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Recent developments in the study of singularities of vectorfields

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§0. Introduction and aknowledgments

In this paper we want to consider certain recent developments in the topological study of singularities of vector fields and their bifurcations. First in § 1 we recall a few definitions in order to fix our language and in § 2 we treat some important techniques in preparing the singularities for further study. From § 3 to § 5 we consider some problems concerning the singularities themselves. In § 6 we introduce the notion of unfolding of a singularity and, in the perspective of the theory of bifurcations, we give an a posteriori motivation for the study of rather degenerate singularities. The paragraphs § 7 and § 8 are devoted to some results about unfoldings. For more information about these and related subjects, we refer to the survey papers of Arnol'd [5] and Takens [21].

We would like to thank F. Takens and J. Sotomayor for many helpful conversations and we aknowledge the "Instituto de Matemática Pura e Aplicada" for the hospitality offered to us during the preparation of this paper.

§1. Definitions

1.1. **Definition.** A "C^k singularity (of a vector field)" is a triple $(\mathbb{R}^m, p, X)$ where $p \in \mathbb{R}^m$ and X is a C^k vector field on $\mathbb{R}^m$ with the property $X(p) = 0$.

To simplify the notations we are going to choose p to be the origin in some of our definitions.

1.2. **Definition.** Two vector fields X and Y on $\mathbb{R}^m$ with $X(0) = Y(0) = 0$ are germ-equivalent at 0 if they coincide on some neighbourhood of 0. The equivalence classes for this relation are called germs of singularities. Let us denote by $\mathscr{G}^m$ the set of germs at 0 of C^∞ singularities.

1.3. **Definition.** Let $\tilde{X}$ and $\tilde{Y} \in \mathscr{G}^m$, then $\tilde{X}$ and $\tilde{Y}$ are said k-jet-equivalent if for some representatives X and Y of respectively $\tilde{X}$ and $\tilde{Y}$, we have $X - Y = 0(\| x \|^{k+1})$ i.e. $\exists\, c > 0,\ \delta > 0 \ni \| (X - Y)(x) \| \leq c \| x \|^{k+1},\ \forall \| x \| < \delta$.

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An equivalence class for this equivalence relation is called a k-jet. We denote the k-jet of X in 0 by $j_k(X)(0)$, and the set of k-jets by $\mathcal{J}^m_k$. Choosing coordinates we see that $j_k(X)(0)$ is just the k^{th}-order polynomial approximation of X in 0, and hence we can induce on $\mathcal{J}^m_k$ a natural Euclidean topology, such that the projections $\mathcal{J}^m_l \longrightarrow \mathcal{J}^m_k$, $j_l(X)(0) \longrightarrow j_k(X)(0)$ $(l \geq k)$ are continuous. This enables us to consider the inverse limit $\mathcal{J}^m_\infty$ of the system $(\mathcal{J}^m_k, \pi_{lk})$, whose elements we call ∞-jets. On $\mathcal{T}^m_\infty$ (resp. $\mathcal{G}^m$) we take now the coarsest topology for which all mappings $\pi_k : \mathcal{J}^m_\infty \longrightarrow \mathcal{J}^m_k$ (resp. $j_k : \mathcal{G}^m \longrightarrow \mathcal{J}^m_k$) are continuous. By a theorem of Borel [17] we know that $j_\infty : \mathcal{G}^m \longrightarrow \mathcal{J}^m_\infty$ defined as the inverse limit of the j_k is surjective.

1.4. **Definition.** Two singularities (R^m, p, X) and (R^m, p', X') are called "(weakly)-C^i-equivalent" if there is a neighbourhood U of p and a C^i-diffeomorphism (homeomorphism in case $i = 0$) $h : U \longrightarrow h(U)$ with $h(p) = p'$ such that:

(a) in case of C^i-equivalence: h maps integral curves of X to integral curves of X', preserving the sense but not necessarily the parametrization

(b) in case of weak-C^i-equivalence: $h \mid K : K \longrightarrow K'$ is a homeomorphism which maps integral curves of X to integral curves of X', sense-preserving. $K = \{x \in U \mid \phi_X(x, t) \in U$ for all $t \geq 0$ or for all $t \leq 0\}$ and K' is defined analogously using $h(U)$ and X' instead of U and X.

Mapping integral curves to integral curves in a sense preserving way means the following: if $x \in U$ and $\phi_X(x, [0, t]) \subset U$ for some $t > 0$, then there exists a $t' > 0$ such that

$$h[\phi_X(x, [0, t])] = \phi_Y(h(x), [0, t'])$$

where $\phi_X : D_X \subset \mathbb{R}^m \times \mathbb{R} \longrightarrow \mathbb{R}^m$ denotes the flow of X.

