A generalization of a maximum principle for the wave-operator with lower order terms

*Wolf Hofmann**

Summary. *It is shown that a classical maximum principle can be extended to continuous functions with piecewise continuous first and second derivatives. A simple application to the numerical solution of an initial value problem for the telegraph equation is presented.*

I. Introduction

We consider initial value problems for the operator

$$L[u] = -u_{x_1x_1} + u_{x_2x_2} - du_{x_1} - eu_{x_2} - hu, \qquad d, e, h \in \mathbb{R}.$$

We call a domain D (i.e. open and connected) in the half-plane $x_2 > 0$ an admissible domain if it has the following property: To each $P \in D$ the corresponding closed characteristic triangle (ABP) (see figure 1) belongs to $D \cup \Gamma_0$, where Γ_0 denotes the portion of the boundary of D situated on the x_1-axis. For such an admissible domain the following theorem holds [3, p. 199].

Theorem 1. *Let* $u \in C^2(D) \cup C^1(D \cup \Gamma_0)$, $h \geq 0$, $e + d \geq 0$, $e - d \geq 0$, *and suppose that* u *satisfies the following inequalities*:

(*i*) *on* $\Gamma_0 : u_{x_2} - eu \leq 0$, $u \leq 0$,

(*ii*) *in* D: $L[u] \leq 0$,

then $u \leq 0$ *in* D.

Such theorems give one of the few possibilities to obtain error estimates. But it is often difficult to satisfy that $u \in C^2(D)$. Therefore we want to extend this theorem to continuous functions which are piecewise continuously differentiable. To do this the domain D is partitioned into subsets and the functions are assumed to be twice differentiable in the interior of these subsets. A similar result for the Laplace-operator has been shown in [2].

Recebido em fevereiro de 1976

*This paper was presented at the "2.º Seminário Brasileiro de Análise" held at the Instituto de Matematica e Estatistica da Universidade de São Paulo in October 1975, during my stay at the program GMD-CNPQ.

Let D be an admissible domain and $\{D_i\}_{i\in I}$ a family of open sets with the following properties:

$D_1)$ $\bar{D} = \bigcup_{i\in I} \bar{D}_i$, $D_i \cap D_j = \phi$ for $i \neq j$, $i, j \in I$.

$D_2)$ For each normal domain $R \subset D$ (i.e. a domain allowing the application of the Gauss-integral-theorem), $R \cap D_i$, $i \in I$, is a normal domain.

$D_3)$ Each compact subset of D intersects only a finite numer of the $\bar{D}_i$.

$D_4)$ For each $i \in I$ and for each $t > 0$, the set $\bar{D}_i \cap \{x_2 \equiv t\}$ consists only of a finite number of intervals.

$D_5)$ Whenever $[\alpha, \beta] \subset \Gamma_0$ is an arbitrary closed and bounded interval, λ the 1-dimensional Lebesgue-measure and $N = \{i \in I; \lambda(\bar{D}_i \cap [\alpha, \beta]) > 0\}$ then

$$\lim_{t\to 0+} \lambda(\{(x_1, t) \in \mathbb{R}^2;\ \alpha \leq x_1 \leq \beta\} \cap \bigcup_{i\in I-N} D_i) = 0.$$

Now denoting by u_i the restriction of a function u to D_i, we shall say that a function u belongs to $F(D_I)$ if

(1) $u \in C(D \cup \Gamma_0)$,

(2) $u_i \in C^1(\bar{D}_i - \tilde{\Gamma})$, $i \in I$ where $\partial D = \tilde{\Gamma} \cup \Gamma_0$, i.e. the derivatives exist in D_i and can be extended continuously to Γ_0 and to those parts of the boundary of D_i, which are situated in the interior of D.

(3) $u_i \in C^2(D_i)$, $i \in I$.

Since the "interior" part of ∂D_i (with respect to D) is sufficiently smooth (accordint to condition D_2), there exists a.e. an interior normal v_i, and a conormal σ_i may be introduced by $\cos(v_i, x_1) = \cos(\sigma_i, x_1)$, $\cos(v_i, x_2) = -\cos(\sigma_i, x_2)$, [1, p. 122].

Now the following generalization of theorem 1 holds.

