# Wave Front Sets, Fourier Integrals and Propagation of Singularities

*Fernando Cardoso*

## Preface

Pseudo-differential operators have been developed as a tool for the study of elliptic differential equations. Suitably extended versions are also applicable to hypoelliptic equations but their value is rather limited in genuinely non-elliptic problems. Many operators arising in the solution of differential equations are not pseudolocal. For instance, if $L$ is a hyperbolic operator, say the wave operator,

$$L = \frac{\partial^2}{\partial t^2} - \Sigma \frac{\partial^2}{\partial x_i^2}$$

the operator $P$ mapping the Cauchy data, $u$ and $\partial u/\partial t$, at time $t = 0$ to their values at time $T$ is not pseudo-local. In [4], Hormander introduced a wider class of operators (the so called Fourier integral operators), no longer pseudo-local, in order to study hyperbolic equations. Pseudo-differential operators as well as the more general Fourier integral operators are intended to make it possible to handle differential operators with variable coefficients roughly as one would handle differential operators with constant coefficients using the Fourier transformation.

It seems clear that still more general operators will play a decisive role in future developments.

This paper is an introduction to some recent ideas which play an important role in the theory of linear partial differential equations, namely the notions of *wave front set of a distribution* and of *Fourier integrals*, and to some of its more beatiful results namely the *propagation of singularities*.

It is aimed at non specialists and at Ph.D. students in Brazil still searching for a subject to write their dissertations.

No proofs are given; they may be found in the references listed in the bibliography.

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## 1. The wave front set of a distribution

The *wave front set of a distribution* is a refinement of the notion of singular support of a distribution.

Let $u \in \mathscr{D}'(X)$, $X$ open in $\mathbb{R}^n$. According to the Paley-Wiener theorem $u$ is $C^\infty$ in a neighborhood of $x$, in other words, $x \notin \text{sing supp } u$, if and only if there exists a neighborhood $U$ of $x$ such that for every $\phi \in C_0^\infty(U)$:

$$\mathscr{F}(\phi u)(\tau\xi) = \langle e^{-i\tau\langle\cdot,\xi\rangle} \phi, u\rangle = 0(\tau^{-N}) \tag{1.1}$$

for $\tau \longrightarrow \infty$, uniformly in $|\xi| = 1$, for all $N$.

It turns out that it is very fruitful not only to localize with respect to $x$ but also with respect to $\xi$ describing not only the location of singularities but also their local harmonic analysis.

This leads to the following definition:

**Definition 1.1** *If* $u \in \mathscr{D}'(X)$, *then the wave front set* $WF(u)$ *of* $u$ *is defined as the complement in* $X\,x(\mathbb{R}^n\backslash\{0\})$ *of the collection of all* $(x_0, \xi^0) \in X\,x(\mathbb{R}^n\backslash\{0\})$ *such that for some neighborhood* $U$ *of* $x_0$, $V$ *of* $\xi^0$ *we have for each* $\phi \in C_0^\infty(U)$ *and each* $N$:

$$\mathscr{F}(\phi\mu)(\tau\xi) = 0(\tau^{-N}) \quad \text{for} \quad \tau \longrightarrow \infty, \quad \text{uniformly in} \quad \xi \in V. \tag{1.2}$$

It is not difficult to see that

$$\text{sing supp } u = \cap\{x, \phi(x) = 0\} \tag{1.3}$$

the intersection being taken over all $\phi \in C^\infty(X)$ with $\phi u \in C^\infty(X)$.

For those familiar with the notion of pseudodifferential operators, we introduce the following equivalent definition of $WF(u)$.

Replacing the function $\phi$ by a pseudodifferential operator $A$ we introduce

$$WF(u) = \bigcap_{Au \in C^\infty} char(A) \tag{1.4}$$

where char $(A)$ is the set of characteristics of $A$.

The definition (1.4) has the advantage of being invariant with respect to change of variables and thus lends itself to defining $WF(u)$ when $X$ is a manifold. (It is also possible to give a variant of Definition 1.1 to obtain a coordinate invariant definition of wave front sets).

**Proposition 1.1** $WF(u)$ *is a closed cone in* $T^*(X)\backslash 0$ *and sing supp* $u = \pi\, WF(u)$.

Here $\pi$ is the bundle projection: $T^*(X) \longrightarrow X$:



A subset $\Gamma \subset T^*(X)\backslash 0$ is called *a cone if*

$$(x, \xi) \in \Gamma \Rightarrow (x, \tau\xi) \in \Gamma \qquad \text{for all} \qquad \tau > 0.$$

**Proposition 1.2** *If* $A$ *is a pseudodifferential operator then*

$$WF(Au) \subset WF(u) \subset WF(Au) \cup char(A). \tag{1.5}$$

The second part, extending the regularity theorem for elliptic operators is obvious; the first part improves the pseudo local property of pseudo-differential operators.

