Region of Motion for a Gravitational System with Negative Total Energy

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1. Introduction and Preliminary Remarks

In the following we obtain the regions of motion of n particles in a Newtonian gravitational system. The motivation for this arose from the author's attempt for a classification of motion of n particles. Previous classification by Saari [4] involves oscillatory and pulsating motions. The existence of the latter still remains an open question. So we wish to attempt to classify the motion by dividing the three dimensional space into regions of motion and forbidden regions of motion. In doing so, we have to consider the sign of the total energy of the system. Physically, the system with negative total energy being most interesting will be considered first. A topological approach may possibly be made to further classify the regions of motion. But, this the author has not been able to accomplish so far.

We assume, first that the center of mass of n particles with masses $m_1, m_2, \dots m_n$ is fixed at the origin with the rectangular coordinate system X, Y, Z centered at 0 and oriented in such a way that the Z-axis coincides with the direction of angular momentum vector C of the system. We shall restrict ourselves to the system for which $C \neq o$ and total energy $h < o$. In such a system it follows from the well-known theorem of Sundman [5] that a simultaneous collision of all particles is impossible. Also it is well-known [5] that a complete disintegration cannot take place since

$$\min_{i,j} r_{ij} = r \leq \delta = \frac{1}{|h|} \sum_{1 \leq i < j \leq n} m_i m_j \tag{1.1}$$

where r_{ij} is the distance between m_i and m_j.

Our results heavily depends on theorem 1.1 below which is analogous to the following well-known inequality of Sundman [5]

$$I\,T \geq \frac{C^2}{2} + \frac{\dot{I}^2}{8}, \tag{1.2}$$

Recebido em 5 de Maio de 1975.

where I, T, and C are respectively the moment of inertia, kinetic energy and absolute value of the angular momentum vector and all quantities are referred to the center of mass of the system.

Theorem 1.1. *Let x, y, z be any rectangular coordinate system with origin at the center of mass 0 of a system of n particles. Let I_z, T_z and C_z be the moment of inertia, kinetic energy and the angular momentum of the system referred to z-axis. Then the following inequality*

$$I_z T_z \geq \frac{1}{2} C_z^2 \tag{1.3}$$

holds.

Proof. Let (x_i, y_i, z_i) be the position of m_i, $i = 1, 2, \ldots n$. By definition, we have

$$I_z = \sum_{i=1}^{n} m_i(x_i^2 + y_i^2),$$

$$T_z = \frac{1}{2} \sum_{i=1}^{n} m_i(\dot{x}_i^2 + \dot{y}_i^2), \tag{1.4}$$

$$C_z = \sum_{i=1}^{n} m_i(x_i\dot{y}_i - y_i\dot{x}_i). \tag{1.5}$$

Then the proof follows using Schwarz's inequality*.

Remark. This is analogous to Sundman's inequality. (I, T and C are referred to the center of mass in the case of Sundman's inequality) and here the quantities I, T and C are referred to a particular direction being the z-axis in this case.

We shall first attempt to find the regions of motion for the case of general three-body problem. The arguments will be suitably modified for the case of four-body problem. For the case of $n(n > 4)$ bodies the approach becomes extremely complex and we are unable to make any progress. However, for the case of three and four bodies we have obtained some precise estimates. This hopefully will help us in our future attempt for a classification of motion for a system of four bodies.

I would like to thank Professor H. Pollard for his helpful suggestions during the preparation of my thesis. This paper forms a part of the thesis.

*This shorter proof was pointed out by Professor Pollard.

2. For the Case of Three Bodies

The motivation to find the regions of motion for the general three-body problem is due to the presente of a related problem in the case of restricted three-body problem. In the latter, the regions of motion for the body of zero mass relative to the two primaries have been established. This fact is well-known and can be found in the textbook of Pollard [3, pp. 58-60] or Moulton [1, pp. 281-286]. We will make a similar approach to begin with. The two primaries will be replaced by the nearer bodies and the body of zero mass will be replaced by the distant body. However, we would not restrict ourselves to a bounded system.

We will call the near pair to form the binary. Since each body is either the distant body, or a member of the binary at any time, the regions of motion in each case will be found and the required region of motion is the intersection of all these possible regions of motion.

