim_R(\theta_0(\varphi)/_{K\varphi}) \leq r + t.$$

But since φ has rank 0, $K\varphi \subset m_0\theta_0(\varphi)$. Thus $\dim_{\mathbb{R}}(m_0\theta_0(\varphi)/_{K\varphi}) \leq r$ Choose $\{v_1, \ldots, v_r\}$ elements of $m_0\theta_0(\varphi)$ whose images in $(m_0\theta_0(\varphi)/_{K\varphi})$ span. Then the map germ:

$$F: \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^r \times \mathbb{R}^t: (u, x) \to (u, \varphi(x) + \langle u, v(x) \rangle)$$

is stable, $\langle u, v(x) \rangle = \sum_{i=1}^{r} u_i v_i(x)$. It is also part of the result that if

$$F: \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^r \times \mathbb{R}^t: (u, x) \to (u, f(u, x))$$

is in $S_r(s, t)$, then the images of $\{\partial f/\partial u_i|_{n=0}, i = 1, \ldots, r\}$ in $m_0\theta_0(f_0)/_{Kf_0}$ form a spanning set $(f_0(x) = f(0, x))$.

Thus F is $\mathscr{A}$-equivalent to the map germ given by:

$$(u, x) \to (u, f(0, x) + \sum_{i=1}^{r} u_i \frac{\partial f}{\partial u_i}(0, x)).$$

Examples

a) *Stable maps of maximal rank* $S_r(s, 0)$ has only one $\mathscr{A}$-orbit, in fact $K_r(s, 0)$ has only one element in it-namely the unique map germ from $(\mathbb{R}^s, 0)$ $t \circ (\{0\}, 0)$. Thus any map in $S_r(s, 0)$ is $\mathscr{A}$-equivalent to the linear projection $(u_1, \ldots, u_r, x_1, \ldots, x_s) \to (u_1, \ldots, u_r)$. Similarly, any map in $S_r(0, t)$ is $\mathscr{A}$-equivalent to the linear injection $(u_1, \ldots, u_r) \to (u_1, \ldots u_r, 0, \ldots, 0)$.

b) *Stable maps of rank* 0. By the preceding, we know that the the set of stable map germs of rank 0, $S_0(s, t)$ is the same as $K_0(s, t)$, the germs of K-codim $\leq t$ and $\mathscr{A}$-orbits and K-orbits coincide. If $f \in K_0(s, t)$, then $Kf = m_0\, \theta_0(f)$. Working modulo m_0^2, we see that $(m_0/m_0^2)^t$ is spanned by the images of $\{\partial f/\partial x_1, \ldots, \partial f/\partial x_s\}$.

Thus $st \leq s$, or $s = 0$ or $t \leq 1$, so the only new case is $s \neq 0$, $t = 1$. So if $f \in K_0(s, 1) = S_0(s, 1)$, m_0 is generated by $\{f, \partial f/\partial x_i\, i = 1, \dots s\}$. But working modulo m_0^2 again, it is easy to see that $(\partial^2 f/\partial x_i \partial x_j(0))$ is non-singular, so by the Morse Lemma [16, p. 6] f is $\mathscr{A}$-(and K–) equivalent to one of:

$$(x_1, \dots, x_s) \to \left(\sum_{j=1}^{s} \pm x_j^2\right).$$

c) *Stable maps of rank* 1 For any $r' \geq r$, $K_r(s, t) \subset K_{r'}(s, t)$, obviously. Thus among the $\mathscr{A}$-orbits of stable map-germs of rank r'; are the $\mathscr{A}$-orbits of the suspensions of the stable germs of rank $r \leq r'$. Thus in case $r = 0 < 1 = r'$, the maps:

$$(u, x_1, \dots, x_s) \to \left(u, \sum_{j=1}^{s} \pm x_j^2\right)$$

are all stable. In addition, we have those $\mathscr{A}$-orbits that correspond to the K-orbits of elements in $K_1(s, t) - K_0(s, t)$, that is, germs f for which

$$\dim_{\mathbb{R}}(m_0\, \theta_0(f)/_{Kf}) = 1.$$

Again working modulo m_0^2, we fined that $st \leq 1 + s$, and if we exclude $s = 0$, we have two sets of (s, t) – values namely $(s, 1)$ and $(1, 2)$. The $(1, 2)$-case is trivial since $f \in K_1(1, 2)$ is K-equivalent to $(x) \to (x^2, 0)$ and the corresponding element in $S_1(1, 2)$ is $(u, x) \to (u, x^2, ux)$, the "Whitney Umbrella".

On the other hand, using the proof of the Morse Lemma it is easy to show that any $f \in K_1(s, 1)$ is K-equivalent to one of:

$$(x_1, \dots, x_{s-1}, z) \to \sum_{i=1}^{s-1} \pm x_i^2 + z^3$$

and so any stable germ in $S_1(s, 1)$ is $\mathscr{A}$-equivalent to one of:

$$(u, x_1, \dots, x_{s-1}, z) \to \left(u, \sum_{i=1}^{s-1} \pm x_i^2 + z^3 + uz\right),$$

the "cusps".
Thus normal forms for stable germs are:

For l-dimensional targets:

$$\begin{cases} (u, x_1, \dots, x_s) \to u, & \text{(Maximal rank)} \\ (x_1, \dots, x_s) \to \sum_{i=1}^{s} \pm x_i^2, & \text{(Morse singularity)} \end{cases}$$

For 2-dimensional targets:

$$\begin{cases} (u_1, u_2, x_1, \dots, x_s) \to (u_1, u_2) & \text{(Maximal rank)} \\ (u, x_1, \dots, x_s) \to \left(u, \sum_{i=1}^{s} \pm x_i^2\right) & \text{(folds)} \\ (u, x_1, \dots, x_{s-1}, z) \to \left(u, \sum_{i=1}^{s-1} \pm x_i^2 + uz + z^3\right), & \text{(cusps)} \end{cases}$$

2. The Relation Between $(S(f)$ and $f(S(f)))$ and the tology of N and P for a stable map $f: N - P$.

Given manifolds N and P and a stable map $f: N \to P$, is there any general relation between the topology of N and P and the singular set of f, $S(f)$ and its image $f(S(f))$? (and $f(N)$, in case $\dim N < \dim P$).

The most famous example of such a relation occurs when $P = \mathbb{R}$, the *Morse inequalities*. If $f: N \to \mathbb{R}$ is stable, then $S(f)$ consist of isolated points, no two of which have a common image. At each critical point $p \in S(f)$, we can choose coordenates $x_1, \dots, x_n$, centered at p, in terms of which f is:

$$(x_1, \dots, x_n) \to \left(-\sum_{k=1}^{i} x_k^2 + \sum_{j=i+1}^{n} x_j^2\right), \text{ for some } i.$$

The point p is called a *critical point of f* of index i. [16, p.6]. (This definition of index is independent of the choice of coordenates in the source; the orientation of $\mathbb{R}$ is fixed).

The Morse Inequalities. [16, p 30] *Let $f: N \to \mathbb{R}$ be stable, N a compact n-manifold. Let $N_i(f)$ be the number of critical points of index i of f and* $B_i =$ $= \dim\ H_i(N; \mathbb{R})$, *the* i^{th} *Betti-number, then:*

$$\sum_{i=0}^{k} (-1)^{k-1} N_i(f) \geq \sum_{i=0}^{k} (-1)^{k-1} B_i \text{ and}$$

$$\sum_{i=0}^{n} (-1)^i N_i(f) = \chi(N), \text{ the Euler characteristic of } N.$$

From this, we easily derive:

Corollary (Thom [22, Theorem 9, p 84]) *Let N be a compact manifold and $f: N \to \mathbb{R}^2$ be stable, then the number of cusps of f is congruent to $\chi(N)$ mod 2.*

Proof Let $\pi: \mathbb{R}^2 \to \mathbb{R}$ be a projection such that no cusp point of f is a critical point of $\pi \circ f$. To choose π, notice that a point $x \in N$ is a critical point of $\pi \circ f$ if $Tf(TN_x)$ is contained in ker $(T\pi)_{f(x)}$ = tangent space to the level line of π. Since there are only finitely many cusp points, we can surely choose a linear projection π whose level lines are transverse to all of the "cusp-lines", that is to each line $Tf(TN_x)$ for x, a cusp of f. Thus, the singular points of $\pi \circ f$ are those non-cusp points x of $S(f)$ such that $f(S(f))$ is tangent to the level line of π at $f(x)$. By replacing f by $\varphi \circ f$, for φ a diffeo of $\mathbb{R}^2$, if necessary, we may assume that the critical points of $\pi \circ f$ are non-degenerate. Thus we have $\chi(N)$ congruent mod 2 to the number of points of tangency of the level lines of π with $fS(f)$). We show that the number of such tangencies is congruent mod 2 to the number of cusps of f. It suffices to show this for each component of $S(f)$, and this is immediate from the two following Lemmas:

Lemma 1. *Let $g: S^1 \to \mathbb{R}^2$ be on immersion with normal crossings and let $\pi: \mathbb{R}^2 \to \mathbb{R}$ be a linear projection with $\pi \circ g$ having only non-degenerate critical points, then the number of such points is even.*

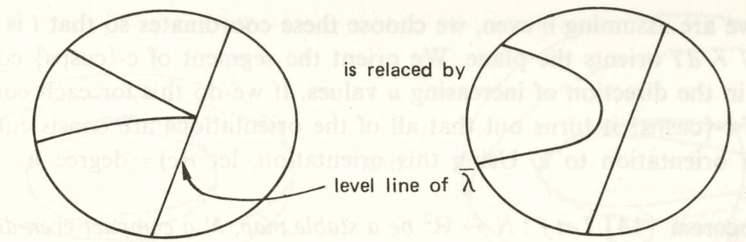
Proof Let $h = g'/|g'|: S^1 \to S^1$. If u is any regular value of h, then deg $h \equiv \# \{h^{-1}(u)\}$ mod 2 (Read $\# X$ as the number of elements in X). Let $\lambda \in S^1$ generate the kernel of π. A point $x \in S^1$ is critical for $\pi \circ g$ iff $g'(x) = \pm$ $\pm |g'(x)| \lambda$ iff $x \in h^{-1}\{\pm\lambda\}$. Thus $\#$ {critical point of $\pi \circ g\} = \# h^{-1}\{\pm\lambda\}$ = $= \# h^{-1}(\lambda) + \# h^{-1}(-\lambda)$. But by our non-degeneracy assumption, $\pm\lambda$ are bath regular values of h, so $\# h^{-1}(\lambda) \equiv \# h^{-1}(-\lambda) \equiv$ deg h (mod 2).

Definition *A C^∞* map $g: S^1 \to \mathbb{R}^2$ is an *immersion with cups and normal crossings* if $S(g)$ is finite and if $x \in S(g)$ (i.e. if $g'(x) = 0$ then $g''(x) \neq 0$ and if $x \in S(g)$ then $g^{-1}g(x) = \{x\} = \{x\}$, and $g \mid S^1 - S(g)$ has normal crossings.

Remark If $f: N \to \mathbb{R}^2$ is stable and S is a component of $S(f)$ then $g = f \mid S$ is an immersion with cusps and normal crossings and $S(g) = S \cap$ {cusps of f}.

Lemma 2. *If $g: S^1 \to \mathbb{R}^2$ is an immersion with cusps and normal crossings and $\bar{\lambda}: \mathbb{R}^2 \to \mathbb{R}$ is a linear projection such that all of the critical points of $\bar{\lambda} \circ g$ are in $S^1 - S(g)$ and are non-degenerate, then* $\#$ {critical points of $\bar{\lambda} \circ g\} \equiv \# S(g)$ mod 2.

Proof. By "rounding off" the cusps of g, we replace g by an immersion $\tilde{g}$, with normal crossings such that $\bar{\lambda} \circ \tilde{g}$ has only non-degenerate critical points.



The critical points of $\bar{\lambda} \circ \tilde{g}$ are all of those of $\bar{\lambda} \circ g$ and, in addition, one new critical points for each "rounded off" cusp. Thus $\#$ critical $(\bar{\lambda} \circ \tilde{g}) = \#$ critical points $(\bar{\lambda} \circ g) + \# S(g)$, and by Lemma 1, we're done.