The concept of wave front sets can be used to define a sheaf $\mathscr{S}$ on $S^*(X)$, the cosphere bundle of $X$, which is analogous to the sheaf $\mathscr{C}$ of Sato in the category of hyperfunctions. Let $U$ be an open subset of $S^*(X)$, which can also be regarded as conic open subset of $T^*(X)\backslash 0$. Call two distributions $u_1$, $u_2$ on $\mathscr{D}'(X)$ *equivalent over* $U$, notation $u_1 \equiv u_2$ in $U$, if $WF(u_1 - u_2) \cap U = \varnothing$. The equivalence classes with respect to this equivalence relation form a space $\mathscr{S}(U)$ and we have a natural mapping $\rho_{u,u'}: \mathscr{S}(U) \longrightarrow \mathscr{S}(U')$ if $U' \subset U$. One can prove that the $\mathscr{S}(U)$ together with the "restriction mappings" $\rho_{u,u'}$ form a presheaf and hence define a sheaf over $S^*(X)$. Sections of this sheaf $\mathscr{S}$ over the whole of $S^*(X)$ are naturally identified with elements of $\mathscr{D}'(X)/C^\infty(X)$ and the support of such a section is equal to the wave front set of the corresponding distribution. Historically, Sato first defined his sheaf $\mathscr{C}$ over $S^*(X)$, $X$ a real analytic monifold. Global sections of $\mathscr{C}$ correspond with hyperfunctions on $X$ modulo real-analytic functions. For the supports of global sections he derived properties analogous to those for the wave front sets. This inspired Hormander to his definition of wave front sets (and the sheaf $\mathscr{S}$).

We shall now list a number of properties of wave front sets.

**Definition 1.2** *Let* $\Gamma$ *be a closed cone in* $T^*(X)\backslash 0$.

*Define* $\mathscr{D}'_\Gamma(X) = \{u \in \mathscr{D}'(X);\ WF(u) \subset \Gamma\}$. *In* $\mathscr{D}'_\Gamma(X)$ *we take the topology defined by the seminorms of the weak topology in* $\mathscr{D}'(X)$ *together with the seminorms given by taking the best possible constants in* (1.2), *where* supp $\phi$ *is contained in a coordinate neighborhood and* $\mathscr{F}(\phi u) =$ *Fourier transform of* $\phi u$ *in the corresponding local coordinates.*

It is not difficult to show that $C^\infty(X)$ is sequentially dense in $\mathscr{D}'_\Gamma(X)$.

**Proposition 1.3** *Let* $X$, $Y$ *be* $C^\infty$ *manifolds,* $\Phi$ *a* $C^\infty$ *mapping:* $X \longrightarrow Y$; *denote by*

$$N_\Phi = \{(y, \eta) \in T^*(Y)\backslash 0;\ y = \Phi(x),\ {}^tD\Phi_x\eta = 0 \quad \text{for some} \quad x \in X\}.$$

*Let* $\Gamma$ *be a closed cone in* $T^*(Y)\backslash 0$ *such that* $L \cap N_\Phi = \varnothing$. *Then the pull-back* $\Phi^*: C^\infty(Y) \rightarrow C^\infty(X)$ *has a unique continuous extension:* $\mathscr{D}'_\Gamma(Y) \rightarrow$

$\mathscr{D}'(X)$, *and supp* $(\Phi^*v) \subset \Phi^{-1}$ *(supp v) for each* $v \in \mathscr{D}'_\Gamma(Y)$. *Moreover, if* $\tilde{\Gamma} = \{(x, \xi) \in T^*(X)\backslash 0;\ \exists\eta : {}^tD\Phi_x\eta = \xi,\ (\Phi(x), \eta) \in \Gamma\}$, *then* $\Phi^*$ *is in fact continuous*: $\mathscr{D}'_\Gamma(Y) \longrightarrow \mathscr{D}'_{\tilde{\Gamma}}(X)$.

Note that the pullback $\Phi^*u$ is defined for all $u \in \mathscr{D}'(X)$ precisely when $\Phi'$ is surjective, and then it is well known that such a definition is possible. In particular we see that if $Y \subset X$ is a submanifold, we can define the restriction of $u$ to $Y$ if the normal bundle $N(Y)$ does not meet $WF(u)$. For example, if $u \in \mathscr{D}'(X)$ and $Au \in C^\infty$ for some pseudo-differential operator $A$, we can define the restriction of $u$ to $Y$ if $Y$ is non characteristics, that is, the normals to $Y$ are non-characteristics with respect to $A$. This is also a well known fact (partial hypoellipticity).

**Proposition 1.4** *The push-forward* $\Phi_* = {}^t(\Phi^*)$ *is a continuous mapping from the space of* $u \in \mathscr{D}'(X)$ *such that* $\Phi$ : *supp u* $\longrightarrow Y$ *is a proper mapping, into* $\mathscr{D}'(Y)$. *For such u we have*:

$$WF(\Phi_*u) \subset \{(y, \eta) \in T^*(Y) \setminus 0;\ y = \Phi(x) \text{ and } (x, {}^tD\Phi_x\eta) \in WF(u) \text{ for some } x \in X\}$$

**Proposition 1.5** *If* $u \in \mathscr{D}'(X)$, $v \in \mathscr{D}'(Y)$ *then*

$$WF(u \otimes v) \subset (WF(u) x\, WF(v)) \cup (WF(u) x\, supp_0 v) \cup (supp_0\, u x WF(v)).$$