1.5. **Definition.** Two singularities $(\mathbb{R}^m, p, X)$ and $(\mathbb{R}^m, p, X')$ are called C^i-conjugated if they are C^i-equivalent by means of a parameter preserving C^i diffeomorphism h; i.e. $h \circ \phi_X(x, t) = \phi_Y(h(x), t)$ whenever $\phi_X(x, t) \in U$ for some $x \in U$, and $t \in \mathbb{R}$.

1.6. **Definition.** Let $A \subset \mathcal{G}^m$ and $X \in A$. We say that X is A-(weakly)-C^i-stable if there is some neighbourhood U of X in $\mathcal{G}^m$ such that every $X' \in A \cap U$ is (weakly)-C^i-equivalent with X.

1.7. We consider also the space $\chi(\mathbb{R}^n \times \mathbb{R}^m)$ of C^∞ n-parameter families of vector fields on R^m i.e. $\chi(\mathbb{R}^n \times \mathbb{R}^m)$ consists of the set of C^∞ vector fields on $\mathbb{R}^n \times \mathbb{R}^m$ of the form

$$\sum_{i=1}^{m} X_i(\lambda_1, \ldots, \lambda_n, x_1, \ldots, x_m) \frac{\partial}{\partial x_i}$$

endowed with the strong Whitney topology [17]. It is a Baire space. Let us recall that a subset of a topological space is called "residual" if it is a countable intersection of open and dense subsets. A property is called "generic" if it is satisfied by all elements of some residual subset.

1.8. As remarked in [5] there are several possibilities of defining the codimension of a singularity, but in view of the results we like to present, we think this one is the most adapted.

Definition. We say that the (topological) codimension of a singularity $(\mathbb{R}^m, p, X)$ is at most k if in all residual subsets $A \subset \chi(\mathbb{R}^k \times \mathbb{R}^m)$ there exists $Y \in A$ and $q \in \mathbb{R}^k \times \mathbb{R}^m$ such that $(\pi^{-1}(\pi(q)), q, Y \mid \pi^{-1}(\pi(q)))$ is C^0-equivalent with $(\mathbb{R}^m, p, X)$, where $\pi : \mathbb{R}^k \times \mathbb{R}^m \longrightarrow \mathbb{R}^k$, $(\lambda, x) \longrightarrow \lambda$. The codimension is exactly k if it is "$\leq k$" and not "$\leq (k-1)$".

§2. Normal form theorem for singularities

In this chapter we are going to describe an important tool in the study of singularities: the normal form theorem. We could classify this as a kind of formal preparation theorem in the sense that it enables us to clean the ∞-jet of a C^∞ singularity of a lot of non-relevant terms.

Preliminary we can use a linear change of coordinates to put the 1-jet of the singularity in "Jordan normal form".

It is now possible to adapt the ∞-jet in the following way: Suppose X is a C^k vector field on $\mathbb{R}^m$ with $X(0) = 0$. Let X_1 be the vector field whose component functions are linear and such that $j_1(X_1)(0) = j_1(X)(0)$. Let H^h denote the vector space of vector fields on $\mathbb{R}^m$ whose component functions are homogeneous polynomials of degree h, and let $[X_1, -]_h : H^h \longrightarrow H^h$ be the linear map which assigns to each $Y \in H^h$ the Lie bracket $[X_1, Y]$. We can consider a splitting $H^h = B^h \oplus G^h$ where $B^h = Im([X_1, -]_h)$ and G^h is some complementary space.

2.1. **Theorem** [23]. *Let X, X_1, B^h and G^h be as above. Then for $l \leq k$, there is a C^∞-diffeomorphism φ: $(\mathbb{R}^m, 0) \longrightarrow (\mathbb{R}^m, 0)$ such that $X' = \varphi_*(X)$ is of the form $X' = X_1 + g_2 + \ldots + g_l + R_l$ where $g_i \in G_i$, $i = 2, \ldots, l$, and $j_l(R_l)(0) = 0$ ($l = k = \infty$ is permitted).*

§3. Finite determinacy

The first questions we want to treat are the following: "Under which conditions is a germ of a singularity topologically (C^0) determined by a finite jet, and which finite jets are determining?"

3.1. **Definition.** A k-jet T_k is called "(weakly)-C^0-determining" if for all germs X and Y with $j_k(X)(0) = j_k(Y)(0) = T_k$, there exists a (weak)-$C^0$-equivalence between X and Y.

3.2. **Definition.** A germ of a singularity is called "(weakly) finitely determined" if there exists some finite k such that $j_k(X)(0)$ is (weakly)-C^o-determining.

A motivation for this study could be as follows: Suppose a phenomenom has a behaviour which makes it hopeful to describe it by means of a differentiable vector field, and suppose also that we are only interested in the asymptotic or qualitative behaviour in the neighbourhood of a certain equilibrium position. If this would be a finitely determined singularity, then to provide a good qualitative local model it would suffice to calculate a determining polynomial approximation.