This result, that the number of cusps of $f: N \to \mathbb{R}^2$ and $\chi(N)$ have the same parity is indeed a pale shadow of the Morse inequalities; we are given two nice functions and all we've determined with the is $\chi(N)$ mod 2. We can make a little improvement on this. In case N is compact, even dimensional, we can compute $\chi(N)$ itself using only the germ of f at $S(f)$ for any stable $f: N \to \mathbb{R}^2$. For any such f, $S(f)$ consists of a disjoint union of embedded circles. If C a component of $S(f)$, $f \mid C$ is an immersion with cusps and normal crossings. Let $g = f \mid c$, then $h = g'/|g'|$: c-cusps $\to S'$. If we compose this map with the map $S^1 \to S^1 : e^{i\theta} \to e^{2i\theta}$ we get a map that extends smoothly to all of c, $k_c: c \to S^1$. To see this choose coordinates at a cusp so that f has the form: $(u, x_1, \ldots, x_{n-2}, z) \to (u, Q(x) + uz + z^3)$, Q a quadratic form, and $S(f)$ has the form $\{x_i = 0, \ u + 3z^2 = 0\}$. Thus restricting f to the component of $S(f)$ near this cusp gives the form of g:

$$z \to (-3z^2, -2z^3),$$

and so h looks like:

$$z \to -(Sgn\, z) \frac{(1, z)}{(1 + z^2)^{1/2}},$$

and k_c is given locally by

$$z \to \frac{(1 - z^2, 2z)}{(1 + z^2)}. *$$

Thus, so for, to each component c of $S(f)$, we have a smooth map $k_c: c \to S^1$. We choose orientations for the components as follows:
At a point $p \in (c - \{\text{cusps}\})$ choose coordinates at p and $f(p)$, so that:

$$\begin{cases} U(f(u, x_1, \ldots, x_{n-1})) = u \\ Y(f(u, x_1, \ldots, x_{n-1})) = -\sum_{j=1}^{i} x_j^2 + \sum_{j=i+1}^{n-1} x_j^2. \end{cases}$$

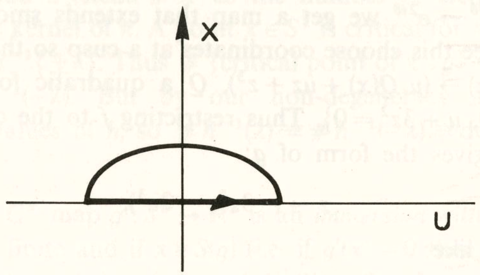
Since we are assuming n even, we choose these coordinates so that i is *even*, and $dU \wedge dY$ orients the plane. We orient the segment of c-{cusps} containing p in the direction of increasing u values. If we do this for each component of c-{cusps} it turns out that all of the orientations are consistent and give an orientation to c. Using this orientation, let $r(c) =$ degree k_c.

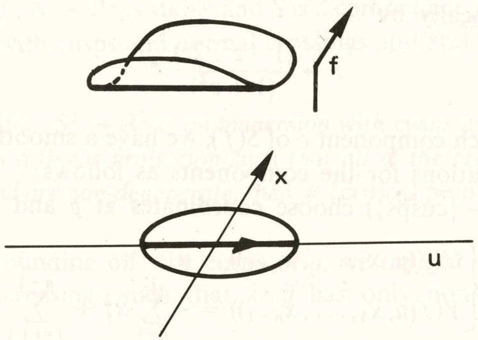
Theorem [14] *Let $f : N \to \mathbb{R}^2$ be a stable map, N a compact even-dimensional manifold, then*

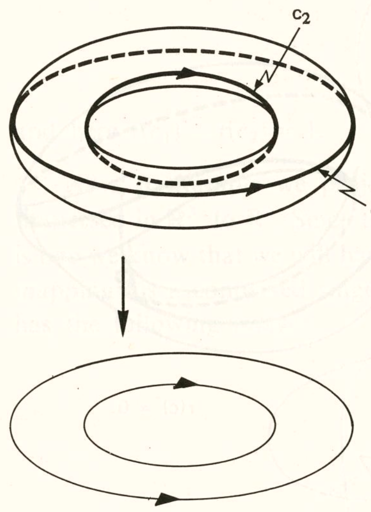
$$\chi(N) = \sum_c r(c),$$

where the sum is taken over all components of $S(f)$.

In case dim $N = 2$, the orientation of the singular curves is easy to describe. At a fold point the map is given by $(u, x) \to (u, x^2)$ and the curve is oriented in the direction of increasing u. That is the curve is oriented so that in the image the surface folds to the left of the fold curve. For example, if we take N to be the *torus* we have stable map:







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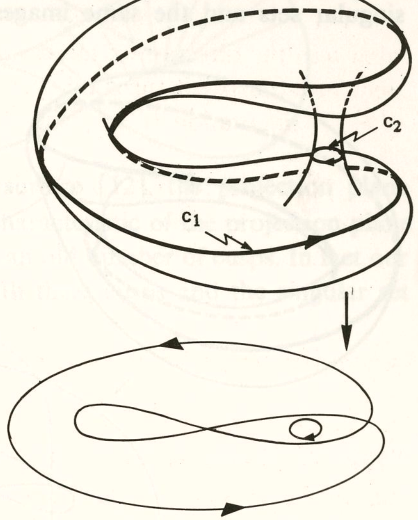
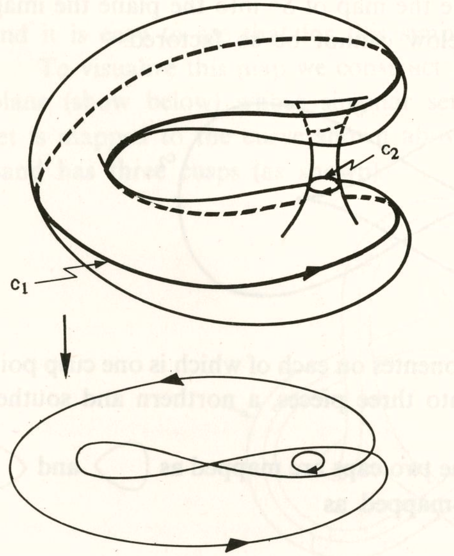
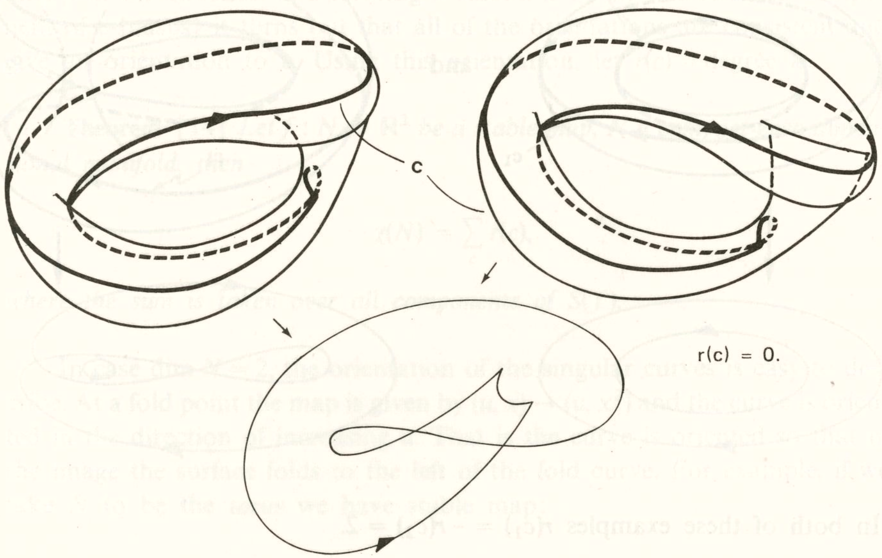


In both of these examples $r(c_1) = -r(c_2) = 2$.

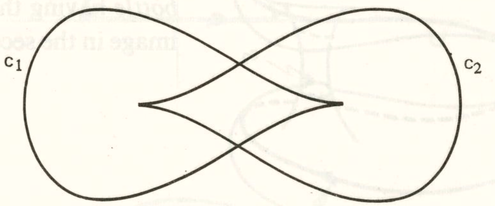


We also get a map of the *Klein bottle* having the some image as the image in the second map if the torus.

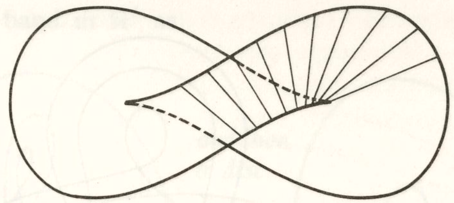
Similarly we can get maps of the torus and Klein bottle having connected singular sets and the same images.



Not all stable maps of surfaces into the plane can be factored through immersions in $\mathbb{R}^3$ [11]. For example the map of S^2 into the plane the image of whose singular set is shown below cannot be so factored.

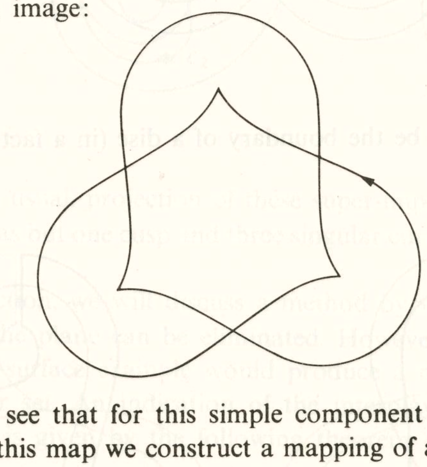


The singular set has two componentes on each of which is one cusp point. The two fold curves cut S^2 into three pieces, a northern and southern cap and a cylinder joining them. The two caps are mapped as  and  and the cylinder joining them is mapped as



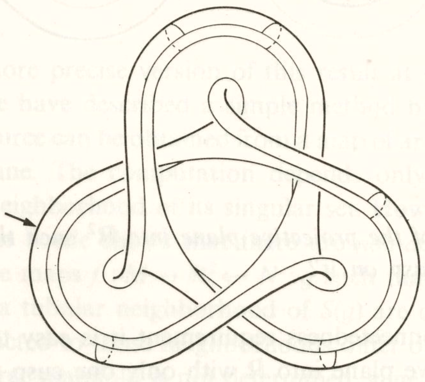
and here $r(c_1) = r(c_2) = 1$.

As a it example we project Boy's surface [12], the projection plane immersed in $\mathbb{R}^3$, to $\mathbb{R}^2$. Since the Euler characteristic of the projection plane is one we know that we will have to have an old number of cusps. In fact our mapping his a connected singular set with three cusps and the singular set has the following image:

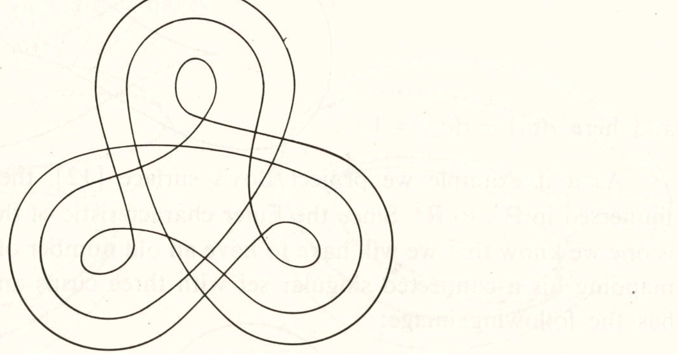


and it is easy to see that for this simple component c, $r(c) = 1$.

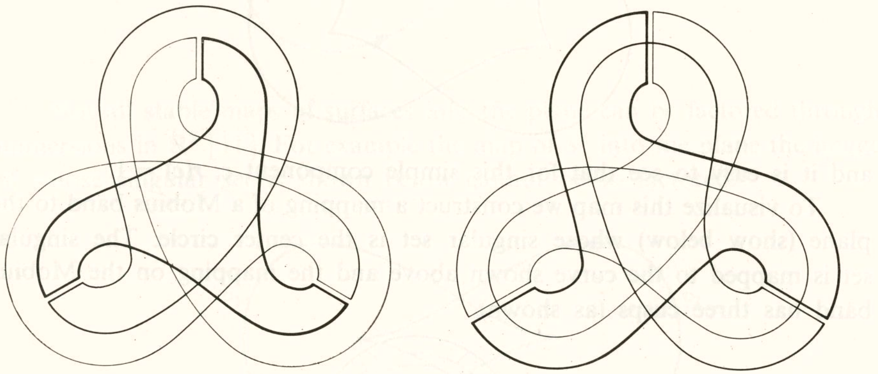
To visualize this map we construct a mapping of a Mobius band to the plane (show below) whose singular set is the center circle. The singular set is mapped to the curve shown above and the mapping on the Mobius band has three cusps (as shown).