Here $supp_0 u = \{(x, 0) \in T^*(X);\ x \in supp\, u\}$ and analogously

$$supp_0 v = \{(y, 0) \in T^*(Y);\ y \in supp\, v\}.$$

We shall now study the multiplication of distributions $u_1$ and $u_2$. Let $\psi \in C_0^\infty(R^n)$, $\int \psi\, dx = 1$ and set $\psi_\varepsilon(x) = \varepsilon^{-n}\, \psi(x/\varepsilon)$. Assuming that $u_j \in \mathscr{E}'(R^n)$ we wish to define $u_1u_2$ as the limit of $(u_1 * \psi_\varepsilon)(u_2 * \psi_\varepsilon)$ as $\varepsilon \longrightarrow 0$. In general this is not possible but the limit does exist (and it is then independent of the choice of coordinates and $\psi$) if

$$(1.6) \qquad WF(u_1) + WF(u_2) = \{(x, \xi_1 + \xi_2);\ (x, \xi_j) \in WF(u_j)\} \subset T^*(X)\backslash 0.$$

Noting that $u_1u_2 = \Delta^*(u_1 \otimes u_2)$ where $\Delta$: $X \longrightarrow X\, x\, X : x \longrightarrow (x, x)$ is the diagonal map, Propositions 1.3 and 1.5 imply:

**Proposition 1.6** *Let* $\Gamma_1$, $\Gamma_2$ *be closed cones in* $T^*(X)\backslash 0$ *such that* $\Gamma_1 + \Gamma_2 = \{(x, \xi_1 + \xi_2);\ (x, \xi_j) \in \Gamma_j\}$ *does not meet the zero section in* $T^*(X)$. *Then there is a unique continuous mapping*: $\mathscr{D}'_{\Gamma_1}(X) x\, \mathscr{D}'_{\Gamma_2}(X) \rightarrow \mathscr{D}'(X)$ *extending the product* $(u_1, u_2) \longrightarrow u_1 \cdot u_2 : C^\infty(X) x\, C^\infty(X) \longrightarrow C^\infty(X)$. *Moreover* $(\Gamma_1 + \Gamma_2) \cup \Gamma_1 \cup \Gamma_2$ *is a closed cone in* $T^*(X) \setminus 0$ *and the product is in fact continuous*: $\mathscr{D}'_{\Gamma_1}(X) x\, \mathscr{D}'_{\Gamma_2}(X) \longrightarrow \mathscr{D}'_{(\Gamma_1+\Gamma_2)\cup\Gamma_1\cup\Gamma_2}(X)$.



Next we consider the linear transformation defined by a distribution $K \in \mathscr{D}'(XxY)$ where $X \subset R^n$, $Y \subset R^m$ are open sets (The results have an obvious extension to manifolds if one works troughout with densities of orders 1/2). Then by the kernel theorem of L. Schwartz we can identity $\mathscr{D}'(X x\, Y)$ with the space of continuous linear operators $C_0^\infty(Y) \longrightarrow \mathscr{D}'(X)$ by means of the formula

$$\langle A\phi, \psi \rangle = K_A(\psi \otimes \phi);\ \phi \in C_0^\infty(Y),\ \psi \in C_0^\infty(X);$$

**Theorem 1.1** *For any* $u \in C_0^\infty(Y)$ *the set*

$$WF_X(A) = \{(x, \xi);\ (x, \xi, y, 0) \in WF(K_A) \quad \text{for some} \quad y \in Y\}$$

*contains* $WF(Au)$. *Thus A maps* $C_0^\infty(Y)$ *into* $C^\infty(X)$ *if* $WF_X(A) = \emptyset$, *that is, if* $WF(K_A)$ *contains no point which is normal to a manifold* $x =$ *constant.*

An essential dual question concerns the definition of $Au$ for general distributions $u$. First note that if $u \in \mathscr{D}'(Y)$ then $WF(1 \otimes u) = X x\, WF(u)$. The product $K_A(1 \otimes u)$ is therefore well defined when $WF(K_A) + (X x\, WF(u))$ does not meet the zero section, that is, $WF(u)$ does not meet

$$\{(y, \eta);\ (x, 0, y, -\eta) \in WF(K_A) \quad \text{for some} \quad x\} = WF'_Y(A)$$

When $u \in \mathscr{E}'_\Gamma(Y)$ for some $\Gamma$ not meeting $WF'_Y(A)$ the product depends continuously on $u$ and so does the integral with respect to $y$. This we define to be $Au$. Explicitly

$$\langle Au, \phi \rangle = \langle K_A(1 \otimes u),\ \phi \otimes 1 \rangle, \quad \phi \in C_0^\infty(X).$$

Therefore we have:

**Theorem 1.2.** *A can be extended to a continuous map* $\mathscr{E}'_\Gamma(Y)$ *to* $\mathscr{D}'(X)$ *when* $\Gamma$ *does not meet* $WF'_Y(A)$. *In particular, when the set* $WF'_Y(A)$ *is empty we have a continuous map* $\mathscr{E}'(Y) \longrightarrow \mathscr{D}'(X)$.