Important in this direction is of course the normal form Theorem 2.1. It diminishes the number of coefficients in the ∞-jet and hence simplifies considerably the research of determinacy criteria. Such a program has for example been carried out in classifying the singularities of codimension at most 2 in all dimensions [23] and the singularities of codimension at most 4 on the plane [9]. But more about in § 4. Here however we would like to remark that calculating the normal form of a singularity is quite a lengthy operation.

In the context of finite determinacy, we should certainly not forget to recall the following result:

3.3. **Theorem of Hartman-Grobman** [12] [13]. A 1-jet is C^o-determining if and only if it is hyperbolic and a germ with a hyperbolic 1-jet is $\mathscr{G}^m$-C^o-stable.

As a matter of fact the underlying notion of C^o-equivalence in this theorem can be replaced by C^o-conjugacy. Hyperbolic means that the eigenvalues have a nonzero real part.

3.4. Inspired by the theory of singularities of mappings [28], the following question was raised: "Can all finite jets be stabilized?" i.e., if T_k is a k-jet, does there exists an l-jet T_l with $\pi_{lk}(T_l) = T_k$ $(l \geq k)$ such that T_l is (weakly)-C^o-determining.

Thom conjectured in 1970 that this question would have a negative answer and indeed Takens first in [22] showed that there exist 5-jets on $\mathbb{R}^4$ which can not be stabilized for weak-C^o-determinacy, and later on in [23] showed the existence of 2-jets on $\mathbb{R}^5$ with similar behaviour. In both cases the program consists roughly in showing by means of the normal form Theorem 2.1 that a certain symmetry on the 1-jet can be extended to the ∞-jet in certain coordinates, but can be broken up in topologically different ways at the germ level. On $\mathbb{R}^3$ the question is still open and on $\mathbb{R}^2$ it has been proven that the question has a positive answer [9] [11].

On $\mathbb{R}^2$ there has also been obtained a partial characterisation of finitely determined germs [9]:

- "if a germ is finitely determined then it satisfies an inequality of Łojasiewicz type"

  i.e. $\exists k \in \mathbb{N},\ \delta > 0,\ c > 0 \ni \| X(x) \| \geq c \| x \|^k\ \forall x$ with $\| x \| < \delta$

- "if a germ satisfies an inequality of Łojawiewicz type and has a characteristic orbit, then it is finitely determined"

By "characteristic orbit" we mean an orbit which tends to the singularity or leaves the singularity with a well defined direction.

§4. Singularities of finite codimension

An other kind of problems which are still of purely local type occur in the context of n-parameter families of vector fields and the related notion of codimension of a singularity (see def. 18).

(1) "Are all singularities of finite codimension finitely determined (up to C^o or weak-C^o-equivalence)?"

(2) "Whenever we fix an n, are the singularities of codimension at most n finite in number (up to some C^o-equivalence)?"

(3) "Try to characterize or classify the codimension n-singularities for n as big as possible."

Some results concerning questions (1) and (2) are the following:

$m \geq 5$: for $n \geq 3$ questions (1) and (2) have negative answer ([23]) even up to weak-C^o-equivalence.

The counter examples are the same as the ones used in § 3 to disprove the stabilizability of all finite jets, because they occur in a residual subset of some codimension 3 manifold in $\mathscr{J}^5_2$.

$m = 4$: there exists some k such that questions (1) and (2) have negative answer for $n \geq k$. Same remark as in the case $m \geq 5$ [22].
$m = 3$: conjecture of Takens: negative answer to questions (1) and (2).
$m = 2$: all singularities of finite codimension are finitely determined and for all n, we only have a finite number of codimension-n singularities up to C^o-equivalence [9] [11].

This result on R^2 verifies the conjecture that Takens has made in [21]. The result also remains valid if we change the notion of C^o-equivalence by that of C^o-conjugacy, because finitely determined singularities on the plane are C^o-equivalent if and only if they are C^o-conjugated [10].

Concerning problem (3) we can give the following results: The singularities of cod $\leq$ 2, and this in any dimension, have been classified by Takens up to weak-C^o-equivalence [23]. Since [21] provides a good survey about this classification we are not going to repeat it here.

For the plane we have extended this classification up to codimension 4 [9]. Except for some problems with codimension 4-singularities without characteristic orbits, we find 10 singularities of codimension $\leq$ 4, up to C^o-conjugacy.

cod 0

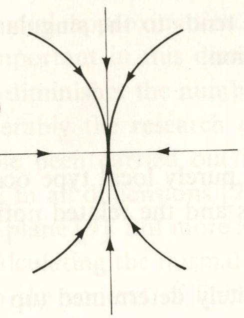


Fig. 1

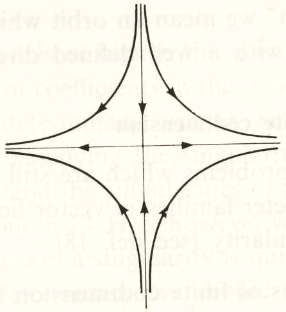


Fig. 2

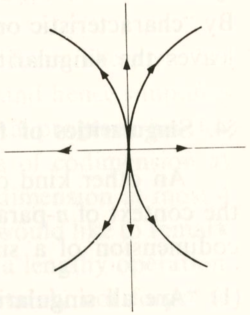
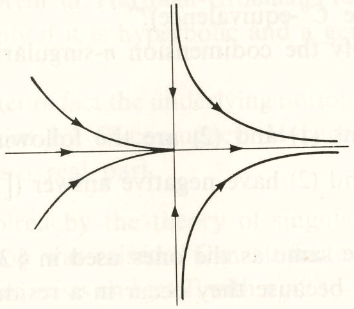