The image of the boundary of this Mobius band is:



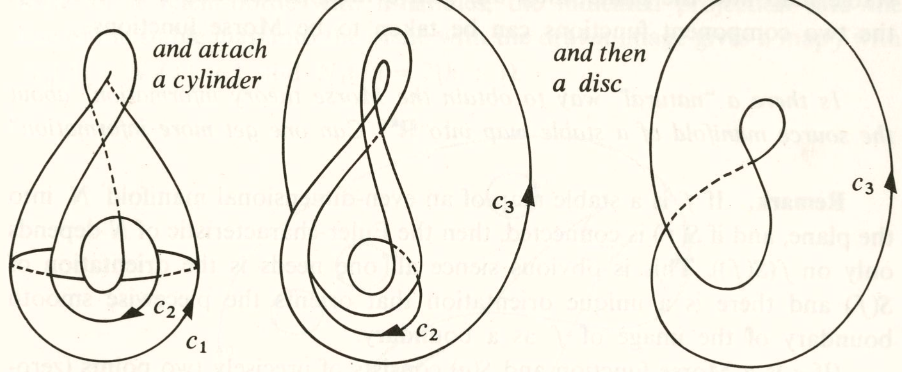
which can be seen to be the boundary of a disc (in a fact in two ways!)



Is there a stable f of the projective plane into $\mathbb{R}^2$ such that $S(f)$ is connected and has only one cusp on it?

Without the connectedness requirement it is easy to construct a stable map of the projective plane into $\mathbb{R}$ with only one cusp which in fact comes from an immersion of the projective plane in $\mathbb{R}^3$.

First embed a Mobius band in $\mathbb{R}^3$ as



Our mapping is, as usual, projection of these super-imposed attached pieces into $\mathbb{R}^2$. The map has but one cusp and three singular curves. $r(c_1) = 1, r(c_3) = = 2 = -r(c_2)$.

In the next section, we will discuss a method by which pairs of cusps of mappings into the plane can be eliminated. However applying the technique to the Boy's-surface example would produce a mapping with a two component singular set. An indication of the interplay between the parts of the singular set is given by the following theorem of Haefliger [11]:

Let f be a stable map of an orientable surface N in the plane. Suppose $S(f)$ separates N into two parts, N_1 and N_2. Then the number of cusps is at least $|\chi(N_1) - \chi(N_2)|$.

We will give a more precise version of this result at the end of § 5.

In this section we have described a simple method by which the Euler characteristic of the source can be obtained from a map of an even-dimensional manifold into the plane. The computation depends only on the behavior of the mapping in a neighborhood of its singular set. However the exemples of the torus and Klein bottle shows that there are *non-diffeomorphic* manifolds N, M, and stable maps $f : M \to \mathbb{R}^2 \leftarrow N : g$ such that a tubular neighborhood of $S(f)$ and a tubular neighborhood of $S(g)$ are diffeomorphic and the maps f and g restricted to those neighborhoods differ only by that diffeomorphism. So the source manifold is not determined, generally, by the behavior of a stable map in a neighborhood of its singular set (at least not when the source and target are of equal dimension of course, more information than just the Euler characteristic of the source must be obtainable from a stable map into the plane: after all, with a little perturbation, if necessary, the two component functions can be taken to be Morse functions.

Is there a "natural" way to obtain the Morse theory information" about the source manifold of a stable map into $\mathbb{R}^k$? Can one get more information?

Remark. If f is a stable map of an even-dimensional manifold N into the plane, and if $S(f)$ is connected, then the Euler-characteristic of N depends only on $f(S(f))$. This is obvious sience all one needs is the orientation of $S(f)$ and there is a unique orientation that orients the piecewise smooth boundary of the image of f as a boundary.

(If g is a Morse function and $S(g)$ consists of precisely two points (zero--sphere), then we know the source completely (it must be a sphere) – so we know its Euler characteristic).

Question. *For which n-manifolds N, do there exist stable maps $f : N \to \mathbb{R}^k$, $n \geq k$, such that $S(f)$ is connected or is a (k-1)-sphere? Does such a map exist if $n < 2k$ or if $H_j(N;\mathbb{Z}) = 0$ for $k : j \leq n-k$?* (For the $n = k$ case, see the of § 5)

3. Simplifying Stable Maps

For any manifold N, there is an obvious meaning to the phrase "the simplest Morse function in N". That is, there are inequalities that are stronger than Morse inequalities [18].

If f is a Morse function on N with $N_k(f)$ critical points of index k, then

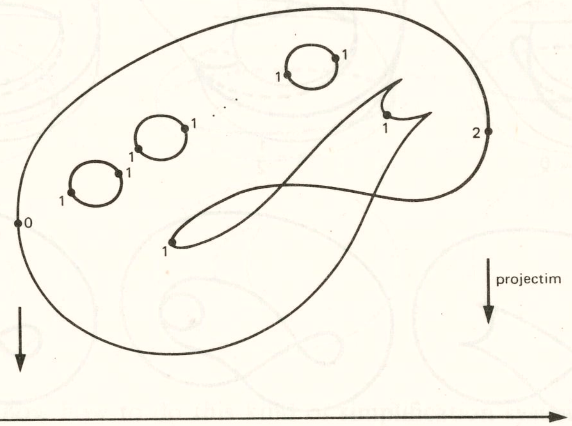
$$N_k(f) \geq B_k + (T_k + T_{k-1}),$$

where B_k is the k^{th} Betti-number of N and T_j is the number of torsion coefficients in $H_j(N;\mathbb{Z})$.

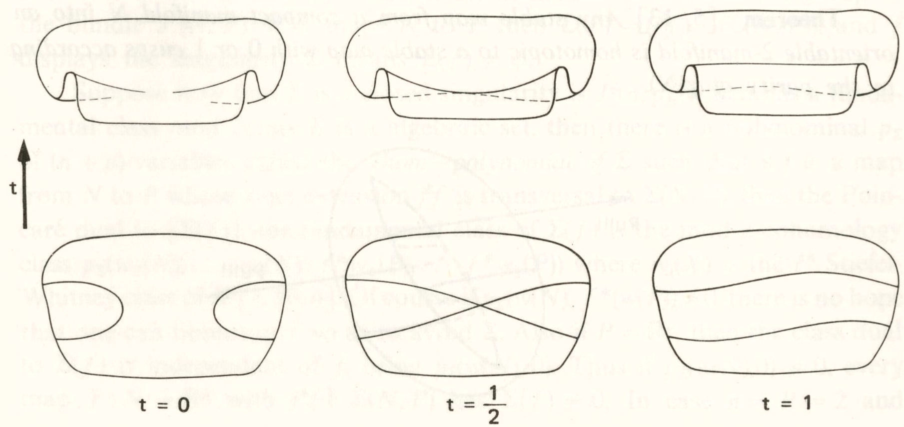
A function for which all of these inequalities are equalities has the simplest possible singularities. In the special case of N, a compact manifold with boundary consisting of two simply connected manifolds V_1, V_2 where dim $V_1 \geq 4$ and each V_i is a deformation retract of N, the proof of the existence of a simplest Morse function on N is the proof of the *h-cobordism theorem of S. Smale* [17]. In this case, the B's and T's appearing in the right-hand sides

of the above inequalities are those of the pair (N, V_1).

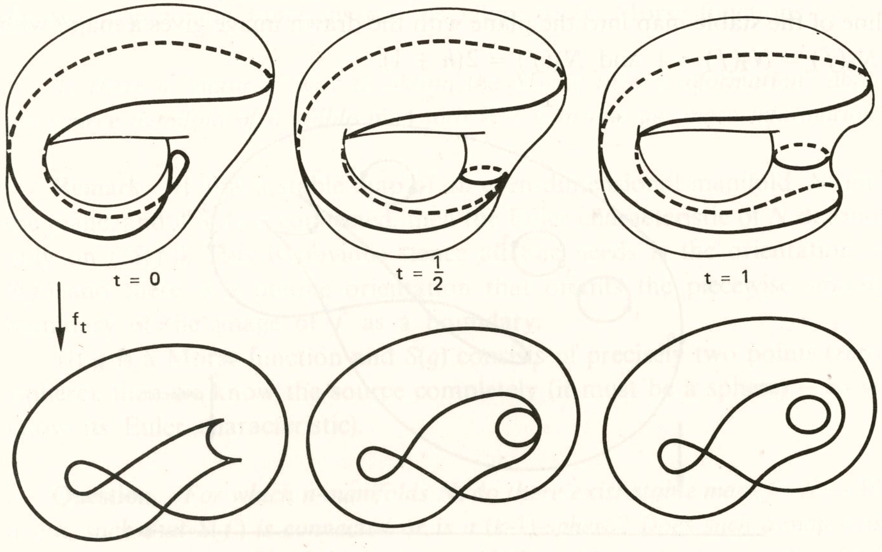
It is easy to give examples of such simplest functions: If N is lither a torus or a Klein-bottle with h-handles, the indicated projection into the line of the stable map into the plane with the drawn image gives a map f with $N_0(f) = N_2(f) = 1$ and $N_1(f) = 2(h+1)$.



Although there is no such natural notion if a simplest stable map into $\mathbb{R}^2$, there are obvious questions of this simplifying type, For example, we know that the Euler characteristic and the number of cusps have the same parity. Is it possible, by a homotopy (through stable maps for almost all values of the homotopy parameter) to reduce the number of cusps to 0 or 1. For dim $N \geq 3$, it is always possible to do this using the following simple geometric deformation, illustrated below in case dim $N = 2$ [13].



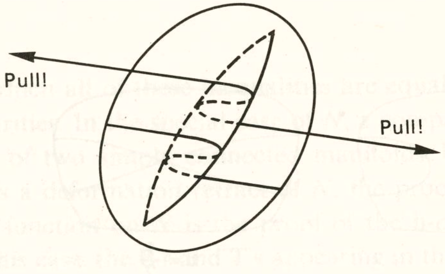
To see this process in a concrete example, consider the map of the torus into the plane:



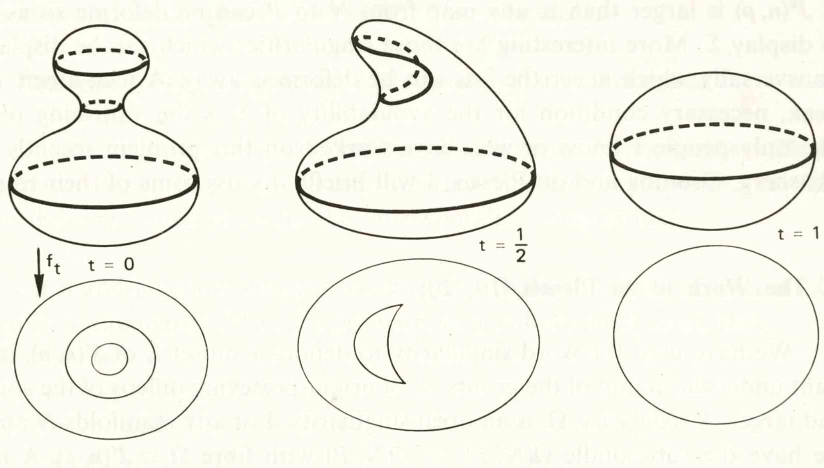
In this example we've gotten rid of a pair of cusps at the expense of adding a component to the singular sethardly an inobjectionable "simplification". Notice that although the simple geometric deformation used to eliminate (or create) a pair of cusps requires a change in the map near a whole arc joining the cusps, the change in the map is, nevertheless, C^0-small (thus is surelly true for dim $N \geq 3$).