Let (i, j, k) be an arbitrary but fixed permutation of $(1, 2, 3)$. Assume the body P_k to be the distant body and P_i and P_j form the binary. That is $r_{ik} \geq r_{ij}$, $r_{jk} \geq r_{ij}$. Recall that the existence of such a binary (not necessarily the same pair) follows from the fact that

$$\min_{i,j} r_{ij} = r \leq \delta,$$

where

$$\delta = \sum_{1 \leq j < k \leq 3} m_j m_k / |h|. \tag{1.6}$$

Since Z-axis coincides with the direction of the angular momentum vector C, XY-plane is the invariable plane. This being a fixed plane of the system, let us make all measurements relative to this plane. Let $\mathscr{I}_k$ be the inclination of the line OP_k with the invariable plane.

Now theorem 1.1 when applied to this three-body case and when quantities I, T and C are referred to the OP_k axis takes the following form:

Lemma 1.2. *For the three-body system*

$$I_k(U + h) \geq \frac{1}{2} C^2 \sin^2 \mathscr{I}^k, \tag{1.7}$$

holds.

The proof readily follows from the fact that $T \geq T_k$ and $T = U + h$, where $U = \sum_{1 \leq i < j \leq 3} \frac{m_i m_j}{r_{ij}}$ (self-potential of the system).

The following lemma relating to the properties of a triangle will be useful in subsequent discussion.

Lemma 1.3. *If r_i is the distance of P_i from the center of mass, and $r_{ij} = r$ is the shortest side, then r_{ik}/r_k and r_{jk}/r_k have the following estimates from above and below:*

$$\frac{1}{2} < r_{ik}/r_k < \frac{2\sqrt{2}\,M - m_j}{2m_i},$$

(1.8)

$$\frac{1}{2} < r_{jk}/r_k < \frac{2\sqrt{2}\,M - m_i}{2m_j},$$

where $M = m_i + m_j + m_k$.

The proof follows from Siegel and Moser [5].

The following theorem obtains a subset of the forbidden regions of motion for the distant body.

Theorem 1.4. *Let P_k be the distant body. If the total energy h of the system satisfies*

$$h < -\frac{m_i{}^2 m_j{}^2 (m_i + m_j)}{2C^2},$$

then there exists a forbidden region of motion for $r_k > D_k(\mathscr{I}_k)$ and $\mathscr{I}_k^* < |\mathscr{I}_k| \leq \pi/2$, where $\mathscr{I}_k^*$ is defined by

$$\mathscr{I}_k^* = \sin^{-1}\left\{\frac{m_i m_j}{C}\left(\frac{m_i + m_j}{2|h|}\right)^{1/2}\right\}$$

and $D_k(\mathscr{I}_k)$ defined by (1.14). $(o < \mathscr{I}_k^* < \pi/2)$

Proof. First let us compute the moment of inertia of this three-body system with respect to OP_k-axis. It is given by

$$I_k = m_i\, d_i{}^2 + m_j\, d_j{}^2,$$

(1.9)

$$< (m_i + m_j)\, r^2$$

where d_i, d_j are the distances of P_i and P_j from the line OP_k, and $r = r_{ij}$.

We next turn to estimate U from above.

$$U = \frac{m_i m_j}{r_{ij}} + \frac{m_i m_k}{r_{ik}} + \frac{m_j m_k}{r_{jk}},$$

$$< \frac{m_i m_j}{r} + \frac{2m_k(m_i + m_j)}{r_k}$$

by lemma 1.3. Therefore, by lemma 1.2 we have

$$(m_i + m_j)\, r^2 \left[\frac{m_i m_j}{r} + \frac{2m_k(m_i + m_j)}{r_k} + h\right] > \frac{1}{2} C^2 \sin^2 \mathscr{I}_k, \quad \text{or}$$

$$\left[\frac{2m_k(m_i + m_j)^2}{r_k} + h(m_i + m_j)\right] r^2 + m_i m_j (m_i + m_j)\, r > \frac{1}{2} C^2 \sin^2 \mathscr{I}_k, \quad \text{or}$$

$$(1.10) \quad \left[|h|(m_i + m_j) - \frac{2m_k(m_i + m_j)^2}{r_k}\right] r^2 - m_i m_j (m_i + m_j)\, r + \frac{1}{2} C^2 \sin^2 \mathscr{I}_k < o.$$