If we have three open sets, $X \subset R^n$, $Y \subset R^m$, $Z \subset R^p$ and distributions $K_A \in \mathscr{D}'(X \times Y)$, $K_B \in \mathscr{D}'(Y \times Z)$ where for simplicity we assume that $A$ and $B$ are properly supported, then $Bu \in \mathscr{E}'(Y)$ and $WF(Bu) \subset WF_Y(B)$ when $u \in C_0^\infty(Z)$. The composition $A(Bu)$ is therefore defined if

$$(1.7) \qquad WF'_Y(A) \cap WF_Y(B) = \emptyset,$$

and it is of the form $(K_A \circ K_B)u$ where $K_A \circ K_B \in \mathscr{D}'(X \times Z)$. It is convenient to introduce the following notation:

$$(1.8) \qquad WF'(A) = \{(x, y, \xi, \eta),\ (x, y, \xi, -\eta) \in WF(K_A)\}.$$

If we note that $K_{A\circ B} = \pi_* \Delta^*(K_A \otimes K_B)$, where

$$\Delta: (x, y, z) \longrightarrow (x, y;\ y, z) : X \times Y \times Z \longrightarrow X \times Y \times Y \times Z \text{ and}$$

$$\pi: (x, y, z) \longrightarrow (x, z) : X \times Y \times Z \longrightarrow X \times Z, \quad \text{we obtain}$$

from propositions 1.3, 1.4 and 1.5

**Theorem 1.3.** *When (1.7) is fulfilled we have:*

$$WF'(A \circ B) \subset WF'(A) \circ WF'(B) \cup (WF_X(A) \times 0_{T^*(Z)}) \cup \cup (0_{T^*(X)} \times WF'_Z(B)). \tag{1.9}$$

*Here $WF'(A)$ and $WF'(B)$ are composed as relations from $T^*(Y)$ to $T^*(X)$ and from $T^*(Z)$ to $T^*(Y)$.*

**Remark 1.1:** *If $R_1 \subset U \times V$, $R_2 \subset V \times W$ are relations then the composition $R_1 \circ R_2 \subset U \times W$ is defined by*

$$R_1 \circ R_2 = \{(u, w) \in U \times W;\ \exists\, v \in V: (u, v) \in R_1 \quad \text{and} \quad (v, w) \in R_2\} \tag{1.10}$$

The special case when $Z$ reduces to a point is worth special notice:

**Theorem 1.4:** *Let $K_A \in \mathscr{D}'(X \times Y)$ and $u \in \mathscr{E}'(Y)$, $WF(u) \cap WF'_Y(A) = \phi$. Then we have*

$$WF(Au) \subset (WF'(A) \circ WF(u)) \cup WF_X(A) \tag{1.11}$$

*where again $WF'(A)$ is interpreted as a relation mapping sets in $T^*(Y)$ to sets in $T^*(X)$.*

**Remark 1.2:** *We shall see in Section 2 that if $X = Y$ and $A$ is a pseudodifferential operator in $X$ then both $WF_X(A)$ and $WF'_X(A)$ are empty and $WF'(A)$ is the identity relation.*

Remark 1.2 and (1.11) imply that if $A$ is a pseudodifferential operator in $X$, then

$$WF(Au) \subset WF(u) \tag{1.12}$$

which is the first part in (1.5).

## 2A. Distributions defined by Fourier integrals.

Formally, the distribution kernel of a pseudodifferential operator $A = a(x, D)$ associated to a symbol $a(x, \theta)$ is given by

$$(x, y) \longrightarrow (2\pi)^{-n} \int e^{i\langle x - y, \theta\rangle} a(x, \theta)\, d\theta \tag{2.1}$$



Similarly the fundamental solution of the wave equation $\partial^2 u/\partial t^2 - \Delta u = 0$ in $n$ space variables ($n > 1$) with pole at $(y, 0)$ is at time $t > 0$ given by

$$(x, y) \longrightarrow (2\pi)^{-n}\left(\int e^{i(\langle x - y, \theta\rangle + t|\theta|)} (2i\,|\theta|)^{-1}\, d\theta - \int e^{i(\langle x - y, \theta\rangle - t|\theta|)} (2i\,|\theta|)^{-1}\, d\theta\right). \tag{2.2}$$

These examples suggest the importance of the classes of distributions which we shall study.

Let $X \subset R^n$ and let $\Gamma$ be an open cone in $X \times (R^N \setminus 0)$ for some $N$.

Assume given a real-valued fuction $\phi \in C^\infty(\Gamma)$ satisfying the following conditions:

(i) $\phi$ is positively homogeneous of degree 1 with respect to the variables in $R^N$.

(ii) $d\phi \neq 0$ everywhere in $\Gamma$.

Such a function will be called a *phase* function. Let $S_0^m(\Gamma)$ be the set of all $a \in S^m(X \times R^N \setminus 0)$ vanishing in a conic neighborhood of $C\Gamma$. We recall that $S^m(X \times R^N \setminus 0) = a \in C^\infty(X \times R^N)$ such that for every compact set $K \subset X$ and all multiorders $\alpha, \beta$ the estimate

$$|D_x^\beta D_\theta^\alpha a(x, \theta)| \leq C_{\alpha.\beta.K}(1 + |\theta|^{m-|\alpha|},\ x \in K,\ \theta \in R^N,$$

is valid for some constant $C_{\alpha.\beta.K}$.