Fig. 3

cod 1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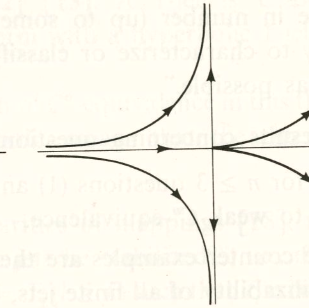
Fig. 5

cod 2

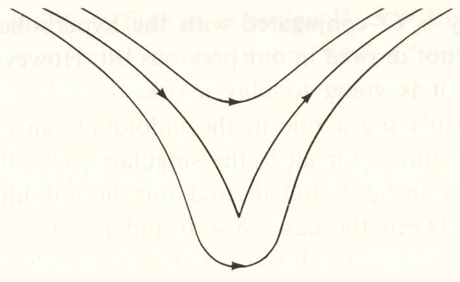
Fig. 6

cod 3

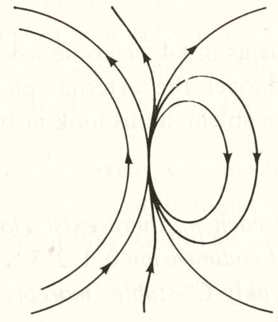
Fig. 7

cod 4

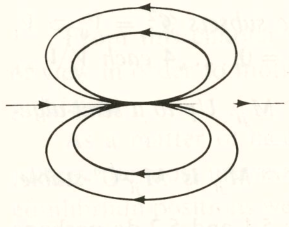
Fig. 8

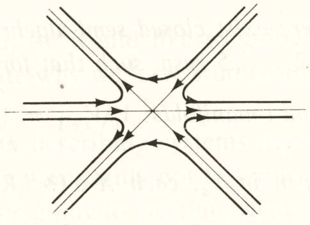
Fig. 9

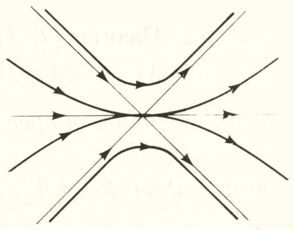
Fig. 10

In fact, the classification in [9] is much finer and reveals also the contact between pairs of characteristic orbits. In codimension 3 for example there appears a singularity of which the phase portrait is as in figure 11.

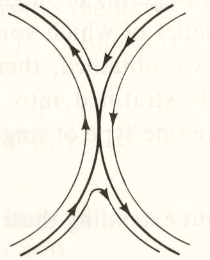
Fig. 11

This singularity is C^0-conjugated with the hyperbolic singularity of figure 2 and hence is not drawed in our previous list. However in the theory of unfoldings (see § 6) it is going to play a role.

The differences playing a role in the unfolding can even be of a more subtle nature; for example, for all b, the singularity $(\mathbb{R}^2, 0, (x + by^2)\,\partial/\partial y + y^2\,\partial/\partial x)$ looks like in fig. 6, but in studying the unfolding there is an essential difference between the cases $b \neq 0$ and $b = 0$.

§5. **Stratifications**

In the same line of ideas as in § 4 we could ask how the different types of singularities are distributed over the external parameterspace.

Before explaining this problem, let us look at two results which are going to clarify what we mean.

5.1. **Theorem** [23]. *For each m, there exist closed semi-algebraic subsets* $\mathscr{G}^m = V_0 \supset V_1 \supset V_2 \supset V_3$ *of codimension* 0, 1, 2, 3 *resp. such that for* $i = 1, 2, 3$ *each* $X \in V_{i-1} \backslash V_i$ *is* V_{i-1}*-weakly-*C^0*-stable; moreover each* $V_{i-1} \backslash V_i$ *is an open codimension* $(i-1)$*-manifold.*

5.2. **Theorem** [9]. *There exist closed semi-algebraic subsets* $\mathscr{G}^2 = V_0 \supset V_1 \supset \ldots \supset V_5$ *of cod.* 0, 1, 2, ..., 5 *resp. such that for* $i = 0, \ldots, 4$ *each* $V_i \backslash V_{i+1}$ *is a finite union of open cod i-manifolds:* $V_i \backslash V_{i+1} = \bigcup_{j=1}^{n_i} M_{ji}$. *Up to a semi-algebraic subset* $R_4 \subset V_4$, *open in* V_4, *each* $X \in (\mathscr{G}^2 \backslash R_4) \cap M_{ji}$ *is* M_{ji}*-*C^0*-stable.*

The manifold-collections used to prove theorems 5.1 and 5.2 do perhaps not define a stratification in the sense of Whitney, but since they are semi-algebraic we know it is possible to refine the manifold collection in order-to obtain a stratification [30].