Theorem 2. *Let $u \in F(D_I)$, $h \geq 0$, $e + d \geq 0$, $e - d \geq 0$, and suppose that u satisfies the following inequalities:*

(i) on Γ_0 $u_{x_2} - eu \leq 0$ a.e. and $u \leq 0$,

(ii) in D_i: $L[u_i] \leq 0$, $i \in I$,

(iii) on $\partial D_i \cap \partial D_j$: $\partial u_i/\partial\sigma_j - \partial u_j/\partial\sigma_j \leq 0$ a.e., $i, j \in I$, $i \neq j$.

then $u \leq 0$ in D.

The last assumption of theorem 2 is a "jump-condition" for the conormal-derivatives of two adjacent sets, and this is for instance satisfied if the interior boundaries of the D_i's are characteristic lines. To prove theorem 2 we shall first establish two lemmas.

II. Auxiliary Lemmas

As usual a subset S of $\mathbb{R}^2$ is called normal if it allows the application of the Gauss-integral-theorem. Then if R is a normal compact subset of D and D is partioned in subsets as stated in the introduction we get $R = \bigcup_{i=1}^{m} R_i$ by putting $R_i = R \cap \bar{D}_i$.

We now define $\Gamma = \partial R$, $\Gamma_i = \partial R_i - \Gamma$, $\Gamma_{ij} = \Gamma_i \cap \Gamma_j$.

With this notation we obtain $\Gamma_i = \bigcup_{j=1, j\neq i}^{m} \Gamma_{ij}$.

By $C_0^\infty(\mathring{R})$ we denote the set of testfunctions on the interior $\mathring{R}$ of R, and by L^* the adjoint operator

$$L^*[v] = -v_{x_1x_1} + v_{x_2x_2} + dv_{x_1} + ev_{x_2} - hv.$$

Now we can prove as in [2, p. 151 f] the following lemma.

Lemma 1. *Let $v \in F(D_I)$, $\varphi \in C_0^\infty(\mathring{R})$, then*

$$\iint_R vL^*[\varphi]\,dx = \sum_{i=1}^{m} \iint_{R_i} \varphi L[v_i]\,dx + \sum_{\substack{i,j=1\\ i>j}}^{m} \int_{\Gamma_{ij}} \varphi\left(\frac{\partial v_i}{\partial\sigma_j} - \frac{\partial v_j}{\partial\sigma_j}\right) d\tau.\text{*}$$

Proof: For each R_i, $i = 1, \ldots, m$, we obtain by application of the Gauss-integral-theorem [1, p. 120 ff]

$$\iint_{R_i} (v_i L^*[\varphi] - \varphi L[v_i])\,dx = -\int_{\partial R_i} v_i M_i[\varphi]\,d\tau - \int_{\partial R_i} \varphi \frac{\partial v_i}{\partial \sigma_i}\,d\tau,$$

$$M_i[\varphi] = \varphi(d\cos(v_i, x_1) + e\cos(v_i, x_2)) - \varphi_{x_1}\cos(v_i, x_1) + \varphi_{x_2}\cos(v_i, x_2).$$

Since φ and all its derivatives vanish on $\Gamma = \partial R$ and $\Gamma_i = \bigcup_{j=1, i\neq j}^{m} \Gamma_{ij}$ it follows that

$$\iint_R vL^*[\varphi]\,dx = \sum_{i=1}^{m} \int_{R_i} \varphi L[v_i]\,dx - \sum_{\substack{i,j=1\\ i>j}}^{m} \int_{\Gamma_{ij}} (v_i M_i[\varphi] + v_j M_j[\varphi])\,d\tau$$
$$- \sum_{\substack{i,j=1\\ i>j}}^{m} \int_{\Gamma_{ij}} \varphi\left(\frac{\partial v_i}{\partial\sigma_i} + \frac{\partial v_j}{\partial\sigma_j}\right) d\tau$$

Obviously we have $v_i = v_j$ on Γ_{ij} and $v_i = -v_j$ a.e. on Γ_{ij}.

Therefore we get a.e. on Γ_{ij}: $M_i[\varphi] = -M_j[\varphi]$. Hence the second sum on the right vanishes. Since in addition $\partial v_i/\partial\sigma_i = -\partial v_i/\partial\sigma_j$ a.e. on Γ_{ij} we obtain the desired formula.

---

*Here and later on dx is the element of the area and $d\tau$ the element of the line.