The coefficient of r^2 is positive for

(1.10 bis) $$r_k > 2m_k(m_i + m_j)^2 / |h|(m_i + m_j),$$

and with these r_k the corresponding equation in r will have two real unequal roots. Henceforth we assume that (1.10 bis) holds. Thus the discriminant is positive. Therefore

$$\frac{4m_k(m_i + m_j)^2}{r_k} + \frac{m_i^2 m_j^2 (m_i + m_j)^2}{C^2 \sin^2 \mathscr{I}_k} - 2\,|h|\,(m_i + m_j) > 0.$$

Set

$$f = f(r_k) = 4m_k(m_i + m_j)^2 / r_k,$$

(1.11)

$$g = g(\mathscr{I}_k) = -\frac{m_i^2 m_j^2 (m_i + m_j)^2}{C^2 \sin^2 \mathscr{I}_k} + 2\,|h|\,(m_i + m_j).$$

Thus we have

(1.12) $$f(r_k) > g(\mathscr{I}_k).$$

We will next analyze the inequality more closely, for this defines the regions of motion for P_k.

From (1.11) it is clear that f is a continuous positive monotonic decreasing function of r_k, and tends to zero as $r_k \to \infty$. And g is a continuous monotonic increasing function for $0 \leq |\mathscr{I}_k| \leq \pi/2$. Since

$$|h| > m_i^2 m_j^2 (m_i + m_j) / 2C^2,$$

considering the function $g(\mathscr{I}_k)$, it follows that there exists $\mathscr{I}_k^*, 0 < \mathscr{I}_k^* < \pi/2$ such that $g(\mathscr{I}_k^*) = 0$.

From the definition of g, we now obtain

$$(1.13) \quad \mathscr{I}_k^* = \sin^{-1}\left\{\frac{m_i m_j}{C}\left(\frac{m_i + m_j}{2\,|h|}\right)^{1/2}\right\}, \quad \text{and} \quad 0 < \mathscr{I}_k^* < \pi/2.$$

We now restrict ourselves to the case $\mathscr{J}_k^* < |\mathscr{J}_k| \le \pi/2$. We have $f(r_k) > g(\mathscr{J}_k) > 0$, and thus r_k must be bounded above by $D_k = D_k(\mathscr{J}_k) < \infty$, because from (1.12) it follows that

$$r_k < \frac{4m_k C^2(m_i + m_j)\sin^2 \mathscr{J}_k}{2|h| C^2 \sin^2 \mathscr{J}_k - m_i^2 m_j^2 (m_i + m_j)} = D_k(\mathscr{J}_k). \tag{1.14}$$

This completes the proof of theorem 1.4.

Remark 1*. It may be noted that $D_k(\mathscr{J}_k)$ is decreasing for $\mathscr{J}_k^* < \mathscr{J}_k \le \pi/2$, but $D_k(\mathscr{J}_k) \longrightarrow \infty$ as $\mathscr{J}_k \longrightarrow \mathscr{J}_k^*$. Now $r_k < D_k(\mathscr{J}_k)$ $(\mathscr{J}_k^* < |\mathscr{J}_k| \le \pi/2)$ implies that the boundary for the region of motion is the curve in figure 1. This curve is to be rotated about the vertical axis (which is determined by vector C). The region is symmetrical with respect to the invariable plane, so we have drawn only the upper half. The shaded area is the region where motion for the distant body P_k cannot occur.

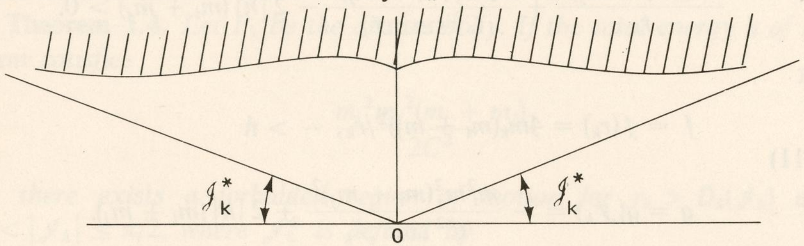

Fig. 1

The following theorem obtains the forbidden regions of motion for a member of the binary.