For $a \in S_0^m(\Gamma)$ we claim that the "integral"

$$A(x) = \int e^{i\phi(x,\theta)} a(x, \theta)\, d\theta \tag{2.3}$$

can be defined, not necessarily as a function of $x$ but as a distribution in $X$. To do so we consider the linear form

$$I_\phi(au) = \iint e^{i\phi(x,\theta)} a(x, \theta)\, u(x)\, dx d\theta, \quad u \in C_0^\infty(X). \tag{2.4}$$

In view of (ii) the fact that $e^{i\phi} = D(e^{i\phi})/Di\phi$ allows one, by successive (formal) partial integrations with no boundary terms, to reduce the growth of the integrand at infinity until it becomes integrable. This gives a precise definition of $I_\phi(au)$ and the linear form $u \longrightarrow I_\phi(au)$ is then a distribution $A \in \mathscr{D}'(X)$. We shall call (2.3) an oscillatory integral but use the standard notation.

**Theorem 2.1:** *The mapping $a \longrightarrow I_\phi(au)$, defined for symbols $a$ which vanish for large $|\theta|$, can for every $u \in C_0^\infty(X)$ be extended to $S^\infty(X \times R^N \setminus 0) = \bigcup_{m \in R} S^m(X \times R^N \setminus 0)$ such that it is continuous on $S^m(X \times R^N \setminus 0)$ for every $m$. Moreover, for every $a \in S^m(X \times R^N \setminus 0)$ the linear form $A: u \longrightarrow I_\phi(au)$ is a distribution of order $k$ if $m - k + N < 0$.*

Using the *method of stationary phase* to investigate the asymptotic behavior of integrals of the form (2.4) we obtain:

**Theorem 2.2:** $WF(A) \subset \{(x, d_x\phi(x,\theta)); (x,\theta)\in\Gamma, d_\theta\phi(x,\theta) = 0\} \subset$ $\subset T^*(X)/0$.

As an example, we see from (2.1) that the wave front set of the kernel of a pseudodifferential operator $A$ in $X$ lies in $\{(x, y, \xi, \eta); x = y, \xi = -\eta\}$ which is the normal bundle of the diagonal $\Delta$ in $X \times X$. This proves Remark 1.2.

**Remark 2.1:** *If $A$ is a pseudodifferential operator in $X$, and if $K_A$ is the kernel of $A$, we can identify $WF(K_A)$ (by the projection of $T^*(X) \times$ $\times T^*(X) \longrightarrow T^*X$ on the first factor) with a closed cone in $T^*(X)\backslash 0$ which we denote by $WF(A)$.*

If $A$ is properly supported we obtain:

**Proposition 2.1:** *The complement of $WF(A)$ is the largest open cone in $T^*(X)\backslash 0$ where $\sigma_A$ (the symbol of $A$) is rapidly decreasing.*

**Proposition 2.2:** *If $u \in \mathscr{D}'(X)$ we have $Au \in C^\infty$ for all pseudodifferential operators $A$ with*

(2.5) $$WF(A) \cap WF(u) = \emptyset.$$

**Remark 2.2:** *A sequence $u_j \in \in \mathscr{D}'_\Gamma(X)$ (see Def. 1.2) converges to $u \in \mathscr{D}'_\Gamma(X)$ if*

i) *$u_j \longrightarrow u$ in $\mathscr{D}'(X)$ (weakly)*
ii) *$Au_j \longrightarrow Au$ in $C^\infty(X)$ if $A$ is a properly supported pseudodifferential operator with $\Gamma \cap WF(A) = \emptyset$.*

As a second example we see that for the two terms in (2.2) the wave front set lies in the set where $x - y = \pm t\,\theta/|\theta|$ and $\xi = -\eta = \theta$. This corresponds to the two components of the normal bundle of $\{(x, y); |x - y|^2 = t^2\}$. In particular the singularities are carried by the light cone.

The notion of wave front set can be used to localize various spaces of distributions not only in $X$ but also in $T^*(X)\backslash 0$. In particular, L. Hormander has shown how to define global spaces of distributions which microlocally (i.e locally in $T^*(X)$) have representations such as (2.3).

We shall restrict ourselves to the case where $\phi$ is *non degenerate*, that is, the differentials of the functions $\partial\phi/\partial\theta_j$ are linearly independent in $C_\phi = \{(x,\theta)\in\Gamma;\ \phi'_\theta(x,\theta) = 0\}$.

Application of the implicit function theorem implies that $C_\phi$ is a conic $C^\infty$ manifold of $X \times R^N\backslash 0$ of dimension $(n + N) - N = n = \dim X$.