An implication of these theorems is the following:

If we take 2-parameter families of germs of singularities (or even 3-parameter families in the case of the plane) of which some jet extension [17] is transversal to these stratifications we obtained, then the space of external parameters $\lambda_1, \ldots, \lambda_n$ can be nicely stratified into manifolds such that in each of these manifolds we only have one type of singularity. Moreover, these families are generic [17].

5.3. What can now be said about extending these stratifications to higher codimension.

$m \geq 5$: It is not possible to find an extension $V_0 \supset V_1 \supset \ldots \supset V_4$ such that the relative stability property for weak-C^0-equivalence holds in $V_3 \backslash V_4$ [23]. Here again the same counterexamples apply as in § 3 and § 4.

$m = 4$: The extension can only be finite [22], by the same remarks as for $m \geq 5$.

$m = 2,3$: It is not clear if the relative stability-property is going to give difficulties, but we have troubles of another kind. Arnold in [4] for $m = 3$ and Il'yashenko in [15] for $m = 2$ have proven that it is not possible to extend indefinitely in a semi-algebraic way. In both cases they show the impossibility of the existence of an algebraic decision method for Lyapunov stability.

Conjecture ($m = 2$): There exists a semi-analytic stratification $\mathscr{G}^2 = V_0 \supset V_1 \supset \ldots \supset V_n \supset V_{n+1} \supset \ldots$ with analogous properties as in theorem 5.2, and with $\bigcap_n V_n$ a set of infinite codimension.

§6. **Unfoldings of singularities; motivation and definitions**

From this chapter on, we would like to globalize somewhat our point of view in order to motivate why we have studied these fairly degenerate singularities.

As a matter of fact, in describing systems depending on external parameters, the most interesting is certainly not to know which kind of degenerate equilibrium positions we are going to meet in a generic way, but to understand how the physical aspects of the system, hence essentially the stable attractors are going to change in function of the external parameters. To give a global description of these changements, which is the aim of bifurcation theory, is a very hard problem. However, a first step in this direction consists in analyzing what happens in the neighbourhood of some degenerate singularity. This is the mean intention of the theory of unfoldings. Let us consider the well-known example of the Hopf-bifurcation: the system described by

$$(*) \qquad \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) + (\mu - (x^2 + y^2))\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$$

has a stable singularity in $(x, y) = (0, 0)$ for $\mu < 0$ and has a stable limitcycle in "$x^2 + y^2 = \mu$" for $\mu > 0$.

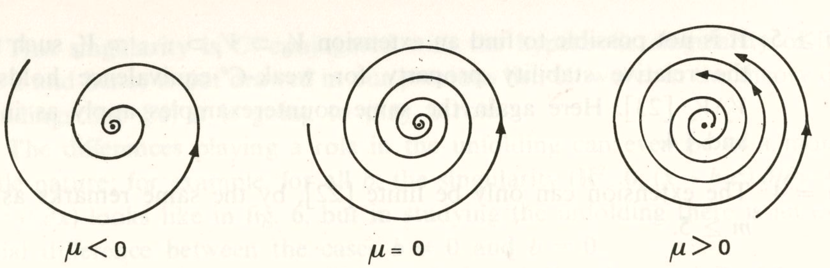

Fig. 12

To understand how the system changes from its stable equilibrium position to a stable periodic motion, we look at its behaviour in the neighbourhood of the degenerate singularity

$$(**) \qquad \left(\mathbb{R}^2 \times \{0\},\ (0,0,0),\ \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) - (x^2+y^2)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\right)$$

We can now prove that generically, an unfolding of the singularity (**) looks topologically always like (*), what we are going to make precise later on.

Hence, this implies that if the singularity (**) occurs in some generic 1-parameter family, then germ of the 1-parameter family in (0, 0, 0) "looks like" (*). This observation is a strong motivation for the following program: first we study the singularities of finite codimension as we have done in § 1 to § 5, and then we look at their unfoldings. For that purpose, instead of working with families of germs, we are now going to consider germs of families or rather germs of unfoldings.

6.1. **Definition.** A C^k unfolding is a quintuple $(\mathbb{R}^n \times \mathbb{R}^m, \pi, \mathbb{R}^n, p, X)$ where $\pi: \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ denotes the projection on the first factor, $\pi(\lambda, x) = \lambda$, $p \in \mathbb{R}^n \times \mathbb{R}^m$, and X is a C^k vector field on $\mathbb{R}^n \times \mathbb{R}^m$ of the form $X = \sum_{i=1}^{m} X_i(\lambda_1, \ldots, \lambda_n, x_1, \ldots, x_m \frac{\partial}{\partial x_i}$ such that $X(p) = 0$.