Theorem 1.5. *Let P_k be a member of the binary. If the total energy h of the system satisfies*

$$h < \min\left[-\frac{m_j^2 m_k^2 (m_j + m_k)}{2C^2}, -\frac{m_i^2 m_k^2 (m_i + m_k)}{2C^2}\right],$$

then there exists a forbidden region of motion.

Proof. Here P_k forms the binary with either P_j or P_i, and P_i or P_j becoming the distant body: If $0'$ be the center of mass of the binary we have

*This remark and the reference below have been pointed out by the referee to the author. The following paper describes a shape of the forbidden region similar our.

Saari, D: Restrictions on the Motion of the Three-Body Problem SIAM J. Appl. Math, Vol 26, N.º 4, 1974.

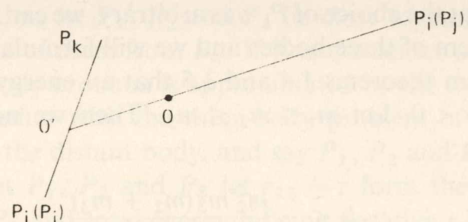

Fig. 2

$$O'P_k = \frac{m_j r}{m_j + m_k} \left(\text{or } \frac{m_i r}{m_i + m_k}\right),$$

$$\le \delta \max\left\{\frac{m_j}{m_j + m_k}, \frac{m_i}{m_i + m_k}\right\}. \tag{1.15}$$

Now by theorem 1.4 since by hypothesis the total energy is in the desired range, treating P_i (or P_j) as the distant body, we obtain

$$00' = \frac{m_i r_i}{m_j + m_k} \left(\text{or } \frac{m_j r_j}{m_k + m_i}\right),$$

$$\le \frac{m_i D_i(\mathscr{J}_i)}{m_j + m_k} \left(\text{or } \frac{m_j D_j(\mathscr{J}_j)}{m_k + m_i}\right),$$

$$\le \max\left\{\frac{m_i D_i(\mathscr{J}_i)}{m_j + m_k}, \frac{m_j D_j(\mathscr{J}_j)}{m_k + m_i}\right\}, \tag{1.16}$$

for $\max(\mathscr{J}_i^*, \mathscr{J}_j^*) < |\mathscr{J}_i|, |\mathscr{J}_j| \le \pi/2$. Set

(1.17) $D_k^{(b)}$ = sum of right-sides of (1.15) and (1.16),

$$= \delta \max\left\{\frac{m_j}{m_j + m_k}, \frac{m_i}{m_i + m_k}\right\} + \max\left\{\frac{m_i D_i(\mathscr{J}_i)}{m_j + m_k}, \frac{m_j D_j(\mathscr{J}_j)}{m_k + m_i}\right\}.$$

Since in this arrangement P_k cannot exceed the limiting position of $0'$ by more than $\delta \max\{m_j/(m_j + m_k), m_i/(m_i + m_k)\}$ in the same direction, it follows from (1.17) that

$$r_k \le D_k^{(b)}, \tag{1.18}$$

whenever $\max(\mathscr{J}_i^*, \mathscr{J}_j^*) < |\mathscr{J}_i|, |\mathscr{J}_j| \le \pi/2$.

This completes the proof of theorem 1.5.

Thus for a given position of P_k, that is for a given value of $|\mathscr{J}_k|$ by choosing the maximum of (1.18) and (1.14) we obtain a forbidden region of motion for the body P_k. It certainly exists for $\mathscr{J}^* < |\mathscr{J}_k| \le \pi/2$, where

$$\mathscr{J}^* = \max(\mathscr{J}_i^*, \mathscr{J}_j^*, \mathscr{J}_k^*) = \max(\mathscr{J}_1^*, \mathscr{J}_2^*, \mathscr{J}_3^*).$$