**Lemma 2.1:** *If $\phi$ is a nondegenerate phase function, then*

(2.6) $$T^{(\phi)} : (x, \theta) \longrightarrow (x, d_x\phi(x,\theta))$$

*is an immersion: $C_\phi \longrightarrow T^*(X)\backslash 0$, commuting with the multiplication with positive real numbers in the fibers. So its image $\Lambda_\phi$ is an immersed n-dimensional conic submanifold of $T^*(X)\backslash 0$.*

Let $(x, \xi)$ denote the standard coordinates in $T^*(X)$ obtained from local coordinates $x_1, \ldots, x_n$ in $X$ by taking $dx_1, \ldots dx_n$ as basis for the cotangent vectors. The form $\Sigma\xi_j dx_j$ is invariant defined in $T^*(X)\backslash 0$ and its restriction to $\Lambda_\phi$ is $\phi'_x dx = d\phi - \phi'_\theta d\theta = 0$ since $\phi'_\theta = 0$ on $C_\phi$ and so $\phi = \langle\theta, \phi'_\theta\rangle = 0$ on $C_\phi$, by Euler's identity. In view of the homogeneity this is equivalent to the vanishing on $\Lambda_\phi$ of the differential which is the symplectic form $\sigma = \Sigma d\xi_j \wedge dx_j$. Thus $\Lambda_\phi$ is a manifold of maximal dimension on which the symplectic two form of $T^*(X)$ vanishes. We shall call such a manifold *Lagrangean*, following Maslov; they play a fundamental role in the classical integration theory of first order differential equations. Locally the class of distributions which can be written in the form (2.3) for some $a \in S_0{}^{m+n/4-N/2}(\Gamma)$, $n = \dim X$, and a nondegenerate real phase function $\phi$, depends only on the Lagrangean manifold $\Lambda_\phi$ corresponding to $\phi$ and on no other properties of this function. Any closed conic Lagrangean submanifold $\Lambda \subset T^*(X)\backslash 0$ can locally be represented as the range of a map (2.6). We can therefore define a space $I^m(X, \Lambda)$ of distributions with wave front set in $\Lambda$ which locally can be written in the form (2.3) with $a \in S_0{}^{m+n/4-N/2}$ and $\phi$ defining a part of $\Lambda$ according to (2.6). With the elements in $I^m(X, \Lambda)$ one can, as for pseudodifferential operators, associate principal symbols on $\Lambda$, which are symbols of order $m + n/4$ modulo symbols of order $m + n/4 - 1$ (with value in certain line bundles). For the kernels of pseudodifferential operators in $X$ which are associated with the normal bundle of the diagonal $X \times X$ this agrees with the standard notion of principal symbol. (Notions such as characteristic points can therefore also be defined). When we take a conic Lagrangean submanifold of $T^*(X \times Y)\backslash 0$ where $X$ and $Y$ are two manifolds we can interpret the distributions in $I^m(X \times Y, \Delta)$ as maps from $C_0^\infty(Y)$ to $\mathscr{D}'(X)$. These maps are called *Fourier integral operators.*

**Examples:**

1) Fourier integrals (or oscillatory integrals) in $X$ can be regarded as Fourier integral operators by taking $Y = \{\text{point}\}$.

2) Let $h$ be a differentiable function: $X \longrightarrow Y$. Then $(u \circ h)(x) = (2\pi)^{-n} \iint e^{i\langle h(x)-y,\eta\rangle} u(y)\,dy\,d\eta$, and it follows that $h^*: C^\infty(Y) \longrightarrow C^\infty(X)$ is a Fourier integral operator defined by a nondegenerate phase function $\phi$ such that

$$\Lambda'_\phi = \{((x,\xi),(y,\eta));\ y = h(x),\ \xi = {}^tDh_x\eta\}.$$

If $h$ is a diffeomorphism then $\Lambda'_\phi$ is the graph of the induced transformation $\tilde{h}: T^*(X)\backslash 0 \longrightarrow T^*(Y)\backslash 0$ defined by $\tilde{h}(x,\xi) = (h(x), ({}^tDh_x)^{-1}(\xi))$. If $X$ is a submanifold of $Y$, $\dim X < \dim Y$ and $h$ is the identity: $X \longrightarrow Y$ then $h^*$ is the restriction operator $\rho: C^\infty(Y) \longrightarrow C^\infty(X)$. In this case

$$\Lambda'_\phi = \{((x,\xi),(y,\eta));\ y = x,\ \xi = \eta|_{Tx(X)}\}$$

which is far from the graph of a map. The operator $\rho$ can be extended continuously to $\mathscr{D}'_\Gamma(Y)$ for any closed cone $\Gamma$ in $T^*(Y)\backslash 0$ which does meet the set

$$\{(y,\eta) \in T^*(Y)\backslash 0;\ y \in X,\ \eta|_{Tx(X)} = 0\},$$

that is the *normal bundle* in $T^*(Y)\backslash 0$ of the *submanifold* $X$.

3) *Pseudodifferential operators in* $X$ are defined as Fourier integral operators with $T = X$ and $\Lambda'_\phi \subset$ diagonal in $T^*(X)\backslash 0 + T^*(X)\backslash 0$ = graph of the identity: $T^*(X)\backslash 0 \longrightarrow T^*(X)\backslash 0$. When $\Lambda \subset (T^*(X)\backslash 0) \times (T^*(Y)\backslash 0)$ we have seen (Theorems 1.1 and 1.2) that the corresponding Fourier Integral operators are actually continuous operators from $C_0^\infty(Y)$ to $C^\infty(X)$ and from $\mathscr{E}'(X)$ to $\mathscr{D}'(Y)$. The set

