Homology Theories and Kan Extensions*

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A memória de Carlos B. de Lyra

1. Introduction. We say that a full subcategory $\mathscr{C}$ of Top_* – the category of topological spaces with non-degenerate base point – is *admissible* if it is non-empty, closed under the formation of mapping cones and, together with any one of its objects, say, X, it contains all objects of Top_* of the same based homotopy type as X. Notice that if $\mathscr{C}$ is admissible, it contains the cone of each one of its objects; these being contractible, $\mathscr{C}$ contains points. Moreover, $\mathscr{C}$ is closed under suspension; let S be the suspension endofunctor.

A *homology theory* on $\mathscr{C}$ is a covariant functor from the homotopy category $\tilde{\mathscr{C}}$ associated with $\mathscr{C}$ into the category $Ab^{\mathbb{Z}}$ of graded abelian groups, satisfying the conditions:

(1) (*Suspension Axiom*) – if ρ is the autofunctor of $Ab^{\mathbb{Z}}$ which shifts index by -1, $\rho h \simeq hS$;

(2) (*Exactness Axiom*) – h takes weak cokernel sequences of $\tilde{\mathscr{C}}$ into exact sequences of $Ab^{\mathbb{Z}}$.

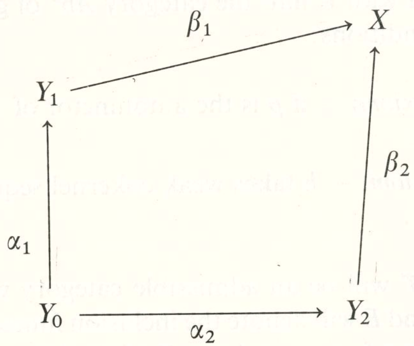
In what follows $\mathscr{C}'$ will be an admissible category which contains $\mathscr{C}$ as a full subcategory and E will denote the inclusion functor $\tilde{\mathscr{C}} \subset \tilde{\mathscr{C}}'$; moreover, we shall assume that for every object X of $\mathscr{C}'$, the comma-category $(E \downarrow X)$ (i.e., the category of $\tilde{\mathscr{C}}$ – objects over X) is small. We then recall from [5] that the (left) Kan extension of the stable homology theory associated to h preceded by stabilization gives rise to a homology theory h' on $\mathscr{C}'$ which extends h. Thus, every homology on $\mathscr{C}$ can be extended to a homology theory on $\mathscr{C}'$ *without requiring the pair* $(\mathscr{C}', \mathscr{C})$ *to satisfy any property other than the conditions for admissibility* (the smallness of the categories $(E \downarrow X)$ is actually only required to avoid foundational problems). We arrived to this conclusion via the following arguments. If $\mathscr{T}_0 \subset \mathscr{T}_1$ are

*Recebido pela SBM em 26 de março de 1975.
[1]Work partially supported by NRC Grant A-8777.

triangulated categories such that $\mathscr{T}_0$ has weak local pushouts rel $\mathscr{T}_1$ (see below) and h is a homology theory on $\mathscr{T}_0$ then a Kan extension of h to $\mathscr{T}_1$ is also homology [4, Theo. 3.1] and [6]; for the stable case, we note that the stable category $St\mathscr{C}$ has weak pushouts (and hence, weak local pushouts) rel $St\ Top_*$, because $\mathscr{C}$ is admissible [7, Theo. III. 1.7].

Observe that at this stage we do not know if the extended homology h' is a Kan extension of h; nor do we know if a Kan extension $'h$ of h to $\tilde{\mathscr{C}}'$ is a homology theory. It is then natural to ask: (i) when is h' a Kan extension of h? (ii) when is a Kan extension $'h$ of h a homology theory? Actually, the two questions are related: $'h \sim h'$ if, and only if, "Kan extension" and "stabilization" commute (2.3). Hence, in order to answer the two questions raised, one can search for conditions which insure the commutativity of Kan extension with stabilization. Not entirely surprising, one such condition is to require $\tilde{\mathscr{C}}$ to have *weak local pushouts* rel $\tilde{\mathscr{C}}'$ that is to say: given any commutative diagram

(1.1)