We also say that $(\mathbb{R}^n \times \mathbb{R}^m, \pi, \mathbb{R}^n, p, X)$ is an unfolding of the singularity $(\pi^{-1}(\pi(p)), p, X \mid \pi^{-1}(\pi(p))$.

A germ of an unfolding is defined similarly as in Def. 1.2 to be an equivalence class in the set of unfoldings. We have of course also a notion of k-jet of an unfolding.

We denote by $\mathcal{G}^{m,n}$ the space of germs of C^∞ unfoldings $(\mathbb{R}^n \times \mathbb{R}^m, \pi, \mathbb{R}^n\ 0, X)$ and by $\mathcal{J}_k^{m,n}$ the space of their k-jets ($k = 0, 1, \ldots, \infty$). The topologies on $\mathcal{J}_k^{m,n}$, $\mathcal{J}_\infty^{m,n}$, and $\mathcal{G}^{m,n}$ are defined as in § 1.

6.2. **Definition** [24]. Suppose we have two unfoldings $(\mathbb{R}^n \times \mathbb{R}^m, \pi_n, \mathbb{R}^n, p, X)$ and $(\mathbb{R}^u \times \mathbb{R}^m, \pi_u, \mathbb{R}^u, q, \gamma)$, then a "$C^i$-admissible pair of mappings" is a pair of C^i-mappings $(F, f): (\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n) \longrightarrow (\mathbb{R}^u \times \mathbb{R}^m, \mathbb{R}^u)$ with the following properties:

(1) $F(p) = q$

(2)
$$\begin{tikzcd} \mathbb{R}^n \times \mathbb{R}^m \arrow[r, "F"] \arrow[d, "\pi_n"] & \mathbb{R}^u \times \mathbb{R}^m \arrow[d, "\pi_u"] \\ \mathbb{R}^n \arrow[r, "f"] & \mathbb{R}^u \end{tikzcd}$$
is a commutative diagram

(3) $\forall x \in \mathbb{R}^n : F \mid \pi_n^{-1}(x): \pi_n^{-1}(x) \longrightarrow \pi_u^{-1}(f(x))$ is a C^i-diffeomorphism.

Note: We speak about a "fiber-C^i-admissible pair of mappings" if we have a pair (F, f) with all the properties as listed above, except that F on itself is only a mapping in the set theoretical sense.

6.3. **Definition.** We say that two unfoldings $(\mathbb{R}^n \times \mathbb{R}^m, \pi, \mathbb{R}^n, p, X)$ and $(\mathbb{R}^n \times \mathbb{R}^m, \pi, \mathbb{R}^n, q, Y)$ are "[fiber] (weakly)-C^i-equivalent" if there exists a [fiber]-C^i-admissible pair of mappings (F, id) from the first to the second, and neighbourhoods U of p and V of q in $\mathbb{R}^n \times \mathbb{R}^m$ with $F(U) = V$, such that $\forall\, y \in \pi(U)$

$F \mid \pi^{-1}(y) \cap U$ is a (weak)-C^i-equivalence between

$X \mid (\pi^{-1}(y) \cap U)$ and $Y \mid (\pi^{-1}(y) \cap V)$.

Remark that for an equivalence, we do not permit changes in the parameter space.

On the other hand, in the next definition, we only change the parameters.

6.4. **Definition.** We say that $(\mathbb{R}^n \times \mathbb{R}^m, \pi_n, \mathbb{R}^n, p, X)$ is "C^i-induced" from $(\mathbb{R}^u \times \mathbb{R}^m, \pi_u, \mathbb{R}^u, q, Y)$ if there exists a neighbourhood U of p in $(\mathbb{R}^n \times \mathbb{R}^m)$ and C^i-mapping $f: \mathbb{R}^n \longrightarrow \mathbb{R}^u$ defined on $\pi_n(U)$ with $f(p) = q$ such that $\forall\, y \in U$: $X(y) = Y((f \times id)(y))$.

Now we can define a very important concept in the theory of unfoldings.

6.5. **Definition.** An unfolding $(\mathbb{R}^N \times \mathbb{R}^m, \pi_N, p, X)$ is called "$(*, C^i)$-versal" if every other unfolding $(\mathbb{R}^n \times \mathbb{R}^m, \pi_n, \mathbb{R}^n, q, Y)$ with germ $(\pi_n^{-1}(\pi_n(q)), q, Y \mid \pi_n^{-1}(\pi_n(q)) =$ germ $(\pi_N^{-1}(\pi_N(p)), p, X \mid \pi_N^{-1}(\pi_N(p)))$ is $*$-equivalent with an unfolding C^i-induced by $(\mathbb{R}^N \times \mathbb{R}^m, \pi_N, \mathbb{R}^N, p, X)$.

The * in this definition runs over all possible equivalences in Def. 6.3 and even other kinds of equivalences using [fiber]-admissible pairs of mappings.