The general theorem about eliminating cusps is:

Theorem [5, 13] *Any stable map from a compact manifold N into an orientable 2-manifold is homotopic to a stable map with* 0 *or* 1 *cusps according to the parity of* $\chi(N)$.



Sometimes a pair of cusps and a whole component of the singular set can be eliminated merely by "smoothing out the wrinkle" Combining smoothing out the wrinkle and creation of a pair of cusps sometimes allows us to cancel a pair of fold curves.



I do not know how to do this kind of simplification systematically.

The general questions analogous to elimination of cusps are: Given a singularity $\Sigma \subset J^k(n, p)$ (i.e. a subset Σ of the k-jets $J^k(n, p)$, invariant under the action of the origin preserving diffeo-germs of the source and target), if N and P are n– and p-dimensional manifolds:

1) *Is there a stable map from N to P that does not display the singularity* Σ?
2) *Is any map from N to P homotopic to one that doesn't display the singularity* Σ?

The sub-bundle $\Sigma(N, P)$ of $J^k(N, P)$ whose fibre is Σ over each point of $N \times P$ is well defined by virtue of the invariance of Σ under the group of the bundle $J^k(N, P)$. If f maps N to P then $\Sigma(f) = (j^k f)^{-1}(\Sigma(N, P))$, and f displays the singularily Σ means $\Sigma(f) \neq \phi$.

Suppose now that Σ is a closed singularity is $J^k(n, p)$, which has a fundamental class mod 2; say Σ is a algebraic set, then there is a polynominal p_Σ of $(n+p)$-variables called the *Thom – polynomial of* Σ such that if f is a map from N to P whose k-jet extension $j^k f$ is transversal to $\Sigma(N, P)$, then the Poincaré dual to $[\Sigma(f)]$, the fundamental class of $\Sigma(f)$ is the mod 2-cohomology class $p_\Sigma(w_1(N), \ldots, w_n(N); f^*w_1(P), \ldots, f^*w_p(P))$ where $w_i(X)$ is the i^{th} Stiefel-Whitney class of X [3, 10, 4]. Of course if $p_\Sigma(w(N), f^*(w(P)) \neq 0$, there is no hope that one can homotop f so as to avoid Σ. Also if $P = \mathbb{R}^p$, then the class dual to $\Sigma(f)$ is independent of f, being $p_\Sigma(w(N), 0)$. Thus if $p_\Sigma(w(N), 0) \neq 0$, every map $f: N \to \mathbb{R}^p$ with $j^k f \pitchfork \Sigma(N, P)$ has $\Sigma(f) \neq 0$. In case $n = P = 2$ and

$P = \mathbb{R}^2$, the class dual to the singular set of f is just $w_1(N)$. Thus if N is orientable, $S(f)$ is a mod 2 boundary but obviously $S(f)$ is never ϕ for compact N.

The general problem of constructing maps without certain singularities is partly solved trivially by transversality. Namely if the codimension of Σ in $J^k(n, p)$ is larger than n, any map from N to P can be deforme so as not to display Σ. More interesting are those singularities which can be displayed transversally which never the less can be deformed away. A just, albeit very weak, necessary condition for the avoidability of Σ is the vanishing of p_Σ. The only people I know of who have worked on this problem recently are Eliasberg, Gromov and du Plessis; I will briefly discuss some of their results.

4. The Work of du Plessis [19, 20]

We have used the word singularity to denote a subset Σ of $J^r(n, p)$, invariant under the action of the group, $\mathscr{A}$ of origin-preseving diffeos of the source and target. A *regularity*, Ω, is an open singularity. For any manifolds N and P we have the sub-bundle $\Omega(N, P) \subset J^r(N, P)$ with fibre $\Omega \subset J^r(n, p)$. A map $f : N \to P$ is *Ω-regular* if $j^r(f) \subset \Omega(N, P)$.

Our questions of the last paragraph become:

1. *Do there exist Ω-regular maps?*
2. *Is any map homotopic to an Ω-regular map?*

Let $\Omega \subset J^r(n, p)$ be a regularity.

Let $C_\Omega(N, P)$ *be the space of smooth, Ω-regular maps* with the C^r-topology and $\Gamma(\Omega(N, P))$ be the space of somooth sections of $\Omega(N, P)$ as a bundle over N with the compact open topology. The map

$$j^r : C_\Omega(N, P) \to \Gamma(\Omega(N, P))$$

is continous. A regularity $\Omega \subset J^r(n, p)$ is integrable for (N, P) if j^r is a weak homotopy equivalence, i.e. if $j^r_* : \pi_i(C_\Omega(N, P) \to \pi_i(\Gamma(\Omega(N, P))$ is an isomorphism for all $i \geq 0$. If Ω is integrable for all (N, P), then Ω is said to be *integrable.* Thus if a regularity, Ω, is integrable for (N, P), any map f from N to P is homotopic to an Ω-regular map iff there is a section in $\Omega(N, P)$ over f. So in case our regularity is integrable, to answer questions about the existence of regular maps it is enough to answer the easier questions about the existence of sections.

A theorem of Gromov [9] says that:

If N is open and P arbitrary every regularity is integrable for (N, P).

However for N compact, integrability cannot be automatic since there are no non-singular maps of N into $\mathbb{R}^p$.

The notion that guarantees integrability of Ω is that of extensibility. $\Omega \subset J^r(n, p)$ is an *extensible regularity* if there is a regularity $\Omega' \subset J^r(n+1, p)$ such that $i^*\Omega' = \Omega$, where $i : \mathbb{R}^n \to \mathbb{R}^{n+1} : x \to (x, 0)$.

Theorem [19] *A regularity is integrable if it is extensible.*

Du Plessis proves the extensibility of two types of regularities: *Thom-Boardman regularities* and certain *Contact-Invariant regularities.* To describe these we must describe the corresponding singularities.

Let n and p be any integers. Given any r-tuple of integers, $I = (i_1, \dots, i_r)$, there is a singularity $\Sigma^I \subset J^r(n, p)$ with is a submanifold such that:

1) $\Sigma^I = \phi$ *unless*
   $n \geq i_1 \geq i_2 \geq \dots \geq i_r \geq 0$
   $i_1 \geq n - p$, and if $i_1 = n - p$ then
   $i_j = n - p$ for all $j \geq 1$.
2) For $r = 1$, Σ^j is the set of all 1-jets of kernel rank j.
3) If $j^r f \pitchfork \Sigma^I(N, P)$, then $\Sigma^{I,j}(f) = \Sigma^j(f \mid \Sigma^I(f))$, where $\Sigma^{I,j}$ is the singularity corresponding to the $(r+1)$-$(i_1, \dots, i_r, j)$.

These singularities are called the *Thom Boardman singularities* [2]. A map f whose jet extensions are transversal to all $\Sigma^I(N, P)$ is called a *Boardman-generic map.* Since there are only countably many non-empty Σ^I, the *Boardman-generic* maps are dense. Order the $\{\Sigma^I\}$ by the lexicographic ordering of the symbols $\{I\}$. For any r-tuple, I, let $\Omega^I = \cup \{\Sigma^K \mid r\text{-tuple}, K \leq I\}$. The regularities, $\{\Omega^I\}$, are called the *Thom-Boardman regularities.* Du Pleiss proves in [19] that Ω^I *is extensible if its symbol* $I = (i_1, \dots, i_r)$ *satisfies*

$$i_r > n - p - d^I, \text{ where}$$

$$d^I = \sum_{j=1}^{r-1} \alpha_j, \text{ and } \alpha_j \begin{cases} 1, & \text{if } i_j - i_{j+1} > 1 \\ 0 & \textit{otherwise} \end{cases}$$

Notice that if $n < p$, all Ω^I are extensible.

A regularity $\Omega \subset J^r(n, p)$ is *contact-invariant* if it is invariant under the action of the group K (see § for the definition of K) the contact-invariant regularities that du Plessis proves extensible are those for which:

If Σ is any K*-orbit in Ω, then there is a* K*-orbit Σ′* such that $\Sigma' \subset \bar{\Sigma}$ and $b(\Sigma') < p$. [20]. The integer $b(\Sigma)$ is defined as follows: Let $f:(\mathbb{R}^n, 0) \to \mathbb{R}^p$ be any polynomial whose r-jet is in Σ′. In $\mathbb{R}[x_1, \ldots, x_n]$, let $\varphi_r(f)$ be the ideal generated by the components of f and all polynomals in $\mathcal{M}^{r+1}$ $b(\Sigma')$ is the minimal number of generators of $\varphi_r(f)$ mod $\mathcal{M}^{r+1}$. Du Plessis remarks in [20] that the result about the extensibility of the Thom-Boardman regularities is included in this are about contact-invariant regularities. From the point of view of our original questions, integrability and a fortiori extensibility are very strong notions. Closer to the modest questions would be criteria for:

Given a section in $\Omega(N, P)$ *over f, there is always an Ω-regular g homotopic to f* (and $j^r g$ is homotopic to the section). Thus du Plessis calls *barely* 0-integrable. In the work if Èliašberg, there is an example of a regularity that is barely 0-integrable – but not integrable; the regularity is Ω^I, where $I = (1, 0)$.

5. Èliašberg's Work [5, 6]

A map from an n-manifold to a p-manifold ($n \geq p$) is called a *map with folds* if at each singular point there are coordinates $(n_1, \ldots, n_{p-1}, x_1, \ldots, x_{n-p+1})$ in the source and $(U_1, \ldots, U_{p-1}, X)$ in the target so that the map has one of the forms:

$$\begin{cases} U = u \\ X = Q(X), \text{ a non-singular quadratic form of index} \end{cases}$$

$$v,\ 0 \leq v \leq \left[\frac{n-p+1}{2}\right].$$

Such a point is called a fold point of reduced index, v. For N, a compact n-manifold and P, a p-manifold, let $M(N, P)$ be the space of smooth maps with folds from N to P (with the C^∞-topology). (In Du Plessis notation this set is $C_{\Omega^I}(N, P)$ where $I = (n-p+1, 0)$). Let $m(N, P)$ be the space of smooth sections of $J^r(N, P)$ such that the germ of $\sigma \in m(N, P)$ at $x \in N$ is the germ at x of $j^r f$ for f a map with folds in a nbhd of x. For $f \in M(N, P)$ and $0 \leq v \leq [n-p+1/2]$, let $\Sigma_v(f)$ be the set of fold points of f of reduced index, v. For $\varphi \in m(N, P)$, we define $\Sigma_v(\varphi)$ in the obvious way. The map $j^r: M(N, P) \to m(N, P)$ is well-defined and obviously $\Sigma_v(f) = \Sigma_v(j^r f)$. Let $k = [n-p+1/2]$ and let $V_0, V_1, \ldots, V_k$ be $(p-1)$-dimensional submanifolds of N. Let

$$M(N, P; V_0, \ldots, V_k) = \{f \in M(N, P) \mid \Sigma_v(f) = V_v;\ v = 0, \ldots, k\}$$

and

$$m(N, P; V_0, \ldots, V_k) = \{\sigma \in m(N, P) \mid \Sigma_v(\sigma) = V_v;\ v = 0, \ldots, k\}$$

Theorem [Èliašberg 6] *If N is a connected* n*-manifold and P is a* p*-manifold* ($n \geq p \geq 2$) and V_v *are non-empty* $(p-1)$*-dimensional submanifolds of N*, $0 \leq v \leq k$, then

$$j^r_*: \pi_0(M(N, P; V_0, \ldots, V_k)) \to \pi_0(m(N, P; V_0, \ldots V_k))$$

is surjective.

This theorem fails for $p = 1$; because of the Morse inequalities it is obviously not sufficient to require that V_v be non-empty. If $n = p$, Èliašberg [6] obtains the sharper:

Theorem *Let N and P be n dimensional manifolds and let V be an* $(n-1)$*-dimensional submanifold of N.*

1. *If V separates N into two diffeomorphic manifolds* N_1 *and* N_2 *with boundary V and if there is a diffeomorphism of* N_1 *to* N_2 *which is the identity on V, then* $M(N, P; V) = M_1 \cup M_2$ *where* $M_1 \cap M_2 = \phi$ *and* $j^r: M_1 \to M(N, P; V)$ *is a weak homotopy equivalence and* M_2 *is homeomorphic to the product of the space of immersions of* N_1 *into P and the space of diffeos of* N_1 *fixed on the boundary, V.*

2. *If either of the hypotheses of* 1 *fails, then* $j^r: M(N, P; V) \to m(N, P; V)$ *is a weak homotopy equivalence.*

These theorems give some geometric insight as to why $\Omega^{(1,0)}$ is not integrable in the sense of du Plessis (in case the source and target have the same dimension).