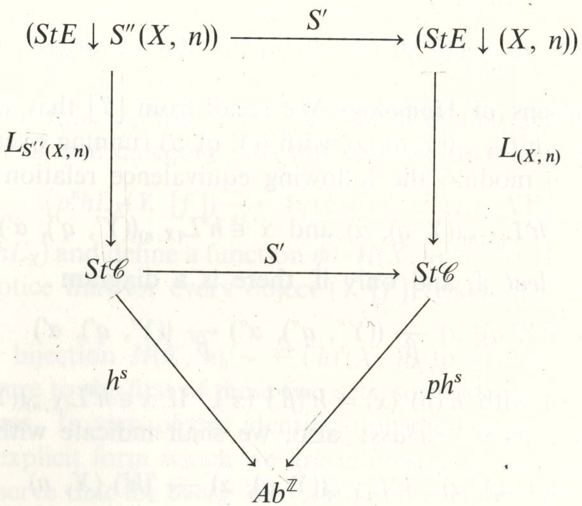
of $\tilde{\mathscr{C}}'$, with $Y_i \in |\mathscr{C}|$ $(i = 0, 1, 2)$ and $X \in |\mathscr{C}'|$, there is an object $W \in |\mathscr{C}|$ and morphisms γ_j: $Y_j \longrightarrow W (j = 1, 2)$ and δ: $W \longrightarrow X$ such that $\gamma_1\alpha_1 = \gamma_2\alpha_2$, $\delta\gamma_j = \beta_j$ $(j = 1, 2)$. We then deduce the following.

(1.2) THEOREM. *Let C and C′ be admissible categories such that C is a full subcategory of C′ and $\tilde{C}$ has weak local pushouts rel $\tilde{C}'$. Let h be a homology theory on C and let ′h be a Kan extension of h to C′. If ′h satisfies the Suspension Axiom, ′h is a homology theory; moreover, ′h is naturally equivalent to ′(h^s) preceded by stabilization.*

2. Kan extension of stable homology. Let StE be the imbedding of $St\mathscr{C}$ into $St\mathscr{C}'$ and, for every $(X, n) \in |St\mathscr{C}'|$, let $L_{(X,n)}$: $(StE \downarrow (X, n)) \longrightarrow St\mathscr{C}$ be the obvious projection. The Kan extension of the stable homology h^s defined by h is given by $'(h^s)(X, n) = colim\ h^s L_{(X,n)}$ on every object (X, n) of $St\mathscr{C}'$.

(2.1) PROPOSITION. *The functor* $'(h^s)$ *satisfies the suspension Axiom.*

PROOF. Recall that the suspension functor S of $St\mathscr{C}'$ (or $St\mathscr{C}$) is naturally equivalent to the autofunctor S'' which takes an object (X, n) into $(X, n+1)$ and maintains morphisms; let S' be the inverse of S''. Construct the commutative (up to natural equivalence) diagram of categories and functors



One then shows that S' induces a natural equivalence

$$\Phi_{S'}: \ colim\ \rho h^s L_{(X,n)} S' \longrightarrow colim\ \rho h^s L_{(X,n)},$$

thereby concluding the proof.

We observe incidentially that this is a much simpler proof of the Suspension Axiom for $'(h^s)$ than the one presented in (III. 3.1) of [7].

Let θ: $\tilde{\mathscr{C}}' \longrightarrow St\tilde{\mathscr{C}}'$ be the functor which takes an object X of $\tilde{\mathscr{C}}'$ into $(X, 0) \in |St\tilde{\mathscr{C}}'|$.

(2.2) COROLLARY. *The functor* $(h^s)\theta : \tilde{\mathscr{C}}' \longrightarrow Ab^{\mathbb{Z}}$ *extends* h *and is a homology theory.*

PROOF. The Suspension Axiom follows trivially from (2.1); as for the Exactness Axiom, see (III. 3.1) and (III. 3.11) of [7].

We close this section with the following simple result.

(2.3) LEMMA. *Let* $'h$ *be a Kan extension of* h *to* $\tilde{\mathscr{C}}'$. *Then if* $'h$ *satisfies the Suspension Axiom (so* $('h)^s$ *can be defined as a functor),* $'(h^s)\,\theta \simeq {}'h$ *if and only if,* $'(h^s) \simeq ('h)^s$.