We also say that $(\mathbb{R}^N \times \mathbb{R}^m, \pi_N, \mathbb{R}^N, p, X)$ is the $(*, C^i)$ versal unfolding of the singularity $(\pi_N^{-1}(\pi_N(p)))$.

The reason for this quite complicated notation is that in most of the known examples of versal families, the changes in the parameter space are differentiable although the changes of the phase portraits have less nice features.

Before stating some results concerning unfoldings, we are going to give an important tool in preparing their study, similar to the one given in § 2.

§7. Normal form theorem for unfoldings

Suppose $(\mathbb{R}^n \times \mathbb{R}^m, \pi, \mathbb{R}^n, 0, X)$ is a C^∞ unfolding and let $\tilde{X}_1$ be the linear vector field on $\mathbb{R}^m$ which has the same 1-jet in 0 as $X \mid \{0\} \times \mathbb{R}^m$. Then $Z \longrightarrow [\tilde{X}_1, Z]$ induces a map $[\tilde{X}_1, -]_k : \mathscr{T}^m_k \longrightarrow \mathscr{T}^m_k$ where $\mathscr{T}^m_k$ denotes the space of k-jets of vector fields on $\mathbb{R}^m$ which are not necessarily zero in the origin. We define $\mathscr{B}^k$ to be the image of $[\tilde{X}_1, -]_k$ and we choose same complementary subspace $\mathscr{G}^k$ of $\mathscr{B}^k$ in $\mathscr{T}^m_k$:

7.1. **Theorem** [25]. *Let $X, \tilde{X}_1$ and $\mathscr{G}^i$ be as defined above, then there exists a C^∞ diffeomorphism Ψ: $\mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n \times \mathbb{R}^m$ with $\pi \circ \Psi = \pi$ such that the ∞-jet of $\Psi_*(X) = \tilde{X}$ has the form:*

$$\tilde{X} = \sum_{j=1}^{m} \hat{X}_j(x_1, \ldots, x_m) \frac{\partial}{\partial x^j} + \sum_{\substack{i_1, \cdots i_n \geq 0 \\ \sum_h i_h \geq 1}} \lambda_1^{i_1} \ldots \lambda_n^{i_n} \chi_{i_1 \ldots i_n} + 0(|\lambda|) \cdot 0(|x|^{k+1})$$

where $X \mid \{0\} \times \mathbb{R}^m = \sum_{j=1}^{m} \hat{X}^j \frac{\partial}{\partial x^j}$ *and all* $\chi_{i_1 \ldots i_n} \in \mathscr{G}^k$.

§8. Results concerning unfoldings

8.1. Just as in the case of singularities, but even of greater interest are the type of questions of § 4, i.e.:

Question: For all $n, m > 0$ given, does there exists an open and dense subset $U_{n,m} \subset \chi(\mathbb{R}^n \times \mathbb{R}^m)$ and a finite list of unfoldings $\{\Gamma^j = (\mathbb{R}^{n_j} \times \mathbb{R}^m, \pi, \mathbb{R}^{n_j}, 0, X^j)\}^j$ such that $\forall X \in U_{n,m}$ and $\forall p \in \mathbb{R}^n \times \mathbb{R}^m$ with $X(p) = 0$, the unfolding $(\mathbb{R}^n \times \mathbb{R}^m, \pi, \mathbb{R}^n, p, X)$ is *-equivalent with an unfolding C^i-induced by Γ^j for some j.

If we take * to be weak-C^0-equivalence and $i = 0$, then the question has a negative answer for $m \geq 3$ and $n \geq 2$ [21]. Again the counterexample uses the theory of normal forms (theorem 7.1) in order to show that some symmetry on the 1-jet of $X \mid \pi^{-1}(\pi(0))$ – where $(\mathbb{R}^2 \times \mathbb{R}^3, \pi, \mathbb{R}^2, 0, X)$ denotes the unfolding – can be extended to the ∞-jet of the unfolding, but can be broken up at the germ level in an infinite number of topologically different ways. Such phenomenum occurs in unfolding the codimension 2-singularity whose 2-jet in (x, y, z)-coordinates is:

$$(1 + dz)\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) + cz\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + (a(x^2 + y^2) + bz^2)\frac{\partial}{\partial z}$$

with $a > 0, b > 0, c < 0$.

As a matter of fact, in the unfolding of this singularity we find invariant tori and the restriction to a family of such invariant tori has to behave like the suspension of a 2-parameter family of diffeomorphisms on S^1.

At the same time, the counterexample shows that it is not always possible to find a finite dimensional (weak-C^0, C^0)-versal family, for a finitely determined singularity.