Let us denote by $\|,\|$ the Euclidean distance. If now R is a normal compact subset of D with $\inf\{\|x-y\|;\ x\in R,\ y\in\partial D\}>\alpha>0$ then we shall denote by φ_α the well-known nonnegative test-function with support $\{x:\|x\|\le\alpha\}$ and $\iint_{\mathbb{R}^2}\varphi_\alpha(x)\,dx=1$. For functions u which are local summable on D we now introduce the mean function u_α which is defined on R by

$$u_\alpha(x)=\iint_D u(t)\,\varphi_\alpha(t-x)\,dt.$$

For any point $P=(r,s)\in D$ we construct the characteristic closed triangle (ABP), (see figure 1). For $x_2\equiv t(t<s)$, we denote by $\overline{A'B'}$ the intersection of this line with the triangle. Now we are able to prove the next lemma.

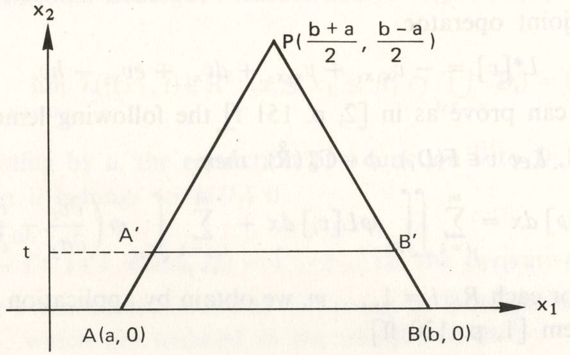


Figure 1.

Lemma 2. *Let* $P=(r,s)\in D$. *Then, if* $v\in F(D_I)$ *and* $e\in\mathbb{R}$ *such that* $v_{x_2}-ev<0$ *a.e. on* $\overline{AB}$, *there exist real numbers* $t,k,\bar\alpha$ *with* $0<t<s$, $k<0$, $\bar\alpha>0$ *such that for all positive* $\alpha\le\bar\alpha$ *we have:*

$$\int_{A'}^{B'}\left(\frac{\partial v_\alpha}{\partial x_2}(x_1,t)-ev_\alpha(x_1,t)\right)dx_1<k.$$

Proof: Let $N=\{i\in I;\ \lambda(\bar D_i\cap\overline{AB})>0\}$, $g(x_1,x_2)=v_{x_2}(x_1,x_2)-ev(x_1,x_2)$, and $D_i^t=D_i\cap(ABP)\cap\{x_2\equiv t\}$ for any $t\ge0$. Since $g(x_1,0)<0$ a.e. on $\overline{AB}$ and because the partitioning satisfies the property D_5, continuity arguments yield the existence of real numbers k_1,k_2,t with $k_1<0$, $k_1+k_2<0$, $t>0$, such that

$$\sum_{i\in N}\int_{D_i^t}g(x_1,t)\,dx_1<k_1\ \text{ and }\ \sum_{i\in I-N}\int_{D_i^t}|g(x_1,t)|\,dx_1<k_2.$$

Now, because D is open, there exists an $\alpha_1>0$ such that

$$\Delta=\{x=(x_1,x_2);\ \sup_{y\in(ABP)}\|x-y\|\le\alpha_1,\ x_2\ge0\}\subset D.$$

But Δ is a compact and $v\in F(D_I)$. Hence there exists an $c>0$ such that $\sup_{x\in\Delta}|g(x)|\le c$ and $\sup_{x\in(ABP)}|g_\alpha(x)|\le c$ whenever $0<\alpha\le\min(t,\alpha_1)$.

Therefore we can choose t so small such that for all α with $0<\alpha\le\min(t,\alpha_1)$ we obtain

$$\sum_{i\in I-N}\int_{D_i^t}|g_\alpha(x_1,t)|\,dx_1\le k_2.$$

For any $\alpha>0$ we put $(D_i)_\alpha=\{x\in D_i;\ \sup_{y\in D_i}\|x-y\|\le\alpha\}$. Then properties D_3 and D_4 of our partitioning show that for sufficiently small α there exists a finite number of compact sets K_μ, $\mu=1,\ldots,n$ with the following two properties:

(1) To each μ there exists an $i\in N$ such that $K_\mu\subset D_i$,

(2) $K:=\bigcup_{\mu=1}^{n}K_\mu=\overline{A'B'}-(\bigcup_{i\in I-N}(D_i^t)_\alpha)$.