PROOF. The condition is sufficient because $('h)^s\theta = {}'h$. Conversely, for every $(X, n) \in |St\tilde{\mathscr{C}}'|$,

$$('h)^s\,(X,\ n) \cong ('(h^s)\theta)^s\,(X,\ n) = \rho^n('h)^s\theta(X) \cong {}'(h^s)\,(X,\ n).$$

3. Kan Extensions of Homology. We recall from [7] that $'(h^s)\,(X, n)$ is isomorphic to $\cup h^s L_{(X,n)}((Y, q), \alpha)$, with $((Y, q), \alpha)$ running over all objects of $(StE \downarrow (X, n))$ modulo the following equivalence relation:

$$x \in h^s L_{(X,n)}((Y,\ q),\ \alpha) \text{ and } x' \in h^s L_{(X,n)}((Y',\ q'),\ \alpha')$$

are $\equiv$ – *equivalent* if, and only if, there is a diagram

$$((Y,\ q),\ \alpha) \xrightarrow[\beta]{} ((Y'',\ q''),\ \alpha'') \xleftarrow[\beta']{} ((Y',\ q'),\ \alpha')$$

in $(StE \downarrow (X, n))$ with $h^s(\beta)\,(x) = h^s(\beta')\,(x')$. If $x \in h^s L_{(X,n)}((Y, q), \alpha)$, we write $[x, \alpha]$ for its $\equiv$ – *class*; also, we shall indicate with

$$k((Y,\ q),\ \alpha){:}\ \ h^s L_{(X,n)}((Y,\ q),\ \alpha) \longrightarrow {}'(h^s)\,(X,\ n)$$

the quotient map which takes x into $[x, \alpha]$. Finally, if ξ: $(X, n) \longrightarrow (X', n')$ is a morphism of $St\tilde{\mathscr{C}}'$, $'(h^s)\,(\xi)\,[x, \alpha] = [x, \xi\alpha]$, for every $[x, \alpha] \in {}'(h^s)\,(X, n)$.

Define the following relation in the set $H(X, n) = \cup \rho^n hL_X(Y, [f])$ where $(Y, [f])$ runs over all objects of $(E \downarrow X)$: $x \in \rho^n hL_X(Y, [f])$ and $x' \in \rho^n hL_X(Y', [f'])$ are $\sim$ – related if, and only if, there is a diagram

$$(Y,\ [f]) \xrightarrow[{[g]}]{} (Y'',\ [f'']) \xleftarrow[{[g']}]{} (Y',\ [f'])$$

in $(E \downarrow X)$ with $h[g]\,(x) = h[g']\,(x')$. This $\sim$ – *relation* is reflexive and symmetric; in order to make it transitive we shall assume that $\mathscr{C}$ *has weak local pushouts rel* $\tilde{\mathscr{C}}'$. (This condition on the pair $(\tilde{\mathscr{C}}', \mathscr{C})$ will be retained until the end of the paper). For any $(Y, [f]) \in |(E \downarrow X)|$, let $k(Y, [f])$: $\rho^n hL_X(Y, [f]) \longrightarrow H(X, n)/\sim$ be the quotient map; it takes x into $\{x, [f]\}$.

(3.1) LEMMA. *There is a bijection between the sets* $H(X, n)/\sim$ *and* $('h)^s(X, n)$.

PROOF. We follow the notation of [7]. Recall that $('h)^s(X, n) = colim\, \rho^n hL_X$ is given by an initial object

$$\{\rho^n hL_X(Y,\ [f]) \longrightarrow ('h)^s(X,\ n)\,|\,(Y,\ [f]) \in |(E \downarrow X)|\}$$

of $I(Ab^{\mathbb{Z}}, \rho^n hL_X)$; for the moment, let us regard it as an initial object of $I(Set^{\mathbb{Z}}, \rho^n hL_X)$. It is trivial to verify that

$$\{\rho^n hL_X(Y,\ [f]) \xrightarrow[k(Y,\ [f])]{} H(X,\ n)/\sim\ |\,(Y,\ [f]) \in |(E \downarrow X)|\}$$

is an object of $I(Set^{\mathbb{Z}}, \rho^n hL_X)$; we want to show that actually, this is an initial object of that category. To this end, we take an arbitrary object

$$\{\rho^n hL_X(Y,\ [f]) \xrightarrow[g_Y]{} A\,|\,(Y,\ [f]) \in |(E \downarrow X)|\}$$

of $I(Set^{\mathbb{Z}}, \rho^n hL_X)$ and define a function ϕ: $H(X, n)/\sim\ \longrightarrow A$ by $\phi\{x, [f]\} =$ $= g_Y(x)$. Notice that for every object $(Y, [f])$, $\phi k(Y, [f]) = g_Y$.