8.2. It is a well-known result, based on the Thom-Mather theory of unfoldings of germs of functions, that on the line all unfoldings are C^0-equivalent (even by means of a C^∞ admissible pair of mappings) with an unfolding C^∞-induced from one of the unfoldings in following list:

$$\left\{X^{(n)} = (x^{n+1} + \lambda_1 x^{n-1} + \ldots + \lambda_{n-1} x + \lambda_n) \frac{\partial}{\partial x}\right\}_{n=0}^{\infty}$$

The bifurcation diagrams are hence exactly the same as for versal unfoldings of singularities of functions [29].

8.3. We prefer not to speak about the reduction to the completely non-hyperbolic part (centrebehaviour), and the associated notion of irreducible unfolding, notions which are well explained in [21] and for which we would have to introduce still more definitions.

Therefore we also refer to [21] for a description of the unfoldings occuring on $\mathbb{R}^m$ $(m \geq 2)$ which are essentially unfoldings of which everything relevant happens on a lower dimensional submanifold, the centremanifold [14]. For more information about extending an equivalence on centremanifolds to an equivalence on the phase space, see [19].

8.4. We shall end this survey by giving two results about unfoldings of singularities on the plane with 2 dimensional centrebehaviour.

A. Generalized Hopf bifurcation

This is the case where the 1-jet has two purely imaginary eigenvalues.

Definition: We call "Standard generalized Hopf bifurcation" an unfolding $(\mathbb{R}^k \times \mathbb{R}^2, \pi, \mathbb{R}^k, 0, X_\pm^{(k)})$, $k = 1, 2, \ldots$, with

$$X_\pm^{(k)} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \pm ((x^2+y^2)^k + \lambda_1(x^2+y^2)^{k-1} + \ldots + \lambda_k)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$$

Theorem [24]: *If $(\mathbb{R}^2, 0, X)$ is a C^∞ singularity of codimension k such that the eigenvalues of $j_1(X)(0)$ are purely imaginary then all C^∞ unfoldings of X are "weak equivalent" with an unfolding C^∞-induced by the standard generalized Hopf bifurcation $X_+^{(k)}$ or $X_-^{(k)}$.*

Remark: "Weak equivalent" here means that there exists a C^∞ admissible pair of mappings (F, id) – let us say from $(\mathbb{R}^k \times \mathbb{R}^2, \pi, \mathbb{R}^k, 0, X)$ to $(\mathbb{R}^k \times \mathbb{R}^2, \pi, \mathbb{R}^k, 0, Y)$ – and a neighbourhood U of $(0, 0)$ in $\mathbb{R}^k \times \mathbb{R}^2$ such that $\forall p \in U$ we have $X(p) = 0$ if and only if $Y(F(p)) = 0$ and the type (sink, saddle, source) is the same, and p lies on an attracting (resp. repelling) closed orbit of X if and only if $F(p)$ lies on an attracting (resp. repelling) closed orbit of Y.
Pictures of the bifurcation diagrams in case $k = 1, 2, 3$ can be found in [24].

3. Unfolding of the "cusp" singularity of codimension 2.

By "cusp" singularity of codimension 2, we mean any C^∞ singularity $(R^2, 0, X)$ such that $j_2(X)(0)$ in normal form is

$$x\frac{\partial}{\partial y} + ay^2\frac{\partial}{\partial x} + by^2\frac{\partial}{\partial y} \qquad a \neq 0,\ b \neq 0 \qquad \text{(see figure 6).}$$

These are the generic singularities between those of which the 1-jet has two zero eigenvalues.

Takens in [25] has proven that such a singularity has a 2-parameter (fiber-C^o, C^∞)-versal unfolding, given by the family

$$x\frac{\partial}{\partial y} + ay^2\frac{\partial}{\partial x} + by^2\frac{\partial}{\partial y} + \lambda_1\frac{\partial}{\partial x} + \lambda_2 y\frac{\partial}{\partial y}$$

Arnold in [5] mentions that Bogdanov has obtained the same result for (C^o, C^o)-versality.

The bifurcation diagram looks as in figure 13, where (1) is a line of saddle-nodes, (2) is a line of Hopfbifurcations and (3) is a line of loops.

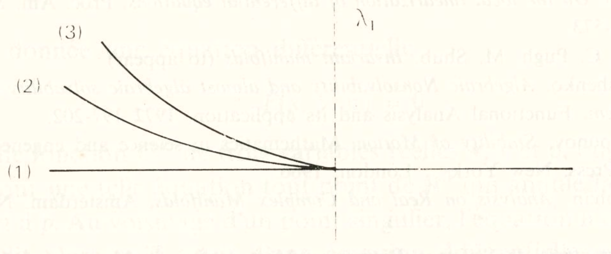

Fig. 13

Nice pictures of the phase portraits in the different regions are provided in [25] and [5].

8.5. The results in this chapter provide a complete list of unfoldings occuring in generic 2-parameter families of C^∞ vector fields on the plane, hence unfoldings of codimension at most 2. Except the unfoldings occuring on the line and the generalized Hopf bifurcations, nothing, as for as I know, has been published about unfoldings on the plane of codimension 3 or higher.

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