Now, recalling that the choice of P_k was arbitrary, we can find the forbidden region of the system of three bodies and we will formulate this in theorem 1.6. It is clear from theorems 1.4 and 1.5 that an energy restriction is necessary besides $h < 0$. Let $m_1 \leq m_2 \leq m_3$. Then we need the following energy restriction:

$$h < -\frac{m_2^2 m_3^2 (m_2 + m_3)}{2C^2} \tag{1.19}$$

Finally setting

$$D(\mathscr{I}) = \frac{4m_3 C^2 (m_2 + m_3) \sin^2 \mathscr{I}}{2|h| C^2 \sin^2 \mathscr{I} - m_2^2 m_3^2 (m_2 + m_3)} \cdot \max\left\{1, \frac{m_3}{m_1 + m_2}\right\} + \frac{\delta m_3}{m_1 + m_2} \tag{1.19a}$$

$$(\mathscr{I}^* < |\mathscr{I}| \leq \pi/2)$$

we have $D(\mathscr{I}) \geq \max\{\max_{1 \leq k \leq 3} \{D_k(\mathscr{I}_k)\}, \max_{1 \leq k \leq 3} \{D_k^{(b)}\}\}$.

Clearly $\mathscr{I}^*$ as defined above must be

$$\mathscr{I}^* = \sin^{-1}\left\{\frac{m_2 m_3}{C}\left(\frac{m_2 + m_3}{2|h|}\right)^{1/2}\right\}. \tag{1.20}$$

It may be noted that throughout this section δ is given by (1.6). Thus finally we formulate the following theorem.

Theorem 1.6. *In the Newtonian gravitational system of three bodies with masses m_1, m_2, m_3 $(m_1 \leq m_2 \leq m_3)$, there always exists a forbidden region of motion which is interior to the cone given by $\mathscr{I}^* < |\mathscr{I}| \leq \pi/2$ and at a distance $D(\mathscr{I})$ defined in (1.19a) from the center of mass (origin), provided the total energy of the system satisfies the relation (1.19).*

3. For the Case of Four Bodies

As pointed out earlier, we will next consider the regions of motion for the case of four bodies. As before the center of mass of the four bodies P_1, P_2, P_3, P_4 with masses m_1, m_2, m_3, m_4 will be taken as the origin and with rectancular coordinate system X, Y, Z oriented in such a way that Z-axis coincides with the direction of the angular momentum vector C. Again, we assume $C \neq o$, and $h < 0$. Thus we do not have a simultaneous collision, or complete disintegration of all bodies.

The inclination of OP_i will be referred to the invariable plane. We will consider in the following only non-planar motions.

At the given time let P_1, P_2 and P_3 lie close together relative to the distance of any one of them from P_4. If $0'$ is the center of mass of P_1, P_2 and P_3, the four-body problem is approximated by the two-body problem for $0'$ and P_4 together with the three-body problem of P_1, P_2 and P_3. We will call P_4 as the distant body, and say P_1, P_2 and P_3 form a triplet.

For the triplet P_1, P_2 and P_3 let $r_{23} = r$ form the shortest side of the triangle $P_1P_2P_3$ and total energy h being negative r cannot exceed δ, given by

$$\delta = \sum_{1 \leq j < k \leq 4} m_j m_k / |h|. \tag{1.21}$$

Throughout this section the quantity δ will have this value. Recall $0'$ is the center of mass of the triplet. Let $P_1 0'$, $P_2 0'$ and $P_3 0'$ be produced to meet the opposite sides of the triangle at Q_1, Q_2, Q_3 respectively. Let $\rho = P_1 Q_1$.

With a view to apply theorem 1.1 we now compute the moment of inertia of the system of four bodies about OP_4. Denoting this by I_4, we get $I_4 = m_1 d_1{}^2 + m_2 d_2{}^2 + m_3 d_3{}^2$, where d_i is the distance of P_i from the straight line OP_4. Let $0'P_i = \rho_i$ $(i = 1, 2, 3, 4)$. Then, we have

$$\begin{aligned} I_4 &\leq m_1 \rho_1{}^2 + m_2 \rho_2{}^2 + m_3 \rho_3{}^2 \\ &< M' \rho^2, \end{aligned} \tag{1.22}$$

where $M' = m_1 + m_2 + m_3$.

Next we obtain an estimate of U as follows:

Lemma 1.7. *For the system of four bodies with $r \geq \alpha > 0$, the self-potential U satisfies*

$$U < \frac{2m_1(m_2 + m_3)}{\rho} + \frac{m_2 m_3}{\alpha} + \frac{2m_4 M'}{r_4}. \tag{1.23}$$

Proof. Suppose $U = U_1 + U_2$,

where $U_1 = \frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}}$ and $U_2 = \frac{m_1 m_4}{r_{14}} + \frac{m_2 m_4}{r_{24}} + \frac{m_3 m_4}{r_{34}}$.