Therefore we can choose by continuity arguments real numbers k_3, k_4, α_2 with $k_3<0$, $k_3+k_4<0$, $0<\alpha_2<\min(t,\alpha_1)$, such that for all positive $\alpha\le\alpha_2$ the inequalities

$$\int_K g(x_1,t)\,dx_1<k_3\ \text{ and }\ \sum_{i\in I-N}\int_{(D_i)_\alpha^t}|g(x_1,t)|\,dx_1<k_4$$

hold. Now there exists for every $\delta>0$ an $\alpha_3=\alpha_3(\delta)>0$ such that for all positive $\alpha\le\alpha_3$ we have

$$\max_{(x_1,t)\in K}\left\{|v_\alpha(x_1,t)-v(x_1,t)|,\ \left|\frac{\partial v_\alpha}{\partial x_2}(x_1,t)-\frac{\partial v}{\partial x_2}(x_1,t)\right|\right\}\le\delta,$$

(see [4, n.° 71: p. 207, theorem 4]). This implies

$$\frac{\partial v_\alpha}{\partial x_2}-ev_\alpha\le\frac{\partial v}{\partial x_2}+\delta-ev+\delta e=\frac{\partial v}{\partial x_2}-ev+\delta(1+e)\ \text{in}\ K.$$

and therefore it is possible to choose such a small δ that for all positive $\alpha\le\min(\alpha_2,\alpha_3(\delta))$ and a $k_5<0$ with $k_5+k_4<0$ the following inequality holds: $\int_K g_\alpha(x_1,t)\,dx_1\le k_5$.

By choosing $k=k_5+k_4$ and $\bar\alpha=\min(\alpha_2,\alpha_3(\delta))$ the proof is complete.

III. Proof of Theorem 2

Under stronger conditions we shall first establish the following.

Lemma 3. *Let* $u\in F(D_I)$, $h\ge0$, $e+d\ge0$, $e-d\ge0$, *and suppose that* u *satisfies the following inequalities:*

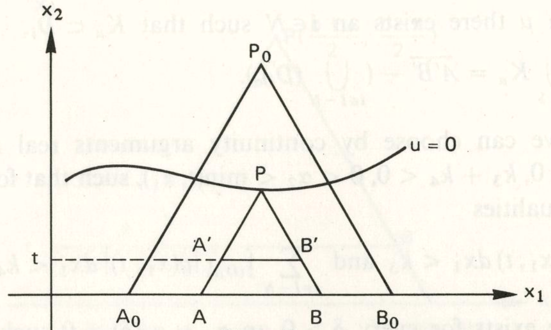
(*i*) *on* Γ_0: $u_{x_2} - eu < 0$ a.e., $u < 0$,

(*ii*) *in* D_i: $L[u_i] \leq 0$, $i \in I$,

(*iii*) *on* $\partial D_i \cap \partial D_j$: $\partial u_i/\partial\sigma_j - \partial u_j/\partial\sigma_j \leq 0$ *a.e.*, $i, j \in I$, $i \neq j$,

then $u < 0$ *in* D.

Proof: Suppose on the contrary that there exists a point $P_0 \in D$ such that $u(P_0) \geq 0$. Let $D_0 = (A_0 B_0 P_0)$ be the corresponding closed characteristic triangle. Since $u \in C(D_0)$ we obtain that $G = \{(x_1, x_2) \in D_0 ; u(x_1, x_2) = 0\}$ is compact. Therefore there exists a point $P = (r, s) \in (A_0 B_0 P_0)$, $s > 0$ with $u(P) = 0$ and such that u is negative for all points $(x_1, x_2) \in (A_0 B_0 P_0)$ with $x_2 < s$ (see figure 2).



Figue 2.

If R denotes the triangle $(A'B'P)$ of Lemma 2, we obtain by applying Stokes' theorem to u_α (with $\alpha < \bar{\alpha}$ (see Lemma 2)) and $D' = (A'B'P)$ the formula [3, p. 198]:

$$2u_\alpha(P) = u_\alpha(A') + u_\alpha(B') + \iint_{D'} h u_\alpha dx + \int_{A'}^{B'} \left(\frac{\partial u_\alpha}{\partial x_2} - e u_\alpha\right) dx_1$$

$$+ \int_{B'}^{P} (e + d)\, u_\alpha dx_2 + \int_{A'}^{P} (e - d)\, u_\alpha dx_2 + \iint_{D'} L[u_\alpha]\, dx.$$

Since D is open, there exists a positive $\tilde{\alpha} < \bar{\alpha}$ such that $\tilde{D} = \overline{(D')}_\alpha \subset D$. Denoting by L_x, L_y^* the operators L, L^* with respect to the variables x and y in $\mathbb{R}^2$, a simple calculation shows that for all positive $\alpha < \tilde{\alpha}$. $L_x\, \varphi_\alpha(y - x) = L_y^*\, \varphi_\alpha(y - x)$ whenever $x \in D'$.