$$\Lambda' = \{(x,\xi,y,-\eta);\ (x,\xi,y,\eta) \in \Lambda\}$$

will then be called a homogeneous *canonical relation*; it is Lagrangean with respect to the symplectic form $\sigma_X - \sigma_Y$. This is the set which occurs in the multiplicative properties of wave front sets described in Theorem 1.3. If we have three manifolds $X$, $Y$, $Z$ and canonical relations $C_1$, $C_2$ from $T^*(Y)$ to $T^*(X)$ resp. $T^*(Z)$ to $T^*(Y)$ one can supplement Theorem 1.3 by proving that the composition $A \circ B$ of properly supported operators $A \in I^{m_1}(X \times Y, C'_1)$ and $B \in I^{m_2}(Y \times Z, C'_2)$ is in $I^{m_1+m_2}(X \times Z, (C_1 \circ C_2)')$ if the appropriate transversality and other conditions are fulfilled which guarantee that $C_1 \circ C_2$ is a manifold. There is a simple formula giving the principal symbol of $A \circ B$ as a product of those of $A$ and $B$ (the normalization of the degree for the operators in $I^m$ was chosen precisely to make the preceding statement valid). We finish this section considering



an important special case due to Yu. V. Egorov which gave rise to much of the work described here. Thus assume that $X$ and $Y$ have the same dimension and that $\Lambda'$ is the graph of a homogeneous *canonical transformation* $\chi$ from $T^*(Y)$ to $T^*(X)$. That $\chi$ is canonical means that $\chi^*\sigma_X - \sigma_Y = 0$ or that $\sigma_X - \sigma_Y$ vanishes on $\Lambda'$, so we have a canonical relation in the sense explained above. If $K \in I^m(X \times Y, \Lambda)$, then the adjoint $K^*$ belongs to the inverse transformation and the compositions $KK^*$ and $K^*K$ belong to the identity, that is, they are pseudodifferential operators in $X$ and $Y$ respectively. If $A$ is a pseudodifferential operator in $X$ of order $\mu$ then the product $AK$ is in $I^{m+\mu}(X \times Y, \Lambda)$ and the principal symbol is the product of the principal symbol of $K$ (considering as living on $\Lambda'$) by that of $A$ lifted from $T^*(X)$ to $\Lambda'$ by the projection $\Lambda' \longrightarrow T^*(X)$. If we multiply to the right instead the result is the same except that we shall use the projection from $\Lambda'$ to $T^*(Y)$. If $A$ and $B$ are pseudodifferential operators in $X$ and in $Y$ respectively and if $AK = KB$ we conclude that for the principal symbols $a$ and $b$ of $A$ and $B$ we must have

$$a(\chi(y,\eta)) = b(y,\eta) \tag{2.7}$$

if the pricipal symbol of $K$ is not zero (i.e if $K$ is elliptic) at $(\chi(y,\eta),(y,-\eta))$. Conversely, (2.7) implies that $AK - KB$ is of lower order. We can therefore successively construct the symbol of $B$ for a given $A$ so that $AK$-$KB$ is of order $-\infty$, provided that the wave front set of $A$ is concentrated near a point where $K$ is elliptic. This argument often allows one to pass from one operator to another with principal symbol modified by a homogeneous canonical transformation and this was a crucial point in the study of local solvability of pseudodifferential equations.

## 3. Propagation of singularities

Let $P$ be a properly supported pseudodifferential operator of order $m$ in a manifold $X$ with homogeneous principal symbol $p$. This means that $p$ is a complex valued $C^\infty$ homogeneous function of degree $m$ on $T^*(X)\backslash 0$ and that for every local coordinate system the full symbol of $P$ differs from $p$ by a symbol in $S^{m-1}$. We shall also require that the characteristics are simple, that is,

$$d_\xi p(x,\xi) \neq 0 \quad \text{if} \quad (x,\xi) \in T^*(X)\backslash 0 \quad \text{and} \quad p(x,\xi) = 0. \tag{3.1}$$

We are interested in the following question:

(3.2) *Given* $(x_0,\xi^0) \in T^*(X)\backslash 0$ *find the biggest subset* $C_{(x_0,\xi^0)} \subset T^*(X)\backslash 0$ *such that the following is true*:

$$\forall u \in \mathscr{D}'(X) \quad \text{such that} \quad (x_0, \xi^0) \notin WF(Pu),$$

we have

$$(x_0, \xi^0) \in WF(u) \Rightarrow C_{(x_0, \xi^0)} \subset WF(u).$$

In the special case where $p$ is *real* the answer is given by

**Theorem 3.1.** (J. J. Duistermaat and L. Hormander) *If $u \in \mathscr{D}'(X)$ and $Pu = f$ it follows that $WF(u) \setminus WF(f)$ is a subset of $p^{-1}(0)$ which is invariant under the flow defined by the Hamilton vector field $H_p$ in $p^{-1}(0) \setminus WF(f)$.*

The fact that $WF(u) \setminus WF(f) \subset p^{-1}(0)$ is precisely the second part of (1.5). The tangent vector $H_p$ to $T^*(X)$ corresponds to the covector $dp$ by the definition: $\langle t, dp \rangle = \sigma(t, H_p)$, $t \in T(T^*(X))$ where $\sigma$ is the symplectic form. In terms of local coordinates $x$ in $X$ and the corresponding coordinates $(x, \xi)$ in $T^*(X)$ the Hamiltonian vector is given by:

$$H_p = \Sigma \left( \frac{\partial p}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j} \frac{\partial}{\partial \xi_j} \right).$$