The most familiar instance of this phenomenon is in case $N = S^2$, $P = \mathbb{R}^2$ and $V = S^1$. Then M_1 and M_2 have the weak homotopy of a disjoint union of two circles. The four components of $M(S^2, \mathbb{R}^2; S^1)$ are represented by: $\{f_1, f_2, f_3, f_4\}$ where: f_1 is any map with folds which commutes with a diffeomorphism that exchanges the northern and southern hemispheres and leaves the equatorial folds S pontwise fixed. f_2 is any map with folds whose restrictions to the northern and southern hemispheres do *not* differ by diffeomorphism between the hemispheres leaving the equator pointwise fixed. (see § 7 for an example) $f_3 = \varphi \circ f_1$ and $f_4 = \varphi \circ f_2$ where $\varphi: \mathbb{R}^2 \circlearrowleft\ : (x, y) \to (-x, y)$.

As a consequence of his results in case $n = p$, Eliasberg [5] obtains:

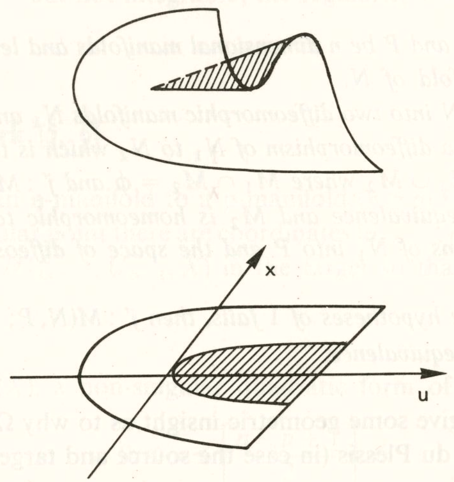
Theorem *Let N and P be n-manifolds ($n > 2$). If f is a map with folds from N to P then there is a homotopic map with folds, g, such that S(g) is connected.*

As the theorem of Haefliger shows (see the end of § 2), this theorem is false for $n = 2$. In that case there is a beautifully precise statement. First one need a definition.

If f is a germ of a map from $\mathbb{R}^2$ to $\mathbb{R}^2$ with a cusp at the origin, the singular curve, $S(f)$, divides the source into two components. Call the component on which f is is one-to-one (the shaded piece in the figure), the *interior of the cusp.* If in coordinates f is given *by:*

$$(u, x) \to (u, ux - x^3)$$

then $\{u > 3x^2\}$ is the interior of the cusp.

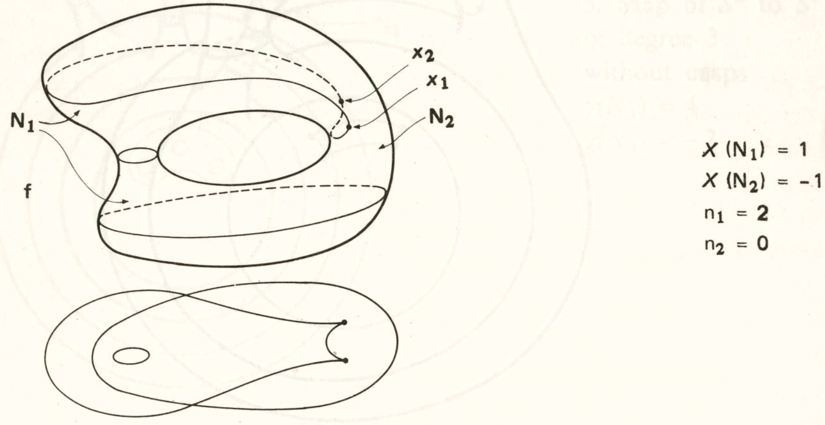


Theorem [11, 5] *Let N and P be oriented 2-dimensional manifolds, N compact. Let C be a 1-dimensional submanifold of N which devides N into two manifolds N_1 and N_2 with cammon boundary, C. Choose $\{x_1, \ldots, x_{n1}\}$ and $\{y_1, \ldots, y_{n2}\}$, points on C and let $\varphi: N \to P$ be any map of degree d ($d = 0$ if P is not compact). Then there is a stable map f, homotopic to φ, such that $S(f) = C$ and x_1 is a cusp whose interior is N_1, ($i = 1, \ldots, n_1$) and y_j is a cusp whose interior is N_2, ($j = 1, \ldots, n_2$) if and only if:*

$$|\chi(N_1) - \chi(N_2) - n_1 + n_2| = |d\chi(P)|.$$

Examples

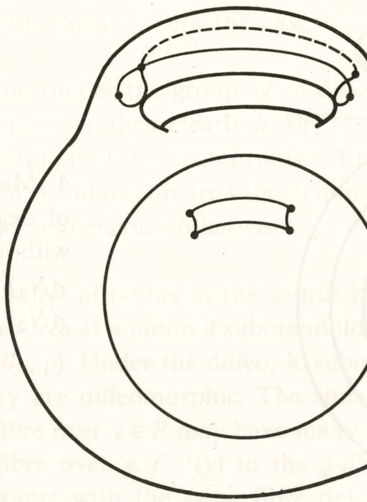
1. f maps the torus to the plane. The singular set of f divides the torus into N_1, a disc and an annulus and N_2, a disc with two holes. f has two cusps both of which have N_1 as interior.



The next examples are maps into S^2. The maps are all radial projections of the outside figures onto the enclosed sphere.

2. Map of the torus to S^2 of degree 1 with

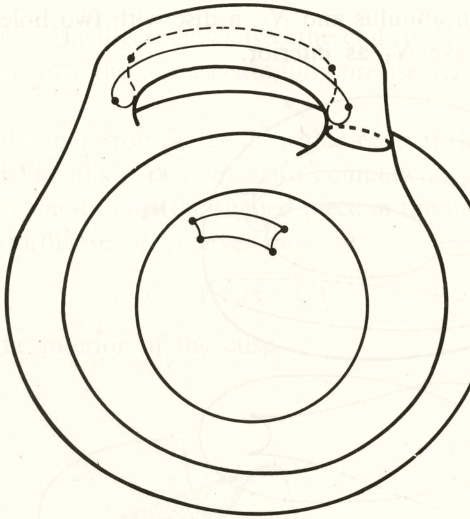
$\chi(N_1) = 1$, $n_1 = 4$
$\chi(N_2) = -1$, $n_2 = 0$



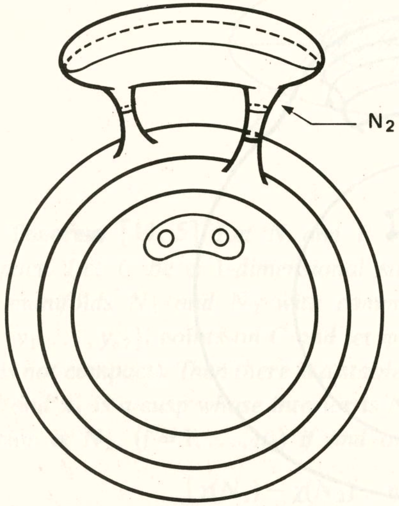
3. Map of S^2 to S^2 of degree 2 with

$\chi(N_1) = \chi(N_2) = 1$

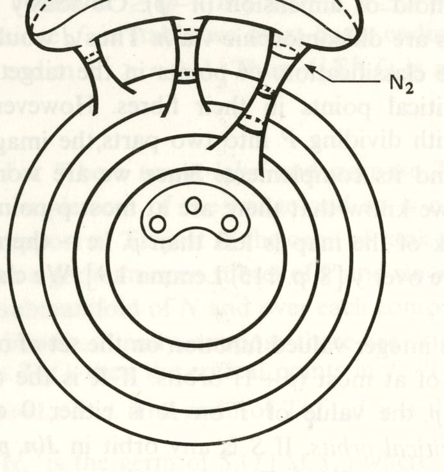
$n_1 = 4, \ n_2 = 0.$



It is clear how to generalize the example giving a map of S^2 to S^2 of degree k with N_1 and N_2, discs, and $n_1 = 2k$ and $n_2 = 0$. Using the elimination of cusps technique, we obtain maps of S^2 to S^2 of any degree having only folds — — no cusps.



4. Map of S^2 to S^2 of degree 2 without cusps $\chi(N_1) = 3$, $\chi(N_2) = -1$



5. Map of S^2 to S^2 of degree 3 without cusps $\chi(N_1) = 4$ $\chi(N_2) = -2$

6. Equivalence of Stable Maps (the work of Leslie Wilson [23]).

Suppose N and P are manifolds of dimension n and p respectively and suppose that f and g are proper, stable maps from N to P which are *equivalent*. That is there are diffeos $h : N \supset$ and $k ; P \supset$ such that $g = kf \circ h^{-1}$.

We first derive some consequences of the equivalence of f and g for arbitrary n and p and then discurs the case $n = p = 2$ (Wilson's Theorem).

(i) If S is an orbit (under the group $\mathscr{A}$ of diffeos of the source and tarjet) in $J(N, P)$ the infinite jets, then clearly $h : S(f) \to S(g)$ is a diffeomorphism and $h : f(S(f)) \to g(S(g))$ is a homeomorphism (in fact, a diffeo almost everywhere since $f \mid S(f)$ and $g \mid S(g)$ are submersions if S is the orbit of jets of rank p (if $n \geq p$), and immersions otherwise).

(ii) The classification of points in the source by the orbit of their jets allows us to write the source as a union if submanifolds each of which bears the name of an orbit in $J(n, p)$. Under the diffeo, h, submanifolds corresponding to the same singularity are diffeomorphic. The situation in the target is different, however. The fibre over $y \in P$ may have many different kinds of points. Since h takes the f-fibre over $y, f^{-1}(y)$ to the g-*fibre* over $k(y)$, $g^{-1}(ky)$), we see that k maps points with the same fibre description to one-another.

To describe the f fibre over y we should first describe the set of points, $R(f^{-1}(g)) = R(f) \cap f^{-1}(y)$, at which the rank of Tf is p = dimension of the target. (Here R is the orbit of rank p-jets in $J(n, p)$). This set of non-critical points is either empty or a manifold of dimension $(n-p)$. Obviously over y and $k(y)$ these parts of the fibres are diffeomorphic via h. Thus d would be reasonable to have as part of the classification. of points in the target, the diffeomorphy-type of the non-critical points in their fibres. However we content ourselves here if $n > p$, with dividing P into two parts, the image of the non-critical points, $f(R(f))$ and its complement. Since we are working with infinitesimally stable maps, we know that there are at most p-points in the fibre over y at which the rank of the map is less than p. i.e – there are at most p critical points in the fibre over y [8, p. 115) Lemma 1.9]. We classify the points in P as follows:

A *target-label*, T, is a non-negative integer valued function on the set of orbits of $J(n, p)$ whose support consists of at most $(p+1)$ orbits. If R is the orbit of the rank p, jets, then if $n > p$, the value of T on R is either 0 or 1. Orbits other than R are called *critical orbits*. If S is any orbit in $J(n, p)$ we write $S \in T$ for $S \in \operatorname{supp} T$. For a target-label T we let:

$$T(f) = \begin{cases} \{y \in P \mid T(S) = \#\{f^{-1}(y) \cap S(f)\}\}, \text{ if } n \leq p \\ \{y \in P \mid T(R) = 1 \text{ iff } y \in f(R(f)), \text{ and if} \\ S \text{ is a critical orbit } \#\{f^{-1}(y) \cap S(f)\} = T(S)\} \text{ if } n > p. \end{cases}$$

(iii) Since points in the source whose f-images have target-label T are mapped by h into points whose g-images also have target label T, the target labels subdevide $S(f)$ and $S(g)$ into h-corresponding subsets. Thus a *source-label* is a pair (S, T) where T is a target label and $S \in T$. We set $(S, T)(f) = S(f) \cap f^{-1}(T(f))$. Since $f(S(f) \supset T(f)$ we have $f((S, T)(f)$. To see that $(S, T)(f)$ and $T(f)$ are manifolds we just show:

Lemma. *If f is stable and S is a orbit in $J(n, p)$ then $f \mid S(f)$ is an immersion except when $S = P$, in which case it is a submersion.*

Proof Since we're working with stable f, we know that every orbit S for which $S(f) \neq \emptyset$ is the pre-image by the projection of an orbit in $J^{p+1}(n, p)$ [15, IV]. There is a unique symbol $I = (i_1, \ldots, i_{p+1})$ such that $S \subset \Sigma^I$ (for the meaning of the Boardman notation see § 4 or [2]). If $I = (n-p, \ldots, n-p)$, then $S = R$ and $S(f)$ is open and $f \mid S(f)$ is a submersion into its open image. On the other hand if S is a critical orbit then $i_1 > n-p$. If suffices to show that $i_{p+1} = 0$. For any $J = (j_1, \ldots, j_{r+1})$ such that $j_1 \geq j_2 \geq j_{r+1} > 0$ and $n \geq j_1 > n-p$, *codim* Σ^J – codim $\Sigma^{J'} \geq 1$ where $J' = (j_1, \ldots, j_r)$; also codim $\Sigma^{j1} = (p-n+j_1)j_1$, [2].