This yields by applying lemma 1 to $R = \tilde{D}$ that we obtain for $x \in D'$:

$$L[u_\alpha](x) = \iint_{\tilde{D}} u(y)\, L_x[\varphi_\alpha](y - x)\, dy = \iint_{\tilde{D}} u(y)\, L_y^*[\varphi_\alpha](y - x)\, dy =$$

$$= \sum_{i=1}^{m} \iint_{R_i} \varphi_\alpha(y - x)\, L_y[u_i]\, y)\, dy + \sum_{\substack{i,j=1 \\ i>j}}^{m} \int_{\Gamma_{ij}} \varphi_\alpha(y - x) \left(\frac{\partial u_i}{\partial \sigma_j}(y) - \frac{\partial u_j}{\partial \sigma_j}(y)\right) d\tau.$$

But $\varphi_\alpha(y - x) \geq 0$ for $x \in D'$, $L[u_i](y) \leq 0$ for $y \in D_i$, $i \in I$, and the jump-condition show that $L[u_\alpha](x) \leq 0$ for $x \in D'$. Therefore, if $\alpha < \min \bar{\alpha}, \tilde{\alpha})$

$$2u_\alpha(P) \leq u_\alpha(A') + u_\alpha(B') + \iint_{D'} h u_\alpha dx + \int_{A'}^{B'} \left(\frac{\partial u_\alpha}{\partial x_2} - e u_\alpha\right) dx_1$$

$$+ \int_{B'}^{P} (e + d)\, u_\alpha dx_2 + \int_{A'}^{P} (e - d)\, u_\alpha dx_2 .$$

Since $u < 0$, $h \geq 0$, $e + d \geq 0$, $e - d \geq 0$ in D', it is possible to choose an α_4 so small that for all positive $\alpha \leq \alpha_4$ there exists a $k_6 \in \mathbb{R}$, such that the following inequality (with k asserted as in lemma 2) holds:

$$\iint_{D'} h u_\alpha dx + \int_{B'}^{P} (e + d)\, u_\alpha dx_2 + \int_{A'}^{P} (e - d)\, u_\alpha dx_2 < k_6, k + k_6 = : k_7 < 0.$$

Hence, $2u_\alpha(P) < u_\alpha(A') + u_\alpha(B') + k_7$, $\alpha < \min(\bar{\alpha}, \tilde{\alpha}, \alpha_4)$.

Choosing in addition α_4 so small that for $\alpha \leq \min \bar{\alpha}, \tilde{\alpha}, \alpha_4)$ $|u_\alpha - u| < < k_7/4$ in D', (see [4, n.° 71, p. 207, theorem 4]), we obtain $2u(P) < u(A') + u(B')$. Since $u(P) = 0$, $u(A') < 0$, $u(B') < 0$ we get a contradiction and the lemma is proved.

To relax the strict inequalities $u_{x_2} - eu < 0$ a.e. on Γ_0 and $u < 0$ on Γ_0, we apply as in [3, p. 199] lemma 3 to the function $u - \varepsilon \exp(\gamma x_2)$, $\varepsilon > 0$. $\gamma > 0$. For $\gamma > e$ the strict inequality

$$\frac{\partial}{\partial x^2} \exp(\gamma x_2) - e \exp(\gamma x_2) > 0$$

holds. $\exp(\gamma x_2)$ is infinitely often differentiable. Hence lemma 1 and $L_x\, \varphi_\alpha(y - x) = L_y^*\, \varphi_\alpha(y - x)$ for $x \in D'$ yield:

$$L_x \iint_{\tilde{D}} \exp(\gamma y_2\, \varphi_\alpha(y - x)\, dy = \iint_{\tilde{D}} \varphi_\alpha(y - x)\, L_y[\exp(\gamma y_2)]\, dy, \quad x \in D'.$$

Since $L_y[\exp(\gamma y_2)] = (+\gamma^2 - e\gamma - h)\exp(\gamma y_2) > 0$ for $\gamma > \frac{1}{2}(e + \sqrt{e^2 + 4h})$, we get $L_x \iint_{\tilde{D}} \exp(\gamma y_2)\, \varphi_\alpha(y - x)\, dy \geq 0$, $x \in D'$.