We use the bijection $H(X, n)/\sim\ \rightleftarrows ('h)^s(X, n)$ to give a graded abelian group structure to the first of these two sets, so to identify them as graded abelian groups. In view of this identification the homomorphisms $('h)^s(\xi)$ assume an explicit form which we are interested in knowing. To begin with, we observe that for every $A \in |Ab^{\mathbb{Z}}|$ and every integer j, there is an isomorphism $\psi^j(A)$: $A \longrightarrow \rho^j(A)$ which takes any element $a \in A_i$ into itself, but as a homogeneous element of degree $i + j$ of $\rho^j(A)$; we shall denote $\psi^j(A)$ simply by ρ^j. Now suppose that ξ is represented by a homotopy class $[g]$: $S^{n+j}X \longrightarrow S^{n'+j}X'$, $n + j, n' + j \geq 0$; on the other hand,

$$\rho^j(x) \in \rho^{n+j} hL_X(Y,\ [f]) \cong hL_X(S^{n+j}Y,\ [S^{n+j}f]),$$

thus

$$\{\rho^j(x),\ [g]\,[S^{n+j}f]\} \in ('h)^s(S^{n'+j}X',\ 0) \cong \rho^j('h)^s(X',\ n').$$

We then have $('h)^s(\xi)\ \{x,\ [f]\} = \rho^{-j}\{\rho^j(x),\ [g \cdot S^{n+j}f]\}$.

Let us study next how the isomorphism ρ^j acts on the graded groups $'(h^s)(X, n)$ and $('h)^s(X, n)$.

(3.2) LEMMA. *Let $[x, \alpha] \in {}'(h^s)(X, n)$, with $\alpha\colon (Y, q) \longrightarrow (X, n)$, be given; then, for every $j \in \mathbb{Z}$, $\rho^j[x, \alpha] = [\rho^j(x), \alpha]$.*

PROOF. By looking at the homogeneous components, we infer that ρ^j commutes with the appropriate quotient maps; on the other hand, since

$$\rho^{j\prime}(h^s)(X, n) \cong {}'(h^s)(S'')^j(X, n) = {}'(h^s)(X, n + j)$$

and $(S'')^j(\alpha) = \alpha$, we can write $\rho^j k((Y, q), \alpha) = k((Y, q + j), \alpha)\rho^j$. Hence, $\rho^j[x, \alpha] = \rho^j k((Y, q), \alpha)(x) = [\rho^j(x), \alpha]$.

(3.3) LEMMA. *Given $\{x, [f]\} \in ('h)^s(X, n)$ and an integer j such that $n + j \geq 0$, $\rho^j\{x, [f]\} = \{\rho^j(x), [S^{n+j}f]\}$.*

PROOF. Again, ρ^j commutes with the appropriate quotient maps; then, because $\rho^{n+j}hL_X(Y, [f])$ is naturally isomorphic to $h\, L_{S^{n+j}X}(S^{n+j}Y, [S^{n+j}f])$ and since $('h)^s(X, n + j)$ is identified to $('h)^s(S^{n+j}X, 0)$ via the equivalence $(X, n+j) = (S'')^{n+j}(X, 0) \cong (S^{n+j}X, 0)$, it follows that $\rho^j k(Y, [f]) = k(S^{n+j}Y, [S^{n+j}f])\rho^j$. From this,

$$\rho^j\{x, [f]\} = \rho^j k(Y, [f])(x) = \{\rho^j(x), [S^{n+j}f]\}.$$

We are now ready to prove the main result of this section.

(3.4) THEOREM. *Let $\mathscr{C}$ and $\mathscr{C}'$ be admissible categories such that $\mathscr{C}$ is a full subcategory of $\mathscr{C}'$ and $\tilde{\mathscr{C}}$ has weak local pushouts rel $\tilde{\mathscr{C}}'$; let h be a homology theory on $\mathscr{C}$. Then, a (left) Kan extension of h to $\tilde{\mathscr{C}}'$, which satisfies the Suspension Axiom, commutes with stabilization.*