Suppose then $S \subset \Sigma^I$, $I = (i_1, \ldots, i_{p+1})$ and $i_1 > n-p$ and $i_{p+1} > 0$, then codim $\Sigma^I \geq (p-n+i_1)i_1 + p \geq n+1$. But by assumption $S(f) \neq \emptyset$ and since f is stable we must have, codim $S \leq n$, which contradicts codim $S \geq$ codim $\Sigma^I \geq n+1$. Thus $f \mid \Sigma^I(f)$ is an immersion, hence so is $f \mid S(f)$.

Let T be a *target-label* and suppose T is the zero function then $T(f)$ is the complement of the image and so is open hence a submanifold of P. If the support of T is just R, the orbit of rank p jets then $T(f)$ is the set of regular values a germ an open subset of P and since $f \mid R(f)$ is a submersion, $(R, T)(f)$ is a submanifold of N and over each component of $T(f)$, $f : (R, T)(f) \to T(f)$ is a bundle. Finally, suppose there are critical orbits in supp T, $\{S_1, \ldots, S_l\}$. If $y \in T(f)$, then the critical points in $f^{-1}(y)$ consist of $\{x_{i_1}, \ldots, x_{i_{T(S_i)}}\} \in S_i(f)$ for $i = 1, \ldots, l$. The germ of $T(f)$ at y is the intersection: $\bigcap_{i,j} f(S_i(f)_{x_{ij}})$ where $S_i(f)x_{i_j}$ is the germ of $S_i(f)$ at x_{i_j}) where $S_i(f)x_{i_j}$ is the germ of $S_i(f)$ at x_{i_j}, and the embedded submanifold germs $f(S_i(f)x_{i_j})$ are in general position at y [8, p. 157 Theorem 5.2]. Thus the germ of $T(f)$ at y is a submanifold whose dimension is independent of $y \in T(f)$.

Further if $x \in (S, T)(f)$ for $x \in f^{-1}(g)$ then if $S = R$, since $f \mid R(f)$ is a submersion $f \mid R(f)$ is transversal to any manifold it meets so the pre-image in $R(f)$ if the germ of $T(f)$ at y is a manifold germ. On the other hand if S is a critical orbit then the germ $S(f)_x$ is embedded in P and the germ $T(f)_y$ is contained in its image as a submanifold germ. Hence the pre-image in $S(f)_x$ of the germ $T(f)_y$ is a submanifold germ. Thus:

If f is stable and (S, T) is a source label then $(S, T)(f)$ and $T(f)$ are submanifolds of the source and target respectively.

If S is a critical manifold or if $n = p$ and $S = R$, then $f : (S, T)(f) \to T(f)$ is a $T(S)$-fold covering.

If $n > p$, and $R \in T$, then over each component of $T(f)$ $f : (R, T)(f) \to T(f)$ is a bundle.

(iv) For any manifold X, a *stratification*, $\mathfrak{U}$, is a set of embedded submanifolds of X, called *strata* of $\mathfrak{U}$, such that:

(1) $\mathfrak{U}$ is locally finite (each point of X has a neighborhood meeting only finitely many strata of $\mathfrak{U}$)

(2) If U and V are strata of $\mathfrak{U}$ and $U \cap \bar{V} \neq \varnothing$ then $U \subset \bar{V}$.

A map, F, from a stratification $\mathfrak{U}$ to a stratification $\mathbb{R}$ is a *stratification map* if

1) $\dim F(U) \leq \dim U$ for $U \in \mathfrak{U}$

2) If $U \subset \bar{V}$ then $F(U) \subset F(V)$ for $U, V \in \mathfrak{U}$.

Moreover F is called an *equivalence* if F is 1:1 and F^{-1} is also a stratification map.

For any proper stable map $f: N \to P$, the collection of the connected components of the manifolds $(S, T)(f)$ (and the collection of connected components of $T(f)$) is a stratification of N (and of P) which we denote by $\mathscr{S}(f)$ (and $\mathscr{T}(f)$). Obviously f induces a stratification map $\bar{f}: \mathscr{S}(f) \to \mathscr{T}(f)$.

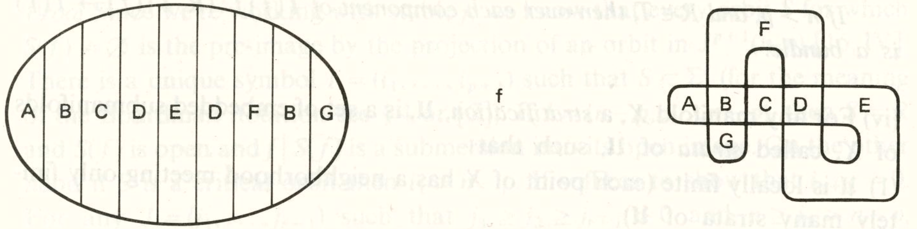
(v) If f and g are equivalent, proper, stable maps from N to P, the diffeomorphisms $h: N \circlearrowleft$ and $k: P \circlearrowleft$ induce equivalences of stratifications $\bar{h}: \mathscr{S}(f) \to \mathscr{S}(y)$ and $\bar{k}: \mathscr{T}(f) \to \mathscr{T}(g)$ such that $\bar{k} \circ \bar{f} = \bar{g} \circ \bar{h}$. Furthermore the pair of equivalences $(\bar{h}, \bar{k})$ *preserve strata-types*, that is, for any source-label, (S, T), and any target label, T, $\bar{h}$ takes components of $(S, T)(f)$ to components of $(S, T)(g)$ and $\bar{k}$ takes components of $T(f)$ to components of $T(g)$.

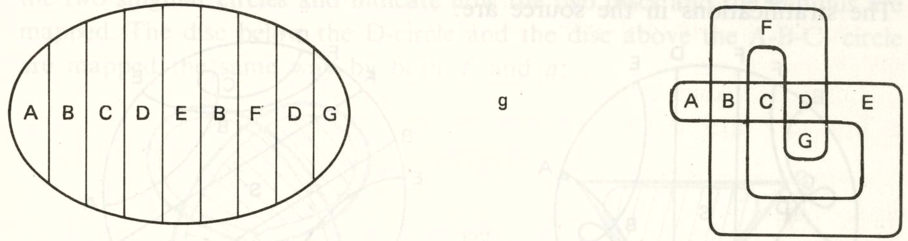
Questions. *Let f and g be proper, stable maps from N to P. Suppose there are stratification equivalences H: $\mathscr{S}(f) \to \mathscr{S}(g)$ and $K: \mathscr{T}(f) \to \mathscr{T}(g)$ which preserve strata-types and such that $K \circ \bar{f} = \bar{g} \circ H$. Are f and g equivalent?*

Is the existence a strata-types-preserving equivalence, H (or K) a consequence of the existence of a strata-type equivalence K (or H)?

We give examples (all of which due to L. Wilson [23]) which show that answer to all of the questions is no.

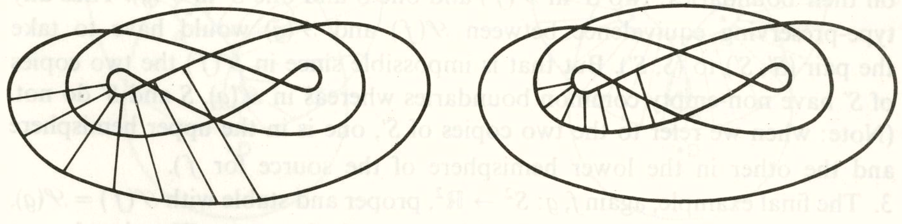
1. $f, g: S^2 \to \mathbb{R}^2$ are stable and $\mathscr{S}(f)$ is equivalent to $\mathscr{S}(g)$ but $\mathscr{T}(f)$ and $\mathscr{T}(g)$ are not equivalent. Both f and g are defined by projecting the sphere to a disc and than mapping the disc as indicated in the figures below. We label the regular strata in the target and use the same letter for the source strata mapping to them.



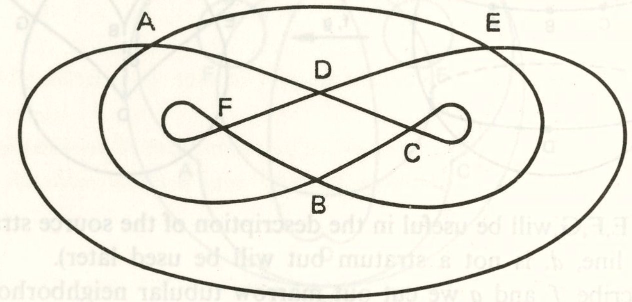


The obvious map between the source stratifications is an equivalence that preserves strata types. If there were a type-preserving K from $\mathscr{T}(f)$ to $\mathscr{T}(g)$ then K must preserve the components of the complement of the image, the components of $0(f)$, and $0(g)$. But one of the components of $0(f)$ has only one 1-stratum in its boundary whereas $0(g)$ has no such component.

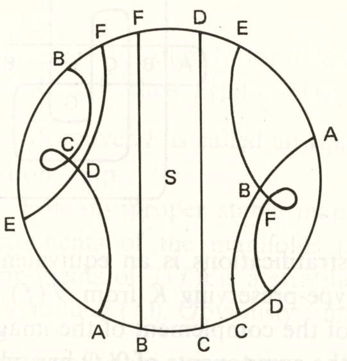
2. In this exemple f and $g: S^2 \to \mathbb{R}^2$ are stable into $\mathscr{T}(f) = \mathscr{T}(g)$ but $\mathscr{S}(f)$ and $\mathscr{S}(g)$ are not equivalent. Consider the two immersions of the disc into the plane given by:



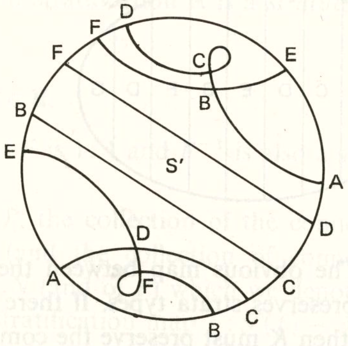
Let f be the doubling of the first immersion and g be the first immersion on the upper hemisphere and the second immersion on the lower hemisphere. The stratification $\mathscr{T}(f)$ is the same as $\mathscr{T}(g)$. We label the 0-strata to help keep track of their pre-images.



The stratifications in the source are:



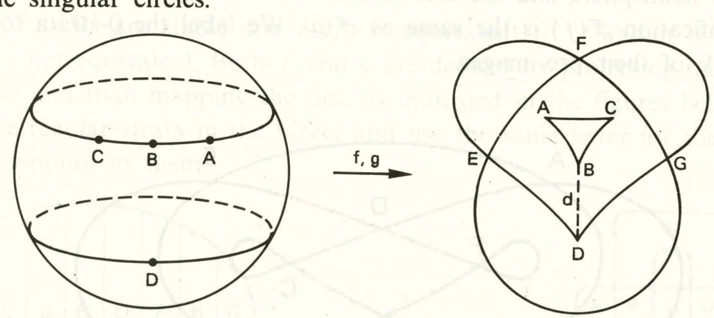
on the lower hemisphere for g and,



on the upper hemisphere for g and on both hemispheres for f.