Therefore, if the conditions of lemma 3 are replaced by the conditions of theorem 2, lemma 3 shows that for sufficiently large γ we have $u - \varepsilon \exp(\gamma x_2) < 0$ in D. Letting $\varepsilon \to 0$, we conclude that $u \leq 0$ in D which completes the proof of theorem 3.

IV. Numerical Example

If the initial value functions are not twice differentiable one encounters difficulties in case one wants to apply a maximum principle of the type of theorem 1. Therefore in these cases we shall resort to our theorem 2. As an example let us consider the telegraph equation

$$L[u] = u_{x_1x_1} - u_{x_2x_2} + u = 0$$

together with the following initial value functions

$$u(x_1, 0) = \frac{1}{(-1)^i x_1 + 1} \quad \text{for } (-1)^i x_1 \geq 0,\ i = 1, 2,\ u_{x_2}(x_1, 0) = 0.$$

Obviously $u(x_1, 0)$ is not differentiable for $x_1 = 0$. Hence we shall apply theorem 2 to obtain lower and upper bounds for the solution u^* in the half-plane $x_2 > 0$.

To get a lower bound we choose

$$D_1 = \{(x, y) \in \mathbb{R}^2;\ x_1 < 0,\ 0 < x_2 < -x_1\},$$
$$D_2 = \{(x, y) \in \mathbb{R}^2;\ x_1 > 0,\ 0 < x_2 < x_1\},$$
$$D_3 = \{(x, y) \in \mathbb{R}^2;\ x_2 > |x_1|\}.$$

The (see [1, p. 402 f]) the function

$$v_i(x_1, x_2) = \frac{1 + 0.5x_2^2}{(-1)^i x_1 + 1},\ i = 1, 2,$$

satisfies the initial conditions and $L[v_i - u^*] = L[v_i] \geq 0$ in D_i, $i = 1, 2$. Therefore v_i is a lower bound for u^* in D_i, $i = 1, 2$. The trial function

$$v_3(x_1, x_2; a, b, c) = c(x_1^2 - x_2^2)\exp(ax_1 + bx_2) + \frac{1 + 0.5x_2^2}{x_2 + 1}, a, b, c \in \mathbb{R},$$

satisfies $v_3 = v_i$ on $\partial D_i \cap \partial D_3$, $i = 1, 2$ and for $1 = 0$, $b = -1$, $c = 0.5 \cdot$ we get $L[v_3] \geq 0$ in D_3. Therefore the function $v = v_i$ in D_i, $i = 1, 2, 3$, is a lower bound for u^* in the upper half-plane.

The function $w_i(x_1, x_2) = \cosh \sqrt{3}x_2/(-1)^i x_1 + 1$ is an upper bound in D_i, $i = 1, 2$, see [1, p. 402 f]).

These functions satisfy $L[u^* - w_i] = L[-w_i] \geq 0$ even for $(-1 x_1 \geq 0$, $i = 1, 2$.

Furthermore $\partial w_1(0, x_2)/\partial x_1 - \partial w_2(0, x_2)/\partial x_1 = 2 > 0$.

Therefore $w = w_i$ for $(-1)^i x_1 \geq 0$, $i = 1, 2$, is an upper bound for u^* in the half-plane $x_2 > 0$.

Improved results can be achieved by using better trial functions with more terms.

References

[1] Collatz, L., *Functional Analysis and Numerical Mathematics.* Academic Press, New York and London, 1966,

[2] Natterer, F. and B. Werner, *Eine Erweiterung des Maximum-prinzips für den Laplace-Operator.* Numr. Math. 22(1974) 149-156.

[3] Protter, M. H. and H. F. Weinberger, *Maximum Principles in Differential Equations.* Prentice-Hall Inc., Englewood Cliffs, New Yersey, 1967.

[4] Smirnov, V. I. *A Course of Higher Mathematics*, V. Pergamon Press, Oxford, London, Edinburgh, New York, Paris, Frankfurt, 1964.

Institut für Angewandte Mathematik der Universitat Harmburg
Rothenbaumchaussee 41
Bundesrepublik Deutschland.