PROOF. For every $(X, n) \in |St\mathscr{C}'|$ define the functor

$$J(X, n)\colon (E \downarrow X) \longrightarrow (StE \downarrow (X, n))$$

which takes any object $Y \xrightarrow{[f]} X$ of $(E \downarrow X)$ into $(Y, n) \xrightarrow{\bar f} (X, n)$, where $\bar f = \underset{j \geq 0}{colim}\ [S^j f]$. Clearly $\rho^n hL_X = h^s L_{(X,n)} J(X, n)$; furthermore, the set

$$\{\rho^n hL_X(Y, [f]) = h^s L_{(X,n)}((Y, n), \bar f) \xrightarrow[k((Y,n),\bar f)]{} {}'(h^s)(X, n) \mid (Y, [f]) \in |(E \downarrow X)|\}$$

is an object of the category $I(Ab^{\mathbb{Z}}, \rho^n hL_X)$. Hence, there is a unique graded group homomorphism

$$\eta(X, n)\colon ('h)^s(X, n) \longrightarrow {}'(h^s)(X, n)$$

such that, for every $(Y, [f])$, $\eta(X, n)k(Y, [f]) = k((Y, n), \bar f)$. In particular, $\eta(X, n)\{x, [f]\} = [x, \bar f]$. It follows that if j is any integer such that $n + j \geq 0$, $\rho^j\eta(X, n) = \eta(S^{n+j}X, 0)\rho^j$. This remark plus (3.2) and (3.3) show that η is natural: given $\xi\colon (X, n) \longrightarrow (X', n')$ represented by

$$[g]\colon S^{n+j}X \longrightarrow S^{n'+j}X' \ (n + j,\ n' + j \geq 0),$$

$$\eta(X', n')('h)^s(\xi)\{x, [f]\} = \eta(X', n')$$

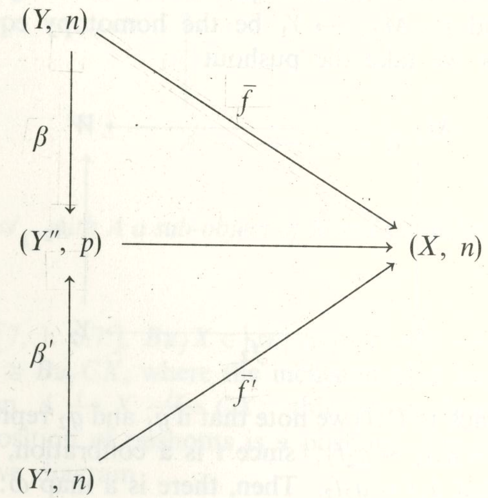
$$\rho^{-j}\{\rho^j(x), [g \,.\, S^{n+j}f]\} = [x, \xi \,.\, \bar f] = {}'(h^s)(\xi)\eta(X, n)\{x, [f]\}.$$

We show next that $\eta(X, n)$ is a bijection. If $[x, \alpha] \in {}'(h^s)(X, n)$ with $x \in h^s L_{(X,n)}((Y, q), \alpha)$ and α is represented by

$$[f]\colon S^{q+j}Y \longrightarrow S^{n+j}X,\ q + j \geq 0,\ n + j \geq 0,$$

$$\eta(X, n)\rho^{-j}\{\rho^j(x), [f]\} = \rho^{-j}\eta(S^{n+j}X, 0)\{\rho^j(x), [f]\} = [x, \alpha].$$

Suppose now that $\{x, [f]\}, \{x', [f']\} \in ('h)^s(X, n)$ with $x \in \rho^n hL_X(Y, [f])$, $x' \in \rho^n hL_X(Y', [f'])$ and $[x, \bar f] = [x', \bar f']$. This last condition implies the existence of a commutative diagram



in $(StE \downarrow (X, n))$ with $h^s(\beta)(x) = h^s(\beta')(x')$. Hence there is a suitable commutative diagram in $\tilde{\mathscr{C}}'$ and representatives $[g]$ and $[g']$ of β and β' respectively, so that $h[g]\rho^j(x) = h[g']\rho^j(x')$ and therefore,

$$\{\rho^j(x), [S^{n+j}f]\} = \{\rho^j(x'), [S^{n+j}f']\}. \text{ By (3.3), } \{x, [f]\} = \{x', [f']\}.$$

The proof of Theorem (1.2) is now very simple: just use (2.3) and the fact that $'(h^s)\theta$ is a homology theory.

We give next examples of pairs of admissible categories which satisfy the weak local pushout condition.

(3.5) EXAMPLE. Let $\mathscr{A}$ be one of the following categories: $\mathscr{CW}_* =$ based CW-complexes, $F\mathscr{CW}_* =$ finite based CW-complexes, $\mathscr{CW}_*(\chi_0) =$ = countable based CW-complexes. We then define $\mathscr{C}(\mathscr{A})$ to be the category of based spaces with the same homotopy type as objects of $\mathscr{A}$; $\mathscr{C}(\mathscr{A})$ is admissible [7, II. 1.2].