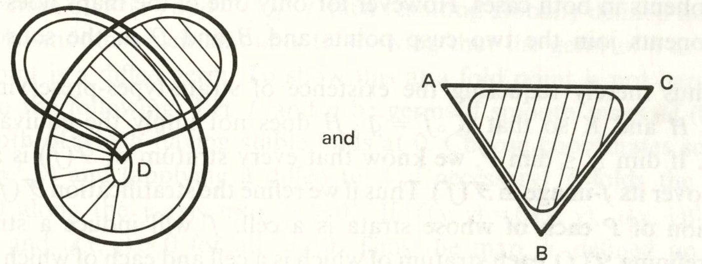
The stratum (CDFB) in the target is covered six times by each map. However only two of the strata for each map have two strata of fold curves on their boundaries: two S' in $\mathscr{S}(f)$ and one S and one S' in $\mathscr{S}(g)$. Thus any type-preserving equivalence between $\mathscr{S}(f)$ and $\mathscr{S}(g)$ would have to take the pair (S', S') to (S, S'). But that is impossible since in $\mathscr{S}(f)$ the two copies of S' have non-empty common boundaries whereas in $\mathscr{S}(g)$, S and S' do not. (Note: when we refer to the two copies of S', one is in the upper hemisphere and the other in the lower hemisphere of the source for f).

3. The final example, again $f, g\colon S^2 \to \mathbb{R}^2$, proper and stable with $\mathscr{S}(f) = \mathscr{S}(g)$, $\mathscr{T}(f) = \mathscr{T}(g)$ and $f = g$ but still f is not equivalent to g. The singular set of f and g cuts S^2 into three pieces as shown. The points A,B,C,D are cusps on the singular circles.

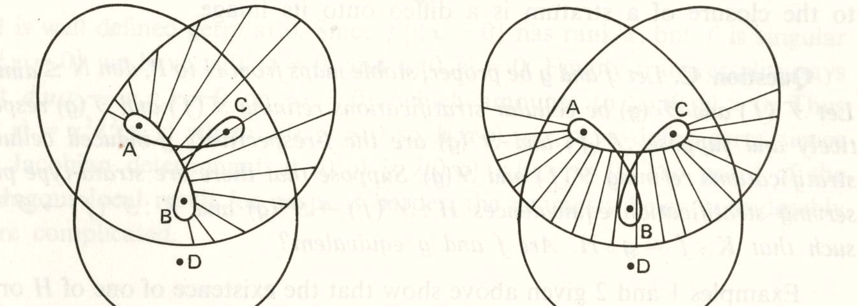


(The letters E,F,G will be useful in the description of the source stratification. The dotted line, d, is not a stratum but will be used later).

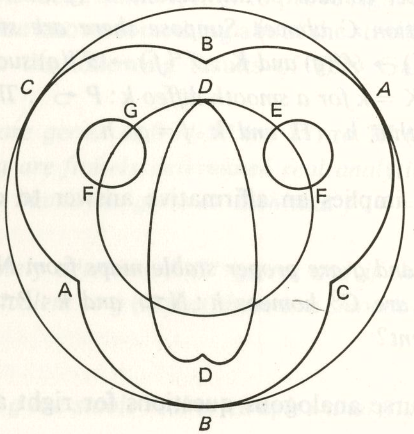
To describe f and g we cut out marrow tubular neighborhoods about the two singular circles and indicate how the two discs and the annulus are mapped. The disc below the D-circle and the disc above the A-B-C- circle are mapped the same way by both f and g:



(Here, the images of the singular curves are indicated in lighter lines). The annulus between can be mapped in two distinct ways; f map one way and g map the other.



The stratification $\mathscr{S}(f)$ and $\mathscr{S}(g)$ consists of the cap above the A-B-C- circle as a stratum and its complementary disc in which the stratification looks like:



Here we have underlined the cusp points.

As described the maps f and g cannot be equivalent since if we consider the pre image under f and g of the closed arc d, we see that it consists of two components in both cases. However for only one of the maps does one of the components join the two cusp points and B and D in the source.

(iv) Thus merely requering the existence of strata-types-preserving equivalences H and K so that $K \circ \bar{f} = \bar{g} \circ H$ does not imply the equivalence of f and g. If $\dim N \leq \dim P$, we know that every stratum of $\mathcal{S}(f)$ is a covering space over its f-image in $\mathcal{T}(f)$. Thus if we refine the stratification $\mathcal{T}(f)$ to a stratification of P each of whose strata is a cell, f will induce a stratification of N, refining $\mathcal{S}(f)$, each stratum of which is a cell and each of which is mapped diffeomorphically by f onto its image stratum in the refinement of $\mathcal{T}(f)$. Call a stratification *cellular* if each stratum is a cell and the closure of each stratum is a closed cell. (The second condition will guarantee that f restricted to the closure of a stratum is a diffeo onto its image.

Question C. *Let f and g be proper, stable maps from N to P, $\dim N \leq \dim P$. Let $\mathcal{T}^c(f)$ and $\mathcal{T}^c(g)$ be cellular stratifications refining $\mathcal{T}(f)$ and $\mathcal{T}(g)$ respectively and suppose $\mathcal{S}^c(f)$ and $\mathcal{S}^c(g)$ are the f-respectively g-induced cellular stratifications refining $\mathcal{S}(f)$ and $\mathcal{S}(g)$. Suppose that there are strata-type preserving stratification equivalences $H: \mathcal{S}^c(f) \to \mathcal{S}^c(g)$ and $K: \mathcal{T}^c(f) \to \mathcal{T}^c(g)$ such that $K \circ f = g \circ H$. Are f and g equivalent?*

Examples 1 and 2 given above show that the existence of one of H or K does not imply the existence of the other.

Theorem. (*Wilson* [23]) *Let $\dim N = \dim P = 2$ and f and g be stable, proper maps from N to P. Let cellular stratifications $\mathcal{S}^c(f)$, $\mathcal{S}^c(g)$, $\mathcal{T}^c(f)$, $\mathcal{T}^c(g)$ be given (as in the question C above). Suppose there are strata-type preserving equivalences $H: \mathcal{S}^c(f) \to \mathcal{S}^c(g)$ and $K: \mathcal{T}^c(f) \to \mathcal{T}^c(g)$ such that $K \circ \bar{f} = \bar{g} \circ H$. Suppose further that $K = \bar{k}$ for a smooth diffeo $k: P \circlearrowleft$. Then there is a smooth diffeo $h: N \circlearrowleft$ such that $\bar{h} = H$ and $k \circ f = g \circ h$.*

A question that implies an affirmative answer to question C is:

Question 0. *If f and g are proper stable maps from N to P which are C^0-equivalent (i.e. there are C^0 homeos $h: N \circlearrowleft$ and $k: P \circlearrowleft$ with $kf = gh$) are f and g C^∞-equivalent?*

There are of course analogous questions for right and left equivalence.

The theorem of Wilson is essentially that C^0 *right equivalence is the same as C^0 right equivalence for proper, stable map between 2-manifolds.*

The proof of Wilson, theorem consistis of showing that if we define h on each stratum of $\mathcal{S}^c(f)$ as $g^{-1} \circ k \circ f$, then resulting globally defined homeomorphism is smooth-which amounts to showing that the germ of h at every singular point is a diffeo germ. To show this at a fold point is not hard and amounts to the following: Let f and g be germs if smooth maps of $(\mathbb{R}^2, 0)$ to itself, both of which having stable folds at 0. Choose coordinates so that $g(x^2, y) = (x^2, y)$ and (applying a diffeo to f if necessary) f folds the germ of $\mathbb{R}^2$ at 0 along the y-axis into $\{x \geq 0\}$. If $f(x, y) = (u(x, y),\ v(x, y))$ then $u(0, y) = 0$, and $u(x, y) > 0$ for all $x \neq 0$. Thus the map, h, defined on each stratum $g^{-1} \circ f$ is just:

$$(x, y) \to \begin{cases} (u^{1/2},\ v)\,(x, y), & x \geq 0 \\ (-u^{1/2},\ v)\,(x, y), & x \leq 0, \end{cases}$$

and is well defined germ at 0. Since $f\,|\,\{x = 0\}$ has rank 1, but f is singular at $\{x = 0\}$ we have $v_y(0, y) \neq 0$, and $u_x(0, y) = 0$. Finally transversality says that $d(u_x v_y - u_y v_x) \neq 0$ on $\{x = 0\}$, which amounts to $u_{xx}(0, y) > 0$. Thus $u(x, y) = x^2 \lambda^2(x, y)$ where $\lambda(0, y) > 0$ so $h(x, y)$, $v(x, y)$, a diffeo germ sience the Jacobian determinant at $(0, g)$ is $\lambda(0, g) \cdot v_y(0, y) \neq 0$. The proof of the analogous local result for cusps is harder, the geometry being considerably more complicated.

7. Right Equivalence – Local and Global.

The work of Wilson suggests both local and global problems. The main difficulty of his proof is local, namely, how to prove right equivalence of germs of stable maps once one knows something about the stratification of the image. In that direction are the following results of Tery Gaffney [7].

Theorem *Here f, g are germs at 0 of $\mathbb{R}^n$ (or $\mathbb{C}^n$) to $\mathbb{R}^p$ (or $\mathbb{C}^p$).*

I. *For $n < p$*, 1) *If f, g are finitely determined real analytic r complex analytic germs, then there is a bianalytic germ or such that $f \circ r = g$ iff the images of f and g are equal.*

2) *If f, g are finitely determined C^∞-germs and if their image germs are equal, then f is $\mathcal{A}$-equivalent to g (in fact there is a smooth diffeo r with $f \circ r = g$ modulo $\mathcal{M}^\infty$).*

II. *For $n = p$. If f and g are stable real or complex analytic germs and $f(S(f) =$*

$= g(S(g))$, *then there is a bianalytic germ r such that either* $f \circ r = g$ *or there is a reflection l across* $f(S(f))$ *such that* $l \circ f \circ r = g$.

The image of the critical (or singular) set does not play as decesive a role in the analogous global question. Consider a proper stable map $f: N \to P$ and its target stratification $\mathcal{T}(f)$. Is there a *right inequivalent* $g: N \to P$ also proper and stable with $\mathcal{T}(f) = \mathcal{T}(g)$ (preserving strata-types). In fact *how many distinct* right equivalence classes of proper, stable maps are there with the some target stratification? Since we know what the stable map germs from $\mathbb{R}^n$ to $\mathbb{R}^p$ all look like we could ask a more primitive question. Suppose $\mathcal{T}$ is a stratifications of a manifold P, each stratum of which is given a target-label. We assume that none of our knowledge about proper, stable maps makes it a priori impossible that $\mathcal{T}$ be $\mathcal{T}(f)$ for some stable, proper f from some manifold N into P.

Question 1. *Gigen a* $\mathcal{T}$, *what are the criteria for the existence of a proper stable f from some N to P with* $\mathcal{T} = \mathcal{T}(f)$?

2. *If these criteria are satisfied how many* different N and f are there?

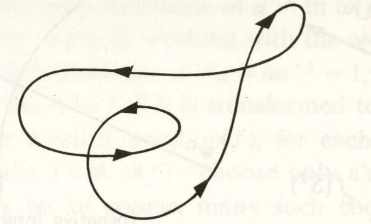
The work of Sandy Blank gives a nice example of this kind of problem in the simplest non-trivial equi-dimensional case $n = p = 2$, [1, 21].