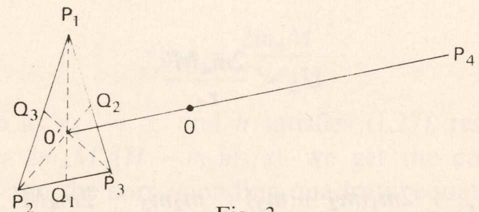

Fig. 3

Now applying triangle inequality to the triangles $P_1P_2Q_1$ and $P_1P_3Q_1$, it follows that

$$U_1 \leq \frac{m_1m_2}{\rho - \dfrac{m_3r}{m_2+m_3}} + \frac{m_1m_3}{\rho - \dfrac{m_2r}{m_2+m_3}} + \frac{m_2m_3}{r},$$

$$= \frac{1}{\rho}\left[\frac{m_1m_2(m_2+m_3)}{(m_2+m_3) - m_3\dfrac{r}{\rho}} + \frac{m_1m_2(m_2+m_3)}{(m_2+m_3) - m_2\dfrac{r}{\rho}}\right] + \frac{m_2m_3}{r}$$

$$< \frac{1}{\rho}[2m_1(m_2+m_3)] + \frac{m_2m_3}{r},$$

since $r < \rho$.
As $r \geq \alpha > o$, it further reduces to

$$U_1 < \frac{2m_1(m_2+m_3)}{\rho} + \frac{m_2m_3}{\alpha}. \tag{1.24}$$

At this point, it may be noted that $C \neq o$ does not rule out the possibility of binary, or triple collisions in this system of four bodies. Since only binary collision singularities are removable and triple collisions are not, it becomes necessary to make an assumption such as $r \geq \alpha > o$. However, we did not need such an assumption in the previous section because we considered system of three bodies.

Now for an estimate of U_2 we need to consider some properties of the triangles $P_4P_1Q_1$, $P_4P_2Q_2$ and $P_4P_3Q_3$. First for the triangle $P_4P_1Q_1$, recalling our arrangements of the bodies P_1, P_2, P_3 and P_4 we have P_1Q_1 is the shortest side.

Now applying lemma 1.3, we obtain $r_{14}/r_4 > 1/2$. Making use of similar inequalities for the remaining triangles, we obtain an estimate of U_2 as

$$U_2 < \frac{2m_1m_4}{r_4} + \frac{2m_2m_4}{r_4} + \frac{2m_3m_4}{r_4},$$

or

$$U_2 < \frac{2m_4M'}{r_4}. \tag{1.25}$$

Thus we have

$$U < \frac{2m_1(m_2+m_3)}{\rho} + \frac{m_2m_3}{\alpha} + \frac{2m_4M'}{r_4}. \tag{1.26}$$

This completes the proof of lemma 1.7.

In the following theorem we obtain explicitly a subset of the forbidden region of motion for the distant body P_4.

Theorem 1.8. *Let P_4 be the distant body. If $r \geq \alpha > o$ and the total energy $h(< o)$ satisfies*

$$h < -\left[\frac{2m_1^2M'(m_2+m_3)^2}{C^2} + \frac{m_2m_3}{\alpha}\right], \tag{1.27}$$

then there exists a forbidden region of motion for $r_4 > D_4(\mathscr{I}_4)$ and $\mathscr{I}_4^* < |\mathscr{I}_4| \leq \pi/2$, where $D_4(\mathscr{I}_4)$ is defined by (1.36) and $\mathscr{I}_4^*$ by

$$\mathscr{I}_4^* = \sin^{-1}\left[\frac{2m_1^2(m_2+m_3)^2M'}{C^2\left(-h-\dfrac{m_2m_3}{\alpha}\right)}\right]^{1/2}. \qquad (o < \mathscr{I}_4^* < \pi/2) \tag{1.28}$$

Proof. From theorem 1.1 referring I and C with respect to OP_4, and noting that $T_4 \leq T = U + h$, we get

$$I_4(U+h) \geq \frac{1}{2}C^2\sin^2\mathscr{I}_4.$$

From lemma 1.7 and (1.22), this becomes

$$M'\rho^2\left[\frac{2m_1(m_2+m_3)}{\rho} + \frac{m_2m_3}{\alpha} + \frac{2m_4M'}{r_4} + h\right] > \frac{1}{2}C^2\sin^2\mathscr{I}_4.$$