Let $\mathscr{C}'$ be any admissible category which contains $\mathscr{C}(\mathscr{A})$ as a full subcategory; we claim that $\tilde{\mathscr{C}}(\mathscr{A})$ has weak local pushouts rel $\tilde{\mathscr{C}}'$. First assume that the objects Y_0, Y_1 and Y_2 of diagram (1.1) actually are objects of $\mathscr{A}$; furthermore, we assume that the morphisms α_1 and α_2 of that diagram are represented by cellular maps f_1 and f_2. By [7, I. 5.7.] the mapping cylinder M_{f_1} can be taken as an object of $\mathscr{A}$; let i: $Y_0 \longrightarrow M_{f_1}$ be the inclusion map and r: $M_{f_1} \longrightarrow Y_1$ be the homotopy equivalence such that $ri = f_1$. Now we take the pushout

$$\begin{array}{ccc} M_{f_1} & \longrightarrow & W \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle h_2} \\ Y_0 & \xrightarrow{f_2} & Y_2 \end{array}$$

in $\mathscr{A}$. Referring back to (1.1) we note that if g_1 and g_2 represent β_1 and β_2 respectively, $g_1ri = g_1f_1 \sim g_2f_2$; since i is a cofibration, we deform g_1r to a map $\bar{g}_1$ so that $\bar{g}_1i = g_2f_2$. Then, there is a map ω: $W \longrightarrow X$ such

that $\omega h_1 = \bar{g}_1$, $\omega h_2 = g_2$. If r^{-1} is a homotopy inverse of r, $[\omega]$, $[h_1r^{-1}]$, $[h_2]$ and $W \in |\mathscr{C}(\mathscr{A})|$ complete (1.1).

(3.6) EXAMPLE. Let **C** be a Serre class of abelian groups and $\mathscr{A}$ be like in (3.5). We say that an object of $\mathscr{C}(\mathscr{A})$ belongs to **C** if its reduced integral homology belongs to **C**. We then define $\mathscr{C}(\mathscr{A})_{\mathbf{C}}$ to be the category of objects of $\mathscr{C}(\mathscr{A})$ which belong to **C**. If f: $X \longrightarrow Y$ is a morphism of $\mathscr{C}(\mathscr{A})_{\mathbf{C}}$, $C_f \in |\mathscr{C}(\mathscr{A})|$; moreover, the exactness of the sequences

$$\tilde{H}_n(Y; Z) \longrightarrow \tilde{H}_n(C_f; Z) \longrightarrow \tilde{H}_{n-1}(X; Z)$$

shows that C_f belongs to **C** and therefore $\mathscr{C}(\mathscr{A})_{\mathbf{C}}$ is admissible. Let $\mathscr{C}'$ be an admissible category of which $\mathscr{C}(\mathscr{A})_{\mathbf{C}}$ is a full subcategory.

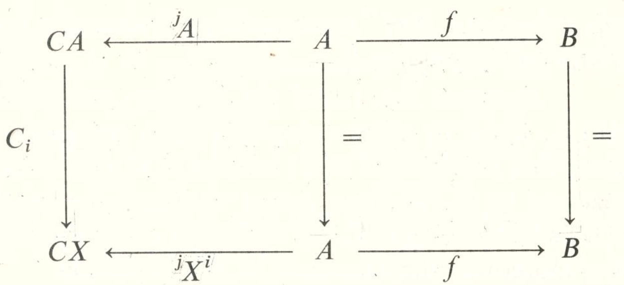
Given a diagram like (1.1), we construct $W \in |\mathscr{C}(\mathscr{A})|$ and morphisms to and from W which complete it; such a construction follows the lines of (3.5). The problem is to show that W belongs to **C**; for this, we prove the following.

(3.7) LEMMA. *Let* $\mathscr{A}_{\mathbf{C}}$ *be the category of objects of* $\mathscr{A}$ *which belong to* **C**. *If*

$$\begin{array}{ccc} X & & \\ {\scriptstyle i}\uparrow & & \\ A & \xrightarrow{f} & B \end{array}$$

is a diagram of $\mathscr{A}_{\mathbf{C}}$ *with* A *a sub-object of* X *and* f *cellular, there is a pushout of it in* $\mathscr{A}_{\mathbf{C}}$.