More generally, it is conjectured that for operators satisfying the local solvability condition $(\psi)$ (grosso modo and microlocally, if Imp is $< 0$ at some point along a null bicharacteristic strip of *Rep*, it remains $\leq 0$ from that point on along $\Gamma$; null bicharacteristic strips of Rep are integral curves of the Hamiltonian field $H_{Rep}$ contained in $(Rep)^{-1}(0)$. They are oriented curves) the set $C(x_0, \xi^0)$ is the following:

Let $z \in \mathbb{C}$ such that $d_\xi(Rezp) \neq 0$ at $(x_0, \xi^0)$. Let $\Gamma_z$ be the bicharacteristic strip of *Rezp* through $(x_0, \xi^0)$ and $\Gamma_z^0$ the greatest closed interval of $\Gamma_z$ containing $(x_0, \xi^0)$ on which $p$ vanishes identically (Note that $\Gamma_z^0$ may reduce to the point $(x_0, \xi^0)$ or even be empty, in which case $p(x_0, \xi^0) \neq 0$). Then

$$C_{(x_0, \xi^0)} = \bigcup_{\substack{z \in \mathbb{C} \\ d_\xi(Rezp)(x_0, \xi^0) \neq 0}} \Gamma_z^0 \tag{3.3}$$

Besides Theorem 3.1, the other cases which tend to confirm (3.3) are:

1) $p(x, \xi)$ complex but $d(Rep)$ and $d(Imp)$ linearly independent in $(x_0, \xi^0)$ (in this case $C_{(x_0, \xi^0)}$ is a two dimension (over $\mathbb{R}$) regular surface on which one can introduce a complex analytic structure where the Hamiltonian of $p$ plays the role of $\partial/\partial z$).

2) The following condition is satisfied:

(R) *Assuming $p(x_0, \xi^0) = 0$, $d_\xi(Rezp)(x_0, \xi^0) \neq 0$, the function Imzp does not change sign in a neighborhood of $(x_0, \xi^0)$ in the hypersurface (in $T^*(X) \setminus 0$) $Rezp(x, \xi) = 0$.*

Under hypothesis $(R)$ F. Treves constructed a suitable parametrix for $P$ from which one can derive the propagation of singularities.

3) $P$ has constant coefficients (In this case, as in the previous when the data are analytic, one may replace the wave front set by the *analytic wave front set*).

In the cases 2) and 3), the "propagator" is not necessarily a regular surface.

We shall now drop the assumption (3.1) and give the following definitions:

**Definition 3.1** *$P$ is said to be of constant multiplicity if p factorizes as*

$$p = q_1^{r_1} \dots q_s^{r_s} \quad \text{with} \quad r_j \in N,$$

*where $q_j$, $j = 1, \dots s$, satisfy (3.1) and are such that $q_j^{-1}(0)$ are disjoint in $T^*(X) \setminus 0$.*

**Definition 3.2** *Assume that p is real and that P has constant multiplicity. P is said to verify the Lévi condition $L_{(x_0, \xi^0)}$ at the point $(x_0, \xi^0) \in p^{-1}(0)$ $\subset T^*(X) \setminus 0$, if for every phase function $\phi(x)$, solution of the equation*

$$q_j(x, d\phi(x)) = 0 \quad (\textit{if } j \textit{ is such that } q_j(x_0, \xi^0) = 0)$$

*in a neighborhood of $x_0$ with $d\phi(x_0) = \xi^0$ and for all amplitude $a \in C_0^\infty(X)$ with support in a neighborhood of $x_0$ where $d\phi \neq 0$, we have*

$$e^{-it\phi} P(a\, e^{it\phi}) = 0(t^{m - r_j}), \qquad t \longrightarrow +\infty.$$

*The operator P verifies the Lévi condition (L) if $L_{(x_0, \xi^0)}$ is satisfied at every point $\in p^{-1}(0)$.*

One can easily verify that $(L)$ is a condition on the terms of degree $m - (\bar{r} - 1)$, where $\bar{r} = \max r_j$; in particular $(L)$ is always satisfied if (3.1) holds. The Lévi contition $(L)$ was introduced by Mizohata-Ohya et al in the study of the Cauchy problem; it implies that the transport equations are ordinary differential equations along the (null) bicharacteristics, whose orders are precisely the multiplicity of the characteristic which contains it. It can be shown that $(L)$ is necessary for the Cauchy problem for a hyperbolic differential operator with constant multiplicity, to be well posed in the $C^\infty$ setting.

**Theorem 3.2** (J. Chazarain). *Let $P$ be of constant multiplicity, $p$ real and assume that the Lévi condition (L) is verified. If $u \in \mathscr{D}'(X)$ and $Pu = f$ it follows that $WF(u) \setminus WF(f)$ is a subset of $p^{-1}(0)$ which is invariant under the bicharacteristic flow (it is understood that on $q_j^{-1}(0)$ one considers the flow defined by the Hamiltonian $H_{q_j}$ of $q_j$).*

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