We consider immersions with normal crossings of a circle in the plane, $f: S^1 \to \mathbb{R}^2$. We want S^1 to be the fold curve of a stable map of a compact, oriented 2-manifold N into $\mathbb{R}^2$. By the result of Haefliger (see the end of § 5) we know that if such a map exists, the fold curve cuts N into two pieces N_1 and N_2 with $\chi(N_1) = \chi(N_2)$ (on N_1, f is orientation reversing and in N_2, f is orientation preserving). Thus our problem is:

Given $f: S^1 \to \mathbb{R}^2$ *an immersion with normal crossings is there an immersion F of a compact 2-manifold M with a one-component boundary,* S^1, *such that* $F \mid \partial M$ *is* f? *Anad further for each M how many inequivalent F are there, and how many such M?*

The last part of the question can be settled immediately: *If there is an immersion,* $F: M \to \mathbb{R}^2$, *with* $F \mid \partial M = f$, *then M is unique.*

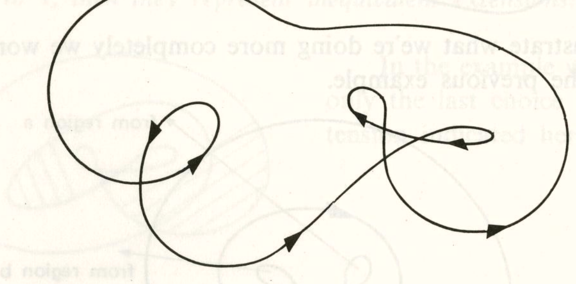
This is obvious since $2\chi(M) = 2r(f)$ where $r(f)$ is the rotation number of f (see the theorem at the end of § 2 before the examples), and any surface immersed in $\mathbb{R}^2$ is oriented. The work of Blank solves the problem completely in case $r(f) = 1$. Thus we are working in the case $M = D$, the disc. To find when and in how many different ways an immersed circle of rotation number 1, bounds an immersed disc. Clearly $r(f) = 1$ is not sufficient. For example, the curve below cannot bound a disc, since there is a region defined by the



curve around which the curve winds onde in a negative sense (clockwise). Thus in addition to $r(f) = 1$, we must have that for each $y \in \mathbb{R}^2 - f(S^1)$, the *winding number* of f about y is non-negative. Recall that if $y \in \mathbb{R}^2 - f(S^1)$, then the winding number of f about y is just the degree of:

$$S^1 \to S^1 : t \to \frac{f(t) - y}{|f(t) - y|}.$$

If f does bound a disc and F is the extension, the winding number of f about y is just the number of points in $F^{-1}(y)$ at which F is orientation preserving minus the number of points in $F^{-1}(y)$ at wich F is orientation reversing. However requiring $r(f) = 1$ and all of these winding numbers to be non-negative is still not sufficient. For example:

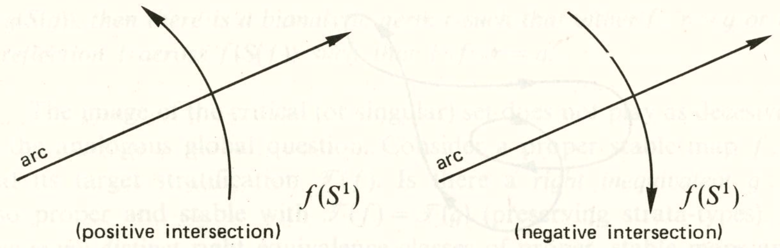


cannot bound.

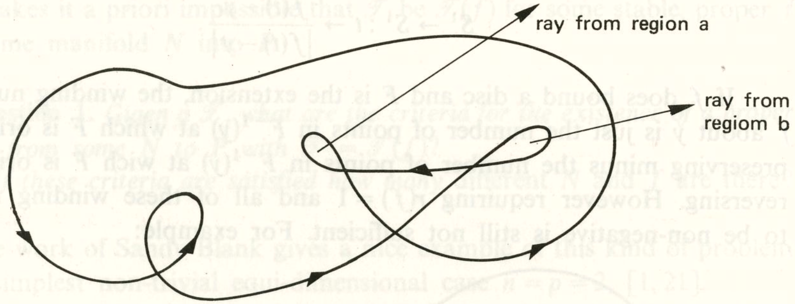
We will checks that this can't bound using the result of Blank, which we now describe.

We restrict ourselves to $f: S^1 \to \mathbb{R}^2$ *immersion with normal* crossing with $r(f) = 1$, which wind non-negatively about each region defined.

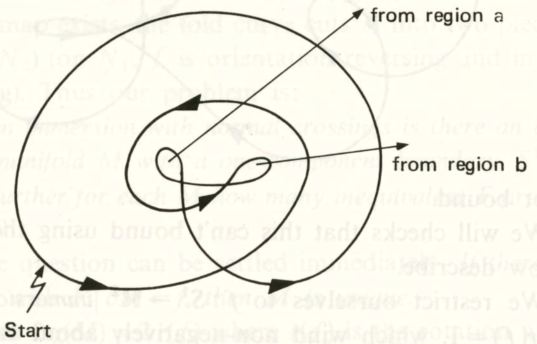
To each bounded component, a of $\mathbb{R}^2 - f(S^1)$ let n_a be the winding number of f about any point in a, and let m_a be the minimum number of times an arc leaving a, meets $f(S^1)$ before reaching the unbounded component of $\mathbb{R}^2 - f(S^1)$. (Remark: n_a can be computed by taking the number of times any arc emanating from a cuts $f(S^1)$ positively and subtracting the number of times that arc cuts $f(S^1)$ negatively.



Thus $m_a \geq n_a$. We restrict our attention to those regions a such that $m_a > n_a$. Thus in the example below we have only two components to consider. Draw a ray from each of these components into the unbounded component of $\mathbb{R}^2 - f(S^1)$ which crosses the immersed $f(S^1)$ normally.



In order to illustrate what we're doing more completely we work with a modification of the previous example.



Pick a starting point in the curve and as you trace the curve write down a word by writing the sequence of names of the regions as you cross their ray; if you cross the ray positively rite the name of the region, if you cross the ray negatively write the $(\text{name})^{-1}$. In the example, above the word is $(b\,a\,b^{-1}\,a^{-1}\,b\,a)$. Call the word so constructed $\omega(f)$. As we remarked above, n_a is the sum of the exponents of the symbol a in $\omega(f)$. since $m_a > n_a \geq 0$,

there must be at least one occurance of a^{-1} in $\omega(f)$ for each region a whose symbol occurs in the word. In working with the words we use a new symbol, 1 and allow the substitutions: $a^{-1}a = aa^{-1} = 1$, $1a = a1 = a$, $1.1 = 1$. We say that a word *reduces to* 1 if it is transformed to 1 by a sequence of these substitutions: Now given a word, $\omega(f)$, for each symbol a that occurs in $\omega(f)$ choose n_a of the a's in $\omega(f)$ – choose only a's – nome of the (a^{-1})'s are chosen. There may be, of course, many such choices; denote the different choices by $c_1, c_2, \ldots$

For example. The word we had constructed was $b\ a\ b^{-1}\ a^{-1}\ b\ a$; notice that $n_a = n_b = 1$.

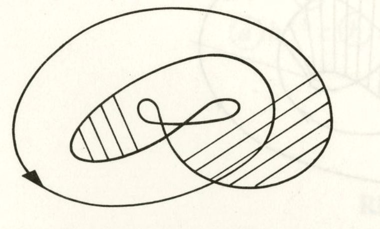
We list the choices:

ⓑ ⓐ $b^{-1}\,a^{-1}\,b\,a$, ⓑ $a\,b^{-1}\,a^{-1}\,b$ ⓐ

$b\,a\,b^{-1}\,a^{-1}$ ⓑ ⓐ b ⓐ $b^{-1}\,a^{-1}\,b$ ⓐ

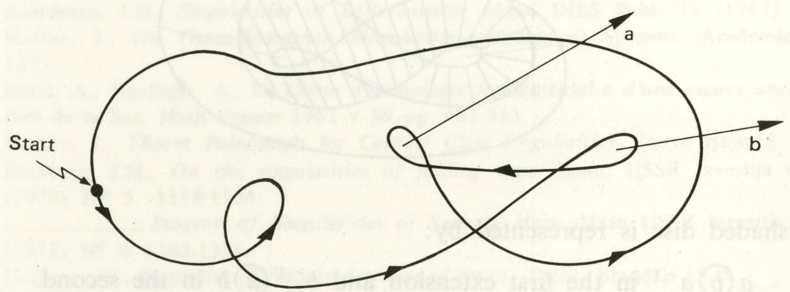
For a given choice C, let $\omega(f)/_C$ be the word obtained by deleting the symbols chosen. Notice that now the sum of the exponents of each symbol is zero, so the reduced word $\omega(f)/_C$ may reduce to 1.

Theorem (Blank) *f is extendable iff there is a choice C such that $\omega(f)/_C$ reduces to 1. If C_1 and C_2 are two different choices such that $\omega(f)/_{C_1}$ and $\omega(f)/_{C_2}$ reduce to 1, then they represent inequivalent extensions.*



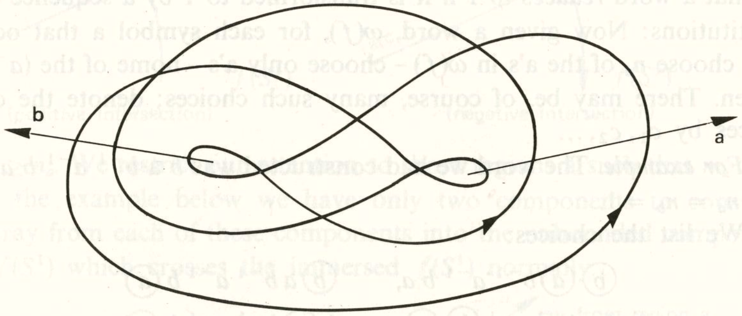
In the example we're been following only the last choice represented the extension indicated here:

The immersion in claimed had no extensions was:



Whose word is: $b^{-1}a^{-1}ba$. Since $n_a = n_o = 0$ we have no choices and obviously this word doesn't reduce to 1, so the immersion doesn't extend.

We check two more examples: First, example 2 of the previous section

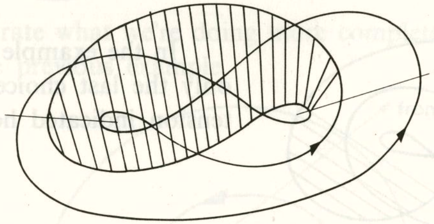


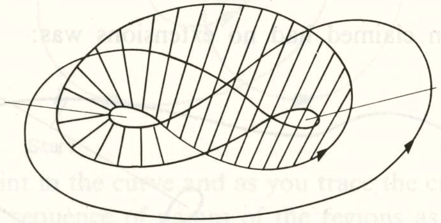
$\omega(f) = ab^{-1}qba^{-1}b$

The only two choices that represent extensions are:

$$\textcircled{a}\, b^{-1} a\, \textcircled{b}\, a^{-1} \text{ and } a b^{-1}\, \textcircled{a}\, b\, a^{-1}\, \textcircled{b}$$

and they represent:

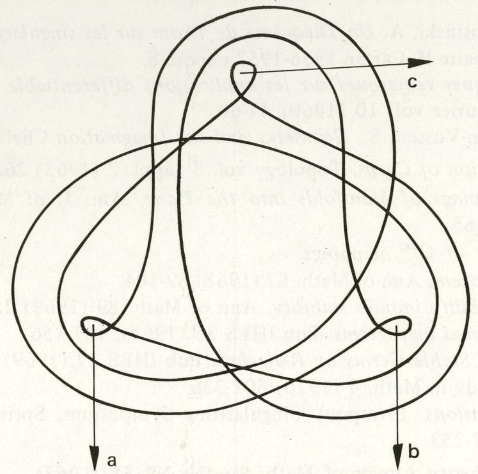




the shaded disc is represented by:

$a\,\textcircled{b}\,a^{-1}$ in the first extension and $b^{-1}\,\textcircled{a}\,b$ in the second.

The last example we check is that given in section 2, when we projected Boy's surface in $\mathbb{R}^2$. The curve we showed bounded a disc was:



$\omega(f) = a b^{-1} c b c^{-1} a c a^{-1} b$

$n_a = n_b = n_c = 1$. The following two choices represent extensions

$$a b^{-1}\, \textcircled{c}\, b c^{-1}\, \textcircled{a}\, c a^{-1}\, \textcircled{b} \quad \text{and}$$
$$\textcircled{a}\, b^{-1} c\, \textcircled{b}\, c^{-1} a\, \textcircled{c}\, a^{-1} b.$$

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