PROOF. By [7, I. 5.7.], $B\mathfrak{U}_fX \in |\mathscr{A}|$ and $\bar{f}$: $X \longrightarrow B\mathfrak{U}_fX$ is cellular. Moreover, $C_{\bar{f}} \cong B\mathfrak{U}_fCX$, where the inclusion of A into CX is given by the composition $A \xrightarrow{i} X \xrightarrow{jX} CX$; this comes from the elementary fact that composition of pushouts is a pushout. On the other hand, in the commutative diagram



the vertical arrows are homotopy-equivalences; hence $C_f \cong B \amalg_f CW$ [3, 7.5.7.]. Since $C_f \in |\mathscr{A}_{\mathbf{C}}|$, it follows that $C_f \in |\mathscr{A}_{\mathbf{C}}|$. The sequence $X \xrightarrow[\bar{f}]{} B \amalg_f X \longrightarrow C_{\bar{f}}$ gives rise to an exact sequence in homology and thus, the Lemma is proved.

Going back to the example, we see that W belongs to $\mathbf{C}$ by applying Lemma (3.7) to the diagram formed by the morphisms $f_2\colon Y_0 \longrightarrow Y_2$ and $i\colon Y_0 \longrightarrow M_{f_1}$.

4. Kan extensions (to *CW*-complexes) of homology theories defined on finite *CW*-complexes. We assume that all CW-complexes used in this section are connected and contained in a fixed Hilbert space $\mathbb{R}^\infty$; then, with the notation of (3.5), we take $\mathscr{C} = \mathscr{C}(F\mathscr{C}\mathscr{W}_*)$ and $\mathscr{C}' = \mathscr{C}(\mathscr{C}\mathscr{W}_*)$.

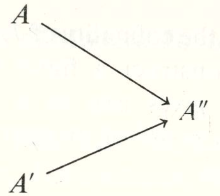
(4.1) THEOREM. *Let h be a homology theory on $\mathscr{C}$ and let h' be an extension of the functor h to $\tilde{\mathscr{C}}'$. The following are equivalent.*

1) *h' is naturally equivalent to a Kan extension of h;*

2) *for every $X \in |\mathscr{C}'|$, $h'(X) \cong \operatorname{colim} h(Y_\alpha)$, where Y_α runs over all finite sub-CW-complexes of a based CW-complex Y with same homotopy type as X;*

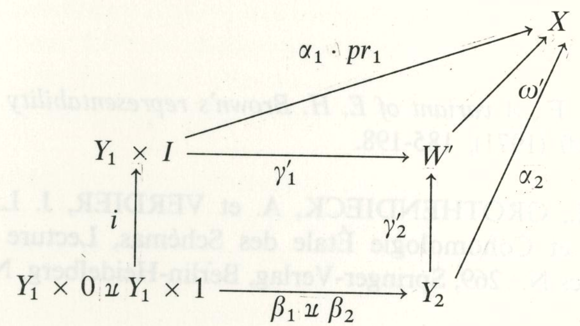
3) *h' is naturally equivalent to a homology theory on $\mathscr{C}'$ defined by a spectrum.*

We show first that 1) and 2) are equivalent. Let $\tilde{\mathscr{Y}}$ be the homotopy category defined by all finite sub-CW-complexes of Y, and let $J_Y\colon \tilde{\mathscr{Y}} \longrightarrow \mathscr{C}'$ be the inclusion. Form the comma-category $(J_Y \downarrow X)$ and let $J\colon (J_Y \downarrow X) \longrightarrow (E \downarrow X)$ be the obvious functor (notice that both comma-categories are small). We recall two definitions from [2, Exposé I]. A category $\mathscr{A}$ is said to be *filtering* if: $F1$) every pair (A, A') of objects of $\mathscr{A}$ can be embedded in a diagram



of $\mathscr{A}$; $F2$) if $A \rightrightarrows A'$ is a pair of morphisms of $\mathscr{A}$, there is a morphism $A' \longrightarrow A''$ such that the two composed morphisms $A \rightrightarrows A''$ are equal. Also, let $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ be a given functor with $\mathscr{A}$ filtering; we say that F is *cofinal* if: $C1$) for every $B \in |\mathscr{B}|$ there is an object $A \in |\mathscr{A}|$ and a morphism $B \longrightarrow F(A)$; $C2$) if $B \in |\mathscr{B}|$, $A \in |\mathscr{A}|$ and $B \rightrightarrows F(A)$ are two morphisms in $\mathscr{B}$, there is a morphism $A \longrightarrow A'$ in $\mathscr{A}$ such that the two composed morphisms $B \rightrightarrows F(A')$ are equal.

We are going to show that $(J_Y \downarrow X)$ is filtering and J is cofinal. Let $Y_1 \xrightarrow{\beta_1} X$ and $Y_2 \xrightarrow{\beta_2} X$ be given objects of $(J_Y \downarrow X)$; take Y_0 as the base point of Y_1, Y_2 and α_1, α_2 to be the homotopy classes of the inclusion maps of Y_0 into Y_1 and Y_2 respectively. As in (3.5) we obtain an object $W \longrightarrow X$ and morphisms $\gamma_i\colon Y_i \longrightarrow W (i = 1, 2,)$, showing $F1$). Let now β_1 and β_2 be morphisms of the object $Y_1 \xrightarrow{\alpha_1} X$ into the object $Y_2 \xrightarrow{\alpha_2} X$ of $(J_Y \downarrow X)$. Notice that $\alpha_2\beta_1 = \alpha_2\beta_2 = \alpha_1$ and hence, if $i\colon Y_1 \times 0 \amalg Y_1 \times 1 \longrightarrow Y_1 \times I$ is the homotopy class of the inclusion, $\alpha_2(\beta_1 \amalg \beta_2) = (\alpha_1 \cdot pr_1)i$. Since $\mathscr{C}$ has weak local pushouts rel $\tilde{\mathscr{C}}'$, there is $W \in |\mathscr{C}|$ and morphisms $\gamma_1\colon Y_1 \times I \longrightarrow W$, $\gamma_2\colon Y_2 \longrightarrow W$, $\omega\colon W \longrightarrow X$ such that $\gamma_1 i = \gamma_2(\beta_1 \amalg \beta_2)$, $\omega\gamma_1 = \alpha_1 \cdot pr_1$, $\omega\gamma_2 = \alpha_2$. Let us take W as a finite CW-complex, $\omega = [w]$, $r\colon X \longrightarrow Y$ a homotopy equivalence. Since $\operatorname{im}(rw)$ is a compact subspace of Y, there is $W' \in |F\mathscr{C}\mathscr{A}_*|$ such that $\operatorname{im}(rw) \subset W' \subset Y$. Thus, we can rearrange matters so to obtain a commutative diagram



With this we show $F2$). As for the cofinality of J, if $Z \xrightarrow{[a]} X \in |(E \downarrow X)|$ (we can assume $Z \in |F\mathscr{C}\mathscr{A}_*|$), construct a finite sub-CW-complex W of Y which contains im(ra); this gives rise to a morphism $(Z \xrightarrow{[a]} X) \longrightarrow (W \longrightarrow X)$, showing $C1$). The proof of property $C2$) follows the lines of the proof of $F2$) given before.

(4.2) LEMMA. *Let A, B be small categories, A filtering, $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ cofinal. Let $\mathscr{C}$ be cocomplete and let $G\colon \mathscr{B} \longrightarrow \mathscr{C}$ be a given functor. Then colim $G \cong$ colim GF.* (This is Proposition 8.1.3, Exposé I of [2]).

Lemma (4.2) completes the proof of the equivalence between 1) and 2): take

$$(J_Y \downarrow X) \xrightarrow{J} (E \downarrow X) \xrightarrow{hY_X} Ab^{\mathbb{Z}}$$

and note that *colim* hL_XJ = *colim* $h(Y_\alpha)$, Y_α running over all finite sub-CW-complexes of Y. Part 2) follows from 3) because of the definition of homology determined by a spectrum (for further details, see [7, II.2.12]).

Finally, 1) $\Rightarrow$ 3). Consider the relation in $\cup hL_X(Y, [f])$ – where $(Y, [f])$ runs over all objects of $(E \downarrow X)$ – similar to that defined in $H(X, n)$ (see (3.1)). Because $\tilde{\mathscr{C}}$ has weak local pushouts rel $\tilde{\mathscr{C}}'$, we have an equivalence relation and $'h(X) \cong \cup\, hL_X(Y, [f])/\sim$. On the other hand, h is equivalent to a homology theory $h(\ ;\mathscr{E})$ determined by a spectrum $\mathscr{E}$, as a homology theory defined on finite CW-complexes (see [1]). We then show that $'h \simeq h(\ ;E)$ by setting $\eta(X)\ \{u, [f]\} = h([f];\ E)u$. (see [7, III.4.4]).

Notice that in this geometric case $'h \simeq '(h^s)\theta$.

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