Arranging in powers of ρ, we get the following quadratic expression in ρ:

$$\left(M'\left(-h - \frac{m_2m_3}{\alpha}\right) - \frac{2m_4M'^2}{r_4}\right)\rho^2 - 2m_1(m_2+m_3)M'\rho + \frac{1}{2}C^2\sin^2\mathscr{I}_4 < o \tag{1.29}$$

Set

$$H = -h,$$

$$f(r_4) = \frac{2m_4M'^2}{r_4}. \tag{1.30}$$

Since $f(r_4) \longrightarrow o$ as $r_4 \longrightarrow \infty$ and h satisfies (1.27), restricting ourselves to the case $r_4 > 2m_4M'/(H - m_2m_3/\alpha)$, we get the coefficient of ρ^2 in (1.29) is positive and the corresponding quadratic equation in ρ will have two real unequal roots. Thus the discriminant is positive. Therefore

$$4m_1^2(m_2+m_3)^2 M'^2 > 4\left(M'H - \frac{M'm_2m_3}{\alpha} - f(r_4)\right)\frac{1}{2}C^2\sin^2\mathcal{J}_4,$$

or

(1.31) $$f(r_4) > M'\left(H - \frac{m_2m_3}{\alpha}\right) - \frac{2m_1^2(m_2+m_3)^2M'^2}{C^2\sin^2\mathcal{J}_4}.$$

Set

(1.32) $$g(\mathcal{J}_4) = M'\left(H - \frac{m_2m_3}{\alpha}\right) - \frac{2m_1^2(m_2+m_3)^2M'^2}{C^2\sin^2\mathcal{J}_4}.$$

Then (1.31) becomes

(1.33) $$f(r_4) > g(\mathcal{J}_4),$$

where $g(\mathcal{J}_4)$ is a continuous monotonic increasing function with $g(o) = -\infty$, and $g(\pi/2) > o$, since $H > 2m_1^2M'(m_2+m_3)^2/C^2 + m_2m_3/\alpha$. Thus there exists $\mathcal{J}_4^*$, $o < \mathcal{J}_4^* < \pi/2$ such that $g(\mathcal{J}_4^*) = o$. This defines $\mathcal{J}_4^*$ as $(o < \mathcal{J}_4^* < \pi/2)$, $\sin^2\mathcal{J}_4^* = 2m_1^2(m_2+m_3)^2M'/C^2(H - m_2m_3/\alpha)$ or

(1.34) $$\mathcal{J}_k^* = \sin^{-1}\left[\frac{2m_1^2(m_2+m_3)^2M'}{C^2\left(H - \frac{m_2m_3}{\alpha}\right)}\right]^{1/2}.$$

For the case $\mathcal{J}_k^* < |\mathcal{J}_4| \le \pi/2$, we have $f(r_4) > g(\mathcal{J}_4) > o$, and then r_4 must be bounded above by $D_4 = D_4(\mathcal{J}_4) < \infty$, which is explicitly obtained as follows.

From (1.30) and (1.31), we get

$$\frac{r_4}{2m_4M'^2} < \frac{1}{M'\left(H - \frac{m_2m_3}{\alpha}\right) - \frac{2m_1^2(m_2+m_3)^2M'^2}{C^2\sin^2\mathcal{J}_4}},$$

or

(1.35) $$r_4 < D_4(\mathcal{J}_4),$$

where

(1.36) $$D_4(\mathcal{J}_4) = \frac{2m_4M'^2}{M'\left(H - \frac{m_2m_3}{\alpha}\right) - \frac{2m_1^2(m_2+m_3)^2}{C^2\sin^2\mathcal{J}_4}M'^2}.$$

$$(\mathcal{J}_4^* < |\mathcal{J}_4| \le \pi/2)$$

It may be noted here that

$$D_4(\mathcal{J}_4) > 2m_4M'/(H - m_2m_3/\alpha).$$

This completes the proof of theorem 1.8.

If P_i is a member of the triplet, an analysis similar to that in theorem 1.5 for the three-body fails because bounds on the triplet are